Chapter 6

Backward Stochastic Differential Equations

For an ordinary differential equation (ODE), under certain regularity conditions, both the initial value and the terminal value problems are well-posed. In fact, for an ODE, the terminal value problem on $[0, T]$ is equivalent to an initial value problem on $[0, T]$ under the time-reversing transformation: $t \to T - t$. However, things are fundamentally different for backward stochastic differential equations (BSDEs) when we are looking for a solution that is adapted to the given filtration. One cannot simply reverse the time to get a solution for a terminal value problem of stochastic differential equations (SDEs), as it would destroy the adaptiveness. Therefore, the first issue one should address in the stochastic case is how to correctly formulate a terminal value problem for SDEs.

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space on which an standard Brownian motion $W(t)$ is defined, such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of $W(t)$, augmented by all the $P$-nullset in $\mathcal{F}$.

6.1 An example

We begin with a simple but illustrative example. Consider the following terminal value problem of SDE:

\[
\begin{cases}
  dZ(t) = 0, & t \in [0, T], \\
  Z(T) = \xi,
\end{cases}
\]

(6.1)
where \( \xi \in L^2_{F_T}(\Omega; R) \). We want to find an \( \{F_t\}_{t \geq 0} \)-adapted solution \( Z(\cdot) \).

However, this is impossible, since the only solution is

\[
Z(t) = \xi, \text{ for all } t \in [0, T],
\]

which is not \( \{F_t\}_{t \geq 0} \)-adapted (unless \( \xi \) is \( F_0 \)-measurable, i.e., \( \xi \) is a constant). So, (6.1) is not well formulated if one expects \( \{F_t\}_{t \geq 0} \)-adapted solution. We should modify or reformulate (6.1).

We start with (6.2). It has been seen that the process \( Z(\cdot) \) given by (6.2) satisfies (6.1) but not necessarily \( \{F_t\}_{t \geq 0} \)-adapted. A natural way of making (6.2) \( \{F_t\}_{t \geq 0} \)-adapted is to redefine \( Z(\cdot) \) as follows:

\[
Z(t) = E(\xi|F_t), \text{ } t \in [0, T].
\]

(6.3)

Then \( Z(\cdot) \) is \( \{F_t\}_{t \geq 0} \)-adapted and satisfies the terminal condition \( Z(T) = \xi \) (since \( \xi \) is \( F_T \)-measurable). But, \( Z(T) \) given by (6.3) no longer satisfies (6.1). Therefore, we need to reformulate (6.1). In other words, we want to find the equation that the process (6.3) satisfies.

Note that the process \( Z(t) \) defined by (6.3) is a square-integrable \( \{F_t\}_{t \geq 0} \)-martingale. By the martingale representation theorem, we can find an \( \{F_t\}_{t \geq 0} \)-adapted process \( \pi(\cdot) \in L^2_F(0, T; R) \) such that

\[
Z(t) = Z(0) + \int_0^t \pi(s)dW(s), \text{ for all } t \in [0, T].
\]

(6.4)

It follows

\[
\xi = Z(T) = Z(0) + \int_0^T \pi(s)dW(s).
\]

(6.5)

Combining (6.4)-(6.5), one has

\[
Z(t) = \xi - \int_t^T \pi(s)dW(s), \text{ for all } t \in [0, T],
\]

whose differential form is

\[
\begin{cases}
  dZ(t) = \pi(t)dW(t), & t \in [0, T] \\
  Z(T) = \xi.
\end{cases}
\]

(6.6)

(6.6) is an appropriate reformulation of (6.1). By comparing (6.6) with (5.13), we see that the term \( \pi(t)dW(t) \) has been added. The process \( \pi(\cdot) \) is not a priori known, and it is a part of the solution. In fact, the presence of
the term $\pi(t)dW(t)$ "corrects" the "non-adaptiveness" of the original $Z(\cdot)$ in (6.2). Thus, by an $\{F_t\}_{t \geq 0}$-adapted solution, we should mean a pair of $\{F_t\}_{t \geq 0}$-adapted processes $(Z(\cdot), \pi(\cdot))$ satisfying (6.6).

The adapted solution of (6.6) is also unique because

$$E|\xi|^2 = E|Z(t)|^2 + \int_0^T E|\pi(s)|^2 ds.$$ 

### 6.2 BSDEs and FBSDs

#### 6.2.1 BSDE

Now we turn to the general situation.

$$ \begin{cases} dZ(t) = h(t, Z(t), \pi(t)) \, dt + \pi(t) \, dW(t), & t \in [0, T], \\ Z(T) = \xi. \end{cases} \quad (6.7) $$

**Theorem 6.12** Under certain regularity conditions and $\xi \in L^2_{\mathbb{F}}$, the BSDE (6.7) admits a unique adapted solution $(Z(t), \pi(t))$.

#### 6.2.2 FBSDs

Let us move on to the system of generally coupled forward-backward stochastic differential equations (FBSDs).

$$ \begin{cases} dX(t) = b(t, X(t), Z(t), \pi(t)) \, dt + \sigma(t, X(t), Z(t), \pi(t)) \, dW(t), \\ dZ(t) = h(t, Z(t), \pi(t)) \, dt + \pi(t) \, dW(t), & t \in [0, T], \\ X(0) = x, \quad Z(T) = \xi. \end{cases} \quad (6.8) $$

In the above, $X, Z$ and $\pi$ are the unknown processes, and they are required to be $\{F_t\}_{t \geq 0}$-adapted. The main feature of the above equation is that process $X(\cdot)$ satisfies a forward SDE, process $Z(\cdot)$ satisfies a BSDE, and they are coupled.

