5. Time consistency

In this section we consider a sequence of convex conditional risk measures

$$\rho_t : L^\infty \to L^\infty_t, \quad t = 0, 1, \ldots$$

In such a dynamic setting the key question is how the risk assessments of a financial position at different times are connected to each other.

**Definition.** A sequence of conditional risk measures $$(\rho_t)_{t=0,1,...}$$ is called time-consistent if for any $X,Y \in L^\infty$ and for all $t \geq 0$ the following condition holds:

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \implies \rho_t(X) \leq \rho_t(Y).$$

**Lemma.** Time-consistency is equivalent to each of the following properties:

a) $$\rho_{t+1}(X) = \rho_{t+1}(Y) \implies \rho_t(X) = \rho_t(Y) \quad \text{for} \quad t = 0, 1, \ldots.$$  

b) recursiveness: $$\rho_t = \rho_t(-\rho_{t+1}) \quad \text{for} \quad t = 0, 1, \ldots.$$  

**Proof.** Time-consistency clearly implies a).

a) ⇒ b): Note that $$\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X),$$ due to conditional cash-invariance and normalization. Applying a) with $$Y := -\rho_{t+1}(X)$$ obtain $$\rho_t(X) = \rho_t(-\rho_{t+1}(X)).$$  

b) implies time-consistency: If $$\rho_{t+1}(X) \leq \rho_{t+1}(Y)$$ then $$\rho_t(-\rho_{t+1}(X)) \leq \rho_t(-\rho_{t+1}(Y))$$ by monotonicity, and so $$\rho_t(X) \leq \rho_t(Y)$$ follows from b).

**Examples.** 1) For fixed $$\beta > 0,$$ the sequence of entropic conditional risk measures

$$\rho_t(X) = \frac{1}{\beta} \log E_P[e^{-\beta X} | \mathcal{F}_t]$$

is time-consistent. Let us check recursiveness:

$$\rho_t(-\rho_{t+1}(X)) = \frac{1}{\beta} \log E_P[\exp(-\beta(-\frac{1}{\beta} \log E_P[\exp(-\beta X)| \mathcal{F}_{t+1}]))| \mathcal{F}_t]$$

$$= \frac{1}{\beta} \log E_P[E_P[e^{-\beta X}| \mathcal{F}_{t+1}] | \mathcal{F}_t]$$

$$= \rho_t(X).$$

Note, however, that time consistency will be lost if the constant risk aversion parameter $$\beta$$ is replaced by an adapted process $$(\beta_t)_{t=0,1,...}.$$
2) The sequence of coherent conditional risk measures

\[ \rho_t(X) = \text{AV@}R_\alpha(X)|\mathcal{F}_t = \frac{1}{\lambda} \int_0^\lambda V@R_\alpha(X)|\mathcal{F}_t d\alpha. \]

given by conditional Average Value at Risk at some level \( \lambda \in (0,1) \) is not time-consistent, and the same is true for conditional Value at Risk and for conditional risk measures defined in terms of Sharpe ratios. To see this, note first that all these risk measures are well defined on \( L^2(P) \), and that they are of the form

\[ \rho_t(X) = -E_P[-X|\mathcal{F}_t] + \gamma E_P[(x - E_P[X|\mathcal{F}_t])^2]^{\frac{1}{2}} \]

if \( X \) has a conditional Gaussian distribution with respect to \( \mathcal{F}_t \). Now consider a position of the form \( X = X_1 + X_2 \) for two independent Gaussian random variables \( X_i \) with distribution \( N(0, \sigma_i^2) \) and assume that \( \mathcal{F}_1 \) is the \( \sigma \)-field generated by \( X_1 \). Then

\[ \rho_1(X) = -X_1 + \rho_1(X_2) = -X_1 + \gamma \sigma_2, \]

hence

\[ \rho_0(-\rho_1(X)) = \rho_0(X_1) + \gamma \sigma_2 = \gamma (\sigma_1 + \sigma_2). \]

On the other hand,

\[ \rho_0(X) = \gamma (\sigma_1^2 + \sigma_2^2)^{\frac{1}{2}}, \]

and so we have \( \rho_0(-\rho_1(X)) > \rho_0(X) \).

