MA4257: Financial Mathematics II

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Syllabus

 Lecturer: Dr. Dai Min  
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Recommended background reading:  

Recommended texts: (Except for the last two, these books have been set as RBR books, available at Science Library)


• Shreve, S.E. (2004), Stochastic Calculus for Finance: The Binomial Asset Pricing Model (Vol I); Continuous-Time Models (Vol II), Springer Verlag, New York

• Oksendal, B. (2003), Stochastic Differential Equations, Springer (for basic theory of stochastic calculus).


We are now able to talk about “mathematical finance”, “financial engineering” or “modern finance” only because of two revolutions that have taken place on Wall Street in the latter half of the twentieth century. The first revolution in finance began with the 1952 publication of “Portfolio Selection”, an early version of the doctoral dissertation of Harry Markowitz, where he employed the so-called “mean-variance analysis” to understand and quantify the trade-off between risk and return inherent in a portfolio
of stocks. The implementation of Markowitz’s idea was aided tremendously by William Sharp who developed the Capital Asset Pricing Model. For their pioneering work, Markowitz and Sharp shared with Merton Millier the 1990 Nobel Prize in economics, the first ever awarded in finance. Markowitz and Sharp’s portfolio selection work is for one-period models. Thanks to Robert Merton and Paul Samuelson, one-period models were replaced by continuous-time (Brownian-motion-driven models), and the quadratic utility function implicit in mean-variance optimization was replaced by more general increasing, concave utility functions. Model-based mutual funds have taken a permanent seat at the table of investment opportunities offered to the public.

The second revolution in finance is regarding what we are going to address in this course, the option pricing theory, founded by Fisher Black, Myron Scholes, and Robert Merton in the early 1970s. This leads to an explosion in the market for derivatives securities. Scholes and Merton won the 1997 Nobel Prize in economics. Black had unfortunately died in 1995.

**Preliminary knowledge**: MA3245 (Financial Mathematics I) or at the least

- Basic concepts: derivatives, options, futures, forward contracts, hedging, Greeks and so on.
- Elementary stochastic calculus: Brownian motion, Ito integral and Ito lemma.
- Derivation of continuous time model (Black-Scholes) and discrete model (Cox-Ross-Rubinstein): delta hedging and no arbitrage.

Even if you know little about the preliminary knowledge, don’t worry too much because I am going to give a review in the first two classes.

**Course Grade**

final exam (80%), mid-term exam (15%), one assignment with tutorials (5%).

**Contents:**

- Preliminary
- Option pricing models for European options (arbitrage pricing can be moved between “beyond Black-Scholes world” and “interest rate
derivatives”; also, we don’t speak of the no-short selling;) One week for chapter 1 and 2.

• American options and early exercise; two weeks;

• Multi-asset options; one week

• Path-dependent options; three weeks;

• Beyond Black-Scholes world; two weeks

• Interest rate derivatives; two weeks

Our philosophy

• We value these derivative products using partial differential equations (PDEs), or equivalently, using binomial tree methods. Probabilistic approach will be discussed in MA4265 (Stochastic Analysis in Financial Mathematics).

• Explicit solutions to the PDE models, if available, will be given. But we don’t care how to get these solutions.

• Since explicit solutions are so rare that fast accurate numerical methods are essential. We shall mainly focus on the binomial tree method which can be regarded as a discrete model and is easy to implement. Other numerical methods are discussed in MA5247 (Computational Methods in Finance).

• Knowledge of Matlab coding is preferable, but not necessary.
Chapter 1

Preliminary

1.1 Basic Financial Derivatives: Forward contracts and Options

A derivative is a financial instrument whose value depends on the values of other, more basic underlying variables such as stocks, indices, interest rate and so on. Typical examples of derivatives include forward contracts, futures, options, swaps, interest rate derivatives and so on. Futures and standard options are traded actively on many exchanges. Forward contracts, swaps, many different types of options are regularly traded by financial institutions, fund managers, and corporations in the over-the-counter market (OTC market).

In this section we introduce two kinds of basic derivative products: forward contracts and options.

1.1.1 Forward Contracts

A forward contract is an agreement between two parties to buy or sell an asset at a certain future time (called the expiry date or maturity) for a certain price (called delivery price). It can be contrasted with a spot contract, which is an agreement to buy or sell an asset today.

One of the parties to the forward contract assumes a long position and agrees to buy the underlying asset at expiry for the delivery price. The other party assumes a short position and agrees to sell the asset at expiry for the delivery price. The payoff from a long position in a forward contract on one unit of an asset is

\[ S_T - K, \]
where $K$ is the delivery price and $S_T$ is the spot price of the asset at maturity of the contract. Similarly the payoff from a short position in a forward contract is $K - S_T$. Observe that the payoff is linear with $S_T$.

At the time the contract is entered into, it costs nothing to take either a long or a short position. This means that on the starting date the value of the forward contract to both sides is zero. A natural question:

\begin{equation}
\text{how to choose the delivery price such that the value of the forward contract is zero when opening the contract?} \quad (1.1)
\end{equation}

### 1.1.2 Options

The simplest financial option, a European call or put option, is a contract that gives its holder the right to buy or sell the underlying at a certain future time (expiry date) for a predetermined price (known as strike price). For the holder of the option, the contract is a right and not an obligation. The other party to the contract, who is known as the writer, does have a potential obligation.

The payoff of a European call option is

\[
(S_T - K)^+,
\]

where $K$ is the strike price and $S_T$ is the spot price of the asset at maturity of the option. Similarly, the payoff a European put option is $(K - S_T)^+$. Note that the payoff of an option is nonlinear with $S_T$.

Since the option confers on its holder a right without obligation it must have some value at the time of opening the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed. So, there is a question:

\begin{equation}
\text{how much would one pay to win the option?} \quad (1.2)
\end{equation}

### 1.2 No Arbitrage Principle

One of the fundamental concepts in derivatives pricing is the no-arbitrage principle, which can be loosely stated as ‘there is no such thing as a free lunch’. More formally, in financial term, there are never any opportunities to make an instantaneous risk-free profit. In fact, such opportunities may exist in a real market. But, they cannot last for a significant length of time before prices move to eliminate them because of the existence of arbitraguer in the market. Throughout this notes, we always admit the no-arbitrage
principle whose application will lead to some elegant modeling. In addition, the market is assumed to be frictionless, i.e., no transaction costs and no taxes.

We often make use of two conclusions below derived from the no-arbitrage principle:

1) Let $\Pi_1(t)$ and $\Pi_2(t)$ be the value of two portfolios at time $t$, respectively.

   If $\Pi_1(T) \leq \Pi_2(T)$ a.s., then $\Pi_1(t) \leq \Pi_2(t)$ for $t < T$.  \hspace{1cm} \text{(1.3)}

Especially,

   if $\Pi_1(T) = \Pi_2(T)$ a.s., then $\Pi_1(t) = \Pi_2(t)$ for $t < T$. \hspace{1cm} \text{(1.4)}

2) All risk-free portfolios must earn the same return, i.e. riskless interest rate. Suppose $\Pi$ is the value of a riskfree portfolio, and $d\Pi$ is its price increment during a small period of time $dt$. Then

   $$\frac{d\Pi}{\Pi} = rdt, \hspace{1cm} \text{(1.5)}$$

where $r$ is the riskless interest rate.

**Remark 1** When applying the no-arbitrage principle (for example, proving the above two conclusions), the assumption of short-selling is needed. Except for special claim, we suppose that short selling is allowed for any assets involved.

In what follows we attempt to derive the price of a forward contract by using the no-arbitrage principle.

### 1.2.1 Pricing Forward Contracts (on traded assets)

Consider a forward contract whose delivery price is $K$. Let $S_t$ and $V(S_t,t)$ be the prices of the underlying asset and the long forward contract at time $t$. The riskless interest rate $r$ is a constant. In addition, we assume that the underlying asset has no storage costs and produces no income.

At time $t$ we construct two portfolios:

- Portfolio A: a long forward contract + cash $Ke^{-r(T-t)}$;
- Portfolio B: one share of underlying asset: $S_t$.

At expiry date, both have the value of $S_T$. At time $t$, portfolio A and B have the values of $V(S_t,t) + Ke^{-r(T-t)}$ and $S_t$, respectively.

We emphasize that the underlying asset discussed here is an investment asset (stock or gold, for example) for which short selling is allowed. Then
we infer from the no-arbitrage principle that the two must have the same value at time \( t \), i.e. (1.4) (otherwise an arbitrage would be caused). So

\[
V(S_t, t) + Ke^{-r(T-t)} = S_t
\]

or

\[
V(S_t, t) = S_t - Ke^{-r(T-t)}. \tag{1.6}
\]

Recall that the delivery price is chosen such that at the time when the contract is opened, the value of the contract to both long and short sides is zero. Let \( t = 0 \) be the time of opening the contract. Then we have

\[
S_0 - Ke^{-rT} = 0,
\]

namely,

\[
K = S_0e^{rT}.
\]

This answers Question 1.1.

Exercise: Distinguish between the forward price and the delivery price. How to determine the forward price of a forward contract?

1.2.2 Pricing forwards contracts on a non-traded underlying

Lack of short selling of underlying assets leads to a different pricing model (we always assume that short selling of derivatives is permitted). Let us look at one example.

If the underlying is not held for investment purposes, we should be careful when using the no-arbitrage principle. For example, assume the underlying to be a consumption commodity: oil for which short selling is not allowed. As in last section, we construct two portfolios A and B in the same way. Due to no-short selling constraint of the underlying, we cannot short sell portfolio B, but we can still short sell portfolio A (a derivative). We claim that at time \( t \) the value of portfolio A is not greater than that of portfolio B, that is

\[
V(S_t, t) + Ke^{-r(T-t)} \leq S_t. \tag{1.7}
\]

Indeed, suppose that instead of equation (1.7), we have

\[
V(S_t, t) + Ke^{-r(T-t)} > S_t. \tag{1.8}
\]

Then one could short sell portfolio A and buy portfolio B at time \( t \). Then the strategy is certain to lead to a riskless positive profit of \( e^{r(T-t)}(V(S_t, t) + Ke^{-r(T-t)} - S_t) \) at expiry \( T \). Therefore, we conclude from the no-arbitrage principle that equation (1.8) cannot hold (for any significant length of time).
If short selling is allowed for the underlying (gold, for example), we are able to similarly deduce that $V(S_t, t) + Ke^{-r(T-t)} < S_t$ cannot hold, and thus we are certain to have equation (1.6). However, all we can assert for the forward contract on a consumption commodity is only equation (1.7), or equivalently,

$$V(S_t, t) \leq S_t - Ke^{-r(T-t)}.$$ 

Corresponding, the delivery price $K \leq S_0e^{rT}$.

### 1.2.3 Properties of Option Prices

Forward contract can be valued by the no-arbitrage principle. Unfortunately, because of the nonlinearity of the payoff of options, arbitrage arguments are not enough to obtain the price function of options. In fact, more assumptions are required to value options. We shall discuss this in Chapter 2.

No-arbitrage principle can only result in some relationships between option prices and the underlying asset price, including (suppose the underlying pays no dividend):

1. $C_E(S_t, t) = C_A(S_t, t)$. In other words, it is never optimal to exercise an American call option on a non-dividend-paying underlying asset before the expiration date.

2. **Put-call Parity (European Options):**

$$C_E(S_t, t) - P_E(S_t, t) = S_t - Ke^{-r(T-t)}$$

3. **Upper and Lower Bound of Option Prices:**

$$(S_t - Ke^{-r(T-t)})^+ \leq C_E(S_t, t) = C_A(S_t, t) \leq S_t$$

$$(Ke^{-r(T-t)} - S_t)^+ \leq P_E(S_t, t) \leq Ke^{-r(T-t)}, \text{ and } (K_t - S_t)^+ \leq P_A(S_t, t) \leq K$$

Here $C_E$– European call; $P_E$– European put, $C_A$– American call, $P_A$– American put.

For details, we refer to Hull (2003).

### 1.3 Brownian Motion, Ito Integral and Ito’s Lemma

In most cases, we assume that the underlying asset price follows an Ito process,

$$dS_t = a(S_t, t)dt + b(S_t, t)dW_t,$$

where $a$ and $b$ are deterministic functions, and $W_t$ is a Brownian motion.
1.3.1 Brownian Motion

Formally, $W$ is a Brownian motion if it has the following two properties:

1. The change $\Delta W$ during a small period of time $\Delta t$ is a random variable, drawn from a normal distribution with zero mean and variance $\Delta t$, i.e.

$$\Delta W = \phi \sqrt{\Delta t},$$

where $\phi$ is a random variable drawn from a standardized normal distribution which has zero mean, unit variance and a density function given by

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ x \in (-\infty, \infty).$$

2. The values of $\Delta W$ for any two different short intervals of time $\Delta t$ are independent.

1.3.2 Ito Process and Ito Integral

Let us go back to (1.9). Thanks to the properties of Brownian motion, we are able to simulate the sample path of $S_t$ in a given period $[0, T]$ by the following procedure: Let $\Delta t = \frac{T}{N}$, $t_n = n\Delta t$, $S_n = S_{t_n}$, $n = 0, 1, ..., N$,

$$S_n = S_{n-1} + a(S_{n-1}, t_n)\Delta t + b(S_{n-1}, t_n)\varepsilon \sqrt{\Delta t}. \quad (1.10)$$

Here $\varepsilon$ should be taken independently for each time interval $[t_{n-1}, t_n]$.

A precise expression of (1.9) is

$$S_t = S_0 + \int_0^t a(S_\tau, \tau)d\tau + \int_0^t b(S_\tau, \tau)dW_\tau,$$

where the first integral is the Riemann integral, and the second is the Ito integral. For a rigorous definition of Ito integral, see Oksendal (2003).

1.3.3 Ito’s Lemma

Ito’s Lemma is essentially the differential chain rule of a function involving random variable.

First let us recall the standard differential chain rule of a function of deterministic variables. Let $V(S(t), t)$ be a function of two variables $S$ and $t$, where

$$dS = a(S, t)dt.$$
Then by Taylor series expansion,

\[ dV(S(t), t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS \]

\[ = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} a(S, t) dt \]

\[ = \left[ \frac{\partial V}{\partial t} + a(S, t) \frac{\partial V}{\partial S} \right] dt \]

Now let us come back to the stochastic process (1.9). Keep in mind that

\[ dW = \phi \sqrt{dt} \quad \text{and} \quad E(dW^2) = dt. \]

So, formally we have

\[ (dS_t)^2 = (adt + bdW)^2 \]

\[ = a^2 dt^2 + 2abdtW + b^2 (dW)^2 \]

\[ = b^2 dt + \cdots. \]

As a result, when applying the Taylor series expansion to \( V(S_t, t) \), we need to retain the second order term of \( dS \). Thus,

\[ dV(S_t, t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS_t)^2 \]

\[ = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} b^2(S_t, t) dt \]

\[ = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} b^2(S_t, t) \frac{\partial^2 V}{\partial S^2} \right] dt + \frac{\partial V}{\partial S} dS_t \]

\[ = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} b^2(S_t, t) \frac{\partial^2 V}{\partial S^2} \right] dt + \frac{\partial V}{\partial S} dS_t \]

\[ + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} b^2(S_t, t) dW. \]

This is the Ito formula, the chain rule of stochastic calculus. Note that it is not a rigorous proof. We refer interested readers to Oksendal (2003) for rigorous proof of Ito’s formula.

A question: now that \( dW \approx O(dt^{1/2}) \), why don’t we omit the first order term of the right hand side in Eq. (1.11)?
Chapter 2

Option Pricing Models for European Options

2.1 Continuous-time Model: Black-Scholes Model

2.1.1 Black-Scholes Assumptions

We list the assumptions that we make for most of this notes.

1. The underlying asset price follows a geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
\]

where \(\mu\) and \(\sigma\) are the expected return rate and volatility of the underlying asset, \(W_t\) is the Brownian motion.

2. There are no arbitrage opportunities. The absence of arbitrage opportunities means that all risk-free portfolios must earn the same return.

3. The underlying asset pay no dividends during the life of the option.

4. The risk-free interest rate \(r\) and the asset volatility \(\sigma\) are known constants over the life of the option.

5. Trading is done continuously. Short selling is permitted and the assets are divisible.

6. There are no transaction costs associated with hedging a position. Also no taxes.

2.1.2 Derivation of the Black-Scholes Model

Let \(V = V(S, t)\) be the value of an European option. To derive the model, we construct a portfolio of one long option position and a short position in
The increment of the value of the portfolio in one time-step is
\[ d\Pi = dV - \Delta dS. \]
To eliminate the risk, we take
\[ \Delta = \frac{\partial V}{\partial S} \]
and then
\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS - \Delta dS. \]
Since there is no random term, the portfolio is riskless. By the no-arbitrage principle, a riskless portfolio must earn a risk free return (i.e. 1.5). So, we have
\[ d\Pi = r\Pi dt = r(V - S \frac{\partial V}{\partial S}) dt. \]
From the above two equalities, we obtain an equation
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \]  
This is the well-known Black-Scholes equation. The solution domain is \( D = \{(S, t) : S > 0, t \in [0, T]\} \). At expiry, we have
\[ V(S, T) = \begin{cases} (S - X)^+, & \text{for call option}, \\ (X - S)^+, & \text{for put option}. \end{cases} \]  
There is a unique solution to the model (2.1-2.2):
\[ V(S, t) = \begin{cases} SN(d_1) - X e^{-r(T-t)} N(d_2) & \text{for call option}, \\ X e^{-r(T-t)} N(-d_2) - SN(-d_1) & \text{for put option} \end{cases} \]
where
\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad d_1 = \frac{\log \frac{S}{X} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]
\textbf{Remark 2} The Black-Scholes equation is valid for any derivative that provides a payoff depending only on the underlying asset price at one particular time (European style).

Exercise: Use the Black-Scholes equation to price a long forward contract, and digital options (binary options).
2.1.3 Risk-Neutral Pricing and Theoretical Basis of Monte-Carlo Simulation

The expected return rate $\mu$ of the underlying asset, clearly depending on risk preference, does not appear in the equation. All of the variables appearing in the Black-Scholes equation are independent of risk preference. So, risk references do not affect the solution to the Black-Scholes equation. This means that any set of risk preferences can be used when evaluating options (or any other derivatives). In particular, we may carry out the evaluation in a risk-neutral world.

In a risk-neutral world, all investors are risk-neutral, namely, the expected return on all securities is the risk-free rate of interest $r$. Thus, the present value of any cash flow in the world can be obtained by discounting its expected value at the risk-free rate. Then the price of an option (a European call, for example) can be represented by

$$V(S, t) = \tilde{E} \left[ e^{-r(T-t)} (S_T - X)^+ | S_t = S \right].$$  \hspace{1cm} (2.3)

Here $\tilde{E}$ denotes the expected value in a risk-neutral world under which the underlying asset price $S_t$ follows

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t.$$  \hspace{1cm} (2.4)

Note that in this situation the expected return rate of the underlying is riskless rate of interest $r$ (suppose the underlying pays no income).

Mathematically, we can provide a rigorous proof for the equivalence of (2.1-2.2) and (2.3). In fact, this is just a corollary of Feynman-Kac formula. We refer interested readers to Oksendal (2003).

Eq (2.3) is the theoretical basis of Monte-Carlo simulation for derivative pricing. The simulation can be carried out by the following procedure:

1. Simulate the price movement of the underlying asset in a risk-neutral world according to (2.4) (see the discrete scheme (1.10));
2. Calculate the expected terminal payoff of the derivative.
3. Discount the expected payoff at the risk-free interest rate.

**Remark 3** It is important to emphasize that risk-neutral valuation (or the assumption that all investors are risk-neutral) is merely an artificial device for obtaining solutions to the Black-Scholes equation. The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral.
2.2 Discrete-time Model: Cox-Ross-Rubinstein Binomial Model

2.2.1 Single-Period Model

Consider an option whose value, denoted by $V_0$ at current time $t = 0$, depends on the underlying asset price $S_0$. Let the expiration date of the option be $T$. Assume that during the life of the option the underlying asset price $S_0$ can either move up to $S_0u$ with probability $p'$, or down to $S_0d$ with probability $1 - p'$ ($u > 1 > d, 0 < p' < 1$). Correspondingly, the payoff from the option will become either $V_u$ (for up-movement in the underlying asset price) or $V_d$ (for down-movement).

