2.3 Properties of Free boundaries

Let us focus on the American option pricing model (2.1)-(2.4). Using a transformation

\[ V(S, t) = u(x, t), \quad x = \log S, \]

we get

\[
\begin{aligned}
\min \left\{ -\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - (r - q - \frac{\sigma^2}{2}) \frac{\partial u}{\partial x} + ru - (e^x - X) \right\} &= 0, \\
u &= (e^x - X)^+, \quad x \in \mathbb{R}, \quad t \in [0, T].
\end{aligned}
\]

First, let us introduce a lemma

**Lemma 2.4** \( V_S \leq 1 \) (or \( u_x \leq e^x \)), and \( V_t \leq 0 \) (or \( u_t \leq 0 \)), for all \( S, t \).

**Proof.** By the comparison principle of variational inequality, it is easy to see \( u(x, t) \leq \bar{u}(x, t) \equiv u(x, t - \delta) \), which leads to \( u_t \leq 0 \). Similarly we can show \( u_x \leq e^x \). Another approach: consider the approximation problem,

\[
\begin{aligned}
-\partial_t u_\varepsilon - \frac{\sigma^2}{2} \partial^2_x u_\varepsilon - (r - q - \frac{\sigma^2}{2}) \partial_x u_\varepsilon + ru_\varepsilon + \beta_\varepsilon (u_\varepsilon - (k - e^x)) &= 0 \\
u_\varepsilon &= \pi_\varepsilon(k - e^x), \quad x \in \mathbb{R}, \quad t \in [0, T),
\end{aligned}
\]

where \( \pi_\varepsilon(x) \) is a smooth function to approximate \( x^+ \). We can show \( \partial_t u_\varepsilon \leq 0 \) and \( \partial_x u_\varepsilon \leq e^x \) (Assignment), then let \( \varepsilon \to 0 \).

**Proposition 2.5** Assume \( q > 0 \). Then there is a continuous, strictly decreasing function \( S^*(t) : [0, T) \to [X, +\infty) \), such that

\[ E = \{(S, t) \in Q : S \geq S^*(t)\}. \]

Moreover,

\[ S^*(t) > S^*(T-) = \max \left\{ X, \frac{r}{q} X \right\}, \quad \text{for any } t < T, \]

and

\[ \lim_{T \to \infty} S^*(t) = S^*_\infty, \]

where \( S^*_\infty = \frac{\alpha_+}{\alpha_+ - 1} X < +\infty, \quad \alpha_+ = \frac{-(r-q-\sigma^2/2) + \sqrt{(r-q-\sigma^2/2)^2 + 2\sigma^2}}{\sigma^2}. \)
Proof: Apparently $V(S, t) > 0$. Then $E \subset \{(S, t) : S \geq X\}$. Note that

$$\partial_S [V - (S - X)] = V_S - 1 \leq 0,$$

which implies the existence of a single-value function $S^*(t) : [0, T) \rightarrow [X, +\infty) \cup +\infty$. Now we prove $S^*(t) < +\infty$. Recall the price function of perpetual American call, denoted by $V_\infty(S)$, has an analytic expression:

$$V_\infty(S) = \begin{cases} (S_\infty - X) \left( \frac{S}{S_\infty} \right)^{\alpha^+}, & \text{if } S < S_\infty^*, \\ S - X, & \text{if } S \geq S_\infty^*, \end{cases}$$

It is easy to show that $V_\infty(S)$ is a supersolution to problem (2.1)-(2.4) (comparison principle). So, for $S > S_\infty^*$, we have $S - X \leq V(S, t) \leq V_\infty(S) \leq S - X$, i.e. $V(S, t) = S - X$. We then deduce $S^*(t) \leq S_\infty^* < +\infty$.

The monotonicity of $S^*(t)$ is an immediate corollary of

$$\frac{\partial}{\partial t} [V - (S - X)] = V_t \leq 0.$$

It is apparent that $S^*(t) \geq X$. By variational inequality, $L(S - X) \leq 0$ yields

$$S^*(t) \geq \frac{r}{q} X.$$

To show $S^*(T-) = \max \{X, \frac{r}{q} X\}$, we use the method of contradiction. Suppose not, then $S^*(T-) > \max \{X, \frac{r}{q} X\}$. For any $S \in (\max \{X, \frac{r}{q} X\}, S^*(T-))$, we have

$$\left. \frac{\partial V}{\partial t} \right|_{(S,T)} = \left[ -\frac{1}{2} \sigma^2 S^2 \frac{\partial}{\partial S^2} - (r - q)S \frac{\partial}{\partial S} + r \right] (S - X) = qS - rX > 0,$$

which is in contradiction with $V_t \leq 0$.