We are now going to characterize time-consistency of the sequence \( (\rho_t)_{t=0,1,...} \) in terms of the corresponding acceptance sets \( (A_t)_{t=0,1,...} \) and in terms of the penalty functions \( (\alpha_t)_{t=0,1,...} \). Our main goal is to prove a supermartingale criterion which characterizes time-consistency in terms of the joint behavior of the stochastic processes \( (\rho_t)_{t=0,1,...} \) and \( (\alpha_t)_{t=0,1,...} \).

To this end we introduce some notation. Suppose that we look just one step ahead and assess the risk only for those positions whose outcome will be known by the end of the next period. This means that we restrict the conditional convex risk measure \( \rho_t \) to the space \( L_{t+1}^\infty \). The corresponding “myopic” acceptance set is given by

\[ A_{t,t+1} := \{ X \in L_{t+1}^\infty \mid \rho_t(X) \leq 0 \}, \]

and

\[ \alpha_{t,t+1}^\min(Q) := \text{ess sup}_{X \in A_{t,t+1}} E_Q[-X|\mathcal{F}_t] \]

is the resulting “myopic” penalty function.

The following lemma holds for any sequence of monetary conditional convex risk measures. The equivalences between set inclusions for the acceptance sets and inequalities for the risk measures can be used as starting points for various departures from the strong notion of time-consistency which we are considering here.
Lemma. Let \((\rho_t)_{t=0,1,...}\) be a sequence of monetary conditional risk measures. Then the following equivalences hold for all \(t \geq 0\) and all \(X \in L^\infty\):

a) \(X \in \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \iff -\rho_{t+1}(X) \in \mathcal{A}_{t,t+1}\)

b) \(\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \iff \rho_t(-\rho_{t+1}) \leq \rho_t\)

c) \(\mathcal{A}_t \supseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} \iff \rho_t(-\rho_{t+1}) \geq \rho_t\)

Proof. a) To prove "\(\Rightarrow\)" take \(X = X_{t,t+1} + X_{t+1}\) with \(X_{t,t+1} \in \mathcal{A}_{t,t+1}\) and \(X_{t+1} \in \mathcal{A}_{t+1}\). Then

\[
\rho_{t+1}(X) = \rho_{t+1}(X_{t+1}) - X_{t,t+1} \leq -X_{t,t+1}
\]

by cash invariance, and monotonicity implies

\[
\rho_t(-\rho_{t+1}(X)) \leq \rho_t(X_{t,t+1}) \leq 0,
\]

hence \(-\rho_{t+1}(X) \in \mathcal{A}_{t+1}\). The converse direction follows immediately from the decomposition \(X = X + \rho_{t+1}(X) - \rho_{t+1}(X)\), since \(X + \rho_{t+1}(X) \in \mathcal{A}_{t+1}\) for all \(X \in L^\infty\).

b) In order to show "\(\Rightarrow\)”, take \(X \in L^\infty\). Since \(X + \rho_t(X) \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}\), we obtain

\[
-\rho_{t+1}(X) + \rho_t(X) = -\rho_{t+1}(X + \rho_t(X)) \in \mathcal{A}_{t,t+1},
\]

by a) and by cash invariance. This implies

\[
\rho_t(-\rho_{t+1}(X)) - \rho_t(X) = \rho_t(-\rho_{t+1}(X) - \rho_t(X)) \leq 0.
\]

To prove "\(\Leftarrow\)" take \(X \in \mathcal{A}_t\). Then \(-\rho_{t+1}(X) \in \mathcal{A}_{t,t+1}\) by the right hand side of b), and hence \(X \in \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}\) by a).

c) Take \(X \in L^\infty\) and assume \(\mathcal{A}_t \supseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}\). Then

\[
\rho_t(-\rho_{t+1}(X)) + X = \rho_t(-\rho_{t+1}(X)) - \rho_{t+1}(X) + \rho_{t+1}(X) + X \in \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} + \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} + \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} + \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}
\]

belongs to \(\mathcal{A}_t\). This implies

\[
\rho_t(X) - \rho_t(-\rho_{t+1}(X)) = \rho_t(X + \rho_t(-\rho_{t+1}(X))) \leq 0
\]

by cash invariance, and so we have shown "\(\Rightarrow\). For the converse direction take \(X \in \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}\). Since \(-\rho_{t+1}(X) \in \mathcal{A}_{t,t+1}\) by a), we obtain

\[
\rho_t(X) \leq \rho_t(-\rho_{t+1}(X)) \leq 0,
\]

hence \(X \in \mathcal{A}_t\).

\[\Box\]

Let us define the set

\[
\mathcal{Q}^* := \{ Q \in \mathcal{M}_1^\infty(P) \mid a_0^\min(Q) < \infty \}.
\]
Note that $Q^*$ is nonempty as soon as we make the very mild assumption that $\alpha_0(P) < \infty$. The following theorem, and in particular the equivalence of 1) and 4), is the main result of this section.