The following argument is similar to that of continuous time case. We construct a portfolio that consists of a long position in the option and a short position in $\Delta$ shares. At time $t = 0$, the portfolio has the value

$$V - \Delta S_0$$

If there is an up movement in the underlying asset price, the value of the portfolio at $t = T$ is

$$V_u - \Delta S_0u.$$  

If there is a down movement in the underlying asset price, the value becomes

$$V_d - \Delta S_0d.$$  

To make the portfolio riskfree, we let the two be equal, that is,

$$V_u - \Delta S_0u = V_d - \Delta S_0d$$

or

$$\Delta = \frac{V_u - V_d}{S_0(u - d)}. \quad (2.5)$$

Again, by the no-arbitrage principle, a risk-free portfolio must earn the risk-free interest rate. As a result

$$V_u - \Delta S_0u = e^{rT}(V - \Delta S).$$

Substituting (2.5) into the above formula, we get

$$V = e^{-rT}[pV_u + (1 - p)V_d],$$

where

$$p = \frac{e^{rT} - d}{u - d}.$$
2.2. DISCRETE-TIME MODEL: COX-ROSS-RUBINSTEIN BINOMIAL MODEL

This is the single-period binomial model. Here $p$ is called the risk-neutral probability. Note that the objective probability $p'$ does not appear in the binomial model, which is consistent with the risk-neutral pricing principle of the continuous time model.

2.2.2 Multi-Period Model

Let $T$ be expiration date, $[0, T]$ be the lifetime of the option. If $N$ is the number of discrete time points, we have time points $n\Delta t$, $n = 0, 1, ..., N$, with $\Delta t = \frac{T}{N}$. At time $t = 0$, the underlying asset price is known, denoted by $S_0$. At time $\Delta t$, there are two possible underlying asset prices, $S_0u$ and $S_0d$. Without loss of generality, we assume

$$ud = 1.$$

At time $2\Delta t$, there are three possible underlying asset prices, $S_0u^2$, $S_0$, and $S_0d^2 = S_0u^{-2}$; and so on. In general, at time $n\Delta t$, $n + 1$ underlying asset prices are considered. These are $S_0u^{-n}, S_0u^{-n+2}, ..., S_0u^n$. A complete tree is then constructed. Let $V^n_j$ be the option price at time point $n\Delta t$ with underlying asset price $S_j = S_0u^j$. Note that $S_j$ will jump either up to $S_{j+1}$ or down to $S_{j-1}$ at time $(n+1)\Delta t$, and the value of the option at $(n+1)\Delta t$ will become either $V_{j+1}^{n+1}$ or $V_{j-1}^{n+1}$. Since the length of time period is $\Delta t$, the discounting factor is $e^{-r\Delta t}$. Then, similar to the arguments in the single-period case, we have

$$V^n = e^{-r\Delta t} \left[pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}\right], \quad j = -n, -n+2, ..., n, \quad n = 0, 1, ..., N-1$$

where

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

At expiry,

$$V^n_j = \begin{cases} 
(S_0u^j - K)^+ \text{ for call,} \\
(K - S_0u^j)^+ \text{ for put,} 
\end{cases} \quad j = -N, -N + 2, ..., N.$$

This is the multi-period binomial model.

To make the binomial process of the underlying asset price match the geometric Brownian motion, we need to choose $u$, $d$ such that

$$p'u + (1-p')d = e^{\mu \Delta t} \quad (2.6)$$

$$p'u^2 + (1-p')d^2 - e^{2\mu \Delta t} = \sigma^2 \Delta t. \quad (2.7)$$
There are three unknowns $u, d$ and $p'$. Without loss of generality, we add one condition

$$ud = 1. \quad (2.8)$$

By neglecting the high order of $\Delta t$, we can solve the system of equations (2.6-2.8) to get

$$u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}.$$

## 2.3 Consistency of Binomial Model and Continuous-Time Model.

### 2.3.1 Consistency

The binomial tree method can be rewritten as

$$V(S, t - \Delta t) = e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)].$$

Here, for the convenience of presentation, we take the current time to be $t - \Delta t$. Assuming sufficient smoothness of the $V(S, t)$, we perform the Taylor series expansion of the binomial scheme at $(S, t)$ as follows

$$0 = -V(S, t - \Delta t) + e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)]$$

$$= -V(S, t) + \frac{\partial V}{\partial t} \Delta t + O(\Delta t^2)$$

$$+ e^{-r\Delta t}V(S, t) + \frac{\partial V}{\partial S}Se^{-r\Delta t}[p(u - 1) + (1 - p)(d - 1)]$$

$$+ \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2 e^{-r\Delta t}[p(u - 1)^2 + (1 - p)(d - 1)^2]$$

$$+ \frac{1}{6} \frac{\partial^3 V}{\partial S^3} S^3 e^{-r\Delta t}[p(u - 1)^3 + (1 - p)(d - 1)^3] + O(\Delta t^2)$$

Observe that

$$e^{-r\Delta t}[p(u - 1) + (1 - p)(d - 1)] = r\Delta t + O(\Delta t^2).$$

$$e^{-r\Delta t}[p(u - 1)^2 + (1 - p)(d - 1)^2] = \sigma^2 \Delta t + O(\Delta t^2)$$

$$e^{-r\Delta t}[p(u - 1)^3 + (1 - p)(d - 1)^3] = O(\Delta t^2).$$

We then get

$$0 = -V(S, t - \Delta t) + e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)]$$

$$= [-rV(S, t) + \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}] \Delta t + O(\Delta t^2)$$
2.3. CONSISTENCY OF BINOMIAL MODEL AND CONTINUOUS-TIME MODEL.

or

\[-rV(S, t) + \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = O(\Delta t).\]

This implies the consistency of two models.

2.3.2 *Equivalence of BTM and an explicit difference scheme

We claim the BTM is equivalent to an explicit difference scheme for the continuous-time model.

Using the transformations \( u(x, t) = V(S, t), S = e^x, \) (2.1-2.2) become the following constant-coefficient PDE problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2}) \frac{\partial u}{\partial x} - ru = 0 \\
u(T, x) = \varphi(x) & \quad \text{in} (-\infty, \infty),
\end{align*}
\]

where \( \varphi(x) = e^x - X \) (call option) or \( \varphi(x) = X - e^x \) (put option).

We now present the explicit difference scheme for (2.9). Given mesh size \( \Delta x, \Delta t > 0, N\Delta t = T, \) let \( Q = \{(j\Delta x, n\Delta t) : 0 \leq n \leq N, j \in \mathbb{Z}\} \) stand for the lattice. \( U^n_j \) represents the value of numerical approximation at \( (j\Delta x, n\Delta t) \) and \( \varphi_j = \varphi(j\Delta x). \) Taking the explicit difference for time and the conventional difference discretization for space, we have

\[
\frac{U^{n+1}_j - U^n_j}{\Delta t} + \frac{\sigma^2 U^{n+1}_{j+1} - 2U^{n+1}_j + U^{n+1}_{j-1}}{\Delta x^2} + (r - \frac{\sigma^2}{2}) \frac{U^{n+1}_{j+1} - U^{n+1}_{j-1}}{2\Delta x} - rU^n_j = 0
\]

or

\[
U^n_j = \frac{1}{1 + r\Delta t} \left( (1 - \frac{\sigma^2 \Delta t}{\Delta x^2})U^{n+1}_j + \frac{\sigma^2 \Delta t}{\Delta x^2} \left( \frac{1}{2} + (r - \frac{\sigma^2}{2}) \frac{\Delta x}{2\sigma^2} \right) U^{n+1}_{j+1} \right.
\]

\[\left. \quad + \frac{\sigma^2 \Delta t}{\Delta x^2} \left( \frac{1}{2} - (r - \frac{\sigma^2}{2}) \frac{\Delta x}{2\sigma^2} \right) U^{n+1}_{j-1} \right),
\]

which is denoted by

\[
U^n_j \approx \frac{1}{1 + r\Delta t} \left[ (1 - \alpha)U^{n+1}_j + \alpha(aU^{n+1}_{j+1} + (1 - a)U^{n+1}_{j-1}) \right],
\]

(2.10)

where

\[
\alpha = \frac{\sigma^2 \Delta t}{\Delta x^2}, \quad a = \frac{1}{2} + (r - \frac{\sigma^2}{2}) \frac{\Delta x}{2\sigma^2}.
\]

By putting \( \alpha = 1 \) in (2.10), namely \( \frac{\sigma^2 \Delta t}{\Delta x^2} = 1, \) we get

\[
U^n_j = \frac{1}{1 + r\Delta t} \left[ aU^{n+1}_{j+1} + (1 - a)U^{n+1}_{j-1} \right].
\]

(2.11)
The final values are given as follows:

\[ U_j^N = \varphi_j^+, \quad j \in \mathbb{Z}. \]

Recall the binomial tree method can be described as follows by adopting the same lattice:

\[
V^n_j = \frac{1}{\rho} \left[ pV^{n+1}_{j+1} + (1 - p)V^{n+1}_{j-1} \right], \quad j = n, n - 2, \ldots, -n, \tag{2.12}
\]

\[
V_j^N = \varphi_j^+, \quad j = N, N - 2, \cdots, -N \tag{2.13}
\]

In view of

\[ \rho = 1 + r \Delta t + O \left( \Delta t^2 \right) \]

and

\[ p = \frac{1}{2} \left( 1 + \sqrt{\Delta t} \sigma \left( r - \frac{\sigma^2}{2} \right) \right) + O(\Delta t^{3/2}), \]

Recall the binomial tree method can be described as follows by adopting the same lattice:

\[
V^n_j = \frac{1}{\rho} \left[ pV^{n+1}_{j+1} + (1 - p)V^{n+1}_{j-1} \right], \quad j = n, n - 2, \cdots, -n, \tag{2.14}
\]

\[
V_j^N = \varphi_j^+, \quad j = N, N - 2, \cdots, -N \tag{2.15}
\]

In view of

\[ \rho = 1 + r \Delta t + O \left( \Delta t^2 \right) \]

and

\[ p = \frac{1}{2} \left( 1 + \sqrt{\Delta t} \sigma \left( r - \frac{\sigma^2}{2} \right) \right) + O(\Delta t^{3/2}), \]

we deduce that the binomial tree method is equivalent to explicit difference scheme (2.11) in the sense of neglecting a higher order of \( \Delta t \).

Question: what’s a trinomial tree method? what about the relation between the trinomial tree method and finite difference schemes?

### 2.4 Continuous-dividend and Discrete-dividend Payments

#### 2.4.1 Continuous-dividend Payment

Let \( q \) be the continuous dividend yield. This means that in a time period \( dt \), the underlying asset pays a dividend \( qS_t dt \). Following a similar argument as
2.4. CONTINUOUS-DIVIDEND AND DISCRETE-DIVIDEND PAYMENTS

in the case no dividend payment, it is not hard to derive the pricing equation.

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0.
\]

For the binomial model, the risk neutral probability is adjusted as

\[ p = \frac{e^{(r-q)\Delta t} - d}{u - d}. \]

We omit the details to readers.

2.4.2 Discrete-dividend Payment

Without loss of generality, suppose that the asset pays dividend just once during the lifetime of the option, at time \( t_d \in (0, T) \), with the known dividend yield \( d_y \). Thus, at time \( t_d \), the holder of the asset receives a payment \( d_y S(t_d^-) \), where \( S(t_d^-) \) is the asset price just before the dividend is paid. To preclude arbitrage opportunities, the asset price must fall by exactly the amount of the dividend payment,

\[ S(t_d^+) = S(t_d^-) - d_y S(t_d^-) = S(t_d^-)(1 - d_y). \]

This means that a discrete dividend payment leads to a jump in the value of the underlying asset across the dividend date.

One important observation for option pricing model is that the value of the option must be continuous as a function of time across the dividend date because the holder of the option does not receive the dividend. So, the value of the option is the same immediately before the dividend date as it is immediately after the date, that is,

\[
V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+) = V(S(t_d^-)(1 - d_y), t_d^+).
\]

Let \( S \) replace \( S(t_d^-) \). We then have the so-called jump condition:

\[
V(S, t_d^-) = V(S(1 - d_y), t_d^+).
\]

For \( t \neq t_d \), there is no dividend payment and thus \( V(S, t) \) still satisfies the equations without dividend payments. Therefore, we have for European options,

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad S > 0, \quad t \in (0, t_d), \quad t \in (t_d, T)
\]

\[
V(S, t_d^-) = V(S(1 - d_y), t_d^+)
\]

\[
V(S, T) = \varphi^+.
\]
2.5 No Arbitrage Pricing: A General Framework

We consider the derivatives on a single underlying variable, $\theta$, which follows

$$\frac{d\theta}{\theta} = \mu(\theta, t)dt + \sigma(\theta, t)dW.$$ 

Here the variable $\theta$ need not be the price of an investment asset. For example, it might be the interest rate, and corresponding derivative products can be bonds or some interest rate derivatives. In this case the shorting selling for the underlying is not permitted and thus we cannot replicate the derivation process of the Black-Scholes equation where the underlying asset is used to hedge the derivative.

Suppose that $f_1$ and $f_2$ are the prices of two derivatives dependent only on $\theta$ and $t$. These could be options or other instruments that provide a payoff equal to some function of $\theta$ at some future time. We assume that during the time period under consideration $f_1$ and $f_2$ provide no income.

Suppose that the processes followed by $f_1$ and $f_2$ are

$$df_1 = a_1 dt + b_1 dW$$

and

$$df_2 = a_2 dt + b_2 dW,$$

where $a_1, a_2, b_1$ and $b_2$ are functions of $\theta$ and $t$. The $W$ is the same Brownian motion as in the process of $\theta$, because this is the only source of the uncertainty in their prices. To eliminate the uncertainty, we can form a portfolio consisting of $b_2$ of the first derivative and $-b_1$ of the second derivative. Let $\Pi$ be the value of the portfolio,

$$\Pi = b_2 f_1 - b_1 f_2.$$ 

Then

$$d\Pi = b_2 df_1 - b_1 df_2$$

$$= (a_1 b_2 - a_2 b_1) dt.$$ 

Because the portfolio is instantaneously riskless, it must earn the risk-free rate. Hence

$$d\Pi = r \Pi dt = r(b_2 f_1 - b_1 f_2) dt$$

Therefore,

$$a_1 b_2 - a_2 b_1 = r(b_2 f_1 - b_1 f_2)$$
or
\[
\frac{a_1 - rf_1}{b_1} = \frac{a_2 - rf_2}{b_2}
\]
Define \( \lambda \) as the value of each side in the equation, so that
\[
\frac{a_1 - rf_1}{b_1} = \frac{a_2 - rf_2}{b_2} = \lambda.
\]
Dropping subscripts, we have shown that if \( f \) is the price of a derivative dependent only on \( \theta \) and \( t \) with
\[
df = adt + bdW
\]
then
\[
\frac{a - rf}{b} = \lambda. \tag{2.16}
\]
The parameter \( \lambda \) is known as the market price of risk of \( \theta \). It may be dependent on both \( \theta \) and \( t \), but it is not dependent on the nature of any derivative \( f \). At any given time, \((a - rf)/b\) must be the same for all derivatives that are dependent only on \( \theta \) and \( t \).

The market price of risk of \( \theta \) measures the trade-offs between risk and return that are made for securities dependent on \( \theta \). Eq. (2.16) can be written
\[
a - rf = \lambda b.
\]
For an intuitive understanding of this equation, we note that the variable \( \sigma \) can be loosely interpreted as the quantity of \( \theta \)-risk present in \( f \). On the right-hand side of the equation we are, therefore, multiplying the quantity of \( \theta \)-risk by the price of \( \theta \)-risk. The left-hand side is the expected return in excess of the risk-free interest rate that is required to compensate for this risk. This is analogous to the capital asset pricing model, which relates the expected excess return on a stock to its risk.

Because \( f \) is a function of \( \theta \) and \( t \), the process followed by \( f \) can be expressed in terms of the process followed by \( \theta \) using Ito’s lemma. The parameters \( \mu \) and \( \sigma \) are given by
\[
a = \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + \mu \theta \frac{\partial f}{\partial \theta} \\
b = \sigma \theta \frac{\partial f}{\partial \theta}.
\]
Substituting these into equation (2.16), we obtain the following differential equation that must be satisfied by \( f \)
\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + (\mu - \lambda \sigma) \theta \frac{\partial f}{\partial \theta} - rf = 0. \tag{2.17}
\]
This equation is structurally very similar to the Black-Scholes equation.

If the variable \( \theta \) is the price of a traded asset, then the asset itself can be regarded as a derivative on \( \theta \). Hence we can take \( f = \theta \) and substitute into Eq. (2.16) to get

\[
\mu f - rf = \lambda \sigma f
\]
or

\[
\mu - r = \lambda \sigma.
\]

Then the equation becomes precisely the Black-Scholes equation:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + r\theta \frac{\partial f}{\partial \theta} - rf = 0.
\]

**Remark 4** Eq (2.17) implies that the risk-neutral process of \( \theta \) is

\[
d\theta = (\mu - \lambda \sigma) \theta dt + \sigma \theta dW.
\]

**Remark 5** Applying Ito’s lemma gives

\[
df = \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} + \mu \theta \frac{\partial f}{\partial \theta} \right] dt + \sigma \theta \frac{\partial f}{\partial \theta} dW
\]

Substituting Eq (2.17) into the above expression, we have

\[
df = \left[ rf + \lambda \sigma \theta \frac{\partial f}{\partial \theta} \right] dt + \sigma \theta \frac{\partial f}{\partial \theta} dW
\]

\[
= rf dt + \sigma \theta \frac{\partial f}{\partial \theta} [\lambda dt + dW].
\]

That is

\[
df - rf dt = \sigma \theta \frac{\partial f}{\partial \theta} [\lambda dt + dW].
\]

Observe that for every unit of risk, represented by \( dW \), there are \( \lambda \) units of extra return. That is why we call \( \lambda \) the market price of risk.

### 2.6 *Option Pricing: Replication*

Consider a market where only two basic assets are traded. One is a bond, whose price process is

\[
dP = rP dt.
\]

The other asset is a stock whose price process is governed by the geometric Brownian motion:

\[
dS_t = \mu S_t dt + \sigma S_t dW_t.
\]
Consider a European call option whose payoff is 

$$(S_T - K)^+.$$ 

The option pricing problem is: what is the fair price of this option at time $t = 0$?

Let us make some observation on this problem. The price of the option at $t = T$ is the amount that the holder of the option would obtain as well as the amount that the writer would lose at that time. Now, suppose this option has a price $z$ at $t = 0$. The writer has to invest this amount of money in some way (called replication) in the market (where there are one bond and one stock available) so that at time $t = T$, his total wealth, denoted by $Z_T$, resulting for the investment of $z$, should at least compensate his potential loss $(S_T - K)^+$, namely

$$Z_T \geq (S_T - K)^+.$$ 

(2.18)

It is clear that for the same investment strategy, the larger the initial endowment $z$, the larger the final wealth $Z_T$. Hence the writer of the option would like to set $z$ large enough so that (2.18) can be guaranteed. On the other hand, if it happens that for some $z$ the resulting final wealth $Z_T$ is strictly larger than the loss $(S_T - K)^+$, then the price $z$ of this option at $t = 0$ is considered to be too high. In this case, the buyer of the option, instead of buying the option, would make his own investment to get the desired payoff $(S_T - K)^+$. As a result, the fair price for the option at time $t = 0$ should be such a $z$ that the corresponding optimal investment would result in a wealth process $Z_T$ satisfying

$$Z_T = (S_T - K)^+.$$ 

Now, let us denote by $Y_t$ the amount that the writer invests in the stock (i.e., the number of shares $\frac{Y_t}{S_t}$). The remaining amount $Z_t - Y_t$ is invested in the bond. The wealth process $Z_t$ is described by

$$dZ_t = [rZ_t + (\mu - r) Y_t] dt + \sigma Y_t dW.$$ 

The option price is given by $Z_0$ at time $0$.

