CONVERGENCE OF BINOMIAL TREE METHODS FOR
EUROPEAN/AMERICAN PATH-DEPENDENT OPTIONS∗

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Abstract. The binomial tree method, first proposed by Cox, Ross, and Rubinstein [Journal of Financial Economics, 7 (1979), pp. 229–263], is one of the most popular approaches to pricing options. By introducing an additional path-dependent variable, such methods can be readily extended to the valuation of path-dependent options. In this paper, using numerical analysis and the notion of viscosity solutions, we present a unifying theoretical framework to show the uniform convergence of binomial tree methods for European/American path-dependent options, including arithmetic average options, geometric average options, and lookback options.

Key words. binomial tree method, European/American path-dependent options, convergence

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1. Introduction. Path-dependent options are options whose payoffs depend on historical values of the underlying asset over a given time period as well as its current price. Well-known examples are Asian arithmetic/geometric average options, lookback options, etc. The binomial tree method (BTM), first proposed by Cox, Ross, and Rubinstein [6], has become one of the most popular approaches to pricing vanilla options due to its simplicity and flexibility. By introducing an additional path-dependent variable at each node, BTM can be readily extended to the valuation of path-dependent options.

Many authors have shown that the prices of European vanilla options computed from BTM converge to their corresponding continuous-time model values (see [12] and references therein). Amin and Khanna [1] and Jiang and Dai [14] produce the convergence proofs for American vanilla options by using the probabilistic approach and the partial differential equation (PDE) approach, respectively. In this paper, using the PDE approach, a unifying framework is given to show uniform convergence of BTMs for both European and American path-dependent options, including Asian arithmetic/geometric average options and lookback options. The basic idea stems from the result of Barles and Souganidis [4], which essentially says that any stable, monotone, and consistent numerical scheme converges, provided that one has a strong comparison principle in the sense of viscosity solution for the limiting equation. For Asian options and lookback options, the BTMs are clearly monotone and the needed strong comparison principles can be deduced from Crandall, Ishii, and Lions [8] and Barles, Daher, and Romano [2]. Hence, in addition to showing consistency, the key point of the proof is to prove stability, namely, to obtain uniform estimates of bounds of the approximate solutions sequences computed by BTMs. We arrive at this by two steps: first, it is shown that values of lookback options (computed

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from BTMs) are the most expensive among those path-dependent options; second, by constructing a suitable auxiliary function, we give a uniform estimate of bounds for the price functions of lookback options. On the basis of the estimates, we then make use of the notion of viscosity solutions and numerical analysis to prove the uniform convergence.

Throughout this paper we only consider continuously monitored path-dependent options. Actually, all results can be generalized to the case of discrete monitoring because the key proof of boundedness follows from the fact that all prices of the options with discrete monitoring are not greater than that of the corresponding continuously monitored lookback option.

The outline for this paper is as follows. In the next section we recall algorithms of BTMs for arithmetic average, geometric average, and lookback options, respectively. Section 3 is devoted to the consistency of BTMs and PDEs in each case. In section 4 we establish the relationship of BTMs and finite difference methods. In sections 5 and 6 we compare prices of the above three path-dependent options and present bounds of solutions of BTMs. We prove the convergence of BTMs in section 7.

2. Algorithms. As is common in the risk neutral world, the underlying asset price \( S \) is assumed to follow the lognormal diffusion process

\[
dS = rSdt + \sigma SdW,
\]

where \( dW \) is a Wiener process and \( r \) and \( \sigma \) represent the interest rate and volatility, respectively. Consider a path-dependent option with the lifetime \([0, T]\) and the payoff

\[
\Lambda(S, A) = \begin{cases} 
(S - A)^+ & \text{for floating strike call}, \\
(A - S)^+ & \text{for floating strike put}, \\
(A - X)^+ & \text{for fixed strike call}, \\
(X - A)^+ & \text{for fixed strike put}, 
\end{cases}
\]

where \( A \) is the path-dependent variable and \( X \) is the strike price.

