Chapter 1
Preliminaries

References:

1.1 Banach and Hilbert Spaces

1.1.1 Metric space

A metric space $X$ is a non-empty set endowed with a metric $\rho(x, y)$:

\begin{align*}
\rho(x, y) &\geq 0, \text{ and } \rho(x, y) = 0 \text{ if and only if } x = y; \\
\rho(x, y) &= \rho(y, x); \\
\rho(x, z) &\leq \rho(x, y) + \rho(y, z), \text{ (for any } x, y, z \in X). 
\end{align*}

The notion of metric is essentially an extension of distance in Euclidean spaces.

On a metric space, we can define
(i) convergence of a sequence $\{v_n\}$ to $v$ in $X$.
(ii) continuity of a mapping from $X$ into another metric space.
(iii) density, compactness or relative compactness of a subset of $X$. 
A subset $B$ is relative compact if any sequence in $B$ has a convergent subsequence. If the convergence point is also in $B$, then the subset is compact.

(iv) completeness or separability of $X$.

If every Cauchy sequence in $X$ converges (Cauchy sequence $\{x_n\}$ if $\rho(x_n, x_m) \to 0$ as $n, m \to +\infty$), then $X$ is complete.

1.1.2 Normed and Banach Spaces

Normed space

A normed space $V$ is a linear space endowed with a norm $\|\cdot\|_V$:

\[
\|\lambda v\|_V = |\lambda| \|v\|_V \text{ for } \lambda \in \mathbb{R}, \ v \in V,
\]

\[
\|v + w\|_V \leq \|v\|_V + \|w\|_V \text{ for } v, w \in V
\]

\[
\|v\| = 0 \text{ if and only if } v = 0.
\]

Since the mapping

\[
u, v \mapsto \|u - v\|_V
\]

is a metric on $V$, we can freely utilize metric notions such as convergence (strong convergence, denoted by $v_n \to v$), continuity, density, compactness, relative compactness, completeness, separability.

Weak convergence

The dual space of $V$ is the linear space $V'$ of continuous, or bounded, linear functionals $F$ on $V$, endowed with the norm

\[
\|F\|_{V'} \equiv \sup_{v \in V, \|v\|_V \leq 1} |Fv|.
\]

By weak convergence of a sequence $\{v_n\}$ to $v$ in $V$, denoted by the symbol $v_n \rightharpoonup v$, we mean convergence of $Fv_n$ to $Fv$ in $R$ whatever $F \in V'$. Strong convergence implies weak convergence, and viceversa if $V$ is finite dimensional.

Assignment: Prove that weakly convergent sequences are bounded, and $F_nv_n \to Fv$ if $F_n \to F$ in $V'$, $v_n \to v$ in $V$. 

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Imbedding

Two normed spaces $V$ and $W$ are (topologically) isomorphic if there exists an injective and surjective linear operator $T : W \rightarrow V$ such that both $T$ and $T^{-1} : V \rightarrow W$ are continuous, i.e., satisfy

$$|Tw|_V \leq C|w|_W \text{ for } w \in W$$

and

$$|T^{-1}v|_V \leq C|v|_V \text{ for } v \in V$$

with some positive constant $C$. $V$ and $W$ are isometrically isomorphic in the particular case when

$$|Tw|_V = |w|_W \text{ for } w \in W.$$  

If the linear operator $T$ is only required to be injective and continuous (which can happen to be the case with $T =$ identity when $W$ is a subspace of $V$ as well as a normed space on its own), we say that $W$ is continuously imbedded, or injected, in $V$ and write

$$W \hookrightarrow V;$$

the particular choice of $T$ is algebraically and topologically irrelevant because $W$ and its image $T(W)$ are isometrically isomorphic when the latter is normed by $z \mapsto |w|_W$ for $z = Tw, w \in W$.

Banach space

When a normed space is complete we call it a Banach space. Simple considerations show that $V'$ is always a Banach space whether the normed space $V$ is complete or not (an exercise).

**Theorem 1.1** Let $V$ be a Banach space. Let $K \subseteq V$ be closed in $V$. If $K$ is convex and $\{v_n\} \subset K$ converges weakly to $v$ in $V$, then $v \in K$.

The linear mapping $I$ defined by

$$\langle Iv, F \rangle = Fv \text{ for } F \in V'$$

is a continuous injection of $V$ in the dual space $V''$ of $V'$, and even more, namely, an isometric isomorphism between $V$ and the image space $I(V)$, by
the Hahn-Banach theorem (an exercise). If \( I \) is surjective, that is, \( I(V) = V'' \), we call \( V \) reflexive.

