Chapter 5

Portfolio Selection with Transaction Costs

5.1 Optimal investment without consumption


Let $X_t$ and $Y_t$ be the dollar value invested in bank and stock, respectively. In the presence of transaction costs, the equations describing their evolution are

\begin{align}
    dX_t &= rX_t dt - (1 + \lambda)dL_t + (1 - \mu)dM_t \quad (5.1) \\
    dY_t &= \alpha Y_t dt + \sigma Y_t dB_t + dL_t - dM_t, \quad (5.2)
\end{align}

where $L_t$ and $M_t$ are right-continuous (with left hand limits), nonnegative, and nondecreasing $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes with $L_0 = M_0 = 0$, representing cumulative dollar values for the purpose of buying and selling stock respectively. The constants $\lambda \in [0, \infty)$ and $\mu \in [0, 1)$ appearing in these equations account for proportional transaction costs incurred on purchase and sale of stock respectively.

We will consider the case of $\alpha > r$.

5.1.1 The investor’s problem

Due to transaction costs, the investor’s net wealth in monetary terms at time $t$ is

\[ W_t = \begin{cases} 
    X_t + (1 - \mu)Y_t & \text{if } Y_t \geq 0, \\
    X_t + (1 + \lambda)Y_t & \text{if } Y_t < 0.
\end{cases} \]

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Since it is required that the investor’s net wealth be positive, following Davis and Norman (1990), we define the solvency region

\[ S = \left\{ (x, y) \in \mathbb{R}^2 : x + (1 + \lambda)y > 0, \ x + (1 - \mu)y > 0 \right\}. \]

Assume that the investor is given an initial position \((x_0, y_0) \in S\). An investment strategy \((L, M)\) is admissible for \((x_0, y_0)\) if \((X_t, Y_t)\) given by (5.1)-(5.2) is in \(S\) for all \(t \in [0, T]\). We let \(A(x_t, y_t, t)\) be the set of admissible investment strategies.

The investor’s problem is to choose an admissible strategy so as to maximize the expected utility of terminal wealth, that is,

\[
\sup_{(L,M) \in A(x_0, y_0)} E_{x_0, y_0}^x [U(W_T)]
\]

subject to (5.1)-(5.2). Here \(E_{t}^{x_{0},y_{0}}\) denotes the conditional expectation at time \(t\) given that initial endowment \(X_t = x_t, Y_t = y_t\).

We define the value function by

\[
\varphi(x, y, t) = \sup_{(L,M) \in A(x,y,t)} E_{t}^{x,y} [U(W_T)] , \ (x_t, y_t) \in S , \ t \in [0, T)
\]

5.1.2 HJB equation

Under certain regularity conditions, the value function satisfies the following HJB equation

\[
\min \{-\varphi_t - \mathcal{L} \varphi, -(1 - \mu)\varphi_x + \varphi_y, (1 + \lambda)\varphi_x - \varphi_y\} = 0, \quad (x, y) \in S , \ t \in [0, T) \tag{5.3}
\]

with the terminal condition

\[
\varphi(x, y, T) = \begin{cases} U(x + (1 - \mu)y) & \text{if } y \geq 0, \\ U(x + (1 + \lambda)y) & \text{if } y < 0, \end{cases} \tag{5.4}
\]

where

\[
\mathcal{L} \varphi = \frac{1}{2} \sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + r x \varphi_x.
\]

I’d like to explain how we get (5.3). We consider a restricted class of policies in which \(L\) and \(M\) are constrained to be absolutely continuous with bounded derivatives, i.e.

\[
L_t = \int_0^t l_s ds, \ M_t = \int_0^t m_s ds, \ 0 \leq l_s, u_s \leq k.
\]
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Then (5.1)-(5.2) become

\[
\begin{align*}
    dX_t &= [rX_t - (1 + \lambda)l + (1 - \mu)m] dt, \\
    dY_t &= [\alpha Y_t + l - m] dt + \sigma Y_t dB_t.
\end{align*}
\]

(5.5)  
(5.6)

Let \( \tilde{\varphi} \) be the value function in this situation. The HJB equation for \( \tilde{\varphi} \) is

\[
\max_{l,u} A^{l,u} \tilde{\varphi} = 0,
\]