If \( N \) is the number of discrete time points, we have time points \( t_n = n\Delta t, \ n = 0, 1, \ldots, N, \) with \( \Delta t = T/N. \) Let \( V^n(S, A) \) be the option price at time \( t_n \) with underlying asset value \( S \) and path-dependent variable \( A. \) Here we might as well assume

\[
A = \begin{cases} 
\frac{1}{n}\sum_{i=1}^{n} S_{t_i} & \text{for arithmetic average}, \\
(\Pi_{i=1}^{n} S_{t_i})^{1/n} & \text{for geometric average}, \\
\max_{0 \leq i \leq n} S_{t_i} & \text{for floating (fixed) strike lookback put (call) and } S \leq A, \\
\min_{0 \leq i \leq n} S_{t_i} & \text{for floating (fixed) strike lookback call (put) and } S \geq A.
\end{cases}
\]

\[
V^n(S, A) = \begin{cases} 
\frac{1}{n}\sum_{i=1}^{n} S_{t_i} & \text{for arithmetic average}, \\
(\Pi_{i=1}^{n} S_{t_i})^{1/(n+1)} & \text{for geometric average}, \\
\max(A, S_{t_i}) & \text{for floating (fixed) strike lookback put (call) and } S \leq A, \\
A & \text{for floating (fixed) strike lookback call (put) and } S \geq A.
\end{cases}
\]
and
\[ A^d = \begin{cases} 
\frac{nA + Sd}{n+1} & \text{for arithmetic average}, \\
(A^nSd)^{1/(n+1)} & \text{for geometric average}, \\
A & \text{for floating (fixed) strike lookback put (call) and } S \leq A, \\
\min(A, Sd) & \text{for floating (fixed) strike lookback call (put) and } S \geq A.
\end{cases} \]

(2.4)

By no-arbitrage argument, one has for European path-dependent options
\[ V^n(S, A) = e^{-r\Delta t}[pV^{n+1}(Su, A^n) + (1-p)V^{n+1}(Sd, A^d)], \]

where \( p = \frac{e^{r\Delta t} - d}{u - d} \). Setting \( ud = 1 \) and combining with stochastic differential equation (2.1), we get
\[ u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}} \]

and thus
\[ p = \frac{e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}. \]

At expiration time \( T = N\Delta t \), we have
\[ V^N(S, A) = \Lambda(S, A). \]

Using the backward induction (2.5)–(2.6), option prices can be calculated. This is the so-called binomial tree model.

For American path-dependent options, (2.5) is replaced by
\[ V^n(S, A) = \max \{ e^{-r\Delta t}[pV^{n+1}(Su, A^n) + (1-p)V^{n+1}(Sd, A^d)], \Lambda(S, A) \}. \]

3. Consistency. For the continuous model, the path-dependent variable is given as follows:
\[ A_t = \begin{cases} 
\frac{1}{t} \int_0^t S(\tau)d\tau & \text{for arithmetic average}, \\
\exp(\frac{1}{2} \int_0^t \ln S(\tau)d\tau) & \text{for geometric average}, \\
\max_{0 \leq \tau \leq t} S(\tau) & \text{for floating (fixed) strike lookback put (call)}, \\
\min_{0 \leq \tau \leq t} S(\tau) & \text{for floating (fixed) strike lookback call (put)}. 
\end{cases} \]

Let \( V(S, A, t) \) be the path-dependent option value. Note that \( S, A, \) and \( t \) are mutually independent from the view point of PDEs. The pricing model of European path-dependent options is (see Kwok [16] or Wilmott, Dewynne, and Howison [17])
\[ \frac{\partial V}{\partial t} + \mathcal{L}V = 0, \quad t \in (0, T), \quad (S, A) \in \mathcal{D}, \]

with the final value condition
\[ V(S, A, T) = \Lambda(S, A), \]

where
\[ \mathcal{L}V = \begin{cases} 
\frac{1}{r}(S - A)\frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV & \text{for arithmetic average}, \\
\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV & \text{for geometric average}, \\
\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV & \text{for lookback}
\end{cases} \]
and
\[ D = \begin{cases} 
(0, \infty) \times (0, \infty) \text{ for arithmetic or geometric average}, \\
\{(S, A) : 0 < S < A < \infty\} \text{ for floating (fixed) strike lookback put (call)}, \\
\{(S, A) : 0 < A < S < \infty\} \text{ for floating (fixed) strike lookback call (put)}. 
\end{cases} \]

