Fair valuation of participating policies with surrender options 
and regime switching

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Abstract

We consider the fair valuation of a participating life insurance policy with surrender options when the market values of the asset are modelled by Markov-modulated Geometric Brownian Motion (GBM). We reduce the dimension of the optimal stopping problem for the policy by changing probability measures. We also provide a decomposition result for the value of the policy. The Barone–Adesi–Whaley approximation has been employed to approximate the solution of the free boundary problem for the policy by second-order piecewise linear ordinary differential equations (ODEs). The fair valuation of participating perpetual American contracts are also considered.

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1. Introduction

Recently, participating (with-profits) life insurance products become more and more popular in the insurance and finance markets due to their lower risk but similar returns compared to other equity-index products. In terms of market size, participating life insurance products are considered the most important modern life insurance products in major insurance and finance markets around the world, such as United Kingdom, United States and Japan, etc. (see Ballotta, 2004) Participating life insurance products are investments plans with associated life insurance benefits, a specified benchmark return, a guarantee of an annual minimum rate of return and a specified rule of

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the distribution of annual excess investment return above the guaranteed return. The policyholder has to pay a single lump sum deposit to the insurer to initialize the contract. The insurer plays the role of a fund manager to manage the investment of funds in a specified reference portfolio. One major feature of these investment plans is the sharing of profits from an investment portfolio between the policyholder and the insurer. Typically, the insurer employs a specified rule of surplus distribution, namely reversionary bonus, to credit interest at or above a specified guaranteed rate to the policyholders every period, say per annum. If the terminal surplus of the fund is positive, the policyholder can also receive a terminal bonus. Grosen and Jørgensen (2000) and Ballotta et al. (2003) provided a comprehensive discussion on different contractual features of participating policies. Since there is a global trend of using the market-based and fair valuation accountancy standards for the implementation of risk management practice for participating policies, it is of practical importance and relevance to develop appropriate and objective models for the fair valuation of these policies.

Wilkie (1987) pioneered the use of modern option pricing theory to investigate the embedded options in bonuses on participating life-insurance policies. Grosen and Jørgensen (2000) developed a flexible contingent claims model to incorporate the minimum rate guarantees, bonus distributions and surrender options. They considered the surrender options as the possibility of early exercise of an American-style option and investigated the pricing behaviors of the policy by varying the levels of interest rates, the parameters of bonus policy and the volatility of the market value of the reference asset. Priel et al. (2001) incorporated the path dependence associated with the rule of the bonus distribution in their contingent claims model. They reduced the dimension of the partial differential equation in their model by similarity transformation of variables. Bacinello (2003) adopted discrete-time binomial models for computing the numerical solutions of the fair valuation problem of participating policies. Grosen and Jørgensen (2002) adopted a barrier option framework to investigate the impact of regulatory intervention rules for reducing the insolvency risk of the policies. Willder (2004) considered the use of modern option pricing approach for investigating the effects of various bonus strategies in unitized participating policies. Chu and Kwok (2005) constructed a contingent claims model for participating policies that can incorporate rate guarantee, bonuses and default risk. They obtained an analytical approximation solution to the problem by perturbation method and the numerical solution by developing effective finite difference algorithms. Ballotta (2004) developed a valuation method based on the Esscher transform for participating policies under a jump-diffusion process for the dynamics of the market values of the reference asset.

In this paper, we consider the fair valuation of a participating life insurance policy with surrender options, bonus distributions and rate guarantees when the dynamics of the market values of the asset is driven by Markov-modulated Geometric Brownian Motion (GBM). As in Buffington and Elliott (2002) and Siu (2005), we suppose that the parameters of the market values of the assets, namely the market interest rates, the expected growth rate and the volatility of the risky asset, depend on unobservable states of an economy which are modelled by a continuous-time hidden Markov chain process. The switching behavior of the economic states can be due to the structural changes in economic conditions and business cycles. In practice, many life insurance products are relatively long-dated compared with financial products. There can be substantial fluctuations in economic variables, which affect the dynamics of the market values of the assets, over a long period of time. Hence, it is of practical importance and relevance to incorporate the switching behavior of the economic states in modelling the dynamics of the market values of the assets for the fair valuation of insurance products. Following Siu (2005), we adopt regime switching Esscher transform developed in Elliott et al. (2005) to determine an equivalent martingale measure for the fair valuation of the policy in an incomplete market described by the Markov-modulated model. The seminal work by Gerber and Shiu (1994) pioneered the use of the Esscher transform for option valuation. By considering the surrender options as the early exercise privilege of an American-style option (see Grosen and Jørgensen, 2000), the fair valuation of the policy can be formulated as an optimal stopping problem from which the value of the policy and the optimal surrender strategy can be determined. We reduce the dimension of the optimal stopping problem for the policy by adopting the method of changing probability measures in Hansen and Jørgensen (2000). We also provide a decomposition for the value of the policy into the value of an identical policy without surrender options and the premium for surrender options. The fair valuation problem can also be formulated as a free boundary
problem. Following Buffington and Elliott (2002), we employ the approximation method due to Barone-Adesi and Whaley (1987) to approximate the solution of the free boundary problem by second-order piecewise linear ordinary differential equations (ODEs). We also consider the fair valuation of participating perpetual American contracts in order to provide some insights in their finite-maturity counterparts. The value of a perpetual contract can also serve as an approximation to its finite-maturity counterpart since most of the participating contracts are relatively long-dated compared with other financial contracts. This paper is outlined as follows.

Section 2 presents the optimal stopping problem associated with the fair valuation of the participating American policies under the Markov-modulated GBM. We also adopt the method of changing probability measures to reduce the dimension of the problem. Section 3 presents a decomposition for the value of the policy. In Section 4, we consider the free-boundary problem for the fair valuation of the policy. In Section 5, we consider the use of the approximation method due to Barone-Adesi and Whaley (1987) and its extension by Buffington and Elliott (2002) to approximate the solution of the free boundary problem by second-order piecewise linear ODEs. Section 6 discusses the fair valuation of participating perpetual American contracts. The final section summarizes the paper and suggests some potential topics for further investigation.

2. The optimal stopping problem

2.1. Introduction

In this section, we consider a continuous-time perfect and frictionless financial market in which there are no taxes, transaction costs and short-sales constraints. We suppose that there are a risk-free money market account and risky assets backing a participating life insurance contract with surrender options. The dynamics of the market values of the reference asset is governed by a Markov-modulated Geometric Brownian Motion. We consider the valuation of the participating life insurance contract with surrender options (participating American-style contract) when the market interest rates, the drift and volatility of the underlying asset base depend on the states of a continuous-time hidden Markov chain process. The states of the hidden Markov chain process represents various states of an economy. The regime switching of the states of the economy can be due to the structural changes in the (macro)-economic conditions, the changes in political situations, the impact of some (macro)-economic news and business cycles. We further assume that there is no expense charges, lapses and mortality risk. The market described by the regime switching model is incomplete in general (see Guo, 2001; Buffington and Elliott, 2002; Elliott et al., 2005; Siu, 2005). Hence, there are infinitely many equivalent martingale measures for the valuation of a participating contract. As in Siu (2005), we employ the regime switching Esscher transform described in Elliott et al. (2005) to determine an equivalent martingale measure for the fair valuation of the participating contract in the incomplete market setting.

The participating American-style contract is different from its European counterpart in that it provides the policyholder the right to terminate (exercise the contract) at any time before the maturity of the contract. In addition to the bonus option and the interest rate credited with guarantee, the policyholder is granted with an option to sell back the policy to the issuer any time before the maturity of the policy. The option that allows the policyholder to sell back the policy to the issuer any time is known as the surrender option in the insurance markets. A participating American-style contract consists of three components, namely the risk-free bond, bonus option and the surrender option while a participating European-style contract can be decomposed into two components, namely the risk-free bond and the bonus option. The presence of the surrender option in a participating American-style contract makes its valuation problem more complicated than that of its European counterpart. As in the valuation of American-style options, the valuation of a participating American-style contract can be formulated as an optimal stopping problem, or equivalently, a free boundary problem. For the valuation of a participating American-style policy, we need to determine the fair value of the policy and its optimal termination or exercise strategy. In practice, some numerical methods, such as Monte-Carlo simulation, finite-difference methods and binomial models, are employed to obtain numerical approximations to the valuation problem. Due to the facts that Monte-Carlo simulation is a forward
method and that the early exercise strategy is not known in advance, Monte-Carlo simulation was considered to be not suitable for the valuation of American-style options in general. However, some recent studies (see, for example, Tilley, 1993; Broadie and Glasserman, 1997) indicated that Monte Carlo simulation is a feasible way to solve the valuation problem of American-style options numerically. The finite-difference methods provide a natural way to solve the free-boundary problems for the valuation of American-style options and the participating American-style contracts. Grosen and Jørgensen (2000) implemented the recursive binomial method for the valuation of the participating American-style contracts. Much of the literature focuses on the valuation of participating American contracts under GBM. There is a relatively little work on the fair valuation of participating American contract under the regime switching models for the asset price dynamics. Here, we provide a valuation method of some particular participating American-style contracts under the Markov-Modulated GBM by modifying the method employed in Hansen and Jørgensen (2000). In the sequel, we introduce the set up of our model.