(5.7)

where

\[
A^{l,u} \tilde{\varphi} = \tilde{\varphi}_t + \frac{1}{2} \sigma^2 \varphi_y^2 \tilde{\varphi}_{yy} + (\alpha y + l - m) \tilde{\varphi}_y + [rx - (1 + \lambda)l + (1 - \mu)m] \tilde{\varphi}_x
\]

\[
= \tilde{\varphi}_t + \frac{1}{2} \sigma^2 \varphi_y^2 \tilde{\varphi}_{yy} + \alpha y \tilde{\varphi}_y + rx \tilde{\varphi}_x
\]

\[
+ l \left[ \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x \right] + m \left[ (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y \right]
\]

\[
= \mathcal{L} \tilde{\varphi} + l \left[ \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x \right] + m \left[ (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y \right].
\]

The maxima are achieved as follows

\[
l = \begin{cases} 
    k, & \text{if } \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x \geq 0 \\
    0, & \text{if } \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x < 0,
\end{cases}
\]

\[
m = \begin{cases} 
    k, & \text{if } (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y \geq 0 \\
    0, & \text{if } (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y < 0.
\end{cases}
\]

Consequently, (5.7) becomes

\[
\tilde{\varphi}_t + \mathcal{L} \tilde{\varphi} + k \left( \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x \right)^+ + k \left( (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y \right)^+ = 0.
\]

We let \( k \) go to infinity. It follows

\[
\max \{ \varphi_t + \mathcal{L} \varphi, \varphi_y - (1 + \lambda) \varphi_x, (1 - \mu) \varphi_x - \varphi_y \} = 0,
\]

which is equivalent to (5.3).

In the following, we will consider the utility function

\[
U(W) = \log W.
\]
Due to the homotheticity of the utility function, it follows that for any positive constant $\rho$,
\[ \varphi(\rho x, \rho y, t) = \varphi(x, y, t) + \log \rho. \]

Let
\[ V(x, t) = \varphi(x, 1, t). \]

Then, for $y > 0$,
\[ \varphi(x, y, t) = V\left(\frac{x}{y}, t\right) + \log y. \]

Accordingly, (5.3)-(5.4) are reduced to
\[
\begin{cases}
\min \{-V_t - \mathcal{L}_0 V, -(x + 1 - \mu)V_x + 1, (x + 1 + \lambda)V_x - 1\} = 0, \\
V(x, T) = \log(x + 1 - \mu) \quad \text{in } \Omega,
\end{cases}
\]
where $\Omega = (- (1 - \mu), +\infty) \times [0, T)$,
\[ \mathcal{L}_0 V = \frac{1}{2} \sigma^2 x^2 V_{xx} - (\alpha - r - \sigma^2) x V_x + \alpha - \frac{1}{2} \sigma^2 \]

**5.1.3 A parabolic double obstacle problem**

We are going to formally derive and study the parabolic double obstacle problem regarding the spatial partial derivative of the value function.

**Derivation**

Eq (5.8) can be rewritten as
\[
\begin{cases}
-V_t - \mathcal{L}_0 V = 0 \quad \text{if } \frac{1}{x+1+\lambda} < V_x < \frac{1}{x+1-\mu} \\
-V_t - \mathcal{L}_0 V \geq 0 \quad \text{if } V_x = \frac{1}{x+1+\lambda} \text{ or } V_x = \frac{1}{x+1-\mu} \\
V(x, T) = \log(x + 1 - \mu)
\end{cases}
\]

Set
\[ v(x, t) = V_x(x, t). \]

Note that
\[
\frac{\partial}{\partial x} \mathcal{L}_0 V = \frac{1}{2} \sigma^2 x^2 v_{xx} - (\alpha - r - 2\sigma^2) x v_x - (\alpha - r - \sigma^2) v \]
\[ \Delta \equiv \mathcal{L} v \]

(5.10)
We then postulate that $v$ satisfies the following parabolic double obstacle problem:

$$
\begin{align*}
- v_t - \mathcal{L}v &= 0 & \text{if } & \frac{1}{x+1+\lambda} < v < \frac{1}{x+1-\mu}, \\
- v_t - \mathcal{L}v &\geq 0 & \text{if } & v = \frac{1}{x+1+\lambda}, \\
- v_t - \mathcal{L}v &\leq 0 & \text{if } & v = \frac{1}{x+1-\mu}, \\
v(x, T) &= \frac{1}{x+1-\mu},
\end{align*}
$$

(5.11)
in $\Omega$. Here $\frac{1}{x+1+\lambda}$ and $\frac{1}{x+1-\mu}$ correspond to lower and upper obstacles, respectively. We stress that $- v_t - \mathcal{L}v \geq 0$ on the lower obstacle and $- v_t - \mathcal{L}v \leq 0$ on the upper obstacle, which has clear physical interpretation.