In addition, for lookback options, one has an additional boundary condition
\[ \frac{\partial V}{\partial A}(S, S, t) = 0. \]  

**Remark 1.** Note that \( \mathcal{L}V \) is not well defined at \( t = 0 \) for Asian options. To remove the singularity, we can take the transformation
\[ I = \begin{cases} 
tA \text{ for arithmetic average}, \\
\ln A \text{ for geometric average}
\end{cases} \]

for arithmetic or geometric average,
\[ \{ (S, A) : 0 < S < A < \infty \} \text{ for floating (fixed) strike lookback put (call)}, \]
\[ \{ (S, A) : 0 < A < S < \infty \} \text{ for floating (fixed) strike lookback call (put)}. \]


\[ (3.4) \quad I = \begin{cases} 
tA \text{ for arithmetic average,} \\
\ln A \text{ for geometric average}
\end{cases} \]

to get
\[ (3.5) \quad \mathcal{L}V = \begin{cases} 
S \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \text{ for arithmetic average,} \\
\ln S \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \text{ for geometric average,}
\end{cases} \]

where \( I \in (0, \infty) \) for the arithmetic average and \( I \in (-\infty, \infty) \) for the geometric average.

**Remark 2.** One does not need to give boundary conditions at \( S = 0 \) that reduce to \( x = -\infty \) by the transformation \( x = \ln S \). Similarly, noting (3.5) and the directions of the characteristic lines, we do not impose boundary conditions at \( A = 0 \) (i.e., \( I = 0 \) or \( -\infty \)) for Asian options. We always assume that option values do not grow too fast at \( S = \infty \) and \( A = \infty \).

For American options, (3.1) is replaced by a variational inequality
\[ (3.6) \quad \min \left\{ \frac{\partial V}{\partial t} - \mathcal{L}V, V - \Lambda \right\} = 0, \quad t \in (0, T), \quad (S, A) \in D. \]

with the final condition (3.2) (and boundary condition (3.3) for lookback options).

**Remark 3.** For American Asian options, even if the transformation (3.4) is employed, one cannot remove the singularity of (3.6) at \( t = 0 \) because \( \Lambda = (S - A)^+ \) (floating strike call, for example) is, as of yet, not well defined at \( t = 0 \). The financial background gives \( S = A \) (i.e., \( I = 0 \)) at \( t = 0 \). However, \( S \) and \( A \) are mutually independent variables in (3.6) and the behavior of the solution at the point \( (S, S, 0) \) remains to be studied further. Throughout this paper we always confine ourselves to the interval \((0, T]\) instead of \([0, T]\), except for special claim options.

In what follows, we will show the consistency of binomial tree methods and PDEs.

**Theorem 3.1.** The binomial tree methods (2.5) (resp., (2.7)) are consistent with the corresponding PDE (3.1) (resp., (3.6)).

**Proof.** We only take the European type arithmetic average option as an example since it is similar for other cases. We need to show that for sufficiently smooth function \( \phi(S, A, t) \) and \( (S_0, A_0, t_0) \in D \times (0, T) \),
\[ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \phi - F_{\Delta t} \phi \right)(S, A, t) = \left. -\frac{\partial V}{\partial t} - \mathcal{L}V \right|_{(S_0, A_0, t_0)}, \]
where
\begin{equation}
F_{\Delta \phi}(S, A, t) = e^{-r\Delta t}[p\phi(Su, A, t) + (1-p)\phi(Sd, A^d, t)],
\end{equation}
\begin{equation}
A^u = \frac{(t-\Delta t)A + Su\Delta t}{t} \quad \text{and} \quad A^d = \frac{(t-\Delta t)A + Sd\Delta t}{t}.
\end{equation}