2.2. Definition of the liabilities

In this section, we describe the dynamics of the liability side of the balance sheet. For each time $t \in T$, let $R_t$ and $D_t$ denote the book value of the policy reserve and the bonus reserve (buffer), respectively. $R_t$ can be considered the policyholder’s account balance and is in general different from the concurrent fair value of the policy. The liabilities of the insurance company consists of two components, namely the policy reserve $R_t$ and the bonus reserve $D_t$. Let $A_t$ denote the market value of the asset backing the policy (asset base). Then, $A_t, R_t$ and $D_t$ satisfy the following accounting identity:

$$A_t = R_t + D_t, \quad t \in T. \tag{2.1}$$

where $R(0) = \alpha A(0)$, $\alpha \in (0, 1]$, and $R(0)$ is the single initial premium paid by the policyholder for acquiring the contract and $\alpha$ is the cost allocation parameter. Note that the $\alpha$-portion of the initial asset portfolio is financed by the policyholder.

The funds are distributed to the two components of liability over time according to the bonus policy described by the continuously compounded interest rate credited to the policy reserve $c_R$; that is,

$$dR_t = c_R R_t dt. \tag{2.2}$$

In practice, the bonus policy and $c_R$ are decided by the management level of an insurance company. The determination of $c_R$ involves many issues, such as the political situations, legal issues and the strategic considerations within the insurance company. There is no consensus on a unified rule for the specification of $c_R$. As in Grosen and Jørgensen (2000), we specify the interest rate crediting mechanism in the form of a mathematical function, which can provide an accurate approximation and realistic description to the "true" bonus policy as much as possible. In this way, as noted by Grosen and Jørgensen (2000), we can adopt arbitrage pricing models in mathematical finance for our analysis. Grosen and Jørgensen (2000) mentioned that the actual interest rate crediting mechanism $c_R$ can be specified as a function $c_R(A, R)$ of the asset base $A$ and the policy reserve $R$. Hence, the interest bonus can be distributed according to the actual investment performance and the current financial condition, such as the degree of solvency, of the insurance company. Typically, the insurance company has specified a constant long-term target buffer ratio $\beta$ of the bonus reserve $D_t$ to the policy reserve $R_t$, where a realistic value of $\beta$ should be between 10 and 15%. The policyholder of the contract can receive a certain proportion, say $\delta \in (0, 1]$, of the excess of the ratio of the bonus reserve $D_t$ to the policy reserve $R_t$ over $\beta$. The proportional constant $\delta$ is called the reversionary bonus distribution rate or distribution ratio. In practice, appropriate values of $\delta$ are chosen to achieve a stable smoothing of the surplus and typical values for $\delta$ are around 20–30%. We also suppose that there is a specified guarantee rate $r_g$ for the minimum interest rate credited to the policyholder’s account. This means that the interest rate $c_R(A, R_t) \geq r_g$. Grosen and Jørgensen (2000), Priel et al. (2001) and Chu and Kwok (2005) provided different specifications for the interest rate crediting scheme. As in Chu and Kwok (2005), we adopt the continuous compounding version of the
interest rate crediting scheme as follows:

\[ cR(A, R_t) = \max \left( r, \delta \left( \ln \frac{A}{R_t} - \beta \right) \right), \]  

(2.3)

where the distribution rate \( \delta \) is assumed to be equal to one; that is, there is a “full participation”.

When the policyholder terminates the policy by exercising the surrender option at time \( t \), the value that is paid to the policyholder by the insurer is called the intrinsic value of the policy. The intrinsic value of the policy depends on the market value of the asset base \( A_t \) and the policy reserve \( R_t \) at time \( t \) and is given as follows:

\[ g(A_t, R_t, t) = \begin{cases} R_t & \text{if } A_t < \frac{R_t}{u} \\ R_t + \gamma (A_t - \frac{R_t}{u}) & \text{if } A_t \geq \frac{R_t}{u} \end{cases}, \]  

(2.4)

where \( \gamma \in (0, 1) \) is the bonus distribution rate.

Let \( T \) denote the maturity of the policy. \( PT = \max(\alpha A_T - R_T, 0) \) is the terminal bonus option, which can be considered a standard European call option that grants the policyholder the right to pay the policy value as a strike price to receive \( \alpha \)-portion of the asset portfolio. The terminal bonus option can also be viewed as a standard European put option that grants the insurer the right to sell the asset for the policy value when the asset value falls below the policy value. The terminal payoff of the policy at time \( T \) is given by:

\[ g(A_T, R_T, T) = \begin{cases} R_T & \text{if } A_T < \frac{R_T}{u} \\ R_T + \gamma (A_T - \frac{R_T}{u}) & \text{if } A_T \geq \frac{R_T}{u} \end{cases} = R_T + \gamma PT. \]  

(2.5)

2.3. Model dynamics

We fix a complete probability space \((\Omega, \mathcal{F}, P)\), where \( P \) is the real-world probability measure. Let \( T \) denote the time index set \([0, T]\) of the model. Let \( \{W_t\}_{t \in T} \) denote a standard Brownian Motion on \((\Omega, \mathcal{F}, P)\) with respect to the \( P \)-augmentation of its natural filtration \( \mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \in T} \). The states of an economy are described by a continuous-time hidden Markov chain \( \{X_t\}_{t \in T} \) with a finite state space \( S = \{s_1, s_2, \ldots, s_N\} \). As in Elliott et al. (1994), we can identify the state space of \( \{X_t\}_{t \in T} \) to be a finite set of unit vectors \( \{e_1, e_2, \ldots, e_N\} \), where \( e_i = (0, \ldots, 1, \ldots, 0) \).

We suppose that \( \{X_t\}_{t \in T} \) and \( \{W_t\}_{t \in T} \) are independent.

Write \( \Pi(t) \) for the generator \( \{\pi_{ij}(t)\}_{i,j} \) of the hidden Markov chain model. Then, from Elliott et al. (1994), we have the following semi-martingale representation theorem for the process \( \{X_t\}_{t \in T} \):

\[ X_t = X_0 + \int_0^t \Pi(s)X_s ds + M_t. \]  

(2.6)

Here \( \{M_t\}_{t \in T} \) is a martingale increment process with respect to the filtration generated by \( \{X_t\}_{t \in T} \).

Let \( \{r_t\}_{t \in T} \) denote the instantaneous market interest rate of the money market account, which depends on the states \( \{X_t\}_{t \in T} \), that is,

\[ r_t = r(t, X_t) = \langle r, X_t \rangle, \quad t \in T, \]  

(2.7)

where \( r = (r_1, r_2, \ldots, r_N) \) with \( r_i > 0 \) for each \( i = 1, 2, \ldots, N \) and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^N \). In this case, the dynamics of the price process \( \{B_t\}_{t \in T} \) for the bank account is described by:

\[ B_t = \exp \left( \int_0^t r_s ds \right). \]  

(2.8)
Now, we assume that the expected growth rate \([\mu_i]_{t\in[T]}\) and the volatility \([\sigma_i]_{t\in[T]}\) of the market value of the asset backing the participating contract (asset base) also depend on \([X_t]_{t\in[T]}\) and are given by:

\[
\mu_i := \mu_i(t, X_t) = (\mu_i, X_i), \quad \sigma_i := \sigma_i(t, X_t) = (\sigma_i, X_i),
\]

(2.9)

where \(\mu := (\mu_1, \mu_2, \ldots, \mu_N)\) and \(\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)\) with \(\sigma_i > 0\) for each \(i = 1, 2, \ldots, N\).