Let us assume

$$Z_t = V(t, S_t).$$

Applying Itô lemma,

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \mu S_t \frac{\partial V}{\partial S} \right) dt + \sigma S_t \frac{\partial V}{\partial S} dW(t).$$
Then we get
\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \mu S_t \frac{\partial V}{\partial S} = r V + (\mu - r) Y_t.
\end{aligned}
\]

In the end, we get
\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0.
\]

This is the Black-Scholes equation.

Especially
\[
\frac{Y_t}{S_t} = \frac{\partial V}{\partial S} \Delta \equiv \Delta.
\]

This means that the strategy for replication is holding \( \Delta \) shares of stock.

Consequently,
\[
dZ_t = [r Z_t + (\mu - r) \Delta S_t] dt + \sigma \Delta S_t dW.
\]

That is
\[
Z_t = Z_0 + \int_0^t [r Z_\tau + (\mu - r) \Delta S] d\tau + \int_0^t \sigma \Delta S_\tau dW_\tau
\]
or
\[
Z_t = Z_0 + \int_0^t r Z_\tau d\tau + \int_0^t \Delta (dS_\tau - r S_\tau d\tau).
\]
American options and early exercise

American options are contracts that may be exercised early, prior to expiry. These options are contrasted with European options for which exercise is only permitted at expiry. Most traded stock and futures options are American style, while most index options are European.

3.1 Pricing models

3.1.1 Continuous-time model for American options

We now consider the pricing model for American options. Here we take into account a put as an example. Let \( V = V(S, t) \) be the option value. At expiry, we still have

\[
V(S, T) = (X - S)^+.
\]  

(3.1)

The early exercise feature gives the constraint

\[
V(S, t) \geq X - S.
\]  

(3.2)

As before, we construct a portfolio of one long American option position and a short position in some quantity \( \Delta \), of the underlying.

\[
\Pi = V - \Delta S.
\]

With the choice \( \Delta = \frac{\partial V}{\partial S} \), the value of this portfolio changes by the amount

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt
\]

(3.3)
In the Black-Scholes argument for European options, we set this expression equal to riskless return, in order to preclude arbitrage. However, when the option in the portfolio is of American style, all we can say is that we can earn no more than the risk-free rate on our portfolio, that is,
\[ d\Pi \leq r\Pi dt = r(V - S \frac{\partial V}{\partial S})dt. \]
The reason is the holder of the option controls the early exercise feature. If he/she fails to optimally exercise the option, the change of the portfolio value would be less than riskless return. Thus we arrive at an inequality
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \leq r(V - S \frac{\partial V}{\partial S})dt \]
or
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0. \]  
(3.3)

**Remark 6** For American options, the long/short position is asymmetrical. The holder of an American option is given more rights, as well as more headaches: when should he exercise? Whereas the writer of the option can do no more than sit back and enjoy the view. The writer of the American option can make more than the risk-free rate if the holder does not exercise optimally. A question: what happens if the portfolio is composed of a long position in some quantity of the underlying and one short American option?

It is clear that (3.1-3.3) are insufficient to form a model because solution is not unique. We need to exploit more information. Note that if \( V(S,t) > X - S \), which implies that the option should not be exercised at the moment, then the equality holds in the inequality (3.3), namely
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \text{ if } V > X - S. \]
If \( V(S,t) = X - S \), of course we still have the inequality, that is
\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0 \text{ if } V = X - S. \]
The above two formulas imply that at least one holds in equality between (3.2-3.3). So we arrive at a complete model:
\[ \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right] [V - (X - S)] = 0, \ (S,t) \in D \]
\[ V(S,T) = (X - S)^+. \]
3.1. PRICING MODELS

It can be shown that there exists a unique solution to the model.

A succinct expression of the above model is

$$\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r S \frac{\partial V}{\partial S} + r V, V - (X - S) \right\} = 0, \ (S, t) \in D$$

$$V(S, T) = (X - S)^+$$

For American call options, we similarly have

$$\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r S \frac{\partial V}{\partial S} + r V, V - (S - X) \right\} = 0, \ (S, t) \in D$$

$$V(S, T) = (S - X)^+$$

We claim the price function of European call option $C(S, t)$ just satisfies the above model. Indeed, $C(S, t) > S - X$ for $t < T$ and $C(S, t)$ clearly satisfies the Black-Scholes equation. So $C(S, t)$ must be the (unique) solution to the American option pricing model. The result $C(S, t) > S - X$ implies that the option should never be exercised before expiry.

**Remark 7** From the view point of probabilistic approach, we have (for an American put)

$$V(S, t) = \max_{t'} \mathbb{E} \left[ e^{-r(t' - t)} (X - S_{t'})^+ | S_t = S \right], \quad (3.4)$$

where $t'$ is a stopping time. Intuitively $t'(.)$ can be thought of as a strategy to exercise the option. The option’s value corresponds to the optimal exercise strategy. Mathematically we can show the equivalence between (3.4) and the above PDE model.

3.1.2 Continuous-dividend payment case

Let $q$ be the continuous dividend yield. Denote by

$$\varphi(S) = \begin{cases} S - X, \\ X - S. \end{cases}$$

Then the American option price function $V$ satisfies

$$\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + r V, V - \varphi \right\} = 0, \ (S, t) \in D$$

$$V(S, T) = \varphi^+$$
3.1.3 Binomial model

Let $T$ be the expiration date, $[0, N]$ be the lifetime of the option. If $N$ is the number of discrete time points, we have time points $n\Delta t$, $n = 0, 1, ..., N$, with $\Delta t = T/N$. Let $V_j^n$ be the option price at time point $n\Delta t$ with underlying asset price $S_j$. Suppose the underlying asset price $S_j$ will move either up to $S_j + 1$ or down to $S_j - 1$ after the next timestep. Similar to the arguments in the continuous time case, we are able to derive the binomial tree method (BTM)

$$
\begin{align*}
V_j^n &= \max\left\{ \frac{1}{p}[pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}], \varphi_j \right\}, \\
\text{for } j &= -n, -n+2, ..., n \text{ and } n = 0, 1, ...N-1 \\
V_j^N &= \varphi_j^+, \text{ for } j = -N, -N+2, ..., N
\end{align*}
$$

where $\varphi_j = \left\{ \begin{array}{ll}
S_0u^j - X \\
X - S_0u^j 
\end{array} \right.$

$$
p = e^{(r-q)\Delta t} - d
$$

$$
\rho = e^{r\Delta t}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad \text{and } d = e^{-\sigma\sqrt{\Delta t}}.
$$

A question: what about the relation between continuous and discrete models for American options?

3.2 Free boundary problems

We still take a put for example. First we give definitions of Stopping Region $E$ (or, Exercise Region) and Holding Region $H$ (or, Continuous Region):

$$
E = \{(S, t) \in D : V(S, t) = X - S\}
$$

$$
H = D \setminus E = \{(S, t) \in D : V(S, t) > X - S\}
$$

3.2.1 *Optimal exercise boundaries

Lemma 1 If $(S_1, t) \in E$, then $(S_2, t) \in E$ for all $S_2 \leq S_1$.

Proof: It suffices to show that

$$
V(S_2, t) + S_2 \leq V(S_1, t) + S_1, \text{ if } S_2 \leq S_1 \quad (3.5)
$$
3.2. FREE BOUNDARY PROBLEMS

Indeed, (3.5) is equivalent to

\[ V(S_2, t) - (X - S_2) \leq V(S_1, t) - (X - S_1) \]

Since \( V(S_1, t) - (X - S_1) = 0 \) and \( V(S_2, t) - (X - S_2) \geq 0 \), we derive \( V(S_2, t) = X - S_2 \), which implies \((S_2, t) \in E\). (3.5) can be proved in terms of the binomial model. We omit the details.

**Remark 8** (3.5) can be rewritten as

\[ \frac{V(S_1, t) - V(S_2, t)}{S_1 - S_2} \geq -1. \]

As \( S_1 \) tends \( S_2 \), we have

\[ \frac{\partial V}{\partial S} \geq -1. \]

**Proposition 2** (i) There exists a boundary \( S^*(t) \), called the optimal exercise boundary hereafter, such that

\[ E = \{(S, t) \in D : S \leq S^*(t)\}, \text{ and } H = \{(S, t) \in D : S > S^*(t)\} \]

(ii) \( S^*(t) \) is monotonically increasing.

(iii) \( S^*(T-) = \min(X, \frac{q}{r}X) \)

Proof: Part i) can be derived from Lemma 1. To show part ii), it is not hard to prove \( V(S, t) \) is monotonically decreasing w.r.t. \( t \) by using the binomial model (financial intuition: the larger the time to expiry, the larger the option value). Thus if \( V(S, t_1) > X - S \), then \( V(S, t_2) \geq V(S, t_1) > X - S \). A complete proof of part iii) requires some knowledge about PDE theory, so we skip it. Numerical experiments can show its validity.

**Remark 9** \( S^*(t) \) is called optimal exercise boundary because it is optimal to exercise the option exactly on the boundary. If \( S < S^*(t) \), then \( V(S, t) = X - S \) and

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV < 0 \]

or

\[ d\Pi < r\Pi dt. \]
3.2.2 Formulation as a free boundary problem

In the continuation region $H = \{S > S^*(t)\}$, the price function of an American put satisfies the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \text{ for } S > S^*(t), \ t \in [0, T) \quad (3.6)$$

On $S = S^*(t)$ we have

$$V(S^*(t), t) = X - S^*(t). \quad (3.7)$$

The final condition is

$$V(S, T) = (X - S)^+. \quad (3.8)$$

However, (3.6-3.8) cannot form a complete model because $S^*(t)$ is not known a priori as a function of time. As a matter of fact, $S^*(t)$ and $V(S, t)$ must be solved simultaneously. Therefore, we need an additional boundary condition

$$\frac{\partial V}{\partial S}(S^*(t), t) = -1. \quad (3.9)$$

This condition means that the hedging ratio $\Delta$ is continuous across the optimal exercise boundary. (3.6-3.9) form a complete model that is called the free boundary problem in PDE theory.

3.2.3 Perpetual American options

Pricing perpetual American options can give us some insights in the understanding of free boundary problems. A perpetual American put can be exercised for a put payoff at any time. There is no expiry; that is why it is called a perpetual option. Note that the price function of such an option is independent of time, denoted by $P_\infty(S)$. It only depends on the level of the underlying. Actually $P_\infty(S)$ can be regarded as the limit of an American put price as the time to expiry tends to infinity, i.e.

$$P_\infty(S) = \lim_{(T-t) \to \infty} V(S, t; T) = \lim_{\tau \to \infty} \tilde{V}(S, \tau).$$

where $\tilde{V}(S, \tau) = V(S, t; T)$, $\tau = T - t$. Thanks to (3.6-3.9), $\tilde{V}(S, \tau)$ satisfies

$$\frac{\partial \tilde{V}}{\partial \tau} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{V}}{\partial S^2} - (r - q)S \frac{\partial \tilde{V}}{\partial S} + r\tilde{V} = 0, \text{ for } S > S_\ast(\tau), \ \tau \in [0, T)$$

$$\tilde{V}(S_\ast(\tau), \tau) = X - S_\ast(\tau), \ \frac{\partial \tilde{V}}{\partial S}(S_\ast(\tau), \tau) = -1$$

$$\tilde{V}(S, 0) = (X - S)^+$$
3.2. FREE BOUNDARY PROBLEMS

where \( S_\tau = S^*(T-\tau) \). Due to part ii) of Proposition 2, \( S_\tau \) is monotone, and we can denote

\[
S_\infty = \lim_{\tau \to \infty} S_\tau(\infty).
\]

Then \( P_\infty(S) \) satisfies

\[
-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_\infty}{\partial S^2} - (r - q)S \frac{\partial P_\infty}{\partial S} + rP_\infty = 0, \quad \text{for } S > S_\infty \tag{3.10}
\]

\[
P_\infty(S_\infty) = X - S_\infty, \quad \frac{\partial P_\infty}{\partial S}(S_\infty) = -1. \tag{3.11}
\]

This is actually a free boundary problem with an ordinary difference equation. We now seek a solution of the form \( S^\alpha \) to Eq. (3.11), where \( \alpha \) satisfies

\[
-\frac{1}{2}\sigma^2 \alpha(\alpha - 1) - (r - q)\alpha + r = 0
\]

or

\[
\frac{1}{2}\sigma^2 \alpha^2 + (r - q - \frac{\sigma^2}{2})\alpha - r = 0.
\]

The two solutions of the above equation are

\[
\alpha_\pm = \frac{-r + \frac{\sigma^2}{2} \pm \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}. \tag{3.12}
\]

So the general solution of Eq. (3.11) is

\[
AS^{\alpha_+} + BS^{\alpha_-}
\]

where \( A \) and \( B \) are arbitrary constants.

Clearly, for the perpetual American put the coefficient \( A \) must be zero; as \( S \to \infty \) then value of the option must tend to zero. What about \( B \)? Here we need to take advantage of the condition (3.11)

\[
BS_\infty^{\alpha_-} = X - S_\infty, \quad \alpha_- BS_\infty^{\alpha_- - 1} = -1
\]

to get

\[
S_\infty = \frac{\alpha_-}{\alpha_- - 1}X, \quad B = \frac{X - S_\infty}{S_\infty^{\alpha_-}}.
\]
So, we obtain the perpetual American option price function

\[ P_\infty(S) = (X - S_\infty) \left( \frac{S}{S_\infty} \right)^{\alpha_-} \]

A by-product of the above calculation is

**Proposition 3**

\[ \lim_{\tau \to \infty} S_\tau(\tau) = \frac{\alpha_-}{\alpha_- - 1} X \]

### 3.2.4 Put-call symmetry relations

For European options, we have the well-known put-call parity

\[ C_E(S, t) - P_E(S, t) = S e^{-q(T-t)} - X e^{-r(T-t)}. \]

However, such a parity doesn’t hold for American options. Let us explain the reason from the viewpoint of PDE. For European options, both \( C_E(S, t) \) and \( P_E(S, t) \) satisfy the Black-Scholes equations. Due to the linearity of the equation, \( C_E(S, t) - P_E(S, t) \) also satisfies the equation, that is

\[
\left[ \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q)S \frac{\partial}{\partial S} - r \right] (C_E - P_E) = 0,
\]

together with the terminal condition

\[ (C_E - P_E)(S, T) = S - X. \]

On the other hand, the Black-Scholes equation with the terminal condition has the unique solution \( S e^{-q(T-t)} - X e^{-r(T-t)} \). This leads to

\[ C_E(S, t) - P_E(S, t) = S e^{-q(T-t)} - X e^{-r(T-t)}. \]

However, for American options, their governing equation is nonlinear so that we cannot obtain such a parity relation.

But, the following put-call symmetry relation holds for both European and American options:

\[ C(S; t; X, r, q) = P(X, t; S, q, r) \quad (3.13) \]

where the underlying price and the strike price in the put formula become the strike price and the underlying price in the call formula, respectively. Also, the roles of \( q \) and \( r \) are interchanged, like \( S \) and \( X \).
The financial explanation is given as follows. We may consider a call option with the payoff \((S - X)^{+}\) as providing the right to exchange one asset \(X\) with dividend yield \(r\) for another asset \(S\) with dividend yield \(q\). Similarly, a put call with payoff \((X_2 - S_2)^{+}\) is regarded as a right to exchange one asset \(S_2\) with dividend yield \(q_2\) for another asset \(X_2\) with dividend yield \(r_2\). Suppose \(X_2 = S, S_2 = X, r_2 = q\) and \(q_2 = r\). then the two have the same payoff and thus the same price, namely,

\[ C(S, t; X, r, q) = P(S_2, t; X_2, r_2, q_2) = P(X, t; S, q, r). \]

Next, we would like to establish the put-call symmetry relation for the optimal exercise prices for American put and call options. Let \(S^*_p(t; r, q)\) and \(S^*_c(t; r, q)\) denote the optimal exercise boundaries for American put and call options on a continuous dividend payment stock, respectively. We assert \(S^*_c(t; r, q)S^*_p(t; q, r) = X^2\) (3.14)

Indeed, due to the homogeneity, Eq. (3.13) can be rewritten as

\[ C(S, t; X, r, q) = \frac{S}{X} P(X^2, t; X, q, r). \]

According to the definition of \(S^*_c(t; r, q)\), we have

\[ \frac{S}{X} P(X^2, t; X, q, r) = C(S, t; X, r, q) = S - X, \text{ for } S \geq S^*_c(t; r, q) \]

\[ \frac{S}{X} P(X^2, t; X, q, r) = C(S, t; X, r, q) > S - X, \text{ for } S < S^*_c(t; r, q). \]

or, equivalently

\[ P(X^2, t; X, q, r) = X - \frac{X^2}{S}, \text{ for } S \geq S^*_c(t; r, q) \]

\[ P(X^2, t; X, q, r) > X - \frac{X^2}{S}, \text{ for } S < S^*_c(t; r, q). \]

By denoting \(S_1 = \frac{X^2}{S}\), we then get

\[ P(S_1, t; X, q, r) = X - S_1, \text{ for } S_1 \leq \frac{X^2}{S^*_c(t; r, q)} \]

\[ P(S_1, t; X, q, r) > X - S_1, \text{ for } S_1 > \frac{X^2}{S^*_c(t; r, q)}. \]
which implies \( \frac{X^2}{S^*_p(t; r, q)} \) is the optimal exercise boundary for American put option with interest rate \( q \) and dividend yield \( r \), namely

\[
S^*_p(t; q, r) = \frac{X^2}{S^*_c(t; r, q)}.
\]

Applying Eqs (3.13-3.14) and Proposition (2) leads to

**Proposition 4** Let \( S^*_c(t; r, q) \) be the optimal exercise boundary for American call options. Then

(i) \( S^*_c(t) \) is monotonically decreasing.

(ii) \( S^*_c(T-; r, q) = \max(X, \frac{r}{q}X) \)

**Remark 10** When \( q = 0 \), then \( S^*_c(T-; r, 0) = \infty \). Because of the monotonicity of \( S^*_c(t) \), we deduce \( S^*_c(t; r, 0) = \infty \) for all \( t \), which means there is no optimal exercise boundary, that is, early exercise should never happen.

In addition, it is not hard to get the closed form price function of the perpetual American option by using the put-call symmetry relation.

**Proposition 5** Suppose \( q > 0 \). Let \( C^\infty(S) \) be the price function of the perpetual American option. Then

\[
C^\infty(S) = (S^\infty,c - X) \left( \frac{S}{S^\infty,c} \right)^{\alpha_+}
\]

where \( \alpha \) is given by (3.12), and

\[
S^\infty,c = \frac{\alpha_+}{\alpha_+ - 1}X.
\]

### 3.3 Bermudan options

In a standard American option, exercise can take place at any time during the life of the option and the exercise price is always the same. In practice, the American options that are traded in the over-the-counter market do not always have these standard features.

One type of nonstandard American option is known as a Bermudan option. In this early exercise is restricted to certain dates \( t_1 < t_2 < ... < t_n \) during the life of the option \( (t_i \in [0, T], i = 1, 2, ..., n) \). Note that at any
interval \((t_i, t_{i+1})\), \(i = 0, 1, ..., n\) (let \(t_0 = 0, t_{n+1} = T\)), there is no early exercise right. So

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \text{ for } t \in (t_i, t_{i+1}), i = 0, 1, ..., n
\]

At \(t = t_i\), due to the early exercise feature, one has

\[
V(S, t^{-}_i) = \max(V(S, t^{+}_i), \varphi(S)), \text{ for } i = 1, 2, ... n
\]

At expiry,

\[
V(S, T) = \varphi^+.
\]

These form a complete model. Also, it is easy to implement by the binomial tree method.

A question: if there is no dividend payment during the life of the option, whether is there a chance to exercise a Bermudan call option prior to the expiry?
Chapter 4

Multi-asset options

In this chapter we introduce the idea of higher dimensionality by describing the Black-Scholes theory for options on more than one underlying asset. This theory is perfectly straightforward; the only new idea is that correlated random walks and the corresponding multifactor version of Ito Lemma.