By Taylor expansions and the identities
\begin{align*}
e^{-r\Delta t}[p(u-1) + (1-p)(d-1)] &= r\Delta t + O(\Delta t^2), \\
e^{-r\Delta t}[p(u-1)^2 + (1-p)(d-1)^2] &= \sigma^2\Delta t + O(\Delta t^2), \\
e^{-r\Delta t}[p(u-1)^3 + (1-p)(d-1)^3] &= O(\Delta t^2),
\end{align*}
(3.7) reduces to
\begin{align*}
(\phi - F_{\Delta \phi})(S, A, t) &= -\left[\frac{\partial \phi}{\partial t}(S, A, t) + rS\frac{\partial \phi}{\partial S}(S, A, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2}(S, A, t) - r\phi(S, A, t)\right] \Delta t \\
&\quad - e^{-r\Delta t}[p(A^u - A) + (1-p)(A^d - A)] \frac{\partial \phi}{\partial A}(S, A, t) \\
&\quad - e^{-r\Delta t}[p(u-1)(A^u - A) + (1-p)(d-1)(A^d - A)] S \frac{\partial^2 \phi}{\partial S \partial A}(S, A, t) \\
&\quad + O(\Delta t^2) + O((A^u - A)\Delta t) + O((A^d - A)\Delta t) + O((A^u - A)^2) + O((A^d - A)^2).
\end{align*}
(3.8)

Noting that $A^u - A = \frac{Su - A}{t} \Delta t$ and $A^d - A = \frac{Sd - A}{t} \Delta t$, we have
\begin{align*}
e^{-r\Delta t}[p(A^u - A) + (1-p)(A^d - A)] &= \frac{S - A}{t} \Delta t + O(\Delta t^2), \\
e^{-r\Delta t}[p(u-1)(A^u - A) + (1-p)(d-1)(A^d - A)] &= O(\Delta t^2).
\end{align*}

Then we get
\begin{align*}
\frac{1}{\Delta t}(\phi - F_{\Delta \phi})(S, A, t) &= -\frac{\partial \phi}{\partial t} - \frac{1}{t}(S - A) \frac{\partial \phi}{\partial A} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2} - rS \frac{\partial \phi}{\partial S} + r\phi \bigg|_{(S, A, t)} \\
&\quad + O(\Delta t).
\end{align*}
(3.9)

The proof is complete.

**Remark 4.** For lookback options, the consistency of BTMs and the boundary condition (3.3) in the sense of the viscosity solution will be shown implicitly in the convergence proof of section 7.

4. Relationship between BTM and finite difference method. It has been pointed out by many authors that, for vanilla options, the BTM is equivalent to certain explicit difference schemes. In this section we establish the relationship between BTMs and finite difference methods for path-dependent options.

To illustrate the basic idea, we confine ourselves to European arithmetic average options. The governing equation is
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{t}(S - A) \frac{\partial V}{\partial A} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= 0.
\end{align*}
Consider the characteristic line of \( \frac{\partial V}{\partial t} + \frac{1}{2}(S - A)\frac{\partial V}{\partial A} = 0 \) in \([t_n, t_{n+1}]\)

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dt}{T} = \frac{dA}{4(S - A)}, \\
A(t_n) = A_n,
\end{array} \right. \quad t_n \leq t \leq t_{n+1},
\end{align*}
\]

whose solution is

\[A(t) = S - \frac{t_n}{t}(S - A_n).\]

The governing equation is thereby rewritten as

\[
\frac{dV}{dt} \left( S, S - \frac{t_n}{t}(S - A_n) \right) + \left( \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right)\bigg|_{A = S + \frac{t_n}{t}(S - A_n)} = 0,
\]

\[t_n \leq t \leq t_{n+1}.
\]