We focus on the asset side of the balance sheet. As in Grosen and Jørgensen (2000), we assume that the insurer maintains the investment of the asset base in a well-diversified and well-specified reference portfolio at any time. For each \(t \in T\), let \(A_t\) denote the market value of the reference portfolio at time \(t\). We do not impose any assumptions on the portfolio compositions with respect to various assets, such as bonds, equities and real estates. No assumptions are imposed on the dynamics of the individual assets in the reference portfolio. We analyze the dynamics of the market value of the assets backing the participating contract in an aggregate level and assume that the dynamics of the market value of the reference portfolio \([A_t]_{t\in[T]}\) is governed by the following Markov-modulated GBM:

\[
dA_t = \mu(A,R,X,t) dt + \sigma_t dW_t, \quad A_0 = a.
\]

(2.10)

2.4. Fair valuation

The fair value of the participating American contract can be decomposed into the fair values of the risk-free bond, the bond option and the surrender option. Suppose the state of the economy \(X_t\) at time \(t\) is \(K\), where \(t \in [0, T]\). Then, when \(A_t = A\) and \(R_t = R\), the intrinsic value of the policy if it is exercised at time \(t\) is given by:

\[
g(A,R,X,t) = \begin{cases} 
R & \text{if } A < \frac{R}{\sigma^2}, \\
R + (\alpha A - R) & \text{if } A \geq \frac{R}{\sigma^2}.
\end{cases}
\]

(2.11)

Let \(P := \max(\alpha A - R, 0)\) represent the payoff of the bonus option. Then, \(g(A, R, X, t)\) can be written in the following form:

\[
g(A, R, X, t) = R + \gamma P.
\]

(2.12)

The fair valuation of the risk-free part of the participating contract, namely the risk-free bond, is straightforward (see Grosen and Jørgensen, 2000). For the fair valuation of the risky part of the policy consisting of the bond option and the surrender option, we need to determine an equivalent risk-neutral martingale measure to ensure that there are no arbitrage opportunities in the market described by the model (see Harrison and Kreps, 1979; Harrison and Pliska, 1981, 1983). We employ the regime switching Esscher transform to determine an equivalent martingale measure for the valuation of the policy by the martingale approach for more discussions on the martingale approach for the valuation of participating policies, see Bacchello (2003). The choice of the equivalent martingale measure by the regime switching Esscher transform can be justified by minimizing the relative entropy of an equivalent martingale measure and the real-world probability \(P\) (see Elliott et al., 2005). The Esscher transform has been adopted by Ballotta (2004) and Siu (2005) for the valuation of participating products in incomplete market settings.

Let \(Y_t\) denote the logarithmic return \(\ln(A_t/A_0)\) from the asset over the time duration \([0, t]\). Then, the dynamics of \(A_t\) can be written as:

\[
A_t = A_0 \exp(Y_t - Y_0).
\]

(2.13)

where

\[
Y_t = \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s.
\]

(2.14)

Write \(\mathcal{F}^P_{t\in[T]}\) and \(\mathcal{F}^\gamma_{t\in[T]}\) for the \(\mathcal{P}\)-augmentation of the natural filtrations generated by \([X_t]_{t\in[T]}\) and \([Y_t]_{t\in[T]}\), respectively. For each \(t \in T\), define \(\mathcal{G}_t\) as the \(\sigma\)-algebra \(\mathcal{F}^\gamma_t \vee \mathcal{F}^\gamma_T\). Define the regime switching parameter \(\theta_t := \ldots\)
θ(t, X_t) as follows:
\[ \theta_t = \langle \theta, X_t \rangle, \] (2.15)
where \( \theta := (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^N. \)

Then, as in Elliott et al. (2005), the regime switching Esscher transform \( \mathbb{P}^\theta \sim \mathbb{P} \) on \( G_t \) is:
\[ d\mathbb{P}^\theta \bigg|_{G_t} = \exp \left( \int_0^t \theta_s \, dY_s \right) E_{\mathbb{P}^\theta} \left[ \exp \left( \int_0^t \theta_s \, dY_s \right) \bigg| F_{X_t} \right], \quad t \in \mathcal{T}. \] (2.16)

Hence, the Radon–Nikodym derivative of the regime switching Esscher transform can be written as:
\[ d\mathbb{P}^\theta \bigg|_{G_t} = \exp \left( \int_0^t \theta_s \sigma_s \, dW_s - \frac{1}{2} \int_0^t \theta_s^2 \sigma_s^2 \, ds \right). \] (2.17)

Write \( \{ \tilde{\theta}_t \}_{t \in \mathcal{T}} \) denote a family of risk-neutral regime switching Esscher parameters. As in Elliott et al. (2005), we assume that \( \{ \tilde{\theta}_t \}_{t \in \mathcal{T}} \) can be determined from the following martingale condition for the discounted market values of the reference asset:
\[ A_0 = E_{\tilde{\mathbb{P}}} \left[ \exp \left( -\int_0^t r_s \, ds \right) A_t \bigg| F_t \right], \quad \text{for any } t \in \mathcal{T}, \] (2.18)
where \( E_{\tilde{\mathbb{P}}} \) denote the expectation operator under \( \tilde{\mathbb{P}} \).

Then, \( \tilde{\theta}_t \) can be determined uniquely by:
\[ \tilde{\theta}_t = r_t - \frac{\mu_t}{\sigma_t^2}, \quad t \in \mathcal{T}. \] (2.19)

See Elliott et al. (2005) for the proof.

Now, we notice that
\[ d\mathbb{P}^\theta \bigg|_{\mathcal{G}_0} = \exp \left( \int_0^t \left( \frac{r_s - \mu_s}{\sigma_s} \right) dW_s - \frac{1}{2} \int_0^t \sigma_s^2 \left( \frac{r_s - \mu_s}{\sigma_s} \right)^2 ds \right). \] (2.20)

By Girsanov’s theorem, \( \tilde{W}_t := W_t + \int_0^t \left( \frac{r_s - \mu_s}{\sigma_s} \right) \, ds \) is a standard Brownian motion with respect to \( \{ \mathcal{G}_t \}_{t \in \mathcal{T}} \) under \( \mathbb{P}^\theta \). Hence, the dynamics of \( A_t \) under \( \mathbb{P}^\theta \) is given by:
\[ dA_t = r_t A_t \, dt + \sigma_t A_t \, d\tilde{W}_t. \] (2.21)

Suppose the trajectory of the hidden process \( X \) from time 0 to time \( t \) is known in advance, where \( t \in \mathcal{T} \); that is, at time \( t \), the enlarged information set \( \mathcal{G}_t \) is accessible to the market’s agent. Write \( \mathcal{T}_{\mathcal{G}_t} \) for the class of \( \{ \mathcal{G}_t \}_{t \in \mathcal{T}} \) stopping times taking values in \([0, T]\). As in Grosen and Jørgensen (2000), we determine the fair value of the participating American contract based on the general theory of the fair valuation of the American-type contingent claims (see Karatzas, 1988, 1989). Then, the fair value of the participating American policy \( V_t \) at time \( t \) can be determined by solving the following optimal stopping problem with respect to the enlarged information set \( \mathcal{G}_t \):
\[ V_t = \text{ess sup}_{\tau \in \mathcal{T}_{\mathcal{G}_t}} E^\mathbb{P} \left[ \exp \left( -\int_{\tau}^t r_s \, ds \right) g(A, R, X, \tau) \bigg| \mathcal{G}_t \right]. \] (2.22)

As in the method of augmenting an additional state variable in the valuation problem of Asian options, we consider an additional state variable \( R_t \), which is a path integral of the process \( A_t \), for determining the fair value of
the participating policy. Since \( R_t \) is a path integral of \( A_t \) and \( A_t \) is a Markov process given that the trajectory of \( X \) is known, \( (A_t, R_t) \) is a two-dimensional Markov process given the trajectory of \( X \). Due to the fact that \( X \) is also a Markov process, \( (A_t, R_t, X_t) \) is a three-dimensional Markov process with respect to \( G_t \). Now, if \( A_t = A, \ R_t = R \) and \( X_t = X \) are given at time \( t \), then by the Markov property of \( (A_t, R_t, X_t) \), the fair value of the participating policy \( V_t \) at time \( t \) is given by:

\[
V_t = V(A, R, X, t) = \text{ess sup}_{\tau \in [t, T]} E^\tilde{\theta} \left[ e^{-\int_{\tau}^{T} r_s ds} V(A, R, X, \tau) \mid (A_t, R_t, X_t) = (A, R, X) \right].
\]

(2.23)

Buffington and Elliott (2002) adopted a similar method to determine the price of an American option.