We defer the proof of the equivalence between the double obstacle problem (5.11) and the original problem (5.3)-(5.4). Now we study the double obstacle problem (5.11). Define

$$
\begin{align*}
\text{SR} &= \left\{ (x, t) \in \Omega : v(x, t) = \frac{1}{x+1-\mu} \right\}, \\
\text{BR} &= \left\{ (x, t) \in \Omega : v(x, t) = \frac{1}{x+1+\lambda} \right\}, \\
\text{NT} &= \left\{ (x, t) \in \Omega : \frac{1}{x+1+\lambda} < v(x, t) < \frac{1}{x+1-\mu} \right\}.
\end{align*}
$$

The three regions defined above stand for the selling region, buying region and no transaction region, respectively.

**Lemma 5.5** Let $x_M$ be the “Merton line” defined as

$$
x_M = \frac{-\alpha - r - \sigma^2}{\alpha - r}
$$

Then

(i) $\text{SR} \subset \{(x, t) \in \Omega : x \leq (1 - \mu)x_M\}$;

(ii) $\text{BR} \subset \{(x, t) \in \Omega : x \geq (1 + \lambda)x_M\}$.

Proof: For any $(x, t) \in \text{SR}$,

$$
0 \geq \left( - \frac{\partial}{\partial t} - \mathcal{L} \right) \left( \frac{1}{x+1-\mu} \right) = - \mathcal{L} \left( \frac{1}{x+1-\mu} \right) = \frac{1 - \mu}{(x+1-\mu)^3} \left[ (\alpha - r) x + (1 - \mu) (\alpha - r - \sigma^2) \right].
$$
Then
\[ x \leq -\frac{\alpha - r - \sigma^2}{\alpha - r} (1 - \mu) = (1 - \mu)x_M, \]
which is desired. Part (ii) can be similarly obtained by
\[ \left( -\frac{\partial}{\partial t} - \mathcal{L} \right) \left( \frac{1}{x + 1 + \lambda} \right) \geq 0. \]
\[ \square \]

5.1.4 Properties of solution and buy / sell boundaries

**Proposition 5.6** Let \( v(x, t) \) be the solution to the double obstacle problem (5.11). Then
\[ v_t(x, t) \geq 0. \quad (5.12) \]

A double obstacle problem usually gives rise to two free boundaries. With regard to the problem (5.11), the two free boundaries stand for the buying and selling boundaries, respectively. To begin with, we study the selling boundary.

**Theorem 5.7** There is a continuous, monotonically increasing function \( x^*_{s}(t), \ t \in [0, T) \), such that
\[ \text{SR} = \{(x, t) \in \Omega : x \leq x^*_{s}(t), \ t \in [0, T)\}. \]
Moreover,
\[ (i) \ x^*_{s}(t) \text{ is monotone}; \]
\[ (ii) \ x^*_{s}(T^-) \triangleq \lim_{t \to T^-} x^*_{s}(t) = (1 - \mu)x_M. \quad (5.13) \]

Now we move on to the buying boundary.

**Theorem 5.8** Let
\[ t_0 = T - \frac{1}{\alpha - r} \log \left( \frac{1 + \lambda}{1 - \mu} \right). \quad (5.14) \]
\[ (i) \ \text{BR} \cap \{t_0 \leq t < T\} = \emptyset. \]
(ii) When \(0 \leq t < t_0\), there is a continuous, strictly monotonically increasing function \(x^*_b(t)\), such that
\[
BR = \{(x, t) \in \Omega : x \geq x^*_b(t), \ 0 \leq t < t_0\}.
\]
Moreover,
\[
\lim_{t \to t_0^-} x^*_b(t) = +\infty.
\] (5.15)