By adding the following three small terms to the above equation at \((S, S + \frac{t_n}{t}(S - A_n), t)\),

\[
\frac{\sigma^2}{2} \left[ \frac{t - t_n}{t} S \right]^2 \frac{\partial^2 V}{\partial A^2} + \sigma^2 S \frac{t - t_n}{t} S \frac{\partial^2 V}{\partial A \partial S} + \left( r - \frac{\sigma^2}{2} \right) S \frac{t - t_n}{t} \frac{\partial V}{\partial A} \quad (t_n \leq t \leq t_{n+1}),
\]

we have

\[
\begin{align*}
\frac{d}{dt} V(S, S - \frac{t_n}{t}(S - A_n), t) + \frac{\sigma^2}{2} S \frac{d}{dS} \left( S \frac{d}{dS} V(S, S - \frac{t_n}{t}(S - A_n), t) \right) & \\
+ \left( r - \frac{\sigma^2}{2} \right) S \frac{d}{dS} V(S, S - \frac{t_n}{t}(S - A_n), t) - rV(S, S - \frac{t_n}{t}(S - A_n), t) & = 0, \\
(t_n \leq t \leq t_{n+1}).
\end{align*}
\]

(4.1)

Noting that \( \frac{d}{dS} \) is a total differential operator, (4.1) can be regarded as a Black–Scholes equation in \([t_n, t_{n+1}]\). By taking the explicit difference scheme for (4.1), we can get

\[
V(S, A_n, t_n) = \frac{1}{1 + r\Delta t} \left[ aV(S_u, A_{n+1}^u, t_{n+1}) + (1 - a)V(S_d, A_{n+1}^d, t_{n+1}) \right],
\]

where

\[a = \frac{1}{2} + \frac{\sqrt{\Delta t}}{2\sigma} \left( r - \frac{\sigma^2}{2} \right).
\]

Since \( e^{r\Delta t} = 1 + r\Delta t + O(\Delta t^2) \) and

\[p = a + O(\Delta t^{3/2}),
\]

we conclude that by neglecting a high order of \( \Delta t \), BTM is equivalent to the above explicit difference scheme with method of characteristic line.

**Remark 5.** For geometric average options and lookback options, we have similar results.
5. Comparison of path-dependent options prices. In this section we will compare prices of arithmetic average options, geometric average options, and lookback options computed from the binomial tree approximation (2.6)–(2.7). To illustrate this method, we will consider the American floating strike put option and the fixed strike call option.

For Δt given and 0 ≤ n ≤ N = T/Δt, we can compute \( V^n(S, A) \) for all \((S, A) \in D\) by (2.6)–(2.7). In the following, \( V^n(S, A) \) is regarded as a function defined by (2.6)–(2.7) in \( D \). In addition, we always suppose

\[
0 < p < 1,
\]

which is a fact for sufficiently small \( Δt \). Under the assumption (5.1), BTMs are monotone schemes.

**Lemma 5.1.** Let \( V^n(S, A) \) be the function defined by (2.6)–(2.7) in \( D \) for the American floating strike put option (or fixed strike call option) with payoff \((A - S)^+\) (or \((A - X)^+\)). If \( A_1 \leq A_2 \), then

\[
V^n(S, A_1) \leq V^n(S, A_2)
\]

for all \( 0 \leq n \leq N \).

**Proof.** The proof is obvious.

**Lemma 5.2.** Let \( V^n_G(S, A) \), \( V^n_A(S, A) \), and \( V^n_L(S, A) \) be the functions defined by (2.6)–(2.7) in \( D \) for American floating strike geometric average, arithmetic average, and lookback put options (or corresponding fixed strike call options) with payoffs \((A - S)^+\) (or \((A - X)^+\)).