### 2.5. Reduction of dimensionality

We notice that the optimal stopping problem for the fair valuation of the participating American policy involves three state variables. In order to simplify the problem, we employ the method of changing probability measures in Hansen and Jørgensen (2000) to reduce the dimension of the problem from three state variables to two state variables. First, we define a new state variable \( Z_t = \ln \frac{A_t}{R_t} \). We further assume that

\[
c_Z(Z_t) = c_R(A_t, R_t), \quad g_Z(Z_t, X_t, t) = g(A_t, R_t, X_t, t).
\]

(2.24)

Note that the intrinsic value at time \( t \) can be written as follows:

\[
g_Z(Z_t, X_t, t) = 1 + \gamma \max(\alpha e^{Z_t} - 1, 0).
\]

(2.25)

By Itô’s lemma, the dynamics of \( Z_t \) under \( \tilde{P} \) is given by:

\[
dZ_t = (\mu_t - c_Z(Z_t) - \frac{1}{2} \sigma^2_t) dt + \sigma_t d\tilde{W}_t.
\]

(2.26)

Now, we define a \( \tilde{P} \)-martingale with respect to \( G_t \):

\[
\xi(t) := \text{exp} \left( -\int_0^t \sigma_s d\tilde{W}_s \right), \quad \frac{d\tilde{P}}{d\tilde{P}} \bigg|_{G_t} = \xi(t). \quad t \in T.
\]

(2.28)

By Girsanov’s theorem,

\[
\tilde{W}_t := W_t - \int_0^t \sigma_s ds,
\]

(2.29)

is a standard Brownian motion under \( \tilde{P} \) with respect to \( G_t \).

Under \( \tilde{P} \), the dynamics of \( A_t \) can be represented by:

\[
dA_t = (r_t + \sigma_t^2) A_t dt + \sigma_t A_t d\tilde{W}_t.
\]

(2.30)

Now, by Itô’s lemma, the dynamics of \( Z_t \) under \( \tilde{P} \) is given by:

\[
dZ_t = \left( r_t + \frac{1}{2} \sigma_t^2 - c_Z(Z_t) \right) dt + \sigma_t d\tilde{W}_t.
\]

(2.31)
Hence, given knowledge of the values of the hidden Markov chain process \( X \), the new state variable \( Z \) is a Markov process on its natural filtration. Write \( \hat{E} \) for the expectation operator under \( \hat{P} \). By Bayes’ rule, the optimal sampling theorem and the Markov property of the state variables,

\[
\hat{E} = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) g(A, R, X, \tau) \right] G_t \, d\hat{P} = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) g(A, R, X, \tau) \right] G_t \, d\hat{P}
\]

when the values of the new state variable \( Z_t \) and the hidden state variable \( X_t \) at time \( t \) are \( Z \) and \( X \), respectively.

Due to the fact that \( Z \) is also a Markov process on its natural filtration, the new optimal stopping problem with two state variables is much easier to solve than the original optimal stopping problem with three state variables. By Øksendal (2003), the optimal stopping time for this problem belongs to the set of stopping times \( \mathcal{T}_K \).

As in Hansen and Jørgensen (2000), let \( \bar{V}_g(Z, X, \tau) \) denote the value of the participating American contract denominated by the asset price \( A \). We also call \( \bar{V}_g(Z, X, \tau) \) the \( A \)-denominated value of the contract. That is,

\[
\bar{V}_g(Z, X, \tau) = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) g(Z, X, \tau) \right] G_t \, d\hat{P} = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) g(Z, X, \tau) \right] G_t \, d\hat{P}
\]

In the sequel, we provide the analysis for the \( A \)-denominated value of the contract \( \bar{V}_g(Z, X, \tau) \) instead of \( V(Z, X, \tau) \). Write \( \tilde{g}_g(Z, X, \tau) = e^{-\tau Z}g(Z, X, \tau) \) for \( e^{-\tau Z}g(Z, X, \tau) \). Then,

\[
\tilde{g}_g(Z, X, \tau) = e^{-\tau Z} + \gamma e^{-\tau Z} \max(\alpha e^\tau - 1, 0) = \begin{cases} e^{-\tau Z} & \text{if } Z \leq \ln 1/\alpha, \\ \gamma e^{-\tau Z} + (1 - \gamma) e^{-\tau} & \text{if } Z > \ln 1/\alpha, \end{cases}
\]

which is a decreasing function of \( Z \).

Let \( \tau^*_d \) denote the optimal stopping rule for the optimal stopping problem (2.32). Then,

\[
\tilde{V}_g(Z, X, \tau) = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) \tilde{g}_g(Z, X, \tau) \right] G_t \, d\hat{P} = \text{ess sup} \int_0^T \left[ \exp \left( -\int_0^t r_s ds \right) \tilde{g}_g(Z, X, \tau) \right] G_t \, d\hat{P}
\]
Let $\mathcal{C}$ and $S$ denote the continuation region and the stopping region of the above optimal stopping problem, respectively. Then,

$$
\mathcal{C} = [(Z, X, s)|s \in [t, T] \wedge \hat{V}_2(Z, X, s) > \hat{g}_2(Z, X, s)],
$$

(2.37)

and

$$
S = [(Z, X, s)|s \in [t, T] \wedge \hat{V}_2(Z, X, s) = \hat{g}_2(Z, X, s)].
$$

(2.38)

In the region $\mathcal{C}$, the $A$-denominated value of the policy is greater than its $A$-denominated intrinsic value. Hence, it is not optimal for the policyholder to surrender the policy. The region $S$ specifies the threshold curve on which it is optimal for the policyholder to surrender the policy. The threshold curve of optimal surrendering can incorporate the dependency of the optimal surrendering decision of the policyholder on the states of the economy.

The optimal stopping rule $\tau^*_s$ is given by the first exit time of $(Z, X, t)$ from $\mathcal{C}$; that is,

$$
\tau^*_s = \inf\{s \in [t, T]|(Z, X, s) \in \hat{C}\},
$$

(2.39)

where $\hat{C}$ is the complement of $C$.

Suppose the number of regimes $N$ for the hidden process $X$ is two. Then, we consider the case that there exists two threshold curves $Z^{i*}_2(s), Z^{i*}_3(s) \leq \ln(1/\alpha)$, for $s \in [t, T]$, such that the continuation region $\mathcal{C}$ can be represented in the following form:

$$
\mathcal{C} = C^1 \cup C^2,
$$

(2.40)

where $C^i := [(Z, \epsilon_i, s)|s \in [t, T] \wedge Z \in (Z^{i*}_i(s), \infty)]$, for $i = 1, 2$.

By Karatzas (1989), Øksendal (2003) and Elliott and Kopp (2004), we require that $Z^{i*}_i(s)$ is a continuous function of time $s \in [t, T]$, for $i = 1, 2$ and $t \in T$. Note also that $Z^{i*}_i(T) = \ln(1/\alpha)$. There are three possible cases, namely $Z^{1*}_2(s) < Z^{2*}_2(s), Z^{1*}_3(s) = Z^{2*}_3(s)$ and $Z^{1*}_2(s) > Z^{2*}_2(s)$, for any given $s \in [t, T]$.  

3. Decomposition of the value

In this section, we provide a decomposition for the $A$-denominated value of the participating American contract into the $A$-denominated value of its European counterpart and the $A$-denominated early exercise premium. For the sake of generality, we suppose that the number of regimes of the economy is $N$ when we derive the decomposition result in this section.

Let $V^E(Z, X, t)$ denote the $A$-denominated value of a participating European contract with identical contractual features with its American counterpart in Section 2, except without surrender options. Following Siu (2005), it can be shown that

$$
V^E(Z, X, t) = \hat{E}(\hat{g}_2(Z, X, T)/Z_t, X_t) = (Z, X)).
$$

(3.1)

Write $V_i^E$ for $V^E(Z, \epsilon_i, t)$, where $i = 1, 2, \ldots, N$, and $V^E := (V_1^E, V_2^E, \ldots, V_N^E)$. Then, it has been shown in Siu (2005) that $V^E$ satisfies the following $N$ coupled PDEs:

$$
L_{Z,X}(V^E_i) + \{V^E_i, \Pi_C\} = 0, \quad i = 1, 2, \ldots, N.
$$

(3.2)

Let $\epsilon(Z, X, t)$ denote the $A$-denominated premium of surrender option or early exercise premium ($A$-denominated) at time $t$ when $Z_t = Z$ and $X_t = X$. The following proposition provides a decomposition of $\hat{V}_2(Z, X, t)$ into the two components $V^E(Z, X, t)$ and $\epsilon(Z, X, t)$, as in Hansen and Jørgensen (2000).
Proposition 3.1. The $A$-denominated value of the participating American contract at time $t$ is given by:

$$\tilde{V}_Z(Z, X, t) = V^E(Z, X, t) + \epsilon(Z, X, t),$$  \hfill (3.3)

where

$$\epsilon(Z, X, t) = \hat{E}\left(\int_t^T e^{-Zu} (c_Z(Z_u) - r_t)I_{\{Z_u \in S\}} \, du | (Z_t, X_t) = (Z, X)\right),$$  \hfill (3.4)

and $I_A$ is the indicator function of an event $A$.