**Theorem.** Let $(\rho_t)$ be a sequence of convex conditional risk measures such that each $\rho_t$ has the Fatou property, and assume that $Q^* \neq \emptyset$. Then the following conditions are equivalent:

1) $(\rho_t)_{t=0,1,...}$ is time-consistent.

2) $A_t = A_{t,t+1} + A_{t+1}$ for all $t = 0,1,...$

3) For any $Q \in M_1(P)$,
   \[ \alpha_{t+1}^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \quad \text{for all } t = 0,1,... \]

4) For any $Q \in Q^*$ and any $X \in L^\infty$, the process
   \[ V_t^{Q,X} := \rho_t(X) + \alpha_t^{\min}(Q), \quad t = 0,1,... \]
   is a $Q$-supermartingale.

In each case the dynamic risk measure admits a robust representation in terms of the set $Q^*$, i.e.,

\[
\rho_t(X) = \text{ess sup}_{Q \in Q^*} \left( E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right)
\]

for all $X \in L^\infty$ and all $t \geq 0$.

The proof of the Theorem will be given in several stages. The equivalence of 1) and 2) follows from the preceding lemma. Let us now gain some first insight into the connection between time-consistency and the supermartingale property appearing in 4) by proving the following proposition. Note that it yields the proof of “4) ⇒ 1)” if we have verified assumption ii) for $Q = Q^*$; this part we skip.

**Proposition.** Let $Q$ be some subset of $M_1(P)$ such that

i) For each $t \geq 0$, the conditional risk measure $\rho_t$ admits a robust representation in terms of $Q$, i.e.
   \[
   \rho_t(X) = \text{ess sup}_{Q \in Q} \left( E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right)
   \]
   for any $X \in L^\infty$,

ii) For any $Q \in Q$ and any $X \in L^\infty$, the process
   \[ V_t^{Q,X} := \rho_t(X) + \alpha_t^{\min}(Q), \quad t = 0,1,... \]
   is a $Q$-supermartingale.
Then \((p_t)_{t=0,1,...}\) is time-consistent.

**Proof.** Take \(X, Y \in L^\infty\) such that \(p_{t+1}(X) \leq p_{t+1}(Y)\); we have to show \(p_t(X) \leq p_t(Y)\).

For each \(Q \in Q\),

\[
\begin{align*}
p_t(Y) + \alpha_t^{\min}(Q) &\geq E_Q[p_{t+1}(Y) + \alpha_{t+1}^{\min}(Q)|F_t] \\
&\geq E_Q[p_{t+1}(X) + \alpha_{t+1}^{\min}(Q)|F_t] \\
&\geq E_Q[E_Q[-X|F_{t+1}]|F_t] \\
&= E_Q[-X|F_t].
\end{align*}
\]

The first inequality follows from the supermartingale property ii), and in the third step we have used the inequality

\[
p_{t+1}(X) + \alpha_{t+1}^{\min}(Q) \geq E_Q[-X|F_{t+1}]
\]

which is valid for any \(Q \in M_1(P)\); see inequality (8) in section 4. Thus

\[
p_t(Y) \geq E_Q[-X|F_t] - \alpha_t^{\min}(Q)
\]

for all \(Q \in Q\), and this implies the desired inequality \(p_t(Y) \geq p_t(X)\) as soon as \(p_t(X)\) admits the representation in i).

\(\Box\)

Let us now look at the properties of the penalty functions.

**Proof of 2) \Rightarrow 3):** Since any \(X \in A_t\) can be written as the sum of some \(X_{t,t+1} \in A_{t,t+1}\) and some \(X_{t+1} \in A_{t+1}\), we obtain

\[
\alpha_t^{\min}(Q) = \text{ess sup}_{X \in A_t} E_Q[-X|F_t]
\]

\[
= \text{ess sup}_{X_{t,t+1} \in A_{t,t+1}} E_Q[-X_{t,t+1}|F_t] + \text{ess sup}_{X_{t+1} \in A_{t+1}} E_Q[-X_{t+1}|F_t]
\]

\[
= \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q)|F_t]
\]

for any \(Q \in M_1(P)\), using in the last step the Lemma in section 4.