4.1 Pricing model

4.1.1 Two-asset options

Consider a European option whose payoff, denoted by $f(S_1, S_2)$, depends on two assets $S_1$ and $S_2$. The basic building block for option pricing with one underlying is the lognormal random walk

$$\frac{dS}{S} = \mu dt + \sigma dW.$$ 

This is readily extended to a world containing two assets via models for each underlying

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dW_1$$

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dW_2$$

As before, we can think of $dW_i$, $i = 1, 2$ as a random number drawn from a Normal distribution with mean zero and standard deviation $dt^{1/2}$ so that

$$E(dW_i) = 0 \text{ and } E[dW_i^2] = dt$$
but the random numbers $dW_1$ and $dW_2$ are correlated:

$$E[dW_1dW_2] = \rho dt$$

Here $\rho$ is the correlation coefficient between the two random walks.

Let $V(S_1, S_2, t)$ be the option value. Since there are two sources of uncertainty, we construct a portfolio of one long option position, two short positions in some quantities of underlying assets:

$$\Pi = V - \Delta_1 S_1 - \Delta_2 S_2.$$ 

Consider the increment

$$d\Pi = dV - \Delta_1 dS_1 - \Delta_2 dS_2$$

Here we need the Ito Lemma involving two variables.

$$dV = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2$$

Actually the two dimensional Ito Lemma can be derived by using Taylor series and the rules of thumb: $dW_i^2 = dt$, $i = 1, 2$, and $dW_1dW_2 = \rho dt$.

Taking $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$ to eliminate risk, we then have

$$d\Pi = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] dt.$$

Then the portfolio is riskless and then earn riskless return, namely

$$d\Pi = r\Pi = r(V - \frac{\partial V}{\partial S_1} S_1 - \frac{\partial V}{\partial S_2} S_2)dt.$$ 

So we arrive at an equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + rS_1 \frac{\partial V}{\partial S_1} + r S_2 \frac{\partial V}{\partial S_2} - rV = 0. \tag{4.1}$$

The solution domain is $\{S_1 > 0, S_2 > 0, t \in [0, T]\}$. The final condition is

$$V(S_1, S_2, T) = f(S_1, S_2) \tag{4.2}$$

(4.1-4.2) form a complete model. Well-known payoffs are the following:
4.1. PRICING MODEL

\[ f(S_1, S_2) = \begin{cases} 
(\max(S_1, S_2) - X)^+, & \text{maximum call} \\
(X - \max(S_1, S_2))^+, & \text{maximum put} \\
(\min(S_1, S_2) - X)^+, & \text{minimum call} \\
(X - \min(S_1, S_2))^+, & \text{minimum put} \\
(S_1 - S_2 - X)^+, & \text{spread option} 
\end{cases} \]  

(4.3)

If the assets pay continuous dividends, then (4.1) is replaced by

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - r V = 0.
\]

(4.4)

where \( q_1 \) and \( q_2 \) are dividend yields of two assets, respectively.

4.1.2 American feature:

Suppose that the option can be exercised early receiving the payoff. Then the pricing model is

\[
\min \{ -L V, V - f(S_1, S_2) \} = 0
\]

\[ V(S, T) = f(S_1, S_2) \]

where

\[
L = \frac{\partial}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + (r - q_1) S_1 \frac{\partial}{\partial S_1} + (r - q_2) S_2 \frac{\partial}{\partial S_2} - r
\]

4.1.3 Exchange option: similarity reduction

An exchange option gives the holder the right to exchange one asset for another. The payoff for this contract at expiry is \((S_1 - S_2)^+\). So the final condition is

\[ V(S_1, S_2, T) = (S_1 - S_2)^+. \]

The governing equation is still (4.1).

This contract is special in that there is a similarity reduction. Let us postulate that the solution takes the form

\[ V(S_1, S_2, t) = S_2 H(\xi, t), \]

where the new variable is

\[ \xi = \frac{S_1}{S_2}. \]
If this is the case, then instead of finding a function $V$ of three variables, we only need find a function $H$ of two variables, a much easier task.

It follows

$$\frac{\partial V}{\partial S_1} = S_2 \frac{\partial H}{\partial \xi} \frac{1}{S_2^2} = \frac{\partial H}{\partial \xi} \frac{\partial^2 V}{\partial S_1^2} = \frac{\partial^2 H}{\partial \xi^2} \frac{1}{S_2^2},$$

$$\frac{\partial V}{\partial S_2} = H + S_2 \frac{\partial H}{\partial \xi} \left( -\frac{S_1}{S_2^2} \right) = H - \xi \frac{\partial H}{\partial \xi},$$

$$\frac{\partial^2 V}{\partial S_2^2} = \frac{\partial H}{\partial \xi} \left( -\frac{S_1}{S_2^2} \right) - \frac{S_1}{S_2^2} \frac{\partial H}{\partial \xi} - \xi \frac{\partial^2 H}{\partial \xi^2} \left( -\frac{S_1}{S_2^2} \right) = \frac{1}{S_2^2} \xi^2 \frac{\partial^2 H}{\partial \xi^2},$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{S_1}{S_2^2} \frac{\partial^2 H}{\partial \xi^2}, \quad \frac{\partial V}{\partial t} = S_2 \frac{\partial H}{\partial \xi}.$$

The partial differential equation now becomes

$$\frac{\partial H}{\partial t} + \left[ \frac{1}{2} \sigma_1^2 \xi^2 - \rho \sigma_1 \sigma_2 \xi^2 + \frac{1}{2} \sigma_2^2 \xi^2 \right] \frac{\partial^2 H}{\partial \xi^2} + (r - q_1) \xi \frac{\partial H}{\partial \xi} + (r - q_2) \left( H - \xi \frac{\partial H}{\partial \xi} \right) - rH = 0$$

or

$$\frac{\partial H}{\partial t} + \frac{1}{2} \sigma' \xi^2 \frac{\partial^2 H}{\partial \xi^2} + (q_2 - q_1) \xi \frac{\partial H}{\partial \xi} - q_2H = 0,$$

where $\sigma' = \sqrt{\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2}$. This equation is just the Black-Scholes equation for a single stock with $q_2$ in place of $r$, $q_1$ in place of the dividend yield on the single stock and with a volatility of $\sigma'$. Note that the final condition is

$$H(\xi, T) = (\xi - 1)^+$$

From this it follows that

$$V(S_1, S_2, t) = S_1 e^{-q_1(T-t)} N(d_1') - S_2 e^{-q_2(T-t)} N(d_2'),$$

where

$$d_1' = \frac{\log(S_1/S_2) + (q_2 - q_1 + \frac{1}{2} \sigma'^2)(T-t)}{\sigma' \sqrt{T-t}}, \quad \text{and} \quad d_2' = d_1' - \sigma' \sqrt{T-t}$$

**Remark 11** An exchange option is a kind of spread option with $X = 0$. If $X \neq 0$, the similarity reduction doesn’t work because the payoff cannot be reduced to a function of $\xi$ and $t$. 
4.1. PRICING MODEL

4.1.4 Options on many underlyings

Options with many underlyings are called basket options, options on baskets or rainbow options. We now extend the two-asset option pricing model to a general case. Suppose

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i.$$ 

Here $S_i$ is the price of the $i$th asset, $i = 1, 2, \ldots, n$, and $\mu_i$ and $\sigma_i$ are the drift and volatility of that asset respectively and $dW_i$ is the increment of a Brownian motion. We can still continue to think of $dW_i$ as a random number drawn from a Normal distribution with mean zero and standard deviation $dt^{1/2}$ so that

$$E(dW_i) = 0 \text{ and } E(dX_i^2) = dt$$

and the random numbers $dW_i$ and $dW_j$ are correlated:

$$E[dW_i dW_j] = \rho_{ij} dt,$$

here $\rho_{ij}$ is the correlation coefficient between the $i$th and $j$th random walks. The symmetric matrix with $\rho_{ij}$ as the entry in the $i$th row and $j$th column is called the correlation matrix. For example, if we have seven underlyings $n = 4$ and the correlation matrix will look like this:

$$D = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & 1 & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & 1 & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & 1
\end{pmatrix}$$

Note that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$. The correlation matrix is positive definite, so that $y^T D y \geq 0$.

To be able to manipulate functions of many random variables we need a multidimensional version of Ito’s lemma. If we have a function of the variables $S_1, S_2, \ldots, S_n$ and $t$, $V(S_1, S_2, \ldots, S_n, t)$, then

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt + \sum_{i=1}^n \frac{\partial V}{\partial S_i} dS_i.$$ 

We can get to this same result by using Taylor series and the rules of thumb:

$$dW_i^2 = dt \text{ and } dW_i dW_j = \rho_{ij} dt.$$ 

The pricing model for basket options is straightforward. Still set up a portfolio consisting of one basket option, and short a number $\Delta_i$ of each of
the asset \( S_i \), employ the multidimensional Ito’s Lemma, take \( \Delta_i = \frac{\partial V}{\partial S_i} \) to eliminate the risk, and set the return of the portfolio equal to the risk-free rate. We are able to arrive at

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} \left( r - q_i \right) S_i \frac{\partial V}{\partial S_i} - rV = 0.
\]

Here \( q_i \) is the dividend yield on the \( i \)th asset. The final condition is

\[
V(S_1, S_2, ..., S_n, t) = f(S_1, S_2, ..., S_n)
\]

The analytic solution to the above model is available, but involves multiply integral, as in the case of two-asset options. (See Page 154, Wilmott (1998))

### 4.2 Quantos

There is one special, and very important type of multi-asset option. This is the cross-currency contract called a quanto. The quanto has a payoff defined with respect to an asset or an index (or an interest rate) in one country, but then the payoff is converted to another currency payment. The general form of its payoff can be expressed as

\[
f(S_\$S, S).
\]

Here \( S_\$S \) is the exchange rate between the domestic currency and the foreign currency (for example, dollar-yen rate, number of dollars per yen), and \( S \) is the level of some foreign asset (for example, the Nikkei Dow index). Note that the quanto contract is measured in domestic currency, but \( S \) is in foreign currency. So this contract is exposed to the exchange rate and the asset. We assume

\[
\begin{align*}
dS_\$S &= \mu_\$S S_\$S dt + \sigma_\$S S_\$dW_\$S \\
dS &= \mu S dt + \sigma S dW
\end{align*}
\]

with a correlation coefficient \( \rho \) between them.

Let \( V(S_\$S, S, t) \) be the quanto option value in US dollar. Construct a portfolio consisting of the quanto, hedged with the foreign currency and the asset:

\[
\Pi = V(S_\$S, S, t) - \Delta_\$S S_\$S - \Delta S S_\$S.
\]

Note that every term in this equation is measured in domestic currency (dollar). \( \Delta_\$S \) is the number of the foreign currency (yen) we hold short, so \(-\Delta_\$S S_\$S \) is the dollar value of that yen. Similarly, with the term \(-\Delta SS_\$S \) we
4.2. QUANTOS

have converted the yen-denominated index \( S \) into dollars, \( \Delta \) is the amount of the index held short.

The change in the value of the portfolio is due to the change in the value of its components and the interest received on the yen:

\[
dV = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S_S^2 \frac{\partial^2 V}{\partial S_S^2} + \rho \sigma_S \sigma_S S \frac{\partial^2 V}{\partial S_S \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt \\
+ \frac{\partial V}{\partial S_S} dS_S + \frac{\partial V}{\partial S} dS \\
- \Delta S_S dS_S - \Delta S_S r_f dt \\
- \Delta S dS - \Delta S r dt \\
= \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S_S^2 \frac{\partial^2 V}{\partial S_S^2} + \rho \sigma_S \sigma_S S \frac{\partial^2 V}{\partial S_S \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \rho \sigma_S \sigma \Delta S_S - \Delta S_S r_f \right] dt \\
+ \left( \frac{\partial V}{\partial S} - \Delta S \right) dS_S + \left( \frac{\partial V}{\partial S_S} - \Delta S_S \right) dS
\]

where the term \(- \Delta S_S r_f dt\) is the interest received by the yen holding, and \(- \rho \sigma_S \sigma \Delta S_S dt\) is due to the increment of the product \(- \Delta S_S\). We now choose

\[
\Delta = \frac{1}{S_S} \frac{\partial V}{\partial S} \quad \text{and} \quad \Delta_S = \frac{\partial V}{\partial S_S} - \frac{S \partial V}{S_S \partial S}
\]

to eliminate the risk in the portfolio. Setting the return on this riskless portfolio equal to the US risk-free rate of interest \( r \), since \( \Pi \) is measured entirely in dollars, yields

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S_S^2 \frac{\partial^2 V}{\partial S_S^2} + \rho \sigma_S \sigma_S S \frac{\partial^2 V}{\partial S_S \partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + (r - r_f) S_S \frac{\partial V}{\partial S_S} + (r_f - \rho \sigma_S) S \frac{\partial V}{\partial S} - r V = 0.
\]

(4.5)

This completes the formulation of the pricing equation. The equation is valid for any contract with underlying measured in one currency but paid in another. The final conditions on \( t = T \):

\[
V(S_S, S, T) = f(S_S, S).
\]

Notice that these parameters correspond to two-asset options with continuous dividend payments (i.e. Eqn (4.4)), where under the risk-neutral world, the underlying assets follow

\[
\frac{dS_1}{S_1} = (r - q_1) dt + \sigma_1 dW_1
\]
\[
\frac{dS_2}{S_2} = (r - q_2)dt + \sigma_2 dW_2
\]

with \( \rho dt = E(dW_1 dW_2) \). For quanto options (i.e. Eqn (4.5)), the underlyings follow in the risk-neutral world

\[
\begin{align*}
\frac{dS_\$}{S_\$} &= (r - f)dt + \sigma_\$ dW \\
\frac{dS}{S} &= (r_f - \rho \sigma_\$ \sigma)dt + \sigma dW
\end{align*}
\]

with \( \rho dt = E(dW_1 W) \). Therefore, in this case, \( q_1 = r_f \) and \( q_2 = r - f_f + \rho \sigma_\$ \sigma \).

### 4.3 Numerical Methods

#### 4.3.1 Binomial tree methods

Suppose \((S_1, S_2)\) will move to \((S_1 u_1, S_2 u_2)\) with probability \(p_1\), \((S_1 u_1, S_2 d_2)\) with probability \(p_2\), \((S_1 d_1, S_2 u_2)\) with probability \(p_3\) and \((S_1 d_1, S_2 d_2)\) with probability \(p_4\) after the next timestep. Then the binomial model for two-asset options is

\[
V(S_1, S_2, t) = e^{-r \Delta t}[p_1 V(S_1 u_1, S_2 u_2, t + \Delta t) + p_2 V(S_1 u_1, S_2 d_2, t + \Delta t) + p_3 V(S_1 d_1, S_2 d_2, t + \Delta t) + p_4 V(S_1 d_1, S_2 u_2, t + \Delta t)]
\]

where \(p_i\) for \(i = 1, 2, 3, 4\), \(u_i, d_i\) for \(i = 1, 2\) are chosen to be consistent with the continuous-time model. One choice for these parameters is given as follows

\[
u_i = e^{\sigma_i \sqrt{\Delta t}}, \quad d_i = \frac{1}{u_i} \quad \text{for } i = 1, 2.
\]

\[
p_1 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1} + \frac{r - q_2 - \frac{\sigma_2^2}{2}}{\sigma_2} \right) \sqrt{\Delta t} + \rho \right]
\]

\[
p_2 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - q_2 - \frac{\sigma_2^2}{2}}{\sigma_2} \right) \sqrt{\Delta t} - \rho \right]
\]

\[
p_3 = \frac{1}{4} \left[ 1 + \left( -\frac{r - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r - q_2 - \frac{\sigma_2^2}{2}}{\sigma_2} \right) \sqrt{\Delta t} + \rho \right]
\]
4.3. NUMERICAL METHODS

\[ p_4 = \frac{1}{4} \left( 1 + \left( \frac{r - q_1 - \sigma_1^2}{\sigma_1^2} + \frac{r - q_2 - \sigma_2^2}{\sigma_2^2} \right) \sqrt{\Delta t} - \rho \right) . \]

We refer interested students to Kwok (1998) [pp 207-208] for derivation of the above parameters. It should be pointed out that we can also use the finite difference method to determine these parameters.

4.3.2 Monte-Carlo simulation

The amount of computation of BTM grows exponentially with the number of underlyings. We will have to give up BTM if the number of underlyings is greater than 3, and instead employ Monte-Carlo simulation which is relatively more efficient as the number of underlyings increases.

Monte-Carlo simulation is based on the risk-neutral valuation result. The expected payoff in a risk-neutral world is calculated using a sampling procedure. It is then discounted at the risk-free interest rate.

Suppose in a risk-neutral world

\[ dS_i = \mu_i S_i dt + \sigma_i S_i dW_i, \quad (1 \leq i \leq n) \]

As in the single-variable case, the life of the derivative must be divided into \( N \) subintervals of length \( \Delta t \). The discrete version of the process for \( S_i \) is then

\[ S_i(t + \Delta t) - S_i(t) = \tilde{\mu}_i S_i \Delta t + \sigma_i S_i \epsilon_i \sqrt{\Delta t}, \quad (4.6) \]

where \( \epsilon_i \) is a random sample from a standard normal distribution. The coefficient of correlation between \( \epsilon_i \) and \( \epsilon_j \) is \( \rho_{ij} \) for \( 1 \leq i, j \leq n \). One simulation trial involves obtaining \( N \) samples of the \( \epsilon_i \) \( (1 \leq i \leq n) \) from a multivariate standardized normal distribution. These are substituted into equation (4.6) to produce simulated paths for each \( S_i \) and enable a sample value for the derivative to be calculated.

Note that correlated samples \( \epsilon_i \) \( (1 \leq i \leq n) \) from standard normal distributions are required. We only give a procedure for \( n = 2 \). For \( n \geq 3 \), we refer interested readers to Appendix. Independent samples \( x_1 \) and \( x_2 \) from a univariate standardized normal distribution are easily obtained. The required samples \( \epsilon_1 \) and \( \epsilon_2 \) are then calculated as follows:

\[ \epsilon_1 = x_1 \]
\[ \epsilon_2 = \rho x_1 + \sqrt{1 - \rho^2} x_2 \]

where \( \rho \) is the coefficient of correlation.
Remark 12  (1) At each time step, we need to find $\epsilon_i$ ($1 \leq i \leq 2$).
(2) The number of simulation trials $M$ carried out depends on the accuracy required. In general, we take $M = 5000$ or $10000$.

Remark 13 The amount of computation of Monte-Carlo simulation grows only linearly with the number of underlyings. The main drawback of Monte Carlo simulation is that it cannot easily handle situations where there are early exercise opportunities.