Proof. Following Hansen and Jørgensen (2000), we consider the dynamics of the values of the participating contract separately on the continuation region $C$ and the stopping region $S$.

First, since $(Z_t, X_t)$ is a two-dimensional Markov process with respect to the enlarged filtration $G_t$,

$$V^E(Z, X, t) = \hat{E}(\tilde{g}_Z(Z, X, T) | (Z_t, X_t) = (Z, X)) = \hat{E}(\tilde{g}_Z(Z, X, T) | G_t),$$  \hfill (3.5)

which is a $\hat{P}$-martingale with respect to $G_t$.

By applying Itô’s rule,

$$V^E(Z, X, t) = V^E(Z, X, 0) + \int_0^t \left\{ \frac{\partial V^E}{\partial u} + (r_t + \frac{1}{2} \sigma^2 u - c_Z(Z_u)) \frac{\partial V^E}{\partial Z_u} + \frac{1}{2} \sigma^2 u \frac{\partial^2 V^E}{\partial Z^2_u} \right\} \, du + \int_0^t \frac{\partial V^E}{\partial Z} \sigma_u \, d\hat{W}_u + \int_0^t \langle V^E, dX_u \rangle,$$  \hfill (3.6)

and

$$dX_t = \Pi X_t \, dt + dM_t.$$  \hfill (3.7)

Since $V^E$ is a $\hat{P}$-martingale with respect to $G_t$,

$$V^E(Z, X, t) = V^E(Z, X, 0) + \int_0^t \frac{\partial V^E}{\partial Z} \sigma_u \, d\hat{W}_u,$$  \hfill (3.8)

or equivalently,

$$dV^E = \frac{\partial V^E}{\partial Z} \sigma_t \, d\hat{W}_t.$$  \hfill (3.9)

On the continuation region $C$, the dynamics of $V^E$ and $\tilde{V}_Z$ are identical. Hence,

$$d\tilde{V}_Z = \frac{\partial \tilde{V}_Z}{\partial Z} \sigma_t \, d\hat{W}_t.$$  \hfill (3.10)

On the stopping region $S$, $\tilde{V}_Z$ is equal to its intrinsic value; that is,

$$\tilde{V}_Z(Z, X, t) = \hat{g}(Z, X, t) = e^{-Z},$$  \hfill (3.11)

since $Z \leq \ln(1/\alpha)$ in $S$.

This implies that

$$d\tilde{V}_Z = \frac{\partial \tilde{V}_Z}{\partial Z} \sigma_t \, d\hat{W}_t + \frac{\partial \tilde{V}_Z}{\partial t} \, dt - e^{-Z} \left( r_t + \frac{1}{2} \sigma_t^2 - c_Z(Z_t) - \frac{1}{2} \sigma_t^2 \right) \, dt + e^{-Z} \sigma_t \, d\hat{W}_t,$$

or

$$d\tilde{V}_Z = \frac{\partial \tilde{V}_Z}{\partial Z} \sigma_t \, d\hat{W}_t + \frac{\partial \tilde{V}_Z}{\partial t} \, dt + \frac{\partial \tilde{V}_Z}{\partial Z} \sigma_t \, d\hat{W}_t.$$  \hfill (3.12)
Hence, in the entire state space,
\[ d\tilde{V}_2 = \frac{\partial \tilde{V}_2}{\partial Z}(r_t - c_t(Z,t)) dt + \frac{\partial \tilde{V}_2}{\partial W} dW. \] (3.13)

Therefore, the result follows by integrating (3.13) from \( t \) to \( T \) and taking expectation. \( \square \)

4. The free boundary problem

We provide a characterisation for the fair valuation of the participating American contract based on a free boundary problem in the context of Markov-modulated GBM. Buffington and Elliott (2002) provided a formulation of the free boundary problem for an American option under Markov-modulated GBM. As in Buffington and Elliott (2002), we suppose that the number of regimes \( N \) for the hidden process \( X \) is two. In this case, we assume that the generator \( \Pi \) of the hidden Markov process \( X \) is \( \{\pi_{ij}\}_{i,j=1,2} \), where \( \pi_{ii} < 0 \) (for \( i = 1, 2 \)), \( \pi_{12} = -\pi_{21} = -\pi_{11} \).

Define a two-dimensional vector \( \tilde{V}_2(Z, t) \) as:
\[ \tilde{V}_2(Z, t) := (\tilde{V}_1(Z, e_1, t), \tilde{V}_1(Z, e_2, t)). \] (4.1)

For simplifying the notations, we write \( \tilde{V}_2 \) for \( \tilde{V}_2(Z, e_i, t) \), where \( i = 1, 2 \), and \( \tilde{V}_2 \) for \( (\tilde{V}_2, \tilde{V}_2) \). For \( X_t = e_i, i = 1, 2 \), write \( \mathcal{C}^i \) and \( \mathcal{S}^i \) for the continuation region and the stopping region, respectively. Then,
\[ \mathcal{C}^i = \{(Z, s)|t \in [t, T] \land Z > Z^i(s)\}, \] (4.2)
and
\[ \mathcal{S}^i = \{(Z, s)|t \in [t, T] \land Z = Z^i(s)\}. \] (4.3)

For each \( s \in [t, T] \), let \( \mathcal{C}^i_s \) denote the \( s \)-section of \( \mathcal{C}^i \). Then, by Buffington and Elliott (2002) and Elliott and Kopp (2004), \( \mathcal{C}^i_s \) is an interval of the form \((Z^i(s), \infty)\). As in Buffington and Elliott (2002), we consider the case that \( Z^i(s) < Z^j(s) \), for \( s \in [t, T] \).

First, we suppose \( Z_t > Z^j(s) \). Then, \((Z, s) \in \mathcal{C}^i \) and \((Z, s) \in \mathcal{C}^j \); that is, \((Z, s) \) is in the continuation region for both states. Now, we define the following partial differential operator \( \mathcal{L}_{t,\alpha}(\cdot) \), for \( i = 1, 2 \):
\[ \mathcal{L}_{t,\alpha}(\tilde{V}_2) = \frac{\partial \tilde{V}_2}{\partial t} + \left( \frac{\partial}{\partial Z} + c_t(Z) \right) \tilde{V}_2 + \frac{\partial^2 \tilde{V}_2}{\partial Z^2}. \] (4.4)

Since \( c_t(Z) = \max(\sigma_t^2 Z, -\beta) \),
\[ \mathcal{L}_{t,\alpha}(\tilde{V}_2) = \begin{cases} \frac{\partial \tilde{V}_2}{\partial t} + \left( \frac{\partial}{\partial Z} + c_t(Z) \right) \tilde{V}_2 + \frac{\partial^2 \tilde{V}_2}{\partial Z^2} & \text{if } Z \leq r_t + \beta, \\ \frac{\partial \tilde{V}_2}{\partial t} + \left( \frac{\partial}{\partial Z} + \left( r_t + \beta + \frac{1}{2} \sigma_t^2 - Z \right) \right) \tilde{V}_2 + \frac{\partial^2 \tilde{V}_2}{\partial Z^2} & \text{if } Z > r_t + \beta. \end{cases} \]

Then, by Buffington and Elliott (2002) and Siu (2005), \( \mathcal{V}_Z := (\tilde{V}_2, \tilde{V}_2) \) satisfies the following pair of coupled PDEs:
\[ \mathcal{L}_{t,\alpha}(\tilde{V}_2) + \mathcal{B}_Z + \mathcal{C}_Z = 0, \quad i = 1, 2. \] (4.6)

Note that \( Z^i(s) \leq \ln(1/\alpha) \), for \( i = 1, 2 \).
When \( Z_t \leq Z^*_1(t) \) and \( Z_t = Z \),
\[
\tilde{V}_1^Z = e^{-Z}.
\] (4.7)

When \( Z_t \leq Z^*_2(t) \) and \( Z_t = Z \),
\[
\tilde{V}_2^Z = e^{-Z}.
\] (4.8)

By the “high contact” principle of an optimal stopping problem, we require that \( \tilde{V}_i^Z \) satisfies both the continuity condition and the smooth pasting condition on the early exercise boundary \( Z^*_i(t) \). Then, for \( i = 1, 2 \), the continuity condition is described by:
\[
\tilde{V}_i(Z^*_i(t), e_{i,t}) = e^{-Z^*_i(t)},
\] (4.9)

and the smooth pasting condition is given by:
\[
\frac{\partial}{\partial Z} \tilde{V}_i(Z^*_i(t), e_{i,t}) = -e^{-Z^*_i(t)}.
\] (4.10)

When \( Z_t \in (Z^*_1(t), Z^*_2(t)) \), which represents the transition region between \( Z^*_1(t) \) and \( Z^*_2(t) \), \( \tilde{V}_1^Z \) satisfies the following partial differential equation:
\[
L_{Z, e_1}(\tilde{V}_1^Z) + \pi_{11} \tilde{V}_1^Z - \pi_{11} e^{-Z} = 0,
\] (4.11)

where \( \pi_{11} < 0 \).