1. For all \( 0 \leq n \leq N \), we have

\[
V^n_G(S, A) \leq V^n_A(S, A) \leq V^n_L(S, \max(S, A)).
\]

2. Let \( A_g \), \( A_a \), and \( A_l \) be values of the path-dependent variable for an identical path. Then for all \( 0 \leq n \leq N \)

\[
V^n_G(S, A_g) \leq V^n_A(S, A_a) \leq V^n_L(S, \max(S, A_l)).
\]

**Proof.** We take floating strike put options for example.

1. Suppose (5.2) is true for \( n + 1 \):

\[
V^n_A(S, A) = \max\{e^{-rΔt}[pV^{n+1}_G(Su, A_u) + (1 - p)V^{n+1}_A(Sd, A_d^u)], (A - S)^+\} \geq \max\{e^{-rΔt}[pV^{n+1}_G(Su, A_u) + (1 - p)V^{n+1}_A(Sd, A_d^u)], (A - S)^+\}.
\]

Here

\[
A_u = \frac{nA + Su}{n + 1} \geq (A^n Su)^{1/(n+1)} = A_G^n,
\]

and similarly

\[
A_d^u \geq A_G^d;
\]

it follows from Lemma 5.1 that

\[
V^n_A(S, A) \geq \max\{e^{-rΔt}[pV^{n+1}_G(Su, A_G^u) + (1 - p)V^{n+1}_G(Sd, A_G^d)], (A - S)^+\} = V^{n+1}_G(S, A),
\]
which is the desired result. Combining with Lemma 5.1 and noticing that $V^n_L(S, A)$ is defined for $S \leq A$, the right inequality follows similarly.

(2) For an identical path, by the definition (2.2), one has

$$A_g \leq A_a \leq A_t,$$

which yields (5.3) due to (5.2) and Lemma 5.1.

**Remark 6.** Lemmas 5.1 and 5.2 remain valid for European path-dependent options. Similar results also hold for floating strike call and fixed strike put options.

6. **Boundedness.** In this section we will present bounds of solutions of BTMs, which is crucial in the proof of convergence.

**Lemma 6.1.** Let $V^n(S, A)$ be the function defined by (2.6)–(2.7) in $D$ for an American fixed strike put option with payoff $(X - A)^+$. Then

$$V^n(S, A) \leq X$$

for all $0 \leq n \leq N$.

**Proof.** By induction, the result is obvious.

**Lemma 6.2.** Let $V^n(S, A)$ be the function defined by (2.6)–(2.7) in $D$ for an American floating strike call option with payoff $(S - A)^+$. Then

$$V^n(S, A) \leq S$$

for all $0 \leq n \leq N$.

**Proof.** Let $V^n(S, A) = S\overline{V^n}(S, A)$ for all $0 \leq n \leq N$. It suffices to show that for all $0 \leq n \leq N$

(6.1) \hspace{1cm} \overline{V^n}(S, A) \leq 1.

Clearly for $n < N$,

$$\overline{V^n}(S, A) = \max \left\{ e^{-r\Delta t}[pu\overline{V^{n+1}}(Su, A^n) + (1 - p)d\overline{V^{n+1}}(Sd, A^d)], \left(1 - \frac{A}{S}\right)^+ \right\}.$$

Since $\overline{V^N}(S, A) = (1 - \frac{A}{S})^+ \leq 1$, one might as well assume that (6.1) holds for $n + 1$ and hence

$$\overline{V^n}(S, A) \leq \max \left\{ e^{-r\Delta t}[pu + (1 - p)d], \left(1 - \frac{A}{S}\right)^+ \right\}$$

$$= \max \left\{ 1, \left(1 - \frac{A}{S}\right)^+ \right\} \leq 1,$$

which arrives at the conclusion.