An important property of reflexive Banach spaces is given by the following theorem.

\textbf{Theorem 1.2} Every bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.

\textbf{Hilbert space}

A special class of normed spaces is that of pre-Hilbert spaces. They are linear spaces \( V \) such that there exists a mapping

\[ u, v \mapsto (u, v)_V \]

from the Cartesian product \( V \times V \) into \( R \), called a scalar product on \( V \), which is linear in each variable and satisfies

\[ (u, v)_V = (v, u)_V \text{ for } u, v \in V \]

as well as

\[ (u, u)_V > 0 \text{ for } u \in V, u \neq 0. \]

On pre-Hilbert spaces the Cauchy-Schwarz inequality holds:

\[ |(u, v)_V| \leq (u, u)_V^{1/2} (v, v)_V^{1/2}. \]

A norm on \( V \) is given by the mapping \( u \mapsto (u, u)_V^{1/2} \equiv |u|_V \). When a pre-Hilbert space is complete (and is therefore a Banach space) we call it a Hilbert space.

\textbf{Theorem 1.3} Hilbert spaces are reflexive.

As a matter of fact, by the Riesz representation theorem, we can show that a Hilbert space is isometrically isomorphic to its image in the dual space \( V' \) under the mapping \( u \mapsto (u, \cdot)_V \).
1.2 Sobolev Spaces

In this section, we always assume that \( \Omega \) is an open subset of \( \mathbb{R}^N \) which is connected as well as bounded. (Openness and connectedness make \( \Omega \) a domain.) Also, we assume the boundary \( \partial \Omega \) is smooth enough. Some results can be extended to unbounded domain.

Before introducing Sobolev spaces, let us recall smooth function spaces and Lebesgue spaces.

1.2.1 \( C^k \) and \( C^{k+\delta} \) spaces

Let \( C^0(\Omega) \) be the linear space of continuous real functions on \( \Omega \). \( C^k(\Omega) \), with \( k \in \mathbb{N} \) (the set of natural numbers), is the linear space of functions on \( \Omega \) have all derivatives of order \( \leq k \) in \( C^0(\Omega) \).

For \( k \in \mathbb{N} \), \( C^k(\Omega) \) is the linear space of functions in \( C^k(\Omega) \) which can be continuously extended to \( \overline{\Omega} \) together with all their derivatives of order \( \leq k \). It is clear that \( C^k(\Omega) \) is a Banach space with the choice of the norm

\[
|u|_{C^k(\Omega)} \equiv \sum_{i=0}^{k} \sum_{|\alpha|=i} |D^\alpha u|_{C^0(\Omega)} \equiv \sum_{i=0}^{k} \sum_{|\alpha|=i} \max_{\Omega} |D^\alpha u|,
\]

where we have used the multi-index notation:

\[
D^\alpha u \equiv \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}
\]

with \( \alpha \equiv (\alpha_1, \ldots, \alpha_N) \), \( |\alpha| = \alpha_1 + \cdots + \alpha_N \).

For \( 0 < \delta \leq 1 \), let

\[
[u]_{C^{\delta}} \equiv \sup_{x,y \in D, \, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\delta}
\]

whenever \( u \) is a function defined on a closed subset \( D \) of \( \mathbb{R}^N \). If \( [u]_{C^{\delta}} < +\infty \), we write \( u \in C^{\delta}(D) \) and say that \( u \) is Holder continuous in \( D \) (with exponent \( \delta \)) when \( 0 < \delta < 1 \), Lipschitz continuous in \( D \) when \( \delta = 1 \). If \( D \) is compact, a norm in the linear space \( C^{0+\delta} \) (sometimes denoted by \( C^{\delta} \)) is defined by

\[
|u|_{C^{0+\delta}(D)} \equiv |u|_{C^{0}(D)} + [u]_{C^{\delta};D}
\].
For $k \in \mathbb{N}$, $C^{k,\delta}(\bar{\Omega})$ is the linear space of functions $u \in C^k(\bar{\Omega})$ such that $D^\alpha u \in C^{0,\delta}(\bar{\Omega})$ whenever $\alpha = k$. When $\Omega$ is bounded, a norm on $C^{k,\delta}(\bar{\Omega})$ is defined by
\[
|u|_{C^{k,\delta}(\bar{\Omega})} \equiv |u|_{C^k(\bar{\Omega})} + \sum_{|\alpha| = k} |D^\alpha u|_{\delta,\bar{\Omega}}.
\]
It is convenient to stipulate the notational convention $C^{k+0} \equiv C^k$. For $k = 0, 1, \ldots$ and $0 \leq \delta \leq 1$, $C^{k+\delta}(\bar{\Omega})$ (with $\Omega$ bounded) is a Banach space which is not reflexive. Note that $C^{k+0}(\bar{\Omega})$, $k = 0, 1, \ldots$, is separable, but $C^{k+\delta}(\bar{\Omega})$, $k = 0, 1, \ldots$, $0 < \delta \leq 1$, is not separable.