5. Second-order piecewise linear ODEs

We adopt the approximation due to Barone-Adesi and Whaley (1987) to provide an approximate solution to the valuation of the participating American contract by second-order piecewise linear ODEs. The Barone-Adesi-Whaley approximation has been adopted by Buffington and Elliott (2002) for obtaining an approximate solution to the valuation of American options under regime switching models. Following the analysis in Buffington and Elliott (2002), we consider the cases of the common continuation region and the transition region.

5.1. Case I

Suppose \((Z, t)\) is in the common continuation region \( \text{CR} := \{(Z, t) | Z > Z^*_2(t)\} \) for both states. Hence, \( \tilde{V}_2^Z \) satisfies the following pair of coupled PDEs:
\[
L_{Z, e_1}(\tilde{V}_2^Z) + \langle \tilde{V}_2^Z, \Pi e_1 \rangle = 0, \quad i = 1, 2.
\] (5.1)

For \( Z \leq Z^*_2(t) \), \( \tilde{V}_2^Z \) satisfies the following boundary condition:
\[
\tilde{V}_2^Z = e^{-Z}.
\] (5.2)

Note that the continuity condition is given by:
\[
\tilde{V}_2(Z^*_2(t), e_2, t) = e^{-Z^*_2(t)},
\] (5.3)

and the smooth pasting condition is given by:
\[
\frac{\partial}{\partial Z} \tilde{V}_2(Z^*_2(t), e_2, t) = -e^{-Z^*_2(t)}.
\] (5.4)

Write \( \epsilon_i \) for \( \epsilon(Z, e_i, t) \) and \( V_i^E := V_i^Z(Z, e_i, t) \), for \( i = 1, 2 \). Then, by Proposition 3.1,
\[
\epsilon_i = \tilde{V}_2^Z - V_i^E.
\] (5.5)
Hence, for $i = (e_1, e_2)$. Since both $V_2$ and $V^\xi$ satisfy the same coupled PDEs in the common continuation region CR, $\epsilon$ also satisfies the same coupled PDEs in CR as follows:

$$L_{x_i}(\epsilon) = 0, \quad i = 1, 2$$  \hfill (5.6)

Now, we adopt the Barone–Adesi–Whaley approximation to find an approximate solution to the valuation of the participating American contract in a separated form. For $i = 1, 2$, we assume that $\epsilon_i$ can be approximated as follows:

$$\epsilon_i := \epsilon(Z, e_i, t) := H(Z, e_i)F(t).$$  \hfill (5.7)

Write $H := H(Z)$ for $H(Z, e_i)$, $H_i := H(Z, e_i)$ for $i = 1, 2$ and $F := F(t)$, then $H_i$ satisfies:

$$H_i \frac{\partial F}{\partial t} + \left(r_i + \frac{1}{2}\sigma_i^2 - c_Z(Z)\right)H_i \frac{\partial F}{\partial Z} + \frac{1}{2}\sigma_i^2 F \frac{\partial^2 H_i}{\partial Z^2} + F(H_i, \Pi e_i) = 0.$$  \hfill (5.8)

We assume that $F(t)$ is given by the following form:

$$F(t) = \hat{F} \left[1 - \exp \left(-\int_0^t r_s \, ds\right)\right].$$  \hfill (5.9)

Then,

$$\frac{\partial F}{\partial t} = r_i(F(t) - 1).$$  \hfill (5.10)

Hence, for $i = 1, 2$,

$$\frac{1}{2}\sigma_i^2 H_i \frac{\partial^2 H_i}{\partial Z^2} + \left(r_i + \frac{1}{2}\sigma_i^2 - c_Z(Z)\right)H_i \frac{\partial H_i}{\partial Z} + \left(r_i + \frac{1}{2}\sigma_i^2 - c_Z(Z)\right)\Pi e_i = \frac{r_i H_i(1 - F(t))}{F(t)},$$  \hfill (5.11)

where the term $\frac{\partial F}{\partial Z}$ is still a function of time $t$.

Now, in the continuation region CR, $H$ satisfies the following two coupled second-order ordinary differential equations (ODEs):

$$\frac{1}{2}\sigma_1^2 \frac{\partial^2 H_1}{\partial Z^2} + \left(r_1 + \frac{1}{2}\sigma_1^2 - c_Z(Z)\right)H_1 \frac{\partial H_1}{\partial Z} + \frac{1}{2}\sigma_1^2 H_1 = \frac{r_1 H_1(1 - F(t))}{F(t)},$$  \hfill (5.12)

$$\frac{1}{2}\sigma_2^2 \frac{\partial^2 H_2}{\partial Z^2} + \left(r_2 + \frac{1}{2}\sigma_2^2 - c_Z(Z)\right)H_2 \frac{\partial H_2}{\partial Z} + \frac{1}{2}\sigma_2^2 H_2 = \frac{r_2 H_2(1 - F(t))}{F(t)}.$$  \hfill (5.13)

Since $c_Z(Z) = \max(r_x, Z - \beta)$, $H$ satisfies the following two second-order piecewise linear ODEs:

$$\frac{1}{2}\sigma_1^2 \frac{\partial^2 H_1}{\partial Z^2} + \left[r_1 - r_8 + \frac{1}{2}\sigma_2^2\right]l_{2(\sigma_1^2 + \beta)} + \left(r_1 + \beta + \frac{1}{2}\sigma_1^2 - Z\right)l_{1(\sigma_1^2 + \beta)} \frac{\partial H_1}{\partial Z} + \frac{1}{2}\sigma_1^2 H_1 = \frac{r_1 H_1(1 - F(t))}{F(t)},$$

$$\frac{1}{2}\sigma_2^2 \frac{\partial^2 H_2}{\partial Z^2} + \left[r_2 - r_8 + \frac{1}{2}\sigma_2^2\right]l_{2(\sigma_2^2 + \beta)} + \left(r_2 + \beta + \frac{1}{2}\sigma_2^2 - Z\right)l_{1(\sigma_2^2 + \beta)} \frac{\partial H_2}{\partial Z} + \frac{1}{2}\sigma_2^2 H_2 = \frac{r_2 H_2(1 - F(t))}{F(t)}.$$
5.2. Case II

Now, we suppose that \((Z, t)\) is in the transition region \(\text{TR} := \{(Z, t) | Z_1^*(t) \leq Z \leq Z_2^*(t)\}\) between the two early exercise boundaries \(Z_1^*(t)\) and \(Z_2^*(t)\). In the transition region \(\text{TR}\), we derive an approximate solution to the valuation of the participating American contract by applying the Barone–Adesi–Whaley approximation only to the state \(X = e_1\).

We suppose that in the transition region \(\text{TR}\), the dynamics of the market values of the asset \(A\) under \(P\) is given by:

\[
\frac{dA_t}{A_t} = \mu_1 dt + \sigma_1 dW_t,
\]

(5.14)

where \(\mu_1\) and \(\sigma_1\) are the constant drift and volatility parameters in the state \(X = e_1\).

We further suppose that the market interest rate for the bank account is \(r_1\), which is the market interest rate in the state \(X = e_1\). Then, \(B_t = \exp(r_1 t)\). Under these assumptions about the market values of the asset \(A\) and the market interest rate, when \(Z > Z_1^*(t)\), the \(A\)-denominated value of the participating American contract \(\tilde{V}_1^Z\) satisfies the following PDE:

\[
L_{Z,e_1}(\tilde{V}_1^Z) = 0,
\]

(5.15)

since this corresponds to the case of no regime switching.