Now we show that a similar analysis leads to the continuity of $S^*(t)$. Suppose not. Then there exists $t_0$ such that

$$\frac{r}{q} X \leq S_1 \equiv S^*(t_0+) < S^*(t_0-) \equiv S_2.$$

So, for $S \in (S_1, S_2)$, $t = t_0$,

$$\left. \frac{\partial V}{\partial t} \right|_{(S,t_0)} = \left[ -\frac{1}{2} \sigma^2 S^2 \frac{\partial}{\partial S^2} - (r - q)S \frac{\partial}{\partial S} + r \right] (S - X)|_{(S,t_0)} = qS - rX > 0.$$
A contradiction.

It remains to show the strict monotonicity. Suppose not, then \( S^*(t) \) has a vertical part. That is, there exist \( t_1 < t_2 < T \), such that \( S^*(t) = S_0 \) for all \( t \in [t_1, t_2] \). Since \( V|_{S=S_0} = S - X \) and \( V_S|_{S=S_0} = 1 \), it follows \( V_t|_{S=S_0} = V_{S_S}|_{S=S_0} = 0 \) for \( t \in [t_1, t_2] \). Applying Hopf lemma for \( V_t \), we deduce \( V_{S_t} > 0 \), a contradiction.

Other approaches to proving the existence of \( S^*(t) \):

a) If we do not have \( \partial_S[V - (S - X)] \leq 0 \) (bear in mind that it is only a sufficient condition to ensure the existence), we may instead make use of \( V_t \leq 0 \), which indicates the existence of a single-value function \( t^*(S) \). Then, we only need to show that \( t^*(S) \) is monotone in \( S \), which can be achieved in some cases by means of the comparison principle and uniqueness of solution.

Two drawbacks: i) it is hard to deal with high-dimension problems; ii) given terminal condition, the obstacle (exercise region) should be above the free boundary.

b) Assume that the solution is convex, namely, \( V(\alpha S_1 + (1 - \alpha)S_2, t) \leq \alpha V(S_1, t) + (1 - \alpha) V(S_2, t) \), for any \( \alpha \in [0, 1] \). In addition, \( \varphi(\alpha S_1 + (1 - \alpha)S_2) = \alpha \varphi(S_1) + (1 - \alpha) \varphi(S_2) \). Then, if \((S_i, t) \in E \), i.e. \( V(S_i, t) = \varphi(S_i), i = 1, 2 \), then we have

\[
\varphi(\alpha S_1 + (1 - \alpha)S_2) \leq V(\alpha S_1 + (1 - \alpha)S_2, t) \\
\leq \alpha V(S_1, t) + (1 - \alpha) V(S_2, t) \\
= \alpha \varphi(S_1) + (1 - \alpha) \varphi(S_2) \\
= \varphi(\alpha S_1 + (1 - \alpha)S_2),
\]

which implies \((\alpha S_1 + (1 - \alpha)S_2, t) \in E \) for any \( \alpha \in [0, 1] \). Thus, if we can further show that \((S, t) \in E \) for \( S \) big enough, then we deduce the existence of a finite \( S^*(t) \).

c) Try \( U(S, t) \equiv \gamma(S, t)V(S, t) \geq \gamma(S, t)\varphi(S, t) \), for some \( \gamma(S, t) > 0 \), in place of \( V(S, t) \geq \varphi(S, t) \).

**Remark 1** It can be shown that \( S^*(t) \in C^\infty \). The proof is similar to Friedman, A.: Parabolic variational inequality in one space dimension and smoothness of the free boundary, Journal of Functional Analysis, 18(3), (1975), 151-176.

**Remark 2** At some occasions, it can be shown that \( S^*(t) \) is convex. See X. Chen, J. Chadam, L. Jiang, and W. Zheng (2008), Convexity of the exercise
boundary of the American put option a zero dividend asset, Mathematical Finance.