Under certain conditions, the problem (6.8) admits a unique adapted solution $(X(t), Z(t), \pi(t))$.

### 6.3 Application to option pricing problems

The theory of BSDEs and/or FBSDEs can be used to handle option pricing problems.
Consider a market where only two basic assets are traded. One is a bond, whose price process is
\[ dX_0(t) = rX_0(t)dt. \]
The other asset is a stock whose price process satisfies the stochastic differential equation:
\[ dX(t) = bX(t)dt + \sigma X(t)dW(t). \]
Consider a European call option whose payoff is
\[ (X(T) - K)^+. \]
The option pricing problem is: what is the fair price of this option at time \( t = 0 \)?

Let us make some observation on this problem. The price of the option at \( t = T \) is the amount that the holder of the option would obtain as well as the amount that the writer would lose at that time. Now, suppose this option has a price \( z \) at \( t = 0 \). He has to invest this amount of money in some way (called replication) in the market (where there are one bond and one stock available) so that at time \( t = T \), his total wealth, denoted by \( Z(T) \), resulting for the investment of \( z \), should at least compensate his potential loss \( (X(T) - K)^+ \), namely
\[ Z(T) \geq (X(T) - K)^+. \quad (6.9) \]
It is clear that for the same investment strategy, the larger the initial endowment \( z \), the larger the final wealth \( Z(T) \). Hence the writer of the option would like to set \( z \) large enough so that (6.9) can be guaranteed. On the other hand, if it happens that for some \( z \) the resulting final wealth \( Z(T) \) is strictly larger than the loss \( (X(T) - K)^+ \), then the price \( z \) of this option at \( t = 0 \) is considered to be too high. In this case, the buyer of the option, instead of buying the option, would make his own investment to get the desired payoff \( (X(T) - K)^+ \). As a result, the fair price for the option at time \( t = 0 \) should be such a \( z \) that the corresponding optimal investment would result in a wealth process \( Z(t) \) satisfying
\[ Z(T) = (X(T) - K)^+. \]

Now, let us denote by \( Y(t) \) the amount that the writer invests in the stock (i.e., the number of shares is \( Y(t)/X(t) \)). The remaining amount \( Z(t) - Y(t) \)
is invested in the bond. Clearly $Y(t)$ determines a strategy of the investment, which is called a portfolio.

Same as in Chapter 4, the wealth process is
\[
dZ(t) = \left[ rZ(t) + (b - r)Y(t) \right] dt + \sigma Y(t) dW.
\]

By setting $\pi(t) = \sigma Y(t)$, we obtain the following FBSDEs:
\[
\begin{align*}
    &dX(t) = bX(t) dt + \sigma X(t) dW(t), \\
    &dZ(t) = \left[ rZ(t) + \frac{b-r}{\sigma} \pi(t) \right] dt + \pi(t) dW(t), \quad t \in [0, T], \\
    &X(0) = x, \quad Z(T) = (X(T) - K)^+.
\end{align*}
\]

According to the theory of BSDEs and FBSDEs, we conclude that there exists a unique adapted solution $(X(t), Z(t), \pi(t))$ to (6.10). The option price is given by $Z(0)$, and the portfolio $Y(t)$ is given by
\[
Y(t) = \frac{\pi(t)}{\sigma}.
\]

To solve (6.10), let us assume
\[
Z(t) = V(t, X(t)).
\]

Applying Ito lemma,
\[
dZ(t) = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X(t)^2 \frac{\partial^2 V}{\partial x^2} + bX(t) \frac{\partial V}{\partial x} \right) dt + \sigma X(t) \frac{\partial V}{\partial x} dW(t).
\]

Then we get
\[
\begin{align*}
    \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X(t)^2 \frac{\partial^2 V}{\partial x^2} + bX(t) \frac{\partial V}{\partial x} &= rV + \frac{b-r}{\sigma} \pi(t).
\end{align*}
\]

In the end, we get
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X(t)^2 \frac{\partial^2 V}{\partial x^2} + rX(t) \frac{\partial V}{\partial x} - rV = 0.
\]

This is the Black-Scholes equation. Especially
\[
Y(t) = \frac{\sigma X(t) \frac{\partial V}{\partial x}}{\sigma} = \frac{\partial V}{\partial x} X(t) \triangleq \Delta X(t).
\]
This means that the strategy for replication is holding $\Delta$ shares of stock. Consequently,

$$dZ(t) = [rZ(t) + (b - r)\Delta X(t)] dt + \sigma \Delta X(t)dW.$$ 

That is

$$Z(t) = Z(0) + \int_0^t [rZ(s) + (b - r)\Delta X(s)] ds + \int_0^t \sigma \Delta X(s)dW(s)$$

or

$$Z(t) = Z(0) + \int_0^t rZ(s)ds + \int_0^t \Delta (dX(s) - rX(s)ds).$$