4.4 *Suggestions for further reading and Appendix

4.4.1 Further reading

Most materials in section 1 is extracted from Wilmott (1998) (Chapter 11) where a general integral representation can be found for multi-asset options.
Section 2 is from Hull (2000 4th) (pp 406-409). We refer interested readers to the book for some skills improving the accuracy of the Monte-Carlo simulation (pp 411-414).
Kwok (1998) presented more examples of quanto options as well as their pricing formulas (pp 115-118).
The pricing formulas for the European options on the maximum or minimum of $n$ risky assets can also be found in Kwok (1998) (P. 123), where the $n$-variate normal distribution function is involved. Especially, we present the formula for the European call option on the minimum of two assets (payoff: $(\min(S_1, S_2) - X)^+$).

$$C_{\min}(S_1, S_2, t) = S_1N_2 \left( \frac{\log \frac{S_1}{X} + (r + \frac{\sigma_1^2}{2})(T - t)}{\sigma_1\sqrt{T - t}}, \frac{\log \frac{S_2}{X} - \frac{\sigma_1^2}{2}(T - t)}{\sigma_1\sqrt{T - t}} ; \frac{\rho\sigma_2 - \sigma_1}{\sigma_1} \right) + S_2N_2 \left( \frac{\log \frac{S_2}{X} + (r + \frac{\sigma_2^2}{2})(T - t)}{\sigma_2\sqrt{T - t}}, \frac{\log \frac{S_1}{X} - \frac{\sigma_1^2}{2}(T - t)}{\sigma_1\sqrt{T - t}} ; \frac{\rho\sigma_1 - \sigma_2}{\sigma_2} \right) - Xe^{-r(T-t)}N_2 \left( \frac{\log \frac{S_1}{X} + (r - \frac{\sigma_1^2}{2})(T - t)}{\sigma_1\sqrt{T - t}}, \frac{\log \frac{S_2}{X} - \frac{\sigma_2^2}{2}(T - t)}{\sigma_2\sqrt{T - t}} ; \rho \right).$$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ and $N_2(a, b; \rho)$ is the standardized bivariate normal distribution function, i.e.

$$N_2(a, b; \rho) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} \exp \left( -\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right) dxdy.$$
A numerical procedure for \( N_2(a, b; \rho) \) can be found at Appendix 11C, page 272, Hull (2000, 4th)

4.4.2 Appendix:

Consider the situation where we require \( n \) correlated samples from normal distributions where the coefficient of correlation between sample \( i \) and sample \( j \) is \( \rho_{ij} \). We first sample \( n \) independent variables \( x_i (1 \leq i \leq n) \), from univariate standardized normal distributions. The required samples are \( \epsilon_i \) (\( 1 \leq i \leq n \)), where

\[
\epsilon_i = \sum_{k=1}^{i} \alpha_{ik} x_k.
\]

For \( \epsilon_i \) to have the correct variance and the correct correlation with the \( \epsilon_j \) (\( 1 \leq j \leq n \)), we must have

\[
\sum_{k=1}^{i} \alpha_{ik}^2 = 1
\]

and, for all \( j \leq i \),

\[
\sum_{k=1}^{j} \alpha_{ik} \alpha_{jk} = \rho_{ij}.
\]

The first sample, \( \epsilon_1 \), is set equal to \( x_1 \). These equations for the \( \alpha \)'s can be solved so that \( \epsilon_2 \) is calculated from \( x_1 \) and \( x_2 \); \( \epsilon_3 \) is calculated from \( x_1, x_2 \) and \( x_3 \); and so on. The procedure is known as the Cholesky decomposition.

For example, when \( n = 3 \),

\[
\begin{align*}
\epsilon_1 &= x_1 \\
\epsilon_2 &= \rho_{21} x_1 + \sqrt{1 - \rho_{21}^2} x_2 \\
\epsilon_3 &= \rho_{31} x_1 + \frac{\rho_{23} - \rho_{31} \rho_{21}}{\sqrt{1 - \rho_{21}^2}} x_2 + \sqrt{\frac{1 + 2 \rho_{23} \rho_{21} \rho_{31} - \rho_{21}^2 - \rho_{31}^2 - \rho_{23}^2}{1 - \rho_{21}^2}} x_3
\end{align*}
\]
Chapter 5

Path-dependent options

The contracts we have seen so far are the most basic and important derivative products. In this chapter, we shall discuss some complex contracts, including barrier options, Asian options, lookback options and so on.

5.1 Barrier options

Barrier (or knock-in, knock-out) options are triggered by the action of the underlying hitting a prescribed value at some time before expiry. For example, as long as the asset remains below a pre-determined barrier price during the whole life of the option, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry, then the option becomes worthless because it has “knocked out”. Barrier options are clearly path dependent. A barrier option is cheaper than a similar vanilla option since the former provides the holder with less rights than the latter does.

5.1.1 Different types of barrier options

There are two main types of barrier option:

1. The out option, that only pays off if a level is not reached. If the barrier is reached then the option is said to have knocked out
2. The in option, that pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have knocked in.

Then we further characterize the barrier option by the position of the barrier relative to the initial value of the underlying:

1. If the barrier is above the initial asset value, we have an up option.
2. If the barrier is below the initial asset value, we have a down option. Finally, the options can also be classified as call and put according to the payoffs. In addition, if early exercise is permitted, the option is called American-style barrier option. In the following, barrier options always refer to European-style options except for special claim.

5.1.2 In-Out parity

For European style barrier option, the relationship between a ‘in’ barrier option and an ‘out’ barrier option (with same payoff and same barrier level) is very simple:

\[ \text{in} + \text{out} = \text{vanilla}. \]

If the ‘in’ barrier is triggered then so is the ‘out’ barrier, so whether or not the barrier is triggered the portfolio of an ‘in’ and ‘out’ option has the vanilla payoff at expiry.

However, the above in-out parity doesn’t hold for American-style barrier options.

5.1.3 Pricing by Monte-Carlo simulation

To illustrate method, we consider an up-out-call option whose terminal payoff can be written as

\[ (S_T - X)^+ I_{\{S_t < H, \text{ for all } t \in [0,T]\}} \]

where \( H \) is the barrier level, \( I \) is the index function, i.e.

\[ I_{\{S_t < H, t \in [0,T]\}} = \begin{cases} 1, & \text{if } S_t < H \text{ for all } t \in [0,T] \\
0, & \text{otherwise} \end{cases} \]

According to the risk neutral pricing principle, the option value is

\[ e^{-rT} \hat{E}[(S_T - X)^+ I_{\{S_t < H, \text{ for all } t \in [0,T]\}}] \]

We then use the Monte-Carlo simulation to get an approximate value of the option.

It should be pointed out that the simulation can only apply to European style options.
5.1. BARRIER OPTIONS

5.1.4 Pricing in the PDE framework

Barrier options are weakly path dependent. We only have to know whether or not the barrier has been triggered, we do not need any other information about the path. This is contrast to some of the contracts we will be seeing shortly, such as the Asian and lookback options, that are strongly path dependent.

We can use \( V(S, t) \) to denote the value of the barrier contract before the barrier has been triggered. This value still satisfies the Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - r V = 0. \tag{5.1}
\]

The details of the barrier feature come in through the specification of the boundary conditions and solution domains.

‘Out’ Barriers

If the underlying asset reaches the barrier in an ‘out’ barrier option then the contract becomes worthless. This leads to the boundary condition

\[ V(H, t) = 0 \text{ for } t \in [0, T]. \]

The final condition is still

\[ V(S, T) = \begin{cases} (S - X)^+ & \text{for call} \\ (X - S)^+ & \text{for put} \end{cases}. \]

The solution domain is \( \{0 < S < H\} \times [0, T) \) for an up-out option, and \( \{H < S < \infty\} \times [0, T) \) for a down-out option.

‘In’ Barriers

An ‘in’ option only has a payoff if the barrier is triggered. Remember that \( V(S, t) \) stands for the option value before the barrier has been triggered. If the barrier is not triggered during the option’s life, the option expires worthless so that we have the final condition

\[ V(S, T) = 0. \]

Once the barrier is triggered, the barrier-in option becomes a vanilla option then on the barrier the contract must have the same value as a vanilla contract:

\[ V(H, t) = \begin{cases} C_E(H, t), & \text{for call} \\ P_E(H, t), & \text{for put} \end{cases}, \text{ for } t \in [0, T], \]
where $C_E(S, t)$ and $P_E(S, t)$ represent the values of the (European-style)
vanilla call and put, respectively.

The solution domain is $\{0 < S < H\} \times [0, T)$ for an up-in option, and
$\{H < S < \infty\} \times [0, T)$ for a down-in option.

*Explicit solutions*

Closed-form solutions for European-style barrier options are available. We
refer interested readers to those references for these formulas. Here we only
list the formula for the down-and-out call option with $H \leq X$:

$$
C_{do}(S, t) = C_E(S, t) - \left( \frac{S}{H} \right)^{1-2(r-q)/\sigma^2} C_E\left( \frac{H^2}{S}, t \right),
$$

(5.2)

where $C_E(S, t)$ is the value of the European vanilla call.

Let us confirm that this is indeed the solution. Clearly, on the barrier
$C_{do}(H, t) = C_E(H, t) - C_E(H, t) = 0$. At expiry

$$
C_{do}(S, T) = (S - X)^+ - \left( \frac{S}{H} \right)^{1-2(r-q)/\sigma^2} \left( \frac{H^2}{S} - X \right)^+
$$

$$
= (S - X)^+, \text{ for } S > H.
$$

So the remaining is to show $C_{do}(S, t)$ satisfies the Black-Scholes equation.
The first term on the right-hand side of (5.2) does. The second term does
also. Actually, if we have any solution $V_{BS}$, of the Black-Scholes equation it
is easy to show that

$$
S^{1-2(r-q)/\sigma^2} V_{BS}\left( \frac{A}{S}, t \right)
$$

is also a solution for any constant $A$.

5.1.5 **American early exercise**

For American knock-out options, the pricing model is a free boundary prob-
lem:

$$
\min \left\{ - \frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - q) S \frac{\partial V}{\partial S} + rV, V - \varphi \right\} = 0, \ (S, t) \in D
$$

$$
V(H, t) = 0
$$

$$
V(S, T) = \varphi^+
$$

where $D = (H, \infty) \times [0, T)$ for a down-out, and $D = (0, H) \times [0, T)$ for an
up-out.
5.1. BARRIER OPTIONS

However, an American knock-in option price is governed still by the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in D$$

(5.3)

The reason is $V(S,t)$ represents the option value before the barrier is triggered. Thus for $(S,t) \in D$, the option cannot be early exercised because it has not been activated. Actually the option should be called the knock-into American option. As in the European case, at expiry we have

$$V(S,T) = 0$$

(5.4)

Once the barrier is triggered, the option becomes an American vanilla option then on the barrier. So,

$$V(H,t) = \begin{cases} 
C_A(H,t), & \text{for call} \\
P_A(H,t), & \text{for put}
\end{cases}, \quad \text{for } t \in [0,T],$$

(5.5)

where $C_A(S,t)$ and $P_A(S,t)$ represent the values of the American vanilla call and put, respectively. (5.3-5.5) forms a complete model for American knock-in options.

5.1.6 BTM

The BTM can be readily extended to cope with barrier options. Recall the Black-Scholes still holds. So the BTM must hold too, that is

$$V(S,t) = \frac{1}{\rho} \left[ pV(Su, t + \Delta t) + (1 - p)V(Sd, t + \Delta t) \right]$$

with appropriate final and boundary conditions. For instance, for a down-out call option,

$$V(H,t) = 0$$

$$V(S,T) = (S - X)^+, \quad S > H$$

and the backward procedure is conducted in the region $\{S > H\} \times [0,T]$.

When implementing the above method, the key point is to set the barrier to pass those nodes of the tree. Otherwise it will lead to a large error. To guarantee that the barrier matches the tree, the starting point of tree, in general, cannot begin with $(S_0,0)$. So, the interpolation technique should be employed: the option value at $(S_0,0)$ will be computed by a linear interpolation of the values of two adjacent tree points.

Also, the method can apply to American barrier options when early exercise is considered.
5.1.7 Hedging

Barrier options have discontinuous delta at the barrier. For example, a up-out call option value is continuous, decreasing approximately linearly towards the barrier then being zero beyond the barrier. This discontinuity in the delta means that the gamma is instantaneously infinite at the barrier. So, it is very hard to hedge a barrier option. We refer interested students to Wilmott (1998) and references therein.

5.1.8 Other features

The barrier option discussed above is the commonest one. Actually, the position of the barrier in the contract can be time dependent. The level may begin at one level and then rise, say. Usually the level is a piecewise-constant function of time.

Another style of barrier option is the double barrier. Here there is both an upper and a lower barrier, the first above and the second below the current asset price. In a double ‘out’ option the contract becomes worthless if either of the barriers is reached. In a double ‘in’ option one of the barriers must be reached before expiry, otherwise the option expires worthless. Other possibilities can be imagined: one barrier is an ‘in’ and the other ‘out’, at expiry the contract could have either and ‘in’ or an ‘out’ payoff.

Sometimes a rebate is paid if the barrier level is reached. This is often the case for ‘out’ barriers in which case the rebate can be thought of as cushioning the blow of losing the rest of the payoff. The rebate may be paid as soon as the barrier is triggered or not until expiry.

5.2 Asian options

Asian options give the holder a payoff that depends on the average price of the underlying during the options’ life.

5.2.1 Payoff types

The payoff from a fixed strike Asian call (or, average call) is $(A_T - X)^+$, and that from a fixed strike Asian put (or, average price put) is $(X - A_T)^+$, where $A_T$ is the average value of the underlying asset calculated over a predetermined average period. Fixed strike Asian options are less expensive than vanilla options and are more appropriate than vanilla options for meeting some of the needs of corporate treasurers. Suppose that a U.S. corporate
treasurer expects to receive a cash flow of 100 million Australian dollars spread evenly over the next year from the company’s Australian subsidiary. The treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. A fixed strike Asian put can achieve this more effectively than vanilla put options.

Another type of Asian options is a floating strike option (or, average strike option). A floating strike Asian call pays off \((S_T - A_T)^+\) and a floating strike Asian put pays off \((A_T - S_T)^+\). Floating strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price.

5.2.2 Types of averaging

The two simplest and obvious types of average are the arithmetic average and the geometric average. The arithmetic average of the price is the sum of all the constituent prices, equally weighted, divided by the total number of prices used. The geometric average is the exponential of the sum of all the logarithms of the constituent prices, equally weighted, divided by the total number of prices used. Furthermore, the average may be based on discretely sampled prices or on continuously sampled prices. Then, we have

\[
A_T = \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} S_{t_i}, & \text{discretely sampled arithmetic} \\
\frac{1}{T} \int_0^T S_r dr, & \text{continuously sampled arithmetic} \\
\exp\left(\frac{1}{n} \sum_{i=1}^{n} \ln S_{t_i}\right) = (S_{t_1}S_{t_2}...S_{t_n})^{1/n}, & \text{discretely sampled geometric} \\
\exp\left(\frac{1}{T} \int_0^T \ln S_r dr\right), & \text{continuously sampled geometric}
\end{cases}
\]

5.2.3 Extending the Black-Scholes equation

Monte-Carlo simulation is easy to cope with the pricing of all kinds of European-style Asian options. However, for American-style Asian options, we have to fall back on PDE framework or BTM. In the following we only consider the continuously sampled Asian options. We refer interested reader to Wilmott et al. (1995), Wilmott (1998) for the PDE formulation of discretely sampled Asian options.

We start by assuming that the underlying asset follows the lognormal random walk

\[
dS_t = \mu S_t dt + \sigma S_t dW_t
\]
The value of an Asian option is not only a function of $S$ and $t$, but also a function of the historical average $A$, that is $V = V(S_t, A_t, t)$. Here $A_t$ will be a new independent variable. Let us focus on the arithmetic average for which

$$A_t = \frac{1}{t} \int_0^t S_\tau d\tau.$$  

In anticipation of argument that will use Ito lemma, we need to know the stochastic differential equation satisfied by $A$. It is not hard to check that

$$dA_t = \frac{tS_t - \int_0^t S_\tau d\tau}{t^2} dt = \frac{S_t - A_t}{t} dt.$$ 

We can see that its stochastic differential equation contains no stochastic terms. So, the Ito lemma for $V = V(S, A, t)$ is

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial A} dA + \frac{\partial V}{\partial S} dS$$

$$= \left( \frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS.$$

To derive the pricing PDE, we set up a portfolio containing one of the path-dependent option and short a number $\Delta$ of the underlying asset:

$$\Pi = V(S, A, t) - \Delta S$$

The change in the value of this portfolio is given by

$$d\Pi = dV - \Delta dS$$

$$= \left( \frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( \frac{\partial V}{\partial S} - \Delta \right) dS.$$ 

Choosing

$$\Delta = \frac{\partial V}{\partial S}$$

to hedge the risk, we find that

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$ 

This change is risk free, thus earns the risk-free rate of interest $r$, leading to the pricing equation

$$\frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (5.6)$$
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The solution domain is \{S > 0, A > 0, t \in [0, T]\}.

This is to be solved subject to

\[ V(S, A, T) = \begin{cases} 
(A - S)^+, \text{ floating put} \\
(S - A)^+, \text{ floating call} \\
(A - X)^+, \text{ fixed call} \\
(X - A)^+, \text{ fixed put} 
\end{cases} \]

The obvious changes can be made to accommodate dividends on the underlying. This completes the formulation of the valuation problem.

For geometric average,

\[ dA = d \exp \left( \frac{1}{t} \int_0^t \ln S_\tau d\tau \right) = \exp \left( \frac{1}{t} \int_0^t \ln S_\tau d\tau \right) \frac{t \ln S - \int_0^t \ln S_\tau d\tau}{t^2} dt \]

So the pde satisfied by geometric Asian options is

\[ \frac{\partial V}{\partial t} + \frac{A \ln S}{t} \frac{\partial V}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \]

The solution domain is \{S > 0, A > 0, t \in [0, T]\}.

5.2.4 Early exercise

The only point to mention is that the details of the payoff on early exercise have to be well defined. The payoff at expiry depends on the value of the average up to expiry; this will, of course, not be known until expiry. Typically, on early exercise it is the average to date that is used. For example, in an American floating strike arithmetic put the early payoff would be

\[ \left( \frac{1}{t} \int_0^t S_\tau d\tau - S_t \right)^+. \]

In general, we denote the exercise payoff at time \(t\) by \(\Lambda(S_t, A_t)\), where

\[ \Lambda(S_t, A_t) = \begin{cases} 
(A_t - S_t)^+, \text{ floating put} \\
(S_t - A_t)^+, \text{ floating call} \\
(A_t - X)^+, \text{ fixed call} \\
(X - A_t)^+, \text{ fixed put} 
\end{cases} \]
Such representation is consistent with the terminal payoff. Then the pricing model is formulated by

\[
\min \left\{ -\frac{\partial V}{\partial t} - LV, V - \Lambda(S, A) \right\} = 0 \\
V(S, A, T) = \Lambda(S, A)
\]

where

\[
L = \begin{cases} 
\frac{S - A}{t} \frac{\partial}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r, & \text{for arithmetic} \\
\frac{\ln(S/A)}{t} \frac{\partial}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r, & \text{for geometric}
\end{cases}
\]

**Remark 14** All European-style Asian geometric options have explicit price formulas. However, that is generally not true for Asian arithmetic options and all American-style Asian options which must be solved by numerical approaches.

### 5.2.5 Reductions in dimensionality

The Asian options value depends on three variables. For some cases, a reduction in dimensionality of the problem is permitted. For example, we consider the a European floating strike geometric Asian option. Adopt the following transformation:

\[
\frac{V(S, A, t)}{S} = W(x, t), \quad x = t \ln \frac{A}{S},
\]

the model reduces to (suppose there are continuous dividend payments)

\[
\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 t^2 \frac{\partial^2 W}{\partial x^2} + (q - r - \frac{\sigma^2}{2}) t \frac{\partial W}{\partial x} - qW = 0, \quad t \in [0, T), \quad x \in (-\infty, \infty)
\]

\[
W(x, T) = \left(1 - e^{x} \right)^{+}.
\]

The problem is relatively easy to solve. Actually it has an explicit solution.

**Remark 15** For any floating strike Asian option (arithmetic or geometric, European or American), one can find an appropriate transformation to reduce the model to a lower dimensional problem. As for fixed strike Asian options, it is true only for European-style.
5.2. ASIAN OPTIONS

5.2.6 Parity relation

We will give one example. Consider European-style floating strike arithmetic call options. The payoff at expiry for a portfolio of one call held long and one put held short is

\[(S - A)^+ - (A - S)^+\]

where \(S\) is simply \(S - A\).

The value of the portfolio then satisfies the equation (??) with the final condition

\[V(S, A, T) = S - A.\]

It can be verified that the solution is

\[V(S, A, t) = S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{t}{T}e^{-r(T-t)}A.\]

Therefore, for floating strike arithmetic Asian options, we have the put-call parity relation

\[V_{fc}(S, A, t) - V_{fp}(S, A, t) = S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{t}{T}e^{-r(T-t)}A.\]

We have similar results for other European Asian options.

5.2.7 Model-dependent and model-independent results

We need to point out that some results depends on assumptions of the model. For example, the above Asian put-call parity holds for the Black-Scholes model, but it might not be true, in general, provided that the geometric Brownian assumption is given up.