**Theorem 5.9** Let \(x^*_s(t)\) and \(x^*_b(t)\) be the free boundaries define in Theorem 5.7 and 5.8. Then,
(i) if \(\alpha - r - \sigma^2 < 0\), then \(x^*_s(t) > 0\), \(x^*_b(t) > 0\) for all \(t\);
(ii) if \(\alpha - r - \sigma^2 = 0\), then \(x^*_s(t) \equiv 0\), \(x^*_b(t) > 0\) for all \(t\);
(iii) if \(\alpha - r - \sigma^2 > 0\), then \(x^*_s(t) < 0\) for all \(t\), and
\[
x^*_s(t) > 0 \text{ for } t \in (t_1, T);
\]
\[
x^*_b(t_1) = 0;
\]
\[
x^*_b(t) < 0 \text{ for } t \in (0, t_1),
\]
where
\[
t_1 = T - \frac{1}{\alpha - r - \sigma^2} \log \frac{1 + \lambda}{1 - \mu}.
\] (5.16)

### 5.1.5 Equivalence

**Proposition 5.10** Let \(v(x, t)\) be the solution to the double-obstacle problem (5.11). Define
\[
V(x, t) = A(t) + \log (x^*_s(t) + 1 - \mu) + \int_{x^*_s(t)}^{x} v(\xi, t) d\xi,
\] (5.17)
where
\[
A(t) = \int_{t}^{T} \left. \frac{rx^2 + (\alpha + r)(1 - \mu)x + \left(\alpha - \frac{1}{2}\sigma^2\right)(1 - \mu)^2}{(x + 1 - \mu)^2} \right|_{x=x^*_s(\tau)} d\tau.
\] (5.18)

Then \(V(x, t)\) is the solution to the problem (5.9).

**Proof:** Since \(v(x, t) = \frac{1}{x+1-\mu}\) for \(x \leq x^*_s(t)\), it is not hard to get
\[
V(x, t) = A(t) + \log(x + 1 - \mu), \ x \leq x^*_s(t).
\] (5.19)
Clearly \( V(x, t) \) satisfies the terminal condition. Therefore, to prove that \( V(x, t) \) is the solution to the problem (5.9), it suffices to show
\[
\begin{align*}
-\frac{\partial}{\partial t} V - \mathcal{L}_0 V &\geq 0 \quad \text{in } \text{SR and } \text{BR}, \\
-\frac{\partial}{\partial t} V - \mathcal{L}_0 V &= 0 \quad \text{in } \text{NT}.
\end{align*}
\] (5.20)

Observe
\[
V_x(x, t) = v(x, t). \quad (5.21)
\]

According to the definition of \( A(t) \), we claim
\[
-\frac{\partial}{\partial x} \left( -\frac{\partial}{\partial t} V - \mathcal{L}_0 V \right) = 0 \quad \text{on } x = x^*_s(t). \quad (5.22)
\]

Indeed, because of (5.21),
\[
\begin{align*}
\mathcal{L}_0 V|_{x=x^*_s(t)} & = \frac{1}{2} \sigma^2 x^2 v_x - (\alpha - r - \sigma^2) x v + \alpha - \frac{1}{2} \sigma^2 |_{x=x^*_s(t)} \\
& = -\frac{1}{2} \sigma^2 x^*_s(t)^2 \left( \frac{1}{(x^*_s(t) + 1 - \mu)^2} - (\alpha - r - \sigma^2) \frac{x^*_s(t)}{x^*_s(t) + 1 - \mu} + \alpha - \frac{1}{2} \sigma^2 \right) \\
& = \frac{1}{(x^*_s(t) + 1 - \mu)^2} \left[ r x^*_s(t)^2 + (\alpha + r) (1 - \mu) x^*_s(t) + \left( \alpha - \frac{1}{2} \sigma^2 \right) (1 - \mu)^2 \right] \\
& = -A'(t) = -\frac{\partial}{\partial t} V(t, x^*_s(t), t), \quad (5.23)
\end{align*}
\]

where the last equality is due to (5.19).