### 1.2.2 Lebesgue spaces

For $1 \leq p < +\infty$ we denote by $L^p(\Omega)$ the linear space of measurable functions $u$ on $\Omega$ such that $|u|^p$ is integrable over $\Omega$, and set
\[
|u|_{p;\Omega} \equiv \left(\int_{\Omega} |u|^p \, dx\right)^{1/p}.
\]
We denote by $L^\infty(\Omega)$ the linear space of measurable functions $u$ on $\Omega$ such that $\text{ess sup}_{\Omega} |u| \equiv \inf \{C \in \mathbb{R} : u \leq C \text{ a.e. in } \Omega\} < +\infty$, and set
\[
|u|_{\infty,\Omega} \equiv \text{ess sup}_{\Omega} |u|.
\]
Note that $|u|_{\infty,\Omega} = |u|_{C^0(\bar{\Omega})}$ if $\Omega$ is bounded and $u \in C^0(\bar{\Omega})$.

(i) For $1 \leq p \leq +\infty$, $L^p(\Omega)$ is a Banach space with respect to the norm $u \mapsto |u|_{p;\Omega}$;

(ii) $L^2(\Omega)$ is a Hilbert space with respect to the scalar product $u, v \mapsto \int_{\Omega} uv \, dx$;

(iii) $L^p(\Omega)$ is separable for $1 \leq p < +\infty$, whereas $L^\infty(\Omega)$ is not;

(iv) $L^p(\Omega)$ is reflexive for $1 < p < +\infty$, whereas $L^1(\Omega)$ and $L^\infty(\Omega)$ are not.

(v) $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < p \leq +\infty$ (if $\Omega$ is bounded).

### 1.2.3 Sobolev spaces

First, let us define weak derivatives. Let $u$ be locally integrable in $\Omega$, i.e. $u \in L^1_{\text{loc}}(\Omega)$. Let $\alpha$ be any multi-index. Then a locally integrable function $v$
1.2. SOBOLEV SPACES

is called the $\alpha$th weak derivative of $u$ if it satisfies

$$
\int_{\Omega} u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_{\Omega} v \varphi dx, \text{ for any } \varphi \in C_0^\infty(\Omega).
$$

We write $v \equiv D^\alpha u$.

For $k \in \mathbb{N}$, Sobolev space $W^{k,p}(\Omega)$ (or $W^k_p(\Omega)$)

$$
W^k_p(\Omega) = \{ u : D^\alpha u \in L^p(\Omega), \text{ for any } 0 \leq \alpha \leq k \},
$$

equipped with the norm

$$
|u|_{W^k_p(\Omega)} = \left( \sum_{|\alpha| \leq k} |D^\alpha u|_{p,\Omega}^p dx \right)^{1/p}, \text{ if } 1 \leq p < +\infty
$$

$$
|u|_{W^k_\infty(\Omega)} = \max_{|\alpha| \leq k} |D^\alpha u|_{\infty,\Omega}
$$
is a Banach space. $W^k_2(\Omega) \equiv H^k(\Omega)$ is a Hilbert space with respect to the scalar product

$$
(u,v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.
$$

$W^k_p(\Omega)$ is separable for $1 \leq p < +\infty$, reflexive for $1 < p < +\infty$. In fact, it can be shown that for $1 \leq p < +\infty$, $C^\infty(\Omega) \cap W^k_p(\Omega)$ is dense in $W^k_p(\Omega)$.

Now let us introduce the imbedding theorem.

**Theorem 1.4** Assume $\Omega \subset \mathbb{R}^N$, $1 \leq p < +\infty$ and the boundary of $\Omega$ has certain regularity.

i) If $kp < N$, then $W^k_p(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q \leq Np/(N - kp)$; The mapping is compact (i.e. the images of bounded sets are compact) if $1 \leq q < Np/(N - kp)$.

ii) If $kp = N$, then $W^k_p(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q < +\infty$ if $kp = N$; The mapping is always compact.

iii) If $kp > N$; then $W^k_p(\Omega) \hookrightarrow C^{(k - N/p) + \delta}(\Omega)$ for any $0 < \delta \leq k - N/p - [k - N/p]$ (where $[a]$ $\equiv$ integer part of $a \in \mathbb{R}$) when $N/p \notin \mathbb{N}$, and the mapping is compact if $\delta < k - N/p - [k - N/p]$; and $W^k_p(\Omega) \hookrightarrow C^{(k - N/p - 1) + \delta}(\Omega)$ for any $\delta \in (0,1)$ when $N/p \in \mathbb{N}$, and the mapping is always compact.
1.3 Fixed Points

It is well known that in a complete metric space (in particular, in a Banach space) a contraction has a unique fixed point. In the following we list more sophisticated existence (not uniqueness) results for fixed points.