**Lemma 6.3.** Let $V^n_L(S, A)$ be the function defined by (2.6)–(2.7) in $D$ for an American floating strike lookback put option (or fixed strike call option) with payoff $(A - S)^+$ (or $(A - X)^+$). Let $\alpha > 0$ and $W^n(S, A)$ be the solution to the following problem:

$$\begin{cases}
W^n(S, A) = \max\{e^{-r\Delta t}[pW^{n+1}(Su, \max(Su, A)) + (1 - p)W^{n+1}(Sd, A)], e^{\alpha(N-n)\Delta t}A\}, & S \leq A, \\
W^N(S, A) = A.
\end{cases}$$

(6.2)
Then for all $0 \leq n \leq N$ and $S \leq A$

\begin{align}
(6.3) \quad V^n_L(S, A) \leq W^n(S, A). \nonumber
\end{align}

Proof. We take floating strike put options for example. Since $W^N(S, A) = A \geq V^n_L(S, A)$, we may suppose (6.3) holds for $n + 1$. Because $\alpha > 0$,

\begin{align}
W^n(S, A) &= \max\{e^{-r\Delta t}[pW^{n+1}(Su, \max(Su, A))+(1-p)W^{n+1}(Sd, A)], e^{\alpha(N-n)\Delta t}A\} \\
&\geq \max\{e^{-r\Delta t}[pV^{n+1}_L(Su, \max(Su, A))+(1-p)V^{n+1}_L(Sd, A)], (A-S)^+\} \\
&= V^{n+1}_L(S, A). \nonumber
\end{align}

The proof is complete.

7. Convergence. In this section, we will employ the notion of viscosity solutions to show the convergence of binomial tree method. Let us first recall the notion of viscosity solutions. For convenience, we use the following notations:

\begin{align}
H(V, S, A, t) &= \left\{ \begin{array}{ll}
-\frac{\partial V}{\partial t} - LV & \text{for European options,} \\
\min\{ -\frac{\partial V}{\partial t} - LV, V - \Lambda & \text{for American options,}
\end{array} \right. \\
(7.1) \quad B(V, S, A, t) &= \left\{ \begin{array}{ll}
-\frac{\partial V}{\partial A} & \text{for floating (fixed) strike lookback put (call),}
\end{array} \right. \\
&\text{for floating (fixed) strike lookback call (put),}
\nonumber
\end{align}

and

\[ \overline{D} = D \cup \partial D, \quad \partial D = \left\{ (S, A) : 0 < S = A < \infty \right\} \text{for lookback options.} \]

Remark 7. In (7.1), the sign before $\frac{\partial V}{\partial A}$ is determined by the outward unit normal to $\partial D \times (0, T)$ (see [8]).

Definition 7.1. A function $V \in USC(\overline{D} \times (0, T))$ (resp., $LSC(\overline{D} \times (0, T))$) is a viscosity subsolution (resp., supersolution) of the problem (3.6), (3.2) (and (3.3) for lookback options) if $V(S, A, T) \leq \Lambda(x)$ (resp., $V(S, A, T) \geq \Lambda(x)$), and whenever $\phi \in C^{2,1}(\overline{D} \times (0, T))$, $V - \phi$ attains its local maximum (resp., local minimum) at $(S_0, A_0, t_0) \in \overline{D} \times (0, T)$ and $(V - \phi)(S_0, A_0, t_0) = 0$, we have

\[ H(\phi, S_0, A_0, t_0) \leq 0 \text{ for } (S_0, A_0, t_0) \in \overline{D} \times (0, T) \]

(resp.,

\[ H(\phi, S_0, A_0, t_0) \geq 0 \text{ for } (S_0, A_0, t_0) \in \overline{D} \times (0, T), \]

and (only for lookback option)

\[ \min\{ H(\phi, S_0, A_0, t_0), B(\phi, S_0, A_0, t_0)\} \leq 0 \text{ for } (S_0, A_0, t_0) \in \partial D \times (0, T) \]

(resp.,

\[ \max\{ H(\phi, S_0, A_0, t_0), B(\phi, S_0, A_0, t_0)\} \geq 0 \text{ for } (S_0, A_0, t_0) \in \partial D \times (0, T). \]

We call $V \in C(\overline{D} \times (0, T))$ a viscosity solution of (3.6), (3.2) (and (3.3) for lookback options) if it is both a viscosity subsolution and a supersolution.
The convergence proof needs the strong comparison principle that holds for Asian options (see Remark 1 and [8], [9], and [11] and references therein). For lookback options where the oblique derivative boundary condition is involved, Barles, Daher, and Romano have shown that the strong comparison principle still remains valid (see [2] and [3]). Then we get the following.