2.4. OTHER EXAMPLES

2.4 Other Examples

2.4.1 Pricing models of strike reset options

The holder of the option has right to reset the strike price. The pricing model (put option) is the following:

\[
\begin{align*}
\min \{ -LV, V - SP(t) \} &= 0, \\
V(S, T) &= (X - S)^+,
\end{align*}
\]

in \( S > 0, t \in [0, T) \). Here \( SP(t) \) is the European vanilla at-the-money put option value (with maturity \( T \)).


Let us first consider the problem with zero terminal condition. Let \( V(S, t) \) be the solution to

\[
\begin{align*}
\min \{ -L\overline{V}, \overline{V} - SP(t) \} &= 0, \\
\overline{V}(S, T) &= 0,
\end{align*}
\]

in \( S > 0, t \in [0, T) \). Since \( \overline{V} \) is homogeneous in \( S \), we can write \( \overline{V}(S, t) = Sg(t) \), where \( g(t) \) satisfies

\[
\begin{align*}
\min \{ -d/dt \left( e^{q(T-t)} g(t) \right), g(t) - P(t) \} &= 0, \text{ for } t \in [0, T) \\
g(T) &= 0.
\end{align*}
\]

Now we introduce a lemma:

**Lemma 2.6** The function \( e^{q(T-t)} P(t) \) has the following properties:

(i) If \( r \leq q \), then \( d/dt \left( e^{q(T-t)} g(t) \right) < 0 \) for all \( t \).

(ii) If \( r > q \), then there is a unique critical value \( t^* \), such that

\[
\begin{align*}
\frac{d}{dt} \left( e^{q(T-t)} P(t) \right) &< 0 \text{ for } t \in (t^*, T), \\
\frac{d}{dt} \left( e^{q(T-t)} P(t) \right) &> 0 \text{ for } t \in (0, t^*).
\end{align*}
\]
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Now we solve (2.14). It is easy to check that
(i) If \( r \leq q \), then \( V(S, t) = SP(t) \);
(ii) If \( r > q \), then

\[
\nabla(S, t) = \begin{cases} 
SP(t), & \text{if } t \in [t^*, T) \\
Se^{q(t-t^*)}P(t^*), & \text{if } t \in (0, t^*) 
\end{cases}
\]

This indicates that \( \nabla(S, t) > SP(t) \) if \( r > q \) and \( t < t^* \). Clearly \( V(S, t) \geq V(S, t) \).

We then deduce

**Proposition 2.7** If \( r > q \) and \( t < t^* \), then one should never reset the strike price.

For other cases, we have the following proposition.

**Proposition 2.8** If \( r \leq q \) or \( r > q \) and \( t \in (t^*, T) \), then there is an optimal reset boundary \( S^*(t) < +\infty \), such that

\[
\{(S, t) : V(S, t) = SP(t)\} = \{S \geq S^*(t), \ t \in (t^*, T) \text{ if } r > q, \ \text{ and } t \in [0, T) \text{ if } r \leq q\}.
\]

Moreover, \( S^*(T-) = X \).

Proof: We only show that \( S^*(t) \) is finite. When \( r \leq q \), we consider \( W(S, t) = e^{r(T-t)}V(S, t) \) satisfying

\[
\min \left\{ -W_t - \frac{\sigma^2}{2}S^2W_{SS} - (r - q)W_S, W - Se^{r(T-t)}P(t) \right\} = 0,
\]

in \( S > 0, \ t \in [0, T) \). Noting that \( \lim_{T \to -\infty} e^{r(T-t)}P(t) = 1 \) for \( r \leq q \), we can further consider the stationary solution (as \( T \to \infty \)) to show that \( \lim_{T \to -\infty} S^*(t) \) exists for \( r \leq q \).

When \( r > q \), it suffices to show \( V(S, t) - SP(t) \) has a compact support for all \( t \in (t^*, T) \). Consider the transformation

\[
D(x, t) = \frac{e^{q(T-t)}[V(S, t) - SP(t)]}{S} \text{ and } S = e^x.
\]
2.4. OTHER EXAMPLES