For finite-dimensional Banach spaces we have at Brouwer’s fixed point theorem:

**Theorem 1.5** Let $V$ be a finite-dimensional Banach space, let $K$ be a closed convex subset of $V$, and let $T$ be a continuous mapping of $K$ into itself such that the image $T(K)$ is bounded. Then $T$ has a fixed point

$$u \in K, \ u = Tu.$$

Brouwer’s theorem utilizes the fact that in Euclidean spaces bounded sets are relatively compact. Its direct extension to infinite-dimensional spaces is Schauder’s theorem:

**Theorem 1.6** Theorem 1.5 remains valid in any Banach space provided the image $T(K)$ is required to be relatively compact.

1.4 Schauder’s Estimates and $L^p$ Estimates

Consider the operator

$$Au = - \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

$A$ is said to be elliptic in $\Omega$ if

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq \lambda_x |\xi|^2 \text{ for all } \Omega, \xi \in \mathbb{R}^n (\lambda_x > 0).$$

It is uniformly elliptic if $\lambda_x \geq \lambda > 0$ for all $x \in \Omega$.

Without causing ambiguity, we will use the following simplified notations:

$$|\cdot|_{C^{k+\alpha}(\Omega)} = \|\cdot\|_{k+\alpha}; \quad |\cdot|_{C^{0+\alpha}(\Omega)} = \|\cdot\|_{\alpha};$$

$$|\cdot|_{W^{k,p}_0(\Omega)} = |\cdot|_{k,p};$$

$$|\cdot|_{L^p(\Omega)} = |\cdot|_p.$$
1.4. **Schauder’s Estimates and $L^p$ Estimates**

**1.4.1 Schauder’s (boundary) estimates**

Suppose that $\partial \Omega$ is locally in $C^{2+\alpha}$, $f \in C^\alpha(\overline{\Omega})$, $\Phi \in C^{2+\alpha}(\overline{\Omega})$, and

$$
\sum_{1 \leq i,j \leq n} \|a_{ij}\|_\alpha + \sum_i \|b_i\|_\alpha + \|c\|_\alpha \leq K,
$$

$$
\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ for all } x \in \Omega, \xi \in \mathbb{R}^n \ (\lambda > 0)
$$

If $u \in C^2(\overline{\Omega})$ and

$$
Au = f \text{ in } \Omega,
$$

$$
u = \Phi \text{ in } \partial \Omega,
$$

then

$$
\|u\|_{2+\alpha} \leq C \left( \|f\|_\alpha + \|u\|_0 + \|\Phi\|_{2+\alpha} \right),
$$

where $C$ is a constant depending only on $\lambda$, $K$, and $\Omega$.

**1.4.2 $L^p$ (boundary) estimates**

Suppose that $\partial \Omega$ is locally in $C^2$, $f \in L^p(\overline{\Omega})$, $\Phi \in W^{2,p}_0(\Omega)$, and

$$
|a_{ij}(x) - a_{ij}(y)| \leq \omega(|x - y|), \ [\omega(t) \to 0 \text{ if } t \to 0],
$$

$$
\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ for all } x \in \Omega, \xi \in \mathbb{R}^n \ (\lambda > 0) \ (1.1)
$$

$$
\sum |a_{ij}| + \sum |b_i| + |c| \leq K. \ (1.2)
$$

If $u \in W^{2,p}_0(\Omega)$ and

$$
Au = f \text{ in } \Omega,
$$

$$
u - \Phi \in H^1_0(\Omega),
$$

then

$$
|u|_{2,p} \leq C \left( |f|_p + |\Phi|_{2,p} \right),
$$

where $C$ is a constant depending only on $\lambda$, $K$, and the modulus of continuity $\omega$, and the domain $\Omega$. 
1.4.3 Strong maximum principle

Assume that (1.1), (1.2) hold and \( c(x) \geq 0 \). Let \( u \) be a function in \( H^2(\Omega) \cap C(\Omega) \) satisfying \( Au \leq 0 \) a.e. in \( \Omega \). If \( u \) assumes a positive maximum at some point \( x_0 \) in \( \Omega \), then \( u \equiv \text{const.} \) in \( \Omega \) (and then \( c = 0 \) a.e.).