Recall some results are model-independent. For example:

1. the put-call parity for vanilla options
2. the in-out parity for barrier options
3. American call options should never be early exercised if there is no dividend payment.

To acquire these results, one only needs no-arbitrage principle.
But some are true only under Black-Scholes framework. For instance:

1. the above Asian put-call parity
2. the put-call symmetry relation
5.2.8 Binomial tree method

The binomial tree method for European-style Asian option is given as follows:

\[
V(S, A, t) = e^{-r\Delta t} [pV(Su, A_u, t + \Delta t) + (1 - p)V(Sd, A_d, t + \Delta t)]
\]

\[
V(S, A, T) = \Lambda(S, A)
\]

where

\[
p = \frac{e^{(r-q)\Delta t} - d}{u - d}
\]

and

\[
A_u = \frac{tA + \Delta tSu}{t + \Delta t}, \quad A_d = \frac{tA + \Delta tSd}{t + \Delta t}.
\]

Early exercise can be easily incorporated into the above algorithm as before to deal with American style options.

5.3 Lookback options

The dream contract has to be one that pays the difference between the highest and the lowest asset prices realized by an asset over some period. Any speculator is trying to achieve such a trade. The contract that pays this is an example of a lookback option, an option that pays off some function of the realized maximum and/or minimum of the underlying asset over some prescribed period. Since lookback options have such an extreme payoff they tend to be expensive.

5.3.1 Types of Payoff

For the basic lookback contracts, the payoff comes in two varieties, like the Asian option: the fixed strike and the floating strike, respectively. These have payoffs that are the same as vanilla options except that in the floating strike option the vanilla exercise price is replaced by the maximum or minimum. In the fixed strike option it is the asset value in the vanilla option that is replaced by the maximum or minimum. That is

\[
\Lambda(S_T, M_T) = \begin{cases} 
M_T - S_T, & \text{floating put} \\
S_T - m_T, & \text{floating call} \\
(M_T - X)^+, & \text{fixed call} \\
(X - m_T)^+, & \text{fixed put} 
\end{cases}
\]
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Here

\[ M_T = \max_{0 \leq \tau \leq T} S_\tau \] and \[ m_T = \min_{0 \leq \tau \leq T} S_\tau. \]

Note that for floating lookback options + can be removed.

5.3.2 Extending the Black-Scholes equations

Let \( V = V(S_t, M_t, t) \) (or. \( V = V(S_t, m_t, t) \)) be the lookback option value, where

\[ M_t = \max_{0 \leq \tau \leq t} S_\tau \] and \[ m_t = \min_{0 \leq \tau \leq t} S_\tau. \]

We anticipate that Ito lemma will be used to derive the model. However, \( \max_{0 \leq \tau \leq t} S_\tau \) (or. \( \min_{0 \leq \tau \leq t} S_\tau \)) is not differentiable. So we have to introduce another variable. (let us consider the fixed call or floating put)

\[ J_{nt} = \left( \frac{1}{t} \int_0^t S^n_t d\tau \right)^{1/n}. \]

It is easy to see

\[ \lim_{n \to \infty} J_{nt} = \max_{0 \leq \tau \leq t} S_t = M_t \]

and

\[ dJ_{nt} = \frac{J_{nt}^{1-n}}{nt} (S^n_t - J^n_{nt}) dt. \]

Next we consider the function \( V(S_t, J_{nt}, t) \), which can be imagined as a product depends on the variable \( J_{nt} \). Using the \( \Delta - \)hedging argument and Ito lemma, we find that \( V(S_t, J_{nt}, t) \) satisfies

\[
\frac{\partial V}{\partial t} + \frac{1}{nt} \frac{S^n - J^n_n}{J^n_n - 1} \frac{\partial V}{\partial J_n} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
\]

We now take the limit \( n \to \infty \). Note that \( S_t \leq \max_{0 \leq \tau \leq t} S_\tau = M_t \). When \( S < M \),

\[ \lim_{n \to \infty} \frac{1}{nt} \frac{S^n - J^n_n}{J^n_n - 1} = 0; \]

Thus we obtain a governing equation for the floating strike lookback put option

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \text{ for } S < M
\]

At expiry we have

\[ V(S, M, T) = \Lambda(S, M) \]
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The solution domain is \( \{ S \leq M \} \times [0, T] \). Here we require an auxiliary condition on \( S = M, \)
\[
\frac{\partial V}{\partial M} \bigg|_{M=S} = 0.
\]

For a floating call or fixed call, we can establish its governing equation in a similar way. And the solution domain is \( \{ S \geq m \} \times [0, T] \).

5.3.3 BTM

Consider a floating put or fixed call. Similarly we have
\[
V(S, M, t) = e^{-r \Delta t} [pV(S_u, M_u, t + \Delta t) + (1 - p)V(S_d, M_d, t + \Delta t)]
\]
where
\[
p = \frac{e^{(r-q)\Delta t} - d}{u - d}
\]
and
\[
M_u = \max(M, S_u), \text{ and } M_d = \max(M, S_d).
\]
Due to \( M \geq S \) and \( d < 1 \), we have
\[
M_d = M.
\]
Then the binomial tree method is given by
\[
V(S, A, t) = e^{-r \Delta t} [pV(S_u, M_u, t + \Delta t) + (1 - p)V(S_d, M, t + \Delta t)], \text{ for } S \leq M
\]
\[
V(S, A, T) = \Lambda(S, A)
\]

5.3.4 Consistency of the BTM and the continuous-time model:

Note that \( M \geq S \). We only need to consider two cases:

(1) \( M \geq S_u \): In this case \( M_u = M_d = M \). Then the binomial tree scheme can be rewritten as
\[
V(S, M, t) = e^{-r \Delta t} [pV(S_u, M, t + \Delta t) + (1 - p)V(S_d, M, t + \Delta t)].
\]
Using the Taylor expansion, we obtain
\[
-rV(S, M, t) + \frac{\partial V}{\partial t}(S, M, t) + (r-q)S \frac{\partial V}{\partial S}(S, M, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, M, t) = O(\Delta t).
\]

(2) \( M = S \): Then \( M_u = S_u \) and \( M_d = S \), and the binomial tree scheme becomes
\[
V(S, S, t) = e^{-r \Delta t} [pV(S_u, S_u, t + \Delta t) + (1 - p)V(S_d, S, t + \Delta t)].
\]
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By virtue of the Taylor expansion, we have

\[
V(S, S, t) - e^{-r\Delta t} [pV(Su, Su, t + \Delta t) + (1 - p)V(Sd, S, t + \Delta t)]
= -\left[ \frac{\partial V}{\partial t}(S, S, t) + (r - q)S \frac{\partial V}{\partial S}(S, S, t) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, S, t) - rV(S, S, t) \right] \Delta t
- e^{-r\Delta t} p(Su - S) \frac{\partial V}{\partial M}(S, S, t) + O(\Delta t)
= \Delta t^{1/2} \frac{\partial V}{\partial M}(S, S, t) + O(\Delta t),
\]

which implies

\[
\frac{\partial V}{\partial M} = 0 \text{ at } M = S.
\]

**Remark 16** For Asian options, the Taylor expansion also gives the consistency. Indeed,

\[
V(S, A, t) - e^{-r\Delta t} [pV(Su, A, t + \Delta t) + (1 - p)V(Sd, A, t + \Delta t)]
= -\left[ \frac{\partial V}{\partial t}(S, A, t) + (r - q)S \frac{\partial V}{\partial S}(S, A, t) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, A, t) - rV(S, A, t) \right] \Delta t
- e^{-r\Delta t} [p(A_u - A) + (1 - p)(A_d - A)] \frac{\partial V}{\partial A}(S, A, t)
- e^{-r\Delta t} [p(u - 1)(A_u - A) + (1 - p)(d - 1)(A_d - A)] S \frac{\partial^2 V}{\partial S \partial A}(S, A, t)
+ O(\Delta t^2) + O((A_u - A)\Delta t) + O((A_d - A)\Delta t) = O((A_u - A)^2) + O((A_d - A)^2).
\]

For the arithmetic average, since \( A_u - A = \frac{Su - A}{t} \Delta t \) and \( A_d - A = \frac{Sd - A}{t} \Delta t \), we have

\[
e^{-r\Delta t} [p(A_u - A) + (1 - p)(A_d - A)] = \frac{S - A}{t} \Delta t + O(\Delta t^2),
\]

\[
e^{-r\Delta t} [p(u - 1)(A_u - A) + (1 - p)(d - 1)(A_d - A)] = O(\Delta t^2).
\]

Then we get

\[
V(S, A, t) - e^{-r\Delta t} [pV(Su, A, t + \Delta t) + (1 - p)V(Sd, A, t + \Delta t)]
= \left[ -\frac{\partial V}{\partial t} - \frac{S - A}{t} \frac{\partial V}{\partial A} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - (r - q)S \frac{\partial V}{\partial S} + rV \right] \Delta t + O(\Delta t^2).
\]

It is similar for the geometric average.
5.3.5 Similarity reduction

The similarity reduction also applies to some lookback options. For example, for floating strike lookback put options, it follows from the transformations \( V(S,M,t) = W(x,t) \) and \( x = \frac{M}{S} \),

\[
\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} + (q-r)x \frac{\partial W}{\partial x} - qW = 0, \quad t \in [0,T), \quad x \in (1, \infty)
\]

\[
\frac{\partial W}{\partial x} \bigg|_{x=1} = 0
\]

\[
W(x,T) = x-1.
\]

The reduction can be extended to the binomial tree model. Taking the same transformations, we have for the floating strike lookback call

\[
SW\left( \frac{M}{S}, t \right) = e^{-r\Delta t} \left[ puW\left( \max\left( \frac{M}{Su}, S\right), t+\Delta t \right) + (1-p)SdW\left( \frac{M}{Sd}, t+\Delta t \right) \right]
\]

or

\[
W(x,t) = e^{-r\Delta t} \left[ puW(\max(xd,1), t+\Delta t) + (1-p)dW(xu, t+\Delta t) \right]
\]

\[
W(x,T) = x-1, \quad \text{for } x \geq 1.
\]

(5.7) can be rewritten as

\[
W(x,t) = e^{-r\Delta t} \left[ puW(xd, t+\Delta t) + (1-p)dW(xu, t+\Delta t) \right], \quad \text{for } x \geq u
\]

\[
W(1,t) = e^{-r\Delta t} \left[ puW(1, t+\Delta t) + (1-p)dW(u, t+\Delta t) \right],
\]

\[
W(x,T) = x-1.
\]

5.3.6 Russian options

The similarity reduction can also be applied to the American-style lookback option. Here, we consider the Russian option, a special perpetual option, whose payoff is \( M \). Since it is a perpetual option, the option value is independent of time. Let \( V = V(S,M) \) denote the option value. The pricing model for the Russian option is given by

\[
\min \left\{ -\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S \frac{\partial V}{\partial S} + rV, V - M \right\} = 0, \quad \text{for } S < M
\]

\[
\frac{\partial V}{\partial M} \bigg|_{S=M} = 0.
\]
Using the transformations

\[ W(x) = \frac{V(S, M)}{S} \text{ and } x = \frac{M}{S}, \]

we have

\[
\min \left\{ -\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q - r)x \frac{\partial W}{\partial x} + qW, W - x \right\} = 0, \text{ for } x > 1
\]

\[
\left. \frac{\partial V}{\partial x} \right|_{x=1} = 0.
\]

For a fixed \( S \), it becomes more attractive to exercise the Russian option if \( M \) is sufficiently large. Then, the above model can be rewritten as

\[
-\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (q - r)x \frac{\partial W}{\partial x} + qW = 0, \text{ for } 1 < x < x^*
\]

\[
W(x^*) = x^*
\]

\[
\frac{\partial W}{\partial x}(x^*) = 1
\]

\[
\frac{\partial W}{\partial x}(1) = 0.
\]

Solving the equations we can obtain the option value (Assignment). A question: what about the option value if \( q = 0 \).

### 5.4 Miscellaneous exotics

#### 5.4.1 Forward start options

As its name suggests, a forward start option is an option that comes into being some time in the future. Let us consider an example: a forward start call option is bought now, at time \( t = 0 \), but with a strike price that is not known until time \( T_1 \), when the strike is set at the asset price on that date, say. The option expires later at time \( T \).

The way to price this contract is to ask what happens at time \( T_1 \). At that time we get an at-the-money option with a time \( T - T_1 \) left to expiry. If the stock price at time \( T_1 \) is \( S_1 \) then the value of the contract is simply the Black-Scholes value with \( S = S_1, X = S_1 \) and with given values for \( r \) and \( \sigma \). For a call option this value, as a function of \( S_1 \), is \( S_1 f(T_1) \), where

\[
f(t) = N(d_{1,t}) - e^{-r(T-t)} N(d_{2,t}), \tag{5.8}
\]
CHAPTER 5. PATH-DEPENDENT OPTIONS

\[ d_{1,t} = \frac{r + \frac{1}{2} \sigma^2 \sqrt{T - t}}{\sigma} \quad \text{and} \quad d_{2,t} = \frac{r - \frac{1}{2} \sigma^2 \sqrt{T - t}}{\sigma}. \]

The value is proportional to \( S \). Thus, at time \( T_1 \), we will hold an asset worth \( S_1 f(T_1) \). Since this is a constant multiplied by the asset price at time \( T_1 \) the value today must be

\[ S f(T_1), \]

where \( S \) is today’s asset price.

We can also formulate the problem as a PDE model. Let \( V = V(S, t) \) be the option value. Then \( V(S, t) \) must satisfy the Black-Scholes equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \]

At time \( t = T_1 \), the option values are known as

\[ V(S, T_1) = S f(T_1). \]

So we will solve the Black-Scholes equation in \((S, t) \in (0, \infty) \times [0, T_1)\) with the above final condition. It is not hard to check that

\[ V(S, t) = S f(T_1) \]

is just the solution.

5.4.2 Shout options

A shout call option is a vanilla call option but with the extra feature that the holder can at any time reset the strike price of the option to the current level of the asset. The action of resetting is called ‘shouting’.

There is clearly an element of optimization in the matter of shouting. One would expect to see a free boundary problem occur quite naturally as with American options. For a shout call option, if the shouting happens at time \( t \), the holder essentially acquires an at-the-money option whose value is

\[ S_t f(t). \]

Here \( f(t) \) is given by 5.8). Then its pricing model is

\[ \min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV, V - S f(t) \right\} = 0, \quad S > 0, t \in [0, T) \]

\[ V(S, T) = (S - X)^+ \]

This problem has no closed-form solution and must be solved numerically. It is not hard to extend to the binomial tree model and find a numerical solution.
5.4.3 Compound options

A compound option is simply an option on an option. There are four main types of compound options, namely, a call on a call, a call on a put, a put on a call and a put on a put. A compound option has two strike prices and two expiration dates. As an illustration, we consider a call on a call where both calls are European-style. On the first expiration date $T$, the holder of the compound option has the right to buy the underlying call option for the first strike price $X$. The underlying call option again gives the right to the holder to buy the underlying asset for the second strike price $X_1$ on a later expiration date $T_1$.

Let $S_t$ be the price of the underlying asset of the underlying option. Suppose $S_t$ satisfies the geometric Brownian motion. Then the value of the underlying call option $C(S,t;X,T)$ can be represented by the Black-Scholes formula. The value of the compound option is also a function of $S_t$ and $t$, denoted by $V(S,t)$. Hence it satisfies the Black-Scholes equations for $S > 0$, $t \in [0,T)$, that is,

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - r V = 0, \text{ for } S > 0, \ t \in [0,T).
$$

At $t = T$, we have the final condition

$$
V(S,T) = (C(S,T;X_1,T_1) - X)^+.
$$

The closed-form solution to the above model can be obtained. We refer interested readers to Hull (1998) or Kwok (1998).

**Remark 17** Recall the pricing model for a future call option. At the expiration date $t = T$, the holder of a future call option has the right to get the cash $(F_T - X)$ and a long position of a future contract that expires at later time $T_1$. Here $F_t$ stands for the future price at time $t$. Since it is free to enter a long position of a future contract, the payoff of the future option is $(F_T - X)^+$. Let $S_t$ denote the underlying asset price of the future contract. Due to $F_t = S_t e^{(r-q)(T_1-t)}$, the payoff of the future option can be rewritten as $(S_T e^{(r-q)(T_1-T)} - X)^+$. Then the option value, denoted by $U(S,t)$, satisfies

$$
\frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + (r - q) S \frac{\partial U}{\partial S} - r U = 0, \text{ for } S > 0, \ t \in [0,T)
$$

$$
U(S,T) = (S e^{(r-q)(T_1-T)} - X)^+.
$$

If we consider the option value as a function of the future price, i.e. $V = V(F,t)$, then we can make the transformations $V(F,t) = U(S,t)$ and $F =
Se^{(r-q)(T_1-t)} to get

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0, \text{ for } F > 0, \ t \in [0, T) \]

\[ V(F, T) = (F - X)^+. \]
Chapter 6

Beyond the Black-Scholes world

6.1 Volatility simile phenomena and defects in the Black-Scholes model

Before pointing out some of the flaws in the assumptions of the Black-Scholes world, we must emphasize how well the model has done in practice, how widespread is its use and how much impact it has had on financial markets. The model is used by everyone working in derivatives whether they are salesmen, traders or quants. It is used confidently in situations for which it was not designed, usually successfully. The value of vanilla options are often not quoted in monetary terms, but in volatility terms with the understanding that the price of a contract is its Black-Scholes value using the quoted volatility. The idea of delta hedging and risk-neutral pricing have taken a formidable grip on the minds of academics and practitioners alike. In many ways, especially with regards to commercial success, the Black-Scholes model is remarkably robust.

Nevertheless, there is room for improvement.

6.1.1 Implied volatility and volatility similes

The one parameter in the Black-Scholes pricing formulas that cannot be directly observed is the volatility of the underlying asset price which is a measure of our uncertainty about the returns provided by the underlying asset. Typically values of the volatility of an underlying asset are in the range of 20% to 40% per annum.
The volatility can be estimated from a history of the underlying asset price. However, it is more appropriate to mention an alternative approach that involves what is termed an implied volatility. This is the volatility implied by an option price observed in the market.

To illustrate the basic idea, suppose that the market price of a call on a non-dividend-paying underlying is 1.875 when $S_0 = 21$, $X = 20$, $r = 0.1$ and $T = 0.25$. The implied volatility is the value of $\sigma$, that when substituted into the Black-Scholes formula

$$c = S_0N\left(\frac{\ln\frac{S_0}{X} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) + Xe^{-rT}N\left(\frac{\ln\frac{S_0}{X} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

gives $c = 1.875$. In general, it is not possible to invert the formula so that $\sigma$ is expressed as a function of $S_0$, $X$, $r$, $T$, and $c$. But, it is not hard to use Matlab to find a numerical solution of $\sigma$ because

$$\frac{\partial c}{\partial \sigma} > 0.$$

In this example, the implied volatility is 23.5%.

Implied volatilities can be used to monitor the market’s opinion about the volatility of a particular stock. Analysts often calculate implied volatilities from actively traded options on a certain stock and use them to calculate the price of a less actively traded option on the same stock.

Black-Scholes assumes that volatility is a known constant. If it is true, then the implied volatility should keep invariant w.r.t. different strike prices. However, in reality, the shape of this implied volatility versus strike curve is often like ‘a smile’, instead of a flat line. This is the so-called ‘volatility smile’ phenomena. In some markets it shows considerable asymmetry, a skew, and sometimes it is upside down in a frown. The general shape tends to persist for a long time in each underlying.

The volatility simile phenomena implies that there are flaws in the Black-Scholes model.

### 6.1.2 Improved models

(1) **Local volatility model:**

Black-Scholes assumes that volatility is a known constant. If volatility is not a simple constant then perhaps it is a more complicated function of time and the underlying.

(2) **Stochastic volatility**
The Black-Scholes formulae require the volatility of the underlying to be a constant (or a known deterministic function of time). The Black-Scholes equation requires the volatility to be a known function of time and asset value (i.e. the local volatility model). Neither of these is true. All volatility time series show volatility to be a highly unstable quantity. It is very variable and unpredictable. It is therefore natural to represent volatility itself as a random variable. Stochastic volatility models are currently popular for the pricing of contracts that are very sensitive to the behavior of volatility.