Furthermore, due to (5.10), (5.21) and the fact that \( v(x, t) \) is the solution to the problem (5.11), we have
\[
\begin{align*}
\frac{\partial}{\partial x} (-\frac{\partial}{\partial t} V - \mathcal{L}_0 V) &\leq 0, \quad V_x = \frac{1}{x + 1 - \mu} \quad \text{in } \text{SR}, \\
\frac{\partial}{\partial x} (-\frac{\partial}{\partial t} V - \mathcal{L}_0 V) &= 0 \quad \text{in } \text{NT}, \\
\frac{\partial}{\partial x} (-\frac{\partial}{\partial t} V - \mathcal{L}_0 V) &\geq 0, \quad V_x = \frac{1}{x + 1 + \lambda} \quad \text{in } \text{BR}.
\end{align*}
\]

Combining with (5.22), we then deduce (5.20). \( \square \)
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5.1.6 **Numerical methods**

For convenience, we introduce several notations. Denote

\[
\tau = \alpha - r - \sigma^2 \quad \text{and} \quad \bar{\tau} = 2(\alpha - r) - 3\sigma^2. \tag{5.24}
\]

Then the differential operator \( Lv \) appearing in (5.11) becomes

\[
Lv = \frac{1}{2} \sigma^2 x^2 v_{xx} + (\bar{\tau} - \bar{\eta}) x v_x - \bar{\tau} v
\]

Let \( \Delta t \) be the time step used in the binomial tree method. Assume that \( x \) will go to either \( xu \) with probability \( p \) or \( xd \) with probability \( 1 - p \), where

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}}, \quad d = \frac{1}{u}, \quad p = \frac{pe^{-\tau \Delta t} - d}{u - d}, \quad \rho = e^{\tau \Delta t}.
\end{align*}
\]

It is straightforward to extend the binomial method for the American option pricing model to the problem (5.11). In fact, the difference lies only in that the former involves one lower obstacle while the latter involves two obstacles. The algorithms are given as follows. Let

\[
\Pi_0 = \frac{1}{\rho} \left[ pv(xu, t) + (1 - p)v(xd, t) \right]
\]

If \( \tau \leq 0 \), then \( x^*_s(t) \geq 0 \). Hence we only need to carry out the binomial method for \( x > 0 \). That is,

\[
\begin{align*}
    v(x, t - \Delta t) &= \min \left\{ \max \left\{ \Pi_0, \frac{1}{x+1+\lambda} \right\}, \frac{1}{x+1-\mu} \right\}, \\
    v(x, T) &= \frac{1}{x+1-\mu}, \quad x > 0, \quad \Delta t \leq t \leq T. \tag{5.25}
\end{align*}
\]

If \( \tau > 0 \), by the Fichera Theorem, we can solve the problem (5.11) in \( x > 0 \) and in \( x < 0 \) independently. When \( x > 0 \), there is only the lower obstacle \( \frac{1}{x+1+\lambda} \) involved. Then

\[
\begin{align*}
    v(x, t - \Delta t) &= \max \left\{ \Pi_0, \frac{1}{x+1+\lambda} \right\}, \\
    v(x, T) &= \frac{1}{x+1-\mu}, \quad x > 0, \quad \Delta t \leq t \leq T. \tag{5.26}
\end{align*}
\]

When \( x < 0 \), there always exist two obstacles and the boundary condition at \( x = x^*_{s,\infty} \) should be imposed. So

\[
\begin{align*}
    v(x, t - \Delta t) &= \min \left\{ \max \left\{ \Pi_0, \frac{1}{x+1+\lambda} \right\}, \frac{1}{x+1-\mu} \right\}, \\
    v(x, T) &= \frac{1}{x+1-\mu}, \quad x^*_{s,\infty} < x < 0, \\
    v(x^*_{s,\infty}, t) &= \frac{1}{x^*_{s,\infty}+1-\mu}, \quad \Delta t \leq t \leq T. \tag{5.27}
\end{align*}
\]
At last, we would like to point out that the obstacle problem discussed is very sensitive to the value of $\lambda$ and $\mu$, requesting that the grid of computation be fine enough. The binomial method, however, only possesses the first order of accuracy. As a consequence, it is worth considering more sophisticated numerical approaches for obstacle problems, such as projected SOR method, penalty method, integration equation method and so on.

5.1.7 Appendix: An important lemma

The proofs of Theorem 5.7 and Theorem 5.8 can be found in Dai and Yi (2006) http://www.math.nus.edu.sg/~matdm/oitc.pdf. Here we only introduce a lemma which plays a critical role in the proofs.