Then
\[
\min \left\{ -D_t - \frac{\sigma^2}{2} D_{xx} - (r - q + \frac{\sigma^2}{2}) D_x + (r - q) D - \frac{d}{dt} \left[ e^{q(T-t)} P(t) \right], D \right\} = 0
\]
\[D(x, 0) = (Xe^{-x} - 1)^+\]
in \(x \in (-\infty, \infty), t \in (t^*, T)\). Note that \(\frac{d}{dt} \left[ e^{q(T-t)} P(t) \right] < 0\) in this case. We construct an auxiliary function
\[
w(x) = \begin{cases} 
\gamma (R_0 - x)^2, & \text{if } x \leq R_0 \\
0, & \text{if } x > R_0,
\end{cases}
\]
where \(R_0 > 0\) and \(\gamma > 0\) are to be determined. Note that for \(x \leq R_0\),
\[
\left[ -\partial_t - \frac{\sigma^2}{2} \partial_{xx} - (r - q + \frac{\sigma^2}{2}) \partial_x + (r - q) \right] w(x) \geq -\gamma \sigma^2,
\]
and for \(x > R_0\),
\[
\left[ -\partial_t - \frac{\sigma^2}{2} \partial_{xx} - (r - q + \frac{\sigma^2}{2}) \partial_x + (r - q) \right] w(x) = 0.
\]
So, for any \(t \in (t^*, T)\), we can choose \(\gamma\) small enough such that
\[
-\gamma \sigma^2 > \frac{d}{dt} \left[ e^{q(T-t)} P(t) \right],
\]
and choose \(R_0\) big enough such that \(w(\log X, t) \geq D(\log X, t)\) for all \(t \in (t^*, T)\). Applying maximum principle, we infer \(w(x, t) \geq D(x, t)\) for any \(t \in (t^*, T)\), which implies that \(D(x, t) = 0\) for \(x > R_0, t \in (t^*, T)\). This is the desired result.

**Remark 3** One question: whether or not \(S^*(t^*) < \infty\) when \(r > q\)?

2.4.2 Russian options with finite maturity

Let \(V = V(S, M, t)\) be the option value.
\[
\begin{cases} 
\min \{-LV, V - M\} = 0, \\
\frac{\partial V}{\partial M} \bigg|_{S=M} = 0 \\
V(S, M, T) = M,
\end{cases}
\]
in $S > 0$, $M > S$, $t \in [0, T)$.
By transformation
\[ u(y, t) = \frac{V(S, M, t)}{S} \text{ and } y = \frac{M}{S}, \]
we have
\[
\begin{cases}
\min \left\{ -u_t - \frac{\sigma^2}{2} y^2 u_{yy} - (q-r)y u_y + qu, u - y \right\} = 0, \\
\frac{\partial u}{\partial y} \bigg|_{y=1} = 0 \\
u(y, T) = y,
\end{cases}
\]
in $Q_y = \{ y > 1 \} \times \{ 0 \leq t < T \}$.

Using a similar argument as in the American call option pricing, we can show that when $q > 0$, there is a continuous, monotonically decreasing function $y^*(t) : [0, T) \times [1, +\infty)$, such that
\[
\{(y, t) \in Q_y : u(y, t) = y\} = \{(y, t) \in Q_y : y \geq y^*(t)\}.
\]

We are interested in the situation $q = 0$. It is well-known that a perpetual Russian option has an infinite value when $q = 0$. A natural question is the existence of a finite $y^*(t)$ in the case of $q = 0$. I leave this as an assignment.

### 2.4.3 American exchange options

Assume the payoff
\[
\varphi(S_1, S_2) = \begin{cases} 
\max(S_1, S_2), & \text{for max option} \\
\min(S_1, S_2), & \text{for min option}.
\end{cases}
\]
where $S_1$ and $S_2$ are the prices of two underlying assets with constant correlation coefficient $\rho$. Let $V(S_1, S_2, t)$ be the option value. Then,
\[
\begin{cases}
\min \{-L_2 V, V - \varphi(S_1, S_2)\} = 0, \\
V(S_1, S_2, T) = \varphi(S_1, S_2),
\end{cases}
\]
in $S_1 > 0$, $S_2 > 0$, $t \in [0, T)$, where
\[
L_2 V = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}
\]
\[+(r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - rV.\]
By similarity reduction
\[
    u(\xi, t) = \frac{V(S_1, S_2, t)}{S_2}, \quad \xi = \frac{S_1}{S_2},
\]
we have
\[
    \begin{cases}
        \min \{-L_0 u, u - \psi(\xi)\} = 0, \\
        u(\xi, T) = \psi(\xi),
    \end{cases}
\]
in $\xi > 0$, $t \in [0, T)$, where
\[
    L_0 u = \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \xi^2 \frac{\partial^2 u}{\partial \xi^2} + (q_2 - q_1) \xi \frac{\partial u}{\partial \xi} - q_2 u,
\]
\[
    \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2 \rho \sigma_1 \sigma_2},
\]
\[
    \psi(\xi) = \begin{cases}
        \max(\xi, 1), & \text{for max option}, \\
        \min(\xi, 1), & \text{for min option}.
    \end{cases}
\]