\(\Box\)

**Remark.** It follows from property 3) of the Theorem that

\[
(1)\quad E_Q[\alpha_t^{\min}(Q)|F_t] \leq \alpha_t^{\min}(Q)\quad \text{for any } Q \in M_1(P).
\]

This in turn implies \(E_Q[\alpha_t^{\min}(Q)] < \infty\) for all \(t \geq 0\) if \(Q \in Q^*\), and so the process \((\alpha_t^{\min}(Q))_{t=0,1,...}\) is a \(Q\)-supermartingale for any \(Q \in Q^*\). Note that property 3) provides more information than just the supermartingale property: It actually yields an explicit description of the Doob decomposition of the supermartingale \((\alpha_t^{\min}(Q))_{t=0,1,...}\) in terms of the “one-step” penalty functions \(\alpha_{t,t+1}^{\min}(Q)\), i.e.,

\[
\alpha_t^{\min}(Q) + \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t = 0, 1, \ldots
\]
is a martingale under $Q$.

The weaker version (1) of property 3) is in fact equivalent to a weaker notion of time-consistency, called weak time-consistency, which is defined by the weaker version

$$\mathcal{A}_{t+1} \subseteq \mathcal{A}_t, \quad t = 0, 1, \ldots$$

of property 2).

Let us now sketch the remaining stage of the proof.

**Proof of 3) ⇒ 4):** First one shows that 3) implies the representation (1), and that the family on the right hand side of (1) is directed upward; this part of the proof we skip. For any $X \in L^\infty$, we can thus find some sequence $Q_n \in Q^*$ such that $\rho_{t+1}(X)$ can be identified as the limit of an increasing sequence:

$$\rho_{t+1}(X) = \lim_n \left( E_{Q_n}[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q_n) \right) \quad P-a.s. \quad (2)$$

Now take $Q \in Q^*$, and use the notation $V_t := V^Q_tX$. The process $(V_t)_{t=0,1,\ldots}$ is adapted, and it is integrable with respect to $Q$, since $\rho_t(X)$ is bounded by $||X||_{\infty}$, and since 3) implies $E_Q[\alpha_{t+1}^{\min}(Q)] \leq \alpha_0(Q) < \infty$. It remains to show that $V_t \geq E_Q[V_{t+1}|\mathcal{F}_t]$. Using property 3) we see that

$$E_Q[V_{t+1}|\mathcal{F}_t] = E_Q[\rho_{t+1}(X) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t]$$

$$= E_Q[\rho_{t+1}(X)|\mathcal{F}_t] - \alpha_{t,t+1}^{\min}(Q) + \alpha_t^{\min}(Q) \quad (3)$$

Now we use the sequence of measures $Q_n$ appearing in (2). We are free to assume that $Q_n = Q$ on $\mathcal{F}_{t+1}$, and this implies

$$\alpha_{t,t+1}^{\min}(Q_n) = \alpha_{t,t+1}^{\min}(Q) \quad (4)$$

Using the approximation (2) and monotone convergence for conditional expectations, and applying property 3) together with equation (4) to each $Q_n$, we obtain

$$E_Q[\rho_{t+1}(X)|\mathcal{F}_t] = \lim_{n \to \infty} \left( E_{Q_n}[-X|\mathcal{F}_t] + \alpha_{t,t+1}^{\min}(Q_n) - \alpha_t^{\min}(Q_n) \right)$$

$$= \lim_{n \to \infty} \left( E_{Q_n}[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q_n) \right) + \alpha_{t,t+1}^{\min}(Q)$$

$$\leq \rho_t(X) + \alpha_{t,t+1}^{\min}(Q).$$

Together with (3) this yields

$$E_Q[V_{t+1}|\mathcal{F}_t] \leq \rho_t(X) + \alpha_t^{\min}(Q) = V_t,$$

and so we have shown that the process $(V_t)_{t=0,1,\ldots}$ is a $Q$-supermartingale. \end{proof}