Lemma 5.11 Let $v(x,t)$ be the solution to the double obstacle problem (5.11). Then

$$v_x + v^2 \leq 0 \text{ in } \Omega.$$  

Proof: It is clear that $v_x + v^2 = 0$ in $\text{BR}$ and $\text{SR}$. So, the rest is to show $v_x + v^2 \leq 0$ in $\text{NT}$. Denote

$$p(x,t) = v_x(x,t) \text{ and } q(x,t) = v^2(x,t).$$

It is not hard to check that

$$-p_t - \frac{1}{2} \sigma^2 x^2 p_{xx} + \left( \alpha - r - 3\sigma^2 \right) x p_x + \left( 2\alpha - 2r - 3\sigma^2 \right) p = 0 \text{ in } \text{NT}$$

and

$$-q_t - \frac{1}{2} \sigma^2 x^2 q_{xx} + \left( \alpha - r - 2\sigma^2 \right) x q_x + \left( 2\alpha - 2r - 2\sigma^2 \right) q = -\sigma^2 x^2 v_x^2 \text{ in } \text{NT}.$$  

Let $H(x,t) = v_x(x,t) + v^2(x,t) = p(x,t) + q(x,t)$. Then $H(x,t)$ satisfies

$$-H_t - \frac{1}{2} \sigma^2 x^2 H_{xx} \left( \alpha - r - 3\sigma^2 \right) x H_x + \left( 2\alpha - 2r - 3\sigma^2 \right) H$$

$$= -\sigma^2 x^2 v_x^2 - \sigma^2 x q_x - \sigma^2 q$$

$$= -\sigma^2 (x^2 v_x^2 + 2x v v_x + \sigma^2 v^2)$$

$$= -\sigma^2 ( xv_x + v)^2 \text{ in } \text{NT},$$
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\[ -H_t + \frac{1}{2} \sigma^2 x^2 H_{xx} + \left( \alpha - r - 3\sigma^2 \right) x H_x + \left( 2\alpha - 2r - 3\sigma^2 \right) H = -\sigma^2 (xv + v)^2 \leq 0 \text{ in } NT. \]

Obviously \( H(x, t) = 0 \) on the parabolic boundary of \( NT \). Applying the maximum principle yields the desired result. \( \square \)

5.2 Optimal investment and consumption

When consumption is involved, the equation for \( X_t \) becomes

\[ dX_t = (rX_t - C_t) dt - (1 + \lambda) dL_t + (1 - \mu) dM_t. \]

The investor’s problem is to choose an admissible strategy so as to maximize the expected utility of terminal wealth, that is,

\[ \sup_{(L_t, M_t, C_t) \in A(x_0, y_0)} E_{x_0, y_0}^x \left[ \int_0^T e^{\beta(T-t)} U(C_s) ds + U(W_T) \right]. \]

Define the value function

\[ \varphi(x_t, y_t, t) = \sup_{(L_t, M_t, C_t) \in A(x_t, y_t)} E_{x_t, y_t}^x \left[ \int_t^T e^{\beta(T-s)} U(C_s) ds + U(W_T) \right]. \quad (5.28) \]

To derive the HJB equation for \( \varphi \), let us still start from a restricted class of policies in which \( L \) and \( U \) are constrained to be absolutely continuous with bounded derivatives,

\[ L_t = \int_0^t l_s ds, \quad M_t = \int_0^t m_s ds, \quad 0 \leq l_s, \quad u_s \leq k. \]

Then the HJB equation becomes

\[ \max_{C, l, u} \left\{ A^{C, l, u} \tilde{\varphi} + e^{\beta(T-t)} U(C) \right\} = 0, \]

where

\[ A^{C, l, u} \tilde{\varphi} = \tilde{\varphi}_t + \frac{1}{2} \sigma^2 y^2 \tilde{\varphi}_{yy} + \alpha y \tilde{\varphi}_y + (rx - C) \tilde{\varphi}_x + e^{\beta(T-t)} U(C) \]

\[ + l \left[ \tilde{\varphi}_y - (1 + \lambda) \tilde{\varphi}_x \right] + m \left[ (1 - \mu) \tilde{\varphi}_x - \tilde{\varphi}_y \right]. \]
Consider the logarithm utility function \( U(C) = \log C \). The maxima are achieved as follows

\[
C = \frac{e^{\beta(T-t)}}{\varphi_x},
\]

\[
l = \begin{cases} 
  k, & \text{if } \varphi_y - (1 + \lambda) \varphi_x \geq 0 \\
  0, & \text{if } \varphi_y - (1 + \lambda) \varphi_x < 0,
\end{cases}
\]

\[
u = \begin{cases} 
  k, & \text{if } (1 - \mu) \varphi_x - \varphi_y \geq 0 \\
  0, & \text{if } (1 - \mu) \varphi_x - \varphi_y < 0.
\end{cases}
\]