This result extends also to \( u \), which is not necessarily continuous in \( \Omega \); “maximum” of \( u \) is replaced by “essential supremum” of \( u \); If \( \text{ess} \sup_{\Omega} u \) is positive and coincides with \( \text{ess} \sup_{B} u \) for any ball with center \( x_0 \) and arbitrarily small radius, then \( u = \text{const.} \). This implies:

\[
\text{If } u \in H^2(\Omega) \cap H^1_0(\Omega), \quad Au \leq 0 \text{ a.e. in } \Omega,
\]
\[
\text{then } u \leq 0 \text{ a.e. in } \Omega.
\]

1.4.4 Parabolic equations

All of the above results can be extended to the parabolic equations.
Chapter 2

Variational Inequalities in Finance

2.1 American Option Pricing and Obstacle Problem

2.1.1 Pricing model of American options

Let $V = V(S, t)$ be the price function of an American call option. The option can be exercise at any time before expiry. We must have

$$V(S, t) \geq \varphi(S) \equiv S - X$$

The pricing model is

$$LV \leq 0 \quad (2.1)$$
$$V \geq \varphi(S) \quad (2.2)$$
$$LV \left[ V - \varphi(S) \right] = 0 \quad (2.3)$$

in $Q = \{(S, t) : S > 0, t \in [0, T]\}$, subject to the terminal condition

$$V(S, T) = \varphi(S)^+, \text{ for } S > 0 \quad (2.4)$$

where

$$LV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV$$
(2.1)-(2.3) is called the variational inequality, which can be rewritten as

\[ \min \{-LV, V - \varphi(S)\} = 0. \]

In physics, the model corresponds to an obstacle problem and \( \varphi(S) \) is called the obstacle function. Notice that \( \varphi(S) \) can be replaced by \( \varphi(S)^+ \) because \( V(S,t) \geq 0 \), for all \( S \) and \( t \).

### 2.1.2 Obstacle problems

To understand the pricing model, we recall the (stationary) obstacle problem. Let us start from a variational problem. Consider the functional

\[ G(u) \equiv \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega fu \, dx, \]

where \( f \in L^2(\Omega) \). Find \( u \) such that

\[ u \in H^1_0(\Omega) \text{ and } G(u) = \min_{v \in H^1_0(\Omega)} G(v). \]

Suppose that \( u \) is a solution of this problem. Then, for any \( \epsilon \in \mathbb{R} \) and \( v \in H^1_0(\Omega) \),

\[ h(\epsilon) = G(u + \epsilon v) \]

\[ = \frac{1}{2} \epsilon^2 \int_\Omega |\nabla v|^2 \, dx + \epsilon \int_\Omega (\nabla u \nabla v - fv) \, dx + \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - fu \right) \, dx \]

achieves its minimum at \( \epsilon = 0 \). It follows

\[ \frac{\partial h}{\partial \epsilon} \bigg|_{\epsilon=0} = \epsilon \int_\Omega |\nabla v|^2 \, dx + \int_\Omega (\nabla u \nabla v - fv) \, dx \bigg|_{\epsilon=0} = \int_\Omega (\nabla u \nabla v - fv) \, dx = 0. \]

That is, find \( u \in H^1_0(\Omega) \),

\[ \int_\Omega \nabla u \nabla v \, dx = \int_\Omega fv \, dx \text{ for any } v \in H^1_0(\Omega). \] \quad (2.5)

If \( u \in H^2(\Omega) \), then

\[ \Delta u + f = 0 \text{ in } \Omega, \]

\[ u|_{\partial \Omega} = 0 \text{ on } \partial \Omega. \]
(2.5) is called the weak form of the above equation.

Now let us confine the solution to a closed convex set in $H^1_0(\Omega)$,
\[ K = \{ u \in H^1_0(\Omega) : u(x) \geq \phi(x) \text{ a.e.} \}, \]
where $\phi$ is a given continuous function in $\Omega$. (Assume $\phi \leq 0$ on $\partial \Omega$). Now we consider the (obstacle) problem: Find $u$ such that
\[ u \in K \text{ and } G(u) = \min_{v \in K} G(v). \quad (2.6) \]