For \(Z \leq Z_1^*(t)\), \(\tilde{V}_1^Z\) satisfies the following boundary condition:

\[
\tilde{V}_1^Z = e^{-Z}.
\]

(5.16)

Note that the continuity condition is given by:

\[
\tilde{V}_1^Z(Z_1^*(t), t) = e^{-Z_1^*(t)},
\]

(5.17)

and the smooth pasting condition is given by:

\[
\frac{\partial}{\partial Z} \tilde{V}_1^Z(Z_1^*(t), t) = -e^{-Z_1^*(t)}.
\]

(5.18)

The regime switching Esscher transform in Section 2 reduces to the continuous-time Esscher transform with constant Esscher parameter. Let \(\mathcal{F}^W := \{\mathcal{F}^W_t, t \in \mathcal{T}\}\) denote the \(P\)-augmentation of the natural filtration generated by the process \(W\). Following the same procedure as in Section 2, we can define the risk-neutral Esscher transform \(Q_1\sim P\) as follows:

\[
\frac{dQ_1}{dP} \bigg|_{\mathcal{F}^W} = \exp \left( \left( \frac{\mu_1 - \mu}{\sigma_1} \right) W_t - \frac{1}{2} \left( \frac{\mu_1 - \mu}{\sigma_1} \right)^2 t \right).
\]

(5.19)

By Girsanov’s theorem, \(\tilde{W}_t := W_t + \left( \frac{\mu_1 - \mu}{\sigma_1} \right) t\) is a standard Brownian motion with respect to \(\mathcal{F}^W\) under \(Q_1\). Hence, the dynamics of \(A_t\) under \(Q_1\) is governed by:

\[
\frac{dA_t}{A_t} = (\mu_1 - r_1) dt + \sigma_1 d\tilde{W}_t,
\]

(5.20)

Now, we define a \(Q_1\)-martingale with respect to \(\mathcal{F}^W\):

\[
\xi_1(t) := e^{-\frac{1}{2} \sigma_1^2 t + \sigma_1 \tilde{W}_t}.
\]

(5.21)

Then, we define a new probability measure \(\hat{Q}_1 \sim Q_1\) as follows:

\[
\frac{d\hat{Q}_1}{dQ_1} \bigg|_{\mathcal{F}^W} := \xi_1(t), \quad t \in \mathcal{T}.
\]

(5.22)
By Girsanov’s theorem,
\[ \hat{W}_t := W_t - \sigma t, \]  
(5.23)
is a standard Brownian motion under \( \hat{Q} \) with respect to \( \mathcal{F}^W \).

Under \( \hat{Q} \), the dynamics of \( A \) can be represented by:
\[ dA_t = (r_1 + \sigma^2_1) A_t \, dt + \sigma_1 A_t \, d\hat{W}_t. \]  
(5.24)

By Itô’s lemma, the dynamics of \( Z_t \) under \( \hat{Q} \) is:
\[ dZ_t = (r_1 + \frac{1}{2} \sigma^2_1 - c_2(Z)) \, dt + \sigma_1 \, d\hat{W}_t. \]  
(5.25)

In the transition region TR, write \( \bar{V}_1^Z(\hat{A}, \hat{R}, t) \) for the \( A \)-denominated value of the participating European contract. Then,
\[ \bar{V}_1^Z(\hat{A}, \hat{R}, t) = e^{-r_1 (T - t)} \frac{1}{A_t} E^1(g(\hat{A}, \hat{R}, e_1, T)(\hat{A}_t, \hat{R}_t) = (A, R)), \]  
(5.26)

where \( E^1 \) is the expectation operator under \( \hat{Q} \).

By using the method of change of measures in Section 2, we can show that
\[ E^1(e^{-r_1 (T - t)} g(\hat{A}, \hat{R}, e_1, T)(\hat{A}_t, \hat{R}_t) = (A, R)) = A_t \hat{E}^1(g(Z, e_1, T) | Z_t = Z), \]  
(5.27)

where \( \hat{E}^1 \) is the expectation operator under \( \hat{Q} \).

Hence,
\[ \bar{V}_1^Z(\hat{A}, \hat{R}, t) = \hat{E}^1(g(Z, e_1, T) | Z_t = Z). \]  
(5.28)

Let \( \bar{V}_1^Z(Z, t) := \hat{E}^1(g(Z, e_1, T) | Z_t = Z) \). Then, \( \bar{V}_1^Z := \bar{V}_1^Z(Z, t) \) satisfies:
\[ \frac{\partial \bar{V}_1^Z}{\partial t} + \left( r_1 + \frac{1}{2} \sigma^2_1 - c_2(Z) \right) \frac{\partial \bar{V}_1^Z}{\partial Z} + \frac{1}{2} \sigma^2_1 \frac{\partial^2 \bar{V}_1^Z}{\partial Z^2} = 0, \]  
(5.29)

with auxiliary condition:
\[ \bar{V}_1^Z(Z, T) = e^{-r_1 T} + ye^{-r_1 T} \max(0, e^{2z} - 1, 0). \]  
(5.30)

Now, we define \( \hat{V}_1 := \frac{\partial \bar{V}_1^Z}{\partial t} - \bar{V}_1^Z \). Then,
\[ \frac{\partial \hat{V}_1}{\partial t} + \left( r_1 + \frac{1}{2} \sigma^2_1 - c_2(Z) \right) \frac{\partial \hat{V}_1}{\partial Z} + \frac{1}{2} \sigma^2_1 \frac{\partial^2 \hat{V}_1}{\partial Z^2} = 0. \]  
(5.31)

As in Case I, we suppose that \( \hat{V}_1 := H_{1}(Z)F(t) \) (5.32)

Then,
\[ \hat{H}_1 \frac{\partial F}{\partial t} + \left( r_1 + \frac{1}{2} \sigma^2_1 - c_2(Z) \right) \frac{\partial \hat{H}_1}{\partial Z} + \frac{1}{2} \sigma^2_1 \frac{\partial^2 \hat{H}_1}{\partial Z^2} = 0. \]  
(5.33)
We further assume that \( \bar{F}(t) = 1 - e^{-r_1(T-t)} \). Then,
\[
\frac{\partial \bar{F}(t)}{\partial t} = r_1 \bar{F}(t) - 1.
\] (5.34)
Hence, \( \bar{H}_1 \) satisfies the following second-order ODE:
\[
\frac{1}{2} \sigma_1^2 \frac{\partial^2 \bar{H}_1}{\partial Z^2} + \left( r_1 + \frac{1}{2} \sigma_1^2 - c(Z) \right) \frac{\partial \bar{H}_1}{\partial Z} + r_1 \left( 1 - \frac{1}{F(t)} \right) \bar{H}_1 = 0,
\] (5.35)
where the term \( r_1 \left( 1 - \frac{1}{F(t)} \right) \bar{H}_1 \) is still a function of time \( t \).

By noticing that \( c(Z) = \max(r_g, Z - \beta) \), \( \bar{H}_1 \) satisfies the following second-order piecewise linear ODE:
\[
\frac{1}{2} \sigma_1^2 \frac{\partial^2 \bar{H}_1}{\partial Z^2} + \left( r_1 - r_g + \frac{1}{2} \sigma_1^2 \right) \bar{H}_1 \{ Z \leq r_g + \beta \} + \left( r_1 + \beta + \frac{1}{2} \sigma_1^2 - Z \right) \bar{H}_1 \{ Z > r_g + \beta \} \frac{\partial \bar{H}_1}{\partial Z} + r_1 \left( 1 - \frac{1}{F(t)} \right) \bar{H}_1 = 0.
\] (5.36)

### 6. Participating perpetual American contracts

"Perpetual" means no final maturity date. The policyholder of a participating perpetual American contract can terminate the contract any time indefinitely. The absence of a finite-maturity date for the contract makes the valuation of a perpetual contract less complicated than a finite-maturity contract. The participating perpetual American policy is only a mathematical idealisation and it is not actually traded in the insurance markets (see Elliott and Kopp, 2004 for related discussions on perpetual American options). However, the investigation of the valuation of participating perpetual American contracts can provide some insights in their finite-maturity counterparts. Since participating contracts are relatively long-dated compared with other financial products, the valuation of participating perpetual American contracts can also serve as a reasonable approximation to its finite-maturity counterpart. Guo and Zhang (2004) provided closed-form solutions for the valuation of perpetual American put options with regime switching. Here, we derive a set of second-order piecewise linear ordinary differential equations (ODEs) for the valuation of participating perpetual American contracts under the Markov-modulated asset price process. As in Guo and Zhang (2004), we consider a two-state hidden Markov chain process with the generator \( \Pi \) given in Section 4.