Then

\[
\varphi_t + \frac{1}{2} \sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + r x \varphi_x - e^{\beta(T-t)} + e^{\beta(T-t)} \log \frac{e^{\beta(T-t)}}{\varphi_x} + k \left[ \varphi_y - (1 + \lambda) \varphi_x \right]^+ + k \left[ (1 - \mu) \varphi_x - \varphi_y \right]^+ = 0
\]

Letting \( k \to +\infty \), we then get

\[
\min \{-\varphi_t - L\varphi, -(1 - \mu)\varphi_x + \varphi_y, (1 + \lambda)\varphi_x - \varphi_y\} = 0,
\]

\[
(x, y) \in \mathcal{S}, \quad t \in [0, T) \quad (5.29)
\]

with the terminal condition

\[
\varphi(x, y, T) = \begin{cases} 
  U(x + (1 - \mu) y), & \text{if } y \geq 0 \\
  U(x + (1 + \lambda) y), & \text{if } y < 0,
\end{cases}
\]

where

\[
L\varphi = \frac{1}{2} \sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + r x \varphi_x + e^{\beta(T-t)} (\beta(T - t) - 1 - \log \varphi_x)
\]

Using the transformation,

\[
z = \frac{x}{y} \quad \text{and} \quad \varphi(x, y, t) = V(z, t) + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} \log y, \quad \text{for } y > 0,
\]

it follows

\[
\varphi_x = \frac{1}{y} V_z
\]

\[
\varphi_y = -\frac{x}{y^2} V_z + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta y}
\]

\[
\varphi_{yy} = \frac{2x}{y^3} V_z + \frac{x^2}{y^4} V_{zz} - \frac{e^{\beta(T-t)} - 1 + \beta}{\beta y^2}.
\]
Then

\[-\varphi_t - \mathcal{L} \varphi\]

\[= -V_t + e^{\beta(T-t)} \log y - \frac{1}{2} \sigma^2 y^2 \left( \frac{2x}{y^3} V_z + \frac{x^2}{y^4} V_{zz} - \frac{e^{\beta(T-t)} - 1 + \beta}{\beta y^2} \right)\]

\[-\alpha y \left( -\frac{x}{y^2} V_z + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta y} \right) - rx \frac{1}{y} V_z\]

\[-e^{\beta(T-t)} \left( \beta(T - t) - 1 - \log \left( \frac{1}{y} V_z \right) \right)\]

\[= -V_t + e^{\beta(T-t)} \log y - \frac{1}{2} \sigma^2 \left( 2zV_z + z^2 V_{zz} - \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} \right)\]

\[-\alpha \left( \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} - vz \right) - rzV_z - e^{\beta(T-t)} (\beta(T - t) - 1 - \log V_z + \log y)\]

\[= -V_t - \frac{1}{2} \sigma^2 z^2 V_{zz} + \left( \alpha - r - \sigma^2 \right) zV_z - \left( \alpha - \frac{\sigma^2}{2} \right) \frac{e^{\beta(T-t)} - 1 + \beta}{\beta}\]

\[-e^{\beta(T-t)} (\beta(T - t) - 1) + e^{\beta(T-t)} \log V_z\]

and

\[-(1 - \mu) \varphi_x + \varphi_y = -(1 - \mu) \frac{1}{y} V_z - \frac{x}{y^2} V_z + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta y}\]

\[= \frac{1}{y} \left[ -(z + 1 - \mu) V_z + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} \right],\]

\[(1 + \lambda) \varphi_x - \varphi_y = \frac{1}{y} \left[ (z + 1 + \lambda) V_z - \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} \right].\]

As a result,

\[
\begin{align*}
\{ & -V_t - \mathcal{L}_0 V, -(z + 1 - \mu) V_z + \frac{e^{\beta(T-t)} - 1 + \beta}{\beta}, (z + 1 + \lambda) V_z - \frac{e^{\beta(T-t)} - 1 + \beta}{\beta} \} = 0, \\
& V(z, T) = \log(z + 1 - \mu) \quad \text{in } \Omega,
\end{align*}
\]