(3) Jump diffusion model.
Black-Scholes assumes that the underlying asset path is continuous. It is common experience that markets are discontinuous: from time to time they 'jump', usually downwards. This is not incorporated in the lognormal asset price model, for which all paths are continuous.

(4) Others:
Discrete hedging: Black-Scholes assumes the delta-hedging is continuous. When we derived the Black-Scholes equation we used the continuous-time Ito’s lemma. The delta hedging that was necessary for risk elimination also had to take place continuously. If there is a finite time between rehedges then there is risk that has not been eliminated.

Transaction costs: Black-Scholes assumes there are no costs in delta hedging. But not only must we worry about hedging discretely, we must also worry about how much it costs us to rehedge. The buying and selling of assets exposes us to bid-offer spreads. In some markets this is insignificant, then we rehedge as often as we can. In other markets, the cost can be so great that we cannot afford to hedge as often as we would like.

6.2 Local volatility model

Suppose the volatility of the underlying asset is a deterministic function of $S$ and $t$, i.e. $\sigma = \sigma(S,t)$. It is not hard to show that the option value still satisfies the Black-Scholes equations,

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S,t) S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S \frac{\partial V}{\partial S} - rV = 0,$$

for $S > 0$, $t \in [0,T)$.

with the final condition

$$V(S,T) = \begin{cases} (S - X)^+ & \text{for call} \\ (X - S) & \text{for put} \end{cases}$$

A natural question is how to calibrate the function $\sigma(S,t)$. In general, there are not closed form solutions to the pricing model provided that volatil-
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ity is a function of \( S \) and \( t \). To identify the volatility function, we need to exploit more information.

Let \( V(S,t;K,T) \) be the option price with strike \( K \) and maturity \( T \). It can be shown (see Wilmott (1998)) that as a function \( K \) and \( T \), the function \( V(.,.;K,T) \) satisfies

\[
-\frac{\partial V}{\partial T} + \frac{1}{2} \sigma^2(K,T)K^2 \frac{\partial^2 V}{\partial K^2} - (r - q)K \frac{\partial V}{\partial K} - qV = 0, \text{ for } K > 0, \ T \geq t^*.
\]

\[
V(S^*,t^*;K,T) = (S^* - K)^+\]

where \( t^* \) is current time, \( S^* \) is current asset price. Remember \( \sigma(.,.) \) is just a function. We can identify the function \( \sigma(.,.) \) by

\[
\sigma(K,T) = \sqrt{\frac{\frac{\partial V}{\partial T} + (r - q)K \frac{\partial V}{\partial K} + qV}{\frac{1}{2}K^2 \frac{\partial^2 V}{\partial K^2}}}.
\]

In practice, one often calibrates the model from the market prices of vanilla options so as to price some exotic options of the OTC market.

6.3 Stochastic volatility model

Volatility doesn’t not behave how the Black-Scholes equation would like it to behave; it is not constant, it is not predictable, it’s not even directly observable. This make it a prime candidate for modeling as a random variable.

6.3.1 Random volatility

We continue to assume that \( S \) satisfies

\[
dS = \mu S dt + \sigma S dW_1,
\]

but we further assume that volatility satisfies

\[
d\sigma = p(S,\sigma,t)dt + q(S,\sigma,t)dW_2.
\]

The two increments \( dW_1 \) and \( dW_2 \) have a correlation of \( \rho \). The choice of functions \( p(S,\sigma,t) \) and \( q(S,\sigma,t) \) is crucial to the evolution of the volatility, and thus to the pricing of derivatives.

The value of an option with stochastic volatility is a function of three variables, \( V(S,\sigma,t) \).
6.3. STOCHASTIC VOLATILITY MODEL

6.3.2 The pricing equation

The new stochastic quantity that we are modeling, the volatility, is not a traded asset. Thus, when volatility is stochastic we are faced with the problem of having a source of randomness that cannot be easily hedged away. Because we have two sources of randomness we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the volatility risk. We therefore must set up a portfolio containing one option, with value denoted by \( V(S, \sigma, t) \), a quantity \(-\Delta\) of the asset and a quantity \(-\Delta_1\) of another option with value \( V_1(S, \sigma, t) \). We have

\[
\Pi = V - \Delta S - \Delta_1 V_1.
\]

The change in this portfolio in a time \( dt \) is given by

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial \sigma^2}{\partial \sigma^2} \right) dt \\
- \Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial \sigma^2}{\partial \sigma^2} \right) dt \\
+ \left( \frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} \right) dS \\
+ \left( \frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} \right) d\sigma
\]

where we have used Ito lemma on functions of \( S, \sigma \) and \( t \).

To eliminate all randomness from the portfolio we must choose

\[
\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0,
\]

to eliminate \( dS \) terms, and

\[
\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0,
\]

to eliminate \( d\sigma \) terms. This leaves us with

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial \sigma^2}{\partial \sigma^2} \right) dt \\
- \Delta_1 \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial \sigma^2}{\partial \sigma^2} \right) dt \\
= r\Pi dt = r(V - \Delta S - \Delta_1 V_1) dt
\]
CHAPTER 6. BEYOND THE BLACK-SCHOLES WORLD

where I have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate.

As it stands, this is one equation in two unknowns, $V$ and $V_1$. This contrasts with the earlier Black-Scholes case with one equation in the one unknowns.

Collecting all terms on the left-hand side and all $V_1$ terms on the right-hand side we find that

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V = \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} + r S \frac{\partial V_1}{\partial S} - r V_1.$$

We are lucky that the left-hand side is a function of $V$ but not $V_1$ and the right-hand side is a function of $V_1$ but not $V$. Since the two options will typically have different payoffs, strikes or expiries, the only way for this to be possible is for both sides to be independent of the contract type. Both sides can only be functions of the independent variables, $S$, $\sigma$ and $t$. Thus we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} + a(S, \sigma, t) \frac{\partial V}{\partial \sigma} - r V = 0$$

for some function $a(S, \sigma, t)$. Reordering this equation, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} + a(S, \sigma, t) \frac{\partial V}{\partial \sigma} - r V = 0.$$

The final condition is

$$V(S, \sigma, T) = \left\{ \begin{array}{ll} (S - X)^+, & \text{for call option} \\ (X - S)^+, & \text{for put option} \end{array} \right.$$

The solution domain is $\{ \sigma > 0, S > 0, t \in [0, T) \}$.

**Remark 18** In the risk-neutral world, the underlying asset $S$ follows the following process:

$$dS_t = rS_t dt + \sigma S_t dw_t.$$

We can similarly get the risk-neutral process of $\sigma$ as follows

$$d\sigma = a(S, \sigma, t) dt + q(S, \sigma, t) dw_t.$$

Here $a(S, \sigma, t)$ is often rewritten as

$$a(S, \sigma, t) = p(S, \sigma, t) - \lambda(S, \sigma, t) q(S, \sigma, t),$$

where $\lambda(S, \sigma, t)$ is called the market price of risk.
6.3.3 Named models


\[
\frac{d \sigma^2}{\sigma^2} = k(b - \sigma^2)dt + c\sigma^2dW_2,
\]

where \( k, b \) and \( c \) are constant.

Using the Ito lemma, we can get

\[
d\sigma = \left[-\frac{1}{8}c^2\sigma + \frac{k}{2}\left(\frac{b}{\sigma} - \sigma\right)\right]dt + c\sigma dW_2.
\]

2. Heston (1993)

\[
d\sigma = -\gamma\sigma dt + \delta dW_2.
\]

Explicit price formulas are available for Heston model. For Hull-White model, explicit formulas exist when \( S \) and \( \sigma \) are uncorrelated.

6.4 Jump diffusion model

There is plenty of evidence that financial quantities do not follow the lognormal random walk that has been the foundation of the Black-Scholes model. One of the striking features of real financial markets is that every now and then there is a sudden unexpected fall or crash. These sudden movements occur far more frequently than would be expected from a Normally-distributed return with a reasonable volatility.

6.4.1 Jump-diffusion processes

The basic building block for the random walks we have considered so far is continuous Brownian motion based on the Normally-distributed increment. We can think of this as adding to the return from one day to the next a Normally distributed random variable with variance proportional to timestep. The extra building block we need for the jump-diffusion model for an asset price is the Poisson process. A Poisson process \( dq \) is defined

\[
dq = \begin{cases} 
0, & \text{with probability } 1 - \lambda dt \\
1, & \text{with probability } \lambda dt 
\end{cases}.
\]

There is therefore a probability \( \lambda dt \) of a jump in \( q \) in the timestep \( dt \). The parameter \( \lambda \) is called the intensity of the Poisson process.
This Poisson process can be incorporated into a model for an asset in the following way:

\[
\frac{dS}{S} = \mu dt + \sigma dW + (J - 1) dq.
\]

This is the jump-diffusion process. We assume that there is no correlation between the Brownian motion and the Poisson process. If there is a jump \((dq = 1)\) then \(S\) immediately goes to the value \(JS\). We can model a sudden 10% fall in the asset price by \(J = 0.9\). We can generalize further by allowing \(J\) to be a random quantity.

A jump-diffusion version of the Ito lemma is

\[
dV(S,t) = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW + \left( V(JS,t) - V(S,t) \right) dq.
\]

The random walk in \(\ln S\) follows

\[
d\ln S = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW + (\ln(JS) - \ln(S)) dq
\]

\[
= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW + \ln J dq
\]

### 6.4.2 Hedging when there are jumps

Hold a portfolio of the option and \(-\Delta\) of the underlying:

\[
\Pi = V(S,t) - \Delta S.
\]

The change in the value of this portfolio is

\[
d\Pi = dV - \Delta dS
\]

\[
= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW + \left( V(JS,t) - V(S,t) \right) dq
\]

\[-\Delta[\mu S dt + \sigma S dW + (J - 1) S dq].
\]

### 6.4.3 Merton’s model (1976)

If we choose

\[
\Delta = \frac{\partial V}{\partial S},
\]
6.4. JUMP DIFFUSION MODEL

We are following a Black-Scholes type of strategy, hedging the diffusive movements. The change in the portfolio value is then

\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( V(JS, t) - V(S, t) - (J - 1)S \frac{\partial V}{\partial S} \right) dq. \]

The portfolio now evolves in a deterministic fashion, except that every so often there is a non-deterministic jump in its value. It can be argued (Merton 1976) if the jump component of the asset price process is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced in the option. Diversifiable risk should not be rewarded. In other words, we can take expectations of this expression and set that value equal to the riskfree return from the portfolio, namely

\[ E(d\Pi) = r\Pi dt. \]

This argument is not completely satisfactory, but is a common assumption whenever there is a risk that cannot be fully hedged.

Since there is no correlation between \( dW \) and \( dq \), and

\[ E(dq) = \lambda dt, \]

we arrive at

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda [V(JS, t) - V(S, t)] - \lambda [J - 1]S \frac{\partial V}{\partial S} = 0. \]

If the jump size \( J \) is a random quantity, we need to take the expectation over the \( J \). It follows

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda E^J [V(JS, t) - V(S, t)] - \lambda E^J [J - 1]S \frac{\partial V}{\partial S} = 0. \]

There is a simple solution of this equation in the special case that the logarithm of \( J \) is Normally distributed. If the logarithm of \( J \) is Normally distributed with standard deviation \( \sigma' \) and if we write

\[ k = E^J [J - 1] \]

then the price of a European non-path-dependent option can be written as

\[ \sum_{n=1}^{\infty} \frac{1}{n!} e^{-\lambda(T-t)} (\lambda(T-t))^n V_{BS}(S, t; \sigma_n, r_n), \]
where

\[ \lambda' = \lambda(1 + k), \quad \sigma_n^2 = (\sigma^2 + \frac{n\sigma'^2}{T-t}) \text{ and } r_n = r - \lambda k + \frac{n \ln(1 + k)}{T-t}, \]

and \( V_{BS} \) is the Black-Scholes formula for the option value in the absence of jumps. This formula can be interpreted as the sum of individual Black-Scholes values, each of which assumes that there have been \( n \) jumps, and they are weighted according to the probability that there will have been \( n \) jumps before expiry.

### 6.4.4 Wilmott et al.’s model

In the above we hedged the diffusive element of the random walk for the underlying. Another possibility is to hedge both the diffusion and jumps as much as we can. For example, we could choose \( \Delta \) to minimize the variance of the hedged portfolio.

The changes in the value of the portfolio with an arbitrary \( \Delta \) is

\[
d\Pi = (...) dt + \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dW + \left( V(JS, t) - V(S, t) - \Delta(J - 1)S \right) dq.
\]

The variance in this change, which is a measure of the risk in the portfolio, is

\[
\text{var} [d\Pi] = \sigma^2 S^2 \left( \frac{\partial V}{\partial S} - \Delta \right)^2 dt + \lambda \left[ (V(JS, t) - V(S, t) - \Delta(J - 1)S)^2 \right] dt + O(dt^2)
\]

If \( J \) is a random quantity, we take expectation over \( J \) to get

\[
\text{var} [d\Pi] = \sigma^2 S^2 \left( \frac{\partial V}{\partial S} - \Delta \right)^2 dt + \lambda E[J \left[ (V(JS, t) - V(S, t) - \Delta(J - 1)S)^2 \right] dt + O(dt^2)
\]

By neglecting \( O(dt^2) \), this is minimized by the choice

\[
\Delta = \frac{\lambda E[J (V(JS, t) - V(S, t))] + \sigma^2 S \frac{\partial V}{\partial S}}{\lambda S E[J [(J - 1)^2] + \sigma^2 S}.
\]

If we value the option as a pure discounted real expectation under this best-hedge strategy, then we have

\[
E[d\Pi] = r\Pi dt
\]
6.4. JUMP DIFFUSION MODEL

or

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left( \mu - \frac{\sigma^2}{d} (\mu + \lambda k - r) \right) \frac{\partial V}{\partial S} - rV \\
+ \lambda E \left[ (V(JS, t) - V(S, t)) \left( 1 - \frac{J - 1}{d} (\mu + \lambda k - r) \right) \right] = 0
\]

where

\[
d = \lambda E (J - 1)^2 + \sigma^2 \quad \text{and} \quad k = E [J - 1]
\]

When \( \lambda = 0 \) this recovers the Black-Scholes equation.

6.4.5 Summary

Jump diffusion models undoubtedly capture a real phenomenon that is missing from the Black-Scholes model. Yet they are rarely used in practice. There are three main reasons for this:

1. difficulty in parameter estimation. In order to use any pricing model one needs to be able to estimate parameters. In the lognormal model there is just the one parameter to estimate. This is just the right number. More than one parameter is too much work. The jump diffusion model in its simplest form needs an estimate of probability of a jump, measured by \( \lambda \) and its size \( J \). This can be made more complicated by having a distribution for \( J \).

2. difficulty in solution. The governing equation is no longer a diffusion equation (about the easiest problem to solve numerically), and is harder than the solution of the basic Black-Scholes equation.

3. impossibility of perfect hedging. Finally, perfect risk-free hedging is impossible when there are jumps in the underlying. The use of a jump-diffusion model acknowledges that one’s hedge is less than perfect.

In fact the above remarks also apply to the stochastic volatility model.
Chapter 7

Interest Rate Derivatives

The riskless interest rate has been assumed to be constant in the option pricing models discussed in previous chapters. Such an assumption is acceptable when the life of an option is typically six to nine months.

This chapter is focused on the interest rate derivatives. Well known examples of such derivatives are bonds, bond futures, bonds options, swap, swaptions, and so on. Note that interest rates are used for discounting as well as defining the payoff for some interest rate derivative products. The values of these derivatives depend sensibly on the level of interest rates. In the construction of valuation models for these securities, it is crucial to incorporate the stochastic movement of interest rates into consideration.

The mathematics of this chapter is no different from what we have seen already. Again we shall apply the ideas of hedging and no arbitrage.

7.1 Short-term interest rate modeling

7.1.1 The simplest bonds: zero-coupon/coupon-bearing bonds

The zero-coupon bond is a contract paying a known fixed amount, the principal, at some given date in the future, the maturity date $T$. For example, the bond pays $100 in 10 years’ time. We’re going to scale this payoff, so that in future all principals will be $1.

This promise of future wealth is worth something now: it cannot have zero or negative value. Furthermore, except in extreme circumstances, the amount we pay initially will be smaller than the amount we receive at maturity.

A coupon-bearing bond is similar to the above except that as well as paying the principal at maturity, it pays smaller quantities, the coupons,
at intervals up to and including the maturity date. These coupons are usually prespecified fractions of the principal. For example, the bond pays $1 in 10 years and 2%, i.e. 2 cents, every six months. This bond is clearly more valuable than the bond in the previous example because of the coupon payments. We can think of the coupon-bearing bond as a portfolio of zero-coupon bearing bonds: one zero-coupon bearing bond for each coupon date with a principal being the same as the original bond’s coupon, and then a final zero-coupon bond with the same maturity as the original.

7.1.2 Short-term interest rate

The interest rate that we will be modeling is known as a short-term interest rate or spot interest rate $r(t)$. This means that the rate $r(t)$ is to apply at time $t$: interest is compounded at this rate at each moment in time but this rate may change with time.

If the short-term interest rate $r(t)$ is a known function of time, then the bond price is also a function of time only: $V = V(t, T)$. We begin with a zero-coupon bond example. Because we receive 1 at time $T$, we know that $V(T) = 1$. I now derive an equation for the value of the bond at a time before maturity, $t < T$.

Suppose we hold one bond. The change in the value of that bond in a time-step $dt$ (from $t$ to $t + dt$) is

$$dV.$$ 

Arbitrage considerations again lead us to equate this with the return from a bank deposit receiving interest at a rate $r(t)$:

$$dV = r(t)V dt.$$ 

The solution of this equation is

$$V(t, T) = e^{-\int_t^T r(\tau)d\tau}.$$ 

As a matter of fact, the short rate interest rate can be thing of as the interest rate from a money market account, which is usually unpredictable. To price interest rate derivatives, we must model the (short-term) interest rate as a stochastic variable. That is, we suppose the short term interest rate is governed by a stochastic differential equation of the form

$$dr = u(r, t)dt + \omega(r, t)dW.$$ 

The functional forms of $u(r, t)$ and $\omega(r, t)$ determine the behavior of the short-term rate $r$. 
7.1.3 The bond pricing equation

When interest rates are stochastic a bond has a price of the form \( V(r, t; T) \). Since the interest rate \( r \) is not a traded asset, there is no underlying asset with which to hedge. The only way to construct a hedged portfolio is by hedging one bond with a bond of a different maturity. This is exactly same as what we have discussed about the option pricing on non-traded assets in Section 2.5.2.

We set up a portfolio containing two bonds with different maturities \( T_1 \) and \( T_2 \). The bond with maturity \( T_1 \) has price \( V_1(r, t; T_1) \) and the bond with maturity \( T_2 \) has price \( V_2(r, t; T_2) \). We hold one of the former and a number \(-\Delta\) of the latter. We have

\[
\Pi = V_1 - \Delta V_2.
\]

The change in this portfolio in a time \( dt \) is given by

\[
d\Pi = dt + \frac{\partial V_1}{\partial r} dr - \Delta \left( \frac{\partial V_2}{\partial t} dt + \frac{1}{2} \omega^2 \frac{\partial^2 V_2}{\partial r^2} dt + \frac{\partial V_2}{\partial r} dr \right),
\]

where we have applied Ito’s lemma to functions of \( r \) and \( t \). By the choice

\[
\Delta = \frac{\partial V_1}{\partial r} \frac{\partial V_2}{\partial r}
\]

eliminates all randomness in \( d\Pi \). We then have

\[
d\Pi = \left( \frac{\partial V_1}{\partial t} + \frac{1}{2} \omega^2 \frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1}{\partial r} \frac{\partial V_2}{\partial t} + \frac{1}{2} \omega^2 \frac{\partial^2 V_2}{\partial r^2} \right) dt
\]

\[
= r\Pi dt = r \left( V_1 - \frac{\partial V_1}{\partial r} V_2 \right) dt,
\]

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. The risk-free rate is just the short-term rate.