Let us first look at the case of the max option. Note that
\[
    -\frac{\partial^2}{\partial \xi^2} \max(\xi, 1) \bigg|_{\xi=1} = -\delta.
\]
So, $-L_0 [\max(\xi, 1)]_{\xi=1} < 0$, which indicates $\{\xi = 1\} \not\subset \text{Exercise Region}$. It can be shown that there exist two boundaries $\xi_1(t) < 1$ and $\xi_2(t) > 1$, such that $\xi_1(T) = \xi_2(T) = 1$, and
\[
    \text{Holding Region} = \{\xi_1(t) < \xi < \xi_2(t), \ t \in [0, T)\}.
\]
Moreover
\[
    \begin{align*}
    \{u = \xi\} &= \{\xi \geq \xi_2(t), \ t \in [0, T)\}, \\
    \{u = 1\} &= \{\xi < \xi_1(t), \ t \in [0, T)\}.
    \end{align*}
\]
Exercise region of American min options is left as an assignment.
2.4.4 Pricing convertible bond with redeemable feature: a double obstacle problem

The issuer has right to redeem the bond at predetermined price \( C(t) \), while the holder has right to convert the bond to one share of stock \( S_t \). The pricing model is a double obstacle problem:

\[
-LV \geq 0, \quad \text{if } V = S \\
-LV \leq 0 \quad \text{if } V = C(t) \\
-LV = 0 \quad \text{if } S < V < C(t) \\
V(S, T) = \max(S, X).
\]

Another example of double obstacle problem is the pricing of game options. See Y. Kifer, Game options, Finance and Stochastic, 4(4), 2000, 443-463.

2.4.5 High-dimensional problems

a) American-style fixed-strike path-dependent (call) options:

\[
\min \left\{ -LV - f(S) \frac{\partial V}{\partial A}, V - (A - X)^+ \right\} = 0, \\
V(S, A, T) = (A - X)^+, \\
in \ t \in [0, T), \ S > 0, \ A > 0 \text{ for Asian options or } A > S \text{ for lookback options,}
\]

where

\[
f(S) = \begin{cases} 
S - A, & \text{for arithmetic average} \\
\frac{A}{t} \log \frac{S}{A}, & \text{for geometric average} \\
0, & \text{for lookback.}
\end{cases}
\]

Note that for lookback options, we need a boundary condition

\[
\left. \frac{\partial V}{\partial A} \right|_{S=A} = 0.
\]

I refer you to


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b) American options on multi-assets:

An example:

\[
\begin{align*}
\min \{-L_2 V, V - (\max(S_1, S_2) - X)^+\} &= 0, \\
V(S_1, S_2, T) &= (\max(S_1, S_2) - X)^+,
\end{align*}
\]

in \( S_1 > 0, S_2 > 0, t \in [0, T) \). Similarity reduction no longer works when \( X > 0 \).

I refer you to


(2) S. Villeneuve, Exercise regions of American options on several assets, Finance and Stochastic, 3 (1999), pp. 42-56.


2.4.6 Multiple exercise rights

Let us consider a strike reset (put) option that gives the holder twice chances to reset the strike price. Denote the option value by \( V_2(S, t) \). On resetting, the option becomes an at-the-money strike reset option with one reset chance, whose value is denoted by \( V_1(S, t; S) = SP_2(t) \). In general, let \( V_n(S, t) \) be the value of the strike reset option with \( n \) reset chances, and \( V_{n-1}(S, t; S) = SP_n(S), n \geq 1 \). Here, \( V_0(S, t) \) denotes the value of a European vanilla put option. Then, the pricing model is the following

\[
\begin{align*}
\min \{-LV_n, V_n - SP_n(S)\} &= 0, \\
V_n(S, T) &= (X - S)^+,
\end{align*}
\]

in \( S > 0, t \in [0, T) \).

It is worth pointing out that the option tends to a floating lookback put as \( n \to +\infty \).

For details, see

Other examples include