Let $u \in K$ be the solution. For any $v \in K$, we have \((1 - \epsilon)u + \epsilon v = u + \epsilon(v - u) \in K\) for any $\epsilon \geq 0$. Denote
\[
H(\epsilon) = G(u + \epsilon(v - u)) \\
= \frac{1}{2} \epsilon^2 \int_{\Omega} |\nabla (v - u)|^2 \, dx + \epsilon \int_{\Omega} [\nabla u \nabla (v - u) - f(v - u)] \, dx \\
+ \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 - fu \right) \, dx.
\]
It follows
\[
\frac{\partial G}{\partial \epsilon} \bigg|_{\epsilon=0} = \int_{\Omega} [\nabla u \nabla (v - u) - f(v - u)] \geq 0
\]
or
\[
\int_{\Omega} \nabla u \nabla (v - u) \, dx \geq \int_{\Omega} f(v - u) \, dx \text{ for any } v \in K.
\]
If also $u \in H^2(\Omega)$, then we obtain
\[
\int_{\Omega} (\Delta u + f)(v - u) \, dx \leq 0 \text{ for any } v \in K, \quad (2.7)
\]
and choosing $v = u + \xi$, $\xi \geq 0$, $\xi \in C^\infty_0(\Omega)$, we get
\[ \Delta u + f \leq 0. \]

If $u$ is continuous, then the set
\[ A = \{ x \in \Omega : u(x) > \phi(x) \} \]
is open. For any $\xi \in C^\infty_0(A)$ the function $v = u + \epsilon \xi$ is in $K$ provided that $|\epsilon|$ is small enough. We then obtain from (2.7)
\[ \Delta u + f = 0 \text{ in } A. \]
Thus we have shown that if \( u \) is a solution of (2.6) which belongs to \( H^2(\Omega) \cap C(\Omega) \), then
\[
\begin{align*}
\Delta u + f & \leq 0 \\
u & \geq \phi \\
(\Delta u + f)(u - \phi) & = 0, \text{ a.e. in } \Omega \\
u|_{\partial\Omega} & = 0,
\end{align*}
\]
which is a variational inequality problem. (2.7) is the corresponding weak form.

The set \( K \) is called the constraint set, and \( \phi \) is the obstacle. The set \( A \) is called the non-coincidence set, and the set
\[
\Lambda = \{ x \in \Omega : u(x) = \phi(x) \}
\]
is called the coincidence set; the boundary of the non-coincidence set in \( \Omega \)
\[
\Gamma = \partial A \cap \Omega
\]
is called the free boundary.

It will be proved later that for suitably smooth \( f \), \( g \), \( \phi \), the solution \( u \) of the obstacle problem is in \( C^1(\Omega) \). Since \( u - \phi \) takes its minimum in \( \Omega \) on the coincidence set, it follows that
\[
u - \phi = 0, \text{ and } \nabla (u - \phi) = 0 \text{ on } \Gamma.
\]

Then we may view \( u \) as a solution of the Dirichlet problem
\[
\begin{align*}
\Delta u + f & = 0 \text{ in } A \\
u & = 0 \text{ on } \partial A \cap \partial \Omega \\
u & = \phi \text{ on } \partial A \cap \Gamma
\end{align*}
\]
with the additional condition
\[
\nabla u = \nabla \phi \text{ on } \partial A \cap \Gamma
\]
compensating for the fact that \( \Gamma \) is not a priori known. This point of view is useful in solving variational inequalities in one space dimension. However, for multi-dimensional problem, \( \Gamma \) can be quite irregular and we shall therefore not attempt to solve the obstacle problem by the approach of (2.8)-(2.9).
2.2. REGULARITY OF SOLUTION TO VARIATIONAL INEQUALITY

Now we come back to the pricing model of American options (2.1)-(2.4). Similar to the coincidence/non-coincidence set in obstacle problem, we define

\[ H = \{(S, t) \in Q : V(S, t) > \varphi(S)\} \]

\[ E = \{(S, t) \in Q : V(S, t) = \varphi(S)\}. \]

We will show that for \( q > 0 \), there is a single-value strictly increasing function \( S^*(t) : [0, T) \to (0, +\infty) \), such that

\[ E = \{(S, t) \in Q : S \geq S^*(t), t \in [0, T)\}. \]

Moreover, \( V(S, t) \in W^{2,1}_{p,\text{loc}}(Q) \), which implies \( V \in C^1 \) in \( S \) and the free boundary condition

\[ \frac{\partial V}{\partial S}\bigg|_{S=S^*(t)} = 1. \]

The condition can be utilized to derive the analytical price formulas of perpetual American options.

2.2 Regularity of Solution to Variational Inequality

2.2.1 Classical solution, strong solution and weak solution

First, we distinguish between classical solution \( (C^2) \), strong solution \( (W^2_p) \) and weak solution \( (H^1) \).