Collecting all \( V_1 \) terms on the left-hand side and all \( V_2 \) terms on the right-hand side we find that

\[
\frac{\partial V_1}{\partial t} + \frac{1}{2} \omega^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1 = \frac{\partial V_2}{\partial t} + \frac{1}{2} \omega^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2.
\]
The left-hand side is a function of \( T_1 \) but not \( T_2 \) and the right-hand side is a function \( T_2 \) but not \( T_1 \). The only way for this to be possible is for both sides to be independent of the maturity date. Dropping the subscript from \( V \), we have
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = -a(r, t)
\]
for some function \( a(r, t) \). As discussed in Chapter 4, we denote \( a(r, t) = u(r, t) - \lambda(r, t)\omega(r, t) \), where \( \lambda(r, t) \) is the market price of risk of \( r(t) \).

The bond pricing equation is therefore
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda \omega) \frac{\partial V}{\partial r} - rV = 0. \tag{7.1}
\]
To find a unique solution we must impose final condition
\[
V(r, T; T) = 1. \tag{7.2}
\]
Note that Eq (7.1) holds for any interest rate derivative whose value is only a function of \( r \) and \( t \). It indicates that the risk-neutral process of the short-term rate is
\[
dr = (u - \lambda \omega)dt + \omega dW \tag{7.3}
\]
7.1.4 Tractable models
We have built up the bond pricing equation for an arbitrary model. That is, we have not specified the risk-neutral drift, \( u - \lambda \omega \), or the volatility, \( \omega \). How can we choose these functions to give us a good model? Like in the stochastic volatility model, a simple lognormal random walk would not be suitable for \( r \) since it would predict exponentially rising or falling rates. This rules out the equity price model as an interest rate model. So we must think more carefully how to choose the drift and volatility.

We hope the random walk (7.3) for \( r \) has the following nice properties:
(1) Positive interest rates: Except for a few pathological cases, interest rates are positive.
(2) Mean reversion: The interest rate should tend to decrease towards the mean for large \( r \) and tend to increase towards the mean for low \( r \).

In addition, we expect the solution of (7.1-7.2) for the zero-coupon bond is of a simple form, for example
\[
Z(r, t; T) = e^{A(t; T) - rB(t; T)}.
\]
Some named models are given as follows:
7.1. SHORT-TERM INTEREST RATE MODELING

Vasicek
This is the first mean reverting model.

\[ dr = (\eta - \gamma r)dt + cdW, \]

where \( \eta, \gamma, \) and \( c \) are constants. The value of a zero-coupon bond is given by

\[ Z(r, t; T) = e^{A(t; T) - rB(t; T)}, \]

where

\[ A = \frac{1}{\gamma} \left( 1 - e^{-\gamma(T-t)} \right) \]
\[ B = \frac{1}{\gamma^2} (B(t; T) - T + t) \left( \eta \gamma - \frac{1}{2} c^2 \right) - \frac{c^2}{4\gamma} B^2. \]

The drawback of this model is that the interest rate can easily go negative.

Cox, Ingersoll & Ross
The risk-neutral process of the short-term interest rate takes the form of

\[ dr = (\eta - \gamma r)dt + \sqrt{\alpha} dW; \]

where \( \eta, \gamma, \) and \( \alpha \) are constants. The short-term interest rate is mean reverting and if \( \eta > \alpha/2 \) the short-term interest rate stays positive. Explicit solutions are also available for the zero-coupon bond (see Wilmott (1998) or Hull (2003, P. 542-543)).

Ho & Lee
The risk-neutral process of the short-term interest rate is

\[ dr = \eta(t)dt + cdW, \]

where \( \eta(t) \) and \( c \) are parameters. The value of zero-coupon bonds is given by

\[ Z(r, t; T) = e^{A(t; T) - rB(t; T)} \]

where

\[ B = T - t. \]
and

\[ A = -\int_t^T \eta(s)(T - s)ds + \frac{1}{6}c^2(T - t)^3. \]

Ho-Lee model is the first ‘no-arbitrage model’. This means that the careful choice of the function \( \eta(t) \) will result in theoretical zero-coupon bonds prices, output by the model, which are the same as market prices. This technique is also called yield curve fitting. This careful choice is

\[ \eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + c^2(t - t^*) \]

where today is time \( t = t^* \). In this \( Z_M(t^*; T) \) is the market price today of zero-coupon bonds with maturity \( T \). Clearly this assumes that there are bonds of all maturities and that the prices are twice differentiable with respect to the maturity. We will see why this should give the ‘correct’ prices later.

The drawback of this model is that the process followed by the interest rate is not mean-reverting.

**Hull & White**

Hull and White have extended the Vasicek model to incorporate time-dependent parameters.

\[ dr = (\eta(t) - \gamma r)dt + cdW. \]

Under this risk-neutral process the value of zero-coupon bonds

\[ Z(r, t; T) = e^{A(t; T) - rB(t; T)}, \]

where

\[ A(t; T) = -\int_t^T \eta(s)B(s; T)ds + \frac{c^2}{2\gamma^2} \left( T - t + \frac{2}{\gamma} e^{-\gamma(T - t)} - \frac{1}{2\gamma} e^{-2\gamma(T - t)} - \frac{3}{2\gamma} \right) \]

and

\[ B(t; T) = \frac{1}{\gamma} \left( 1 - e^{-\gamma(T - t)} \right). \]

Here the time-dependent parameter \( \eta(t) \) can also be identified from the technique of yield curve fitting given as follows.
7.1. SHORT-TERM INTEREST RATE MODELING

7.1.5 Yield Curve Fitting

Let us consider the Ho & Lee short term model. It is the simplest that can be used to fit the yield curve. It will be useful to examine this model in detail to see one way in which fitting is done in practice.

In this model, the solution of bond pricing equation for a zero-coupon bond is simply

\[ Z(r, t; T) = e^{A(t; T) - r(T - t)} \]

where

\[ A = -\int_t^T \eta(s)(T - s)ds + \frac{1}{6}c^2(T - t)^3. \]

If we know \( \eta(t) \) then the above gives us the theoretical value of zero-coupon bonds of all maturities. Now turn this relationship around and ask the question “What functional form must we choose for \( \eta(t) \) to make the theoretical values of the zero-coupon bonds for all maturities equal to the market values” Call this special choice for \( \eta, \eta^*(t) \).

The yield curve is to be fitted today, \( t = t^* \), when the spot interest rate is \( r^* \) and values of the zero-coupon bonds in the market are \( Z_M(t^*; T) \). To match the market and theoretical bond prices, we must have

\[ Z_M(t^*; T) = e^{A(t^*; T) - r^*(T - t^*)}. \]

Taking logarithms of this and rearranging slightly we get

\[ \int_{t^*}^T \eta^*(s)(T - s)ds = -\log(Z_M(t^*; T)) - r^*(T - t^*) + \frac{1}{6}c^2(T - t^*)^3. \]  

(7.4)

Observe that we are carrying around in the notation today’s date \( t^* \). This is a constant but we want to emphasize that we are doing the calibration to today’s yield curve. If we calibrate again tomorrow, the market yield curve will have changed.

Differentiate Eq (7.4) with respect to \( T \) to get

\[ \int_{t^*}^T \eta^*(s)(T - s)ds = -\frac{\partial}{\partial T}\log(Z_M(t^*; T)) - r^* + \frac{1}{2}c^2(T - t^*)^2. \]

Differentiating the above equation again with respect to \( T \), we have

\[ \eta^*(T) = -\frac{\partial^2}{\partial T^2}\log(Z_M(t^*; T)) + c^2(T - t^*). \]

With this choice for the time-dependent parameter \( \eta^*(t) \), the theoretical and actual market prices of zero-coupon bonds are the same.

The same arguments can be applied to the Hull-White Model.

We refer Wilmott (1998) to a discussion about the advantages and disadvantages of yield curve fitting (Chapter 34).
7.1.6 Empirical behavior of the short rate and other models

Observe that all of above named models take the form

\[ dr = (...)dt + cr^\beta dW \]

and give zero-coupon bond prices of the form

\[ Z(r, t; T) = e^{A(t,T) - rB(t,T)}. \]

Examples are Ho & Lee (\( \beta = 0 \)), Vasicek (\( \beta = 0 \)) and Cox, Ingersoll & Ross (\( \beta = \frac{1}{2} \)). Since the short rate in the risk-neutral world shares the same volatility as in the real world, we are able to estimate the coefficient \( \beta \) from historical data of the short rate. Chan, Karolyi, Longstaff & Sanders (1992) obtain the estimate \( \beta = 1.36 \) from US data. This agrees with the experiences of many practitioners, who say that in practice the relative spot rate change \( dr/r \) is insensitive to the level of \( r \).

We can think of models with

\[ dr = ... + cdW \]

as having a Normal volatility structure and those with

\[ dr = ... + crdW \]

as having a lognormal volatility structure. In reality it seems that the short rate is closer to the lognormal than to the Normal models. This puts the following BDT model ahead of the options.

In the Black, Derman & Toy (BDT) model, the risk-neutral short rate satisfies

\[ d (\ln r) = \left( \eta(t) - \sigma'(t) \ln r \right) dt + \sigma(t)dW. \]

The two functions of time \( \sigma \) and \( \eta \) allow both zero-coupon bonds and their volatilities to be matched. There are no explicit solutions of the zero-coupon bond prices for this model, but the yield fitting can be done by numerical schemes.

An even more general model is the Black & Karasinski model

\[ d (\ln r) = (\eta(t) - a(t) \ln r) dt + \sigma(t)dW. \]
7.2. HJM MODEL

7.1.7 Coupon-bearing bond pricing

Suppose coupons are paid at time $t_i$, $i = 1, 2, \ldots, n$. (Let $t_0 = 0$, $t_n = T$). Let $V = V(r, t)$ be the value of the bond. Then

$$\frac{\partial V}{\partial t} + \frac{1}{2} \omega^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda \omega) \frac{\partial V}{\partial r} - rV = 0, \text{ for } t_{i-1} < t < t_n, \; i = 1, \ldots, n$$

$$V(r, t_i-) = V(r, t_i+) + c_i, \; \text{for } i = 1, \ldots, n - 1$$

$$V(r, T) = 1 + c_n$$

Here $c_i$ is the coupon at time $t_i$.

Note that the values of zero-coupon bonds with scaled payoff are essentially the discounting factors. If we have known the values of the zero-coupon bonds with all maturities, then the value of the coupon-bearing bond can be represented as

$$V(r; t; T) = Z(r, t; T) + \sum_{t_i \geq t} c_i Z(r; t_i; t_i).$$

Actually, we never use the above formula to value coupon-bearing bonds. Conversely, we take advantage of it to find the values of zero-coupon bonds because all long term bonds are coupon-bearing. That is the so-called bootstrap method (see Hull (2003), page 96).

7.2 HJM Model

The short-term interest models are widely used for pricing interest rate instruments when the simpler Black-Scholes models are inappropriate. One limitation of the model is that they do not give the user complete freedom in choosing the volatility structure.

The HJM model was a major breakthrough in the pricing of interest rate products. Instead of modeling a short-term interest rate and deriving the forward rates (or, equivalently, the yield curve) from that model, Heath, Jarrow & Morton boldly start with a model for the whole forward rate curve. Since the forward rates are known today, the matter of yield-curve fitting is contained naturally within their model, it does not appear as an afterthought. Moreover, it is possible to take real data from the real data for the random movement of the forward rates and incorporate them into the derivative-pricing methodology.
7.2.1 The forward rate

Let $Z(t; T)$ be the price of a zero-coupon bond at time $t$ and maturing at time $T$, when it pays $1$. Let $Y(t; T)$ be the continuously compounded yield of the zero-coupon bond. Then, by definition,

$$Z(t; T) = \exp (-Y(t; T)(T - t))$$

or

$$Y(t; T) = -\frac{\ln Z(t; T)}{T - t}.$$

Let $f(t; T, T + \delta)$ be the forward rates as seen at time $t$ for the period between time $T$ and time $T + \delta$. By definition, we have

$$Z(t; T + \delta) = Z(t; T) \exp (-f(t; T, T + \delta)\delta)$$

or

$$f(t; T, T + \delta) = -\frac{\ln Z(t; T + \delta) - \ln Z(t; T)}{\delta}.$$

The (instantaneous) forward rate curve $F(t; T)$ at time $t$ is defined by

$$F(t; T) = \lim_{\delta \to 0} f(t; T, T + \delta) = -\frac{\partial}{\partial T} \ln Z(t; T). \quad (7.5)$$

So

$$Z(t; T) = \exp \left( -\int_t^T F(t; s)ds \right).$$

The spot rate can be simply represented by the forward rate for a maturity equal to the current date, i.e.

$$r(t) = F(t; t).$$

7.2.2 HJM model

The key concept in the HJM model is that we model the evolution of the whole forward rate curve, not just the short end.

Let us assume that in a real world all zero-coupon bonds evolve according to

$$dZ(t; T) = \mu(t, T)Z(t; T)dt + \sigma(t, T)Z(t; T)dW$$

At this point $d\cdot$ means that time $t$ evolves but the maturity date $T$ is fixed. This is not much of an assumption.
Note that since \( Z(t; t) = 1 \), we must have \( \sigma(t, t) = 0 \).

From (7.5) we have
\[
dF(t; T) = \frac{\partial}{\partial T} \left( \frac{1}{2} \sigma^2(t, T) - \mu(t, T) \right) dt - \frac{\partial \sigma(t, T)}{\partial T} dW.
\]

All of the variables we have introduced above have been real variables. But when we come to pricing derivatives we must do so in the risk-neutral world. Since a zero-coupon bond is a traded asset, the drift of \( Z(t; T) \) in a risk-neutral world must satisfy
\[
\mu(t, T) = r(t),
\]
where \( r(t) \) is the spot rate in a risk-neutral world. In other words, the risk-neutral dynamics for discount factors must be
\[
dZ(t; T) = r(t)Z(t; T)dt + \sigma(t, T)Z(t; T)dW.
\]
Note that the spot rate \( r(t) \) is itself determined by the \( Z(t; T) \).

The risk-neutral process of the forward rate is then given as
\[
dF(t; T) = \sigma(t, T) \frac{\partial \sigma(t, T)}{\partial T} dt - \frac{\partial \sigma(t, T)}{\partial T} dW.
\]
Let us denote
\[
v(t, T) = -\frac{\partial \sigma(t, T)}{\partial T}.
\]
Then
\[
\sigma(t, T) \frac{\partial \sigma(t, T)}{\partial T} = v(t, T) \int_t^T v(t, s) ds,
\]
where we have used \( \sigma(t, t) = 0 \).

So, in the risk-neutral world, the forward rate curve follows
\[
dF(t; T) = m(t, T)dt + v(t, T)dW, \tag{7.6}
\]
where
\[
m(t, T) = v(t, T) \int_t^T v(t, s) ds.
\]
This is known as the HJM model. It indicates that there is a link between the drift of and the standard deviation (volatility) of the risk-neutral forward rate process.
CHAPTER 7. INTEREST RATE DERIVATIVES

7.2.3 The non-Markov nature of HJM

In the short-term interest rate model, we assume the spot rate follows an Ito process. Let us look at what process the spot rate follows in the HJM model.

Suppose today is \( t^* \) and that we know the whole forward rate curve today, \( F(t^*; T) \). We can write the spot rate for any time \( t \) in the future as

\[
\begin{align*}
r(t) &= F(t; t) = F(t^*; t) + \int_{t^*}^{t} dF(s; t) \\
&= F(t^*; t) + \int_{t^*}^{t} m(s, t)ds + v(s, t)dW.
\end{align*}
\]

Differentiating this with respect to time \( t \) we arrive at the stochastic differential equation for \( r \)

\[
\begin{align*}
dr &= \left( F(t^*; t) + m(t, t) + \int_{t^*}^{t} \hat{m} (s, t)ds + \int_{t^*}^{t} \hat{v} (s, t)dW \right) dt + v(t, t)dW.
\end{align*}
\]

In a Markov process it is only the present state of a variable that determines the possible future state. Note that in the above expression the last term in the drift \( \int_{t^*}^{t} \hat{m} (s, t)dW \) is random, depending on the history of the stochastic increments \( dW \). Therefore, for a general HJM model it makes the motion of the spot rate non-Markov.

Having a non-Markov model may not matter to us if we can find a small number of extra state variables that contain all the information that we need for predicting the future. Recall the pricing of an Asian option where the historic average is involved. But we can define the average as a new variable, and then derive the pricing PDE equations or the binomial tree methods. Unfortunately, the general HJM model requires an infinite number of such variables to define the present state; if we were to write the HJM model as a PDE we would need an infinite number of independent variables. If we attempt to use a binomial method to implement the HJM model, the amount of computation will grow exponentially.

7.2.4 Pricing derivatives

Pricing derivatives is all about finding the expected present value of all cash flows in a risk-neutral framework with the number of time step. Explicit solutions of derivatives’ prices are always rare. Because of the non-Markov nature of HJM in general a PDE approach and a BTM are unfeasible. As a result, we have to adopt the Monte-Carlo simulation.
To price a derivative using a Monte Carlo simulation perform the following steps. Assume that we have chosen a model for the forward rate volatility, \( v(t, T) \) for all \( T \). Today is \( t^* \) when we know the forward rate curve \( F(t^*; T) \).

1. Simulate a realized evolution of the risk-neutral forward rate for the necessary length of time until \( T^* \), say. This requires a simulation of Eq (7.6). After this simulation we will have a realization of \( F(t; T) \) for \( t^* \leq t \leq T \) and \( T \geq t \). We will have a realization of the whole forward rate path.

2. Using this forward rate path calculate the value of all the cashflows that would have occurred.

3. Using the realized path for the spot interest rate \( r(t) \) calculate the present value of these cashflows. Note that we discount at the continuously compounded risk-free rate, not at any other rate. In the risk-neutral world all assets have expected return \( r(t) \).

4. Return to Step 1 to perform another realization, and continue until we have a sufficiently large number of realizations to calculate the expected present value as accurately as required.

The disadvantage of the above algorithm is that such a simulation can be very slow. In addition, it cannot easily be used for American-style derivatives.

### 7.2.5 A special case of HJM model: Ho & Lee

We make a comparison between the spot rate modelling of the HJM model and the spot rate model.

In Ho & Lee the risk-neutral spot rate satisfies

\[
dr = \eta(t)dt + cdW
\]

for a constant \( c \). The value of zero-coupon bonds is

\[
Z(t; T) = e^{A(t; T) - rB(t; T)}
\]

where

\[
B = T - t
\]

and

\[
A = -\int_t^T \eta(s)(T - s)ds + \frac{1}{6}c^2(T - t)^3.
\]

So

\[
F(t; T) = -\frac{\partial}{\partial T} \log Z(t; T) = -\frac{\partial}{\partial T} [A(t; T) - rB(t; T)]
\]

\[
= \int_t^T \eta(s)ds - \frac{1}{2}c^2(T - t)^2 + r.
\]
Then

\[
\begin{align*}
\frac{dF(t; T)}{dt} &= \left[ -\eta(t) + c^2(T - t) \right] dt + dr \\
&= \left[ -\eta(t) + c^2(T - t) \right] dt + \eta(t)dt + cdW \\
&= c^2(T - t)dt + cdW \\
&= \left( c \int_t^T cdS \right) dt + cdW.
\end{align*}
\]

This confirms that the forward rate in the Ho-Lee model has the structure of HJM model. Hence Ho-Lee model is a special case of HJM model. Actually, most of popular models discussed before have HJM representations.

7.2.6 Concluding remarks

There are some drawbacks of the HJM model: (1) The instantaneous forward rates involved are not directly observable in the market; (2) It is not easy to calibrate it to prices of actively traded instruments such as caps. An alternative model proposed by Brace, Gatarek, and Musiela (BGM) overcomes this problem. It is known as the LIBOR market model (see Hull (1998) or Wilmott (2000)). It can be thought of as a discrete version of the HJM model in the LIBOR market. To understand the model, the knowledge of stochastic differential equations and martingales is required. As a matter of fact, to have better understanding of interest rate models, we must learn more about stochastic calculus.