First, let \( V^p(A, R, X) \) denote the value of the participating perpetual American contract. Then, the optimal stopping problem for the perpetual contract can be obtained by taking \( T \to \infty \) in the corresponding problem for the finite-maturity contract and is given as follows:
\[
V^p(A, R, X) = \text{ess} \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^\theta \left[ \exp \left( - \int_{\tau}^{T} r_s ds \right) g(A, R, X, \tau)(A, R, X) \right] \] (A, R, X).
\] (6.1)
Write
\[
\mathcal{T}_T^\infty := \{ \tau(\omega, u) \in [t, \infty]: \tau := f^\theta(Z_u, u), f^\theta \text{ is measurable} \}.
\] (6.2)
By adopting the method of changing probability measures as in Section 2, the optimal stopping problem for the perpetual contract can be written as follows:
\[
V^p(A, R, X) = \text{ess} \sup_{\tau \in \mathcal{T}_T^\infty} A_e \tilde{g}(Z, X, \tau)(Z, X) = (Z, X).
\] (6.3)
Let \( V_p(Z, X) \) denote the \( A \)-denominated value of the perpetual contract, that is,
\[
V_p(Z, X) = \text{ess sup}_{\tau \in \mathcal{T}} E(\bar{g}_T(Z, X)(Z, X_t)) = (Z, X_t).
\]
(6.4)

Let \( \bar{g}_T(Z, X) := e^{-Z} + \gamma e^{-Z} \max(\alpha e^Z - 1, 0) \). Following Guo and Zhang (2004), the continuation region \( \mathcal{C}^p \) and the stopping region \( \mathcal{S}^p \) are given by:
\[
\mathcal{C}^p = \{(Z, X) | V_p(Z, X) > \bar{g}_T(Z, X)\},
\]
(6.5)
and
\[
\mathcal{S}^p = \{(Z, X) | V_p(Z, X) = \bar{g}_T(Z, X)\}.
\]
(6.6)
Write \( \tau^p \) for the first exit time of \((Z, X)\) from \( \mathcal{C}^p \); that is, \( \tau^p := \inf\{s \geq t | (Z_s, X_s) \in \mathcal{C}^p\} \), where \( \mathcal{C}^p \) is the complement of \( \mathcal{C} \). Due to the fact that \((Z_t, X_t)\) is a two-dimensional Markov Chain process with respect to the enlarged filtration \( \mathcal{G}_t \), the optimal stopping rule is given by \( \tau^p \). Then,
\[
V_p(Z, X) = E(\bar{g}_T(Z, X, \tau^p)(Z, X_t)) \quad \text{for fixed } X.
\]
(6.7)

Note that both \( V_p(Z, X) \) and \( \bar{g}_T(Z, X) \) are decreasing functions of \( Z \) for fixed \( X \). Then, as in Guo and Zhang (2004), we suppose that there exists two thresholds \( Z_{11}^p, Z_{22}^p \leq \ln(1/\alpha) \) such that the continuation region \( \mathcal{C}^p \) can be represented by:
\[
\mathcal{C}^p = \{(Z, X) | Z \in (Z_{11}^p, Z_{22}^p) \} \cup \{(Z, X) | Z \in (Z_{11}^p, \infty) \}.
\]
(6.8)

There are three possible cases, namely \( Z_{11}^p < Z_{12}^p = Z_{22}^p \) and \( Z_{11}^p > Z_{12}^p \). As in Section 4, we consider the case that \( Z_{11}^p < Z_{12}^p \). Following the analysis of Guo and Zhang (2004), we consider the following three cases:

Case I: \( Z \in [Z_{11}^p, Z_{12}^p) \)

Let \( V_p := V_p(Z, \epsilon_1) \), for \( i = 1, 2; V^i := (V^1_i, V^2_i) \). Then, by Itô’s differentiation rule (see Guo and Zhang, 2004), \( V^i \) satisfies:
\[
0 = \left( r_i + \frac{1}{2}\sigma_i^2 - c_Z(Z) \right) \frac{\partial V^i_s}{\partial Z} + \frac{1}{2}\sigma_i^2 \frac{\partial^2 V^i_s}{\partial Z^2} + \pi_{11}V^i_s + \pi_{11}e^{-Z}, \quad V^i_{Z} = e^{-Z}.
\]
(6.9)

Hence, \( V^i \) satisfies the following second-order piecewise linear ODE:
\[
0 = \frac{1}{2}\sigma_i^2 \frac{\partial^2 V^i_s}{\partial Z^2} + \left( r_i - r_e + \frac{1}{2}\sigma_i^2 \right) h_{(Z<\epsilon_1)} + \left( r_i + \beta + \frac{1}{2}\sigma_i^2 - Z \right) h_{(Z=\epsilon_1)} + \pi_{11}V^i_s - \pi_{11}e^{-Z}.
\]
(6.10)

Case II: \( Z \in [Z_{12}^p, \infty) \)

Again, by Itô’s differentiation rule, \( V^i \) satisfies:
\[
0 = \left( r_i + \frac{1}{2}\sigma_i^2 - c_Z(Z) \right) \frac{\partial V^i_s}{\partial Z} + \frac{1}{2}\sigma_i^2 \frac{\partial^2 V^i_s}{\partial Z^2} + \pi_{11}V^i_s - \pi_{11}V^i_{Z}.
\]
(6.11)
Hence, $V_p$ satisfies the following two coupled second-order piecewise linear ODEs:

\[
0 = \frac{1}{2}\sigma^2 \frac{\partial^2 V_p^1}{\partial Z^2} + \left( \left( r_1 - r_2 + \frac{1}{2}\sigma_1^2 \right) I_{\{Z \leq r_g + \beta\}} + \left( r_1 + \beta + \frac{1}{2}\sigma_1^2 - Z \right) I_{\{Z > r_g + \beta\}} \right) \frac{\partial V_p^1}{\partial Z} + \pi_{11} V_p^1 - \pi_{11} V_p^2,
\]

\[
0 = \frac{1}{2}\sigma^2 \frac{\partial^2 V_p^2}{\partial Z^2} + \left( \left( r_2 - r_g + \frac{1}{2}\sigma_2^2 \right) I_{\{Z \leq r_g + \beta\}} + \left( r_2 + \beta + \frac{1}{2}\sigma_2^2 - Z \right) I_{\{Z > r_g + \beta\}} \right) \frac{\partial V_p^2}{\partial Z} + \pi_{22} V_p^2 - \pi_{22} V_p^1.
\]

\[\text{(6.12)}\]

Case III: $Z \in (-\infty, Z_p_1]$

In this case,

\[V_p^1 = V_p^2 = e^{-Z}.\]

\[\text{(6.13)}\]

7. Summary and further investigation

We have considered the fair valuation of a participating life insurance policy with surrender options, bonus distributions and rate guarantees when the dynamics of the market values of the asset is driven by a Markov-modulated Geometric Brownian Motion (GBM). The method of changing probability measures in Hansen and Jørgensen (2000) has been employed to reduce the dimension of the optimal stopping problem for the valuation of the participating American contract. We have provided a decomposition for the value of the policy into the value of an identical contract without surrender options and the premium of surrender options. The valuation problem has also been formulated as a free boundary problem. We have employed the approximation method due to Barone-Adesi and Whaley (1987) to approximate the solution of the free boundary problem by second-order piecewise linear ODEs. We have also considered the fair valuation of participating perpetual American contracts.

For further investigation, it is worth investigating the fair valuation of other modern insurance products with option-embedded features and early exercise feature or surrender option, in the context of Markov-modulated diffusion processes. We may also investigate the fair valuation of participating American contracts when the dynamics of the market values of the reference portfolio is governed by other types of regime switching models, such as the Markov switching jump-type models and the Markov-modulated Lévy processes. It is of practical importance and relevance to develop some methods and techniques for measuring and managing risk inherent from trading the participating contracts. The risk measurement and management of participating American contracts are more challenging than their European counterparts due to the presence of surrender options. It would be interesting to investigate the risk measurement and management of participating perpetual American contracts in order to gain some insights in the risk measurement and management of their finite-maturity counterparts.

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