Then we point out that the classical solution to a standard variational inequality equation does not exist in general. Let us take the American option pricing model as an example. Indeed, we can show later that \( V, V_S \) and \( V_t \) are continuous. We assert that \( V_{SS} \) is not continuous across the free boundary. Indeed, applying the equation,

\[ V_{SS}|_{S=S^*(t)-} = -\left( V_t + (r - q)SV_S - rV \right)|_{S=S^*(t)} = \frac{qS^*(t) - rX}{\frac{1}{2}\sigma^2S^2(t)}, \]

whereas

\[ V_{SS}|_{S=S^*(t)+} = 0. \]

This implies discontinuity.
2.2.2 Existence and regularity of strong solution: Penalized method

To illustrate method, we consider the obstacle problem with \( \phi \in C^2(\Omega) \), \( f \in C^\alpha(\Omega) \) and smooth \( \partial \Omega \). This is a nonlinear problem. To show the existence of solution, we need fixed point theorems. We plan to make use of the Schauder fixed point theorem. Since all of a priori estimates are for equations, we approximate the obstacle problem using

\[
\begin{align*}
-\Delta u + \beta\epsilon(u - \phi) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \beta\epsilon(t) \) (0 < \( \epsilon \) < 1) be \( C^2 \) in \( t \), satisfying

\[
\beta\epsilon(t) \leq 0, \\
\beta\epsilon(0) = -C_0 \quad (C_0 > 0 \text{ independent of } \epsilon, \text{ to be given}). \\
\beta'\epsilon(t) \geq 0, \beta''\epsilon \leq 0,
\]

and

\[
\lim_{\epsilon \to 0}\beta\epsilon(t) = \begin{cases} 
0, & \text{for } t > 0 \\
-\infty, & \text{for } t < 0.
\end{cases}
\]

The problem (2.10) is called the penalized problem. The financial interpretation is the following. Consider an option that gives a constrained early exercise right subject to a Poisson process with intensity \( \lambda = \frac{1}{\epsilon} \) [see Dai, Kwok and You, Intensity-based framework and penalty formulation of optimal stopping problems, Journal of Economic Dynamics and Control (2007), 31(12):3860-3880]. Then, the pricing model is

\[
-LV = \frac{1}{\epsilon}(\varphi - V)^+
\]

or

\[
-LV + \frac{1}{\epsilon} \min(V - \varphi, 0) = 0
\]

Apparently the option value converges to that of an American option as the intensity \( \lambda \) goes to infinity. It is easy to see that \( \beta\epsilon(t) \) is a smoothing function of \( \frac{1}{\epsilon} \min(t, 0) \).

**Lemma 2.1** There exists a solution \( u_\epsilon \) of (2.10), and \( u_\epsilon \geq \phi \).
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Proof: Since $\beta_\epsilon(t)$ is unbounded, we set, for any $M > 0$,

$$\beta_{\epsilon,M} = \max \{\beta_\epsilon(t), -M\},$$

and consider the problem

$$\begin{array}{l}
-\Delta u + \beta_{\epsilon,M}(u - \phi) = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{array} \tag{2.11}$$

For each $v \in L^p(\Omega)$ ($1 < p < +\infty$) there exists a unique solution $w$ in $W^2_0(\Omega)$ of

$$\begin{array}{l}
-\Delta w = f - \beta_{\epsilon,M}(v - \phi) \text{ in } \Omega, \\
w = 0 \text{ on } \partial \Omega.
\end{array}$$

and

$$|w|_{2,p} \leq R,$$

where $R$ is a constant independent of $v$. Set $w = T(v)$. Consider a closed convex set $D$ in $L^p(\Omega)$:

$$D = \{v \in L^p(\Omega) : |v|_p \leq R\}.$$

It is easy to see

(i) $TD \subset D$;

(ii) $T$ is compact, by the embedding theorem;

(iii) $T$ is continuous. That is, for any $v_j, v \in D$, $v_j \to v$ in $D$, $T(v_j) = w_j$, $T(v) = w$, we need to show

$$w_j \to w \text{ in } D.$$

Note that $w_j - w$ satisfies

$$-\Delta (w_j - w) = \beta'_{\epsilon,M}(\cdot)(v_j - v) \text{ in } \Omega,$$

where $\beta'_{\epsilon,M}(\cdot)$ has a bound independent of $v_j$ and $v$. Therefore, by $L^p$ estimate, we obtain the desired result.