(5.30)

where \(\Omega = (-1 - \mu, +\infty) \times [0, T),\)

\[
\mathcal{L}_0 V = \frac{1}{2} \sigma^2 z^2 V_{zz} - \left( \alpha - r - \sigma^2 \right) zV_z + \left( \alpha - \frac{\sigma^2}{2} \right) \frac{e^{\beta(T-t)} - 1 + \beta}{\beta}
\]

\[+ e^{\beta(T-t)} (\beta(T - t) - 1) - e^{\beta(T-t)} \log V_z.\]
CHAPTER 5. PORTFOLIO SELECTION WITH TRANSACTION COSTS

5.3 Optimal investment and consumption: infinite horizon

In (5.28), the discounted value is at $T$. In order to consider the value at time $t$, we introduce the value function

$$
\psi(x_t, y_t, t) = \sup_{(L_t, M_t, C_t) \in A(x_t, y_t)} E^{x_t, y_t} \left[ \int_t^T e^{-\beta(s-t)} U(C_s) \, ds + e^{-\beta(T-t)}U(W_T) \right].
$$

Clearly

$$
\psi(x, y, t) = e^{-\beta(T-t)}\varphi(x, y, t).
$$

It is not hard to verify that $\psi$ satisfies

$$
\min \{-\psi_t - \mathcal{L}\psi, -(1-\mu)\psi_x + \psi_y, (1+\lambda)\psi_x - \psi_y\} = 0, \\
(x, y) \in \mathcal{S}, \quad t \in [0, T) 
$$

with

$$
\psi(x, y, T) = \begin{cases} 
U(x + (1-\mu)y), & \text{if } y \geq 0 \\
U(x + (1+\lambda)y), & \text{if } y < 0,
\end{cases}
$$

where

$$
\mathcal{L}\psi = \frac{1}{2}\sigma^2 y^2 \psi_{yy} + \alpha y \psi_y + rx \psi_x - \beta \psi - (1 + \log \psi_x).
$$

Letting $T - t \to +\infty$, we rule out $\psi_t$ and then get a stationary problem

$$
\min \{-\mathcal{L}\psi^\infty, -(1-\mu)\psi_x^\infty + \psi_y^\infty, (1+\lambda)\psi_x^\infty - \psi_y^\infty\} = 0.
$$

in $\Omega$. Essentially,

$$
\psi^\infty(x, y) = \sup_{L, M, C} E^{x,y} \left[ \int_0^\infty e^{-\beta s} U(C_s) \, ds \right].
$$

Consequently, the counterpart of (5.30) is

$$
\begin{cases} 
\min \left\{-V_t - \mathcal{L}V, -(z + 1 - \mu)V_z + \frac{e^{-\beta(T-t)(\beta-1)-1}}{\beta}, (z + 1 + \lambda)V_z - \frac{e^{-\beta(T-t)(\beta-1)+1}}{\beta}\right\} = 0, \\
V(z, T) = \log(z + 1 - \mu) \quad \text{in } \Omega,
\end{cases}
$$

(5.32)
5.3. OPTIMAL INVESTMENT AND CONSUMPTION: INFINITE HORIZON

where $\Omega = (- (1 - \mu), +\infty) \times [0, T)$,

$$L_1 V = \frac{1}{2} \sigma^2 z^2 V_{zz} - \left(\alpha - r - \sigma^2\right) z V_z - \beta V + \left(\alpha - \frac{\sigma^2}{2}\right) \frac{\beta - 1}{\beta} e^{-\beta(T-t)} + 1 - (1 + \log V_z).$$

Its stationary counterpart is

$$\min \left\{ -L_{1,\infty} V, -(z + 1 - \mu) V_z + \frac{1}{\beta} (z + 1 + \lambda) V_z - \frac{1}{\beta} \right\} = 0,$$

in $\Omega$, where

$$L_{1,\infty} V = \frac{1}{2} \sigma^2 z^2 V_{zz} - \left(\alpha - r - \sigma^2\right) z V_z - \beta V + \left(\alpha - \frac{\sigma^2}{2}\right) \frac{1}{\beta} - (1 + \log V_z).$$