Then, we apply Schauder’s fixed-point theorem to get the existence of a solution to (2.11), denoted by $u_{\epsilon,M}$. Since $u = u_{\epsilon,M}$ is in $W^2_p(\Omega)$, for any $p < +\infty$, $\beta_{\epsilon,M}(u - \phi)$ is Holder continuous. By a general regularity result for elliptic equations with $C^\alpha$ coefficients, it follows that $u \in C^{2+\alpha}(\Omega)$. We shall now estimate the function

$$\xi(x) = \beta_{\epsilon,M}(u - \phi).$$
By definition of $\beta_\epsilon$,
\[ \xi(x) \leq 0. \]
Consider now the minimum $\gamma$ of $\xi(x)$. Suppose that
\[ \gamma = \xi(x_0), \quad \gamma < \beta_\epsilon(0). \]
Then $x_0 \notin \partial \Omega$. On the other hand, if $x_0 \in \Omega$, then since $\beta_{\epsilon,M}(t)$ is monotone in $t$, $u - \phi$ also takes a minimum at $x_0$. It follows by the standard maximum principle that
\[ -\Delta (u - \phi) \leq 0 \text{ at } x_0. \]
Take $-C_0 \leq \min_{x \in \Omega} (\Delta \phi + f)$. Combining with (2.11), we deduce that
\[ \xi(x_0) = f + \Delta u|_{x=x_0} \geq f(x_0) + \Delta \phi(x_0) \geq -C_0. \]
We have thus shown that
\[ \beta_{\epsilon,M}(u_{\epsilon,M} - \phi) \geq -C_0, \quad (C_0 \text{ independent of } \epsilon, N) \]
which implies
\[ u_{\epsilon,M} \geq \phi \]
provided that $M > C_0$. Then $u_\epsilon = u_{\epsilon,M}$ (for $M$ big enough) is a solution of the penalized problem (2.10). Moreover,
\[ \|\Delta u_\epsilon\|_0 \leq C, \quad (C \text{ independent of } \epsilon) \]
and by the $L^p$ estimates,
\[ |u_\epsilon|_{2,p} \leq C. \quad (C \text{ independent of } \epsilon) \]
We now take a sequence $\epsilon = \epsilon_n \to 0$ such that
\[ u_\epsilon \rightharpoonup u \text{ weakly in } W^2_p(\Omega), \quad \text{for any } p < +\infty. \]
It follows that $u_\epsilon \to u$ uniformly in $\Omega$. We then deduce that
\[ u \geq \phi, \]
\[ \beta_\epsilon(u_\epsilon - \phi) \to 0 \text{ on the set } \{u > \phi\}. \]
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\[
\lim_{\epsilon \to 0} \beta_{\epsilon}(u_\epsilon - \phi) \leq 0.
\]

We conclude that
\[
-\Delta u = f \text{ a.e. on } \{u > \phi\}
\]
\[
-\Delta u \geq f \text{ a.e. in } \Omega.
\]

We have thus proved:

**Theorem 2.2** Assume \( f \in C^\alpha, \phi \in C^2, \partial \Omega \in C^{2+\alpha} \). Then the obstacle problem has a solution \( u \in W^2_p(\Omega) \) for any \( p < +\infty \).

We point out that the conditions in the above theorem can be weakened. For example, \( \phi \in C^2 \) can be replaced by
\[
\phi \in C^{0,1}(\Omega_0), \quad \phi_{\xi\xi} \geq -C \text{ in the sense of distribution, for any direction } \xi,
\]
where \( \Omega_0 \) is a neighborhood of \( \overline{\Omega} \). [see Friedman (1982)]

Now let us look at comparison principle for variational inequality.

**Theorem 2.3** Let \( u_1 \) and \( u_2 \) be solutions in \( H^2(\Omega) \cap C(\overline{\Omega}) \) of the variational inequality corresponding to \( (f_1, \phi_1) \) and \( (f_2, \phi_2) \), respectively. If \( f_1 \geq f_2 \) and \( \phi_1 \geq \phi_2 \), then \( u_1 \geq u_2 \) a.e.

Proof: Suppose the open set \( G = \{x \in \Omega : u_2(x) > u_1(x)\} \) is nonempty. Since \( u_2 > u_1 \geq \phi_1 \geq \phi_2 \) in \( G \),
\[
-\Delta u_2 = f_2, \quad -\Delta u_1 \geq f_1.
\]
Consequently,
\[
-\Delta(u_2 - u_1) \leq 0 \text{ in } G.
\]
Also, \( u_2 - u_1 = 0 \) on \( \partial G \). Hence, by the maximum principle, \( u_2 - u_1 \leq 0 \) in \( G \), a contradiction.

The above comparison principle implies the uniqueness of solution of variational inequality.

**Assignment:** Show the \( W^{2,1}_{p,\text{loc}} \) regularity of solution to the American option pricing model. (Hints: first confine to a bounded domain, and smoothen the terminal condition)