**Lemma 7.2.** The strong comparison principle holds for problem (3.6), (3.2) (and (3.3) for lookback options); namely, if \( u \) and \( v \) are the viscosity subsolution and supersolution of the problem, respectively, then \( u \leq v \).

Let \( V_n(S, A) \) be the function defined by (2.6)–(2.7) in \( D \) for American path-dependent option. We now define the extension function \( V_{\Delta t}(S, A, t) \) as follows: for \( t \in [n\Delta t, (n+1)\Delta t] \), \( n = 0, 1, \ldots, N-1 \),

\[
V_{\Delta t}(S, A, t) = \frac{(n+1)\Delta t - t}{\Delta t} V^n(S, A) + \frac{t - n\Delta t}{\Delta t} V^{n+1}(S, A).
\]

**Theorem 7.3.** Suppose that \( V(S, A, t) \) is the viscosity solution to the problem (3.6), (3.2) (and (3.3) for lookback options). Then, as \( \Delta t \to 0 \), we have \( V_{\Delta t}(S, A, t) \) converges uniformly to \( V(S, A, t) \) in any bounded closed subdomain of \( D \times (0, T) \).

In order to prove this theorem, we have to show \( V^*(S, A, t) \) and \( V^<(S, A, t) \) are well defined at first, where

\[
V^*(S, A, t) = \limsup_{\Delta t \to 0, (x, y, z) \to (S, A, t)} V_{\Delta t}(x, y, z),
\]

\[
V^<(S, A, t) = \liminf_{\Delta t \to 0, (x, y, z) \to (S, A, t)} V_{\Delta t}(x, y, z).
\]

In fact, due to Lemmas 6.1 and 6.2, it is true for fixed strike put options and floating strike call options. As for fixed strike call options and floating strike put options, by Lemma 6.3, it suffices to show the following.

**Lemma 7.4.** Let \( W_n(S, A) \) be the solution to (6.2) with \( \alpha > 0 \). Then we have

\[
W^n(S, A) \leq e^{\alpha T} \left( \max \left( A, \left( \frac{\lambda_-(\lambda_+ - 1)}{\lambda_+ - 1} \right)^{1/(\lambda_+ - \lambda_-)} S \right) + 1 \right)
\]

for sufficiently small \( \Delta t \), where

\[
\lambda_\pm = \frac{r}{\sigma^2} + \frac{1}{2} \pm \sqrt{\left( \frac{r}{\sigma^2} + \frac{1}{2} \right)^2 + \frac{2\alpha}{\sigma^2}}.
\]

**Remark 8.** In Lemma 7.4, \( \alpha > 0 \) guarantees \( \frac{\lambda_-(\lambda_+ - 1)}{\lambda_+ - 1} )^{1/(\lambda_+ - \lambda_-)} < \infty \).

Before proving Lemma 7.4 we inquire into some properties of the solution to the problem (6.2). By transformations

\[
x = \ln \frac{A}{S} \quad \text{and} \quad W^n(x) = e^{-\alpha(N-n)\Delta t} \frac{W^n(S, A)}{S},
\]

the numerical scheme (6.2) is reduced to

\[
\begin{cases}
W^n(x) = \max \{ e^{-(r+\alpha)\Delta t} [puW^{n+1}((x - \sigma \sqrt{\Delta t})^+) + (1-p)dW^{n+1}(x + \sigma \sqrt{\Delta t})], e^x \}, \\
x \geq 0,
\end{cases}
\]

\[
W^N(x) = e^x.
\]
Lemma 7.5. Let $\bar{W}_n(x)$ be the solution to (7.5). Then we have
(a) $\bar{W}_{n+1}(x) \leq \bar{W}_n(x)$,
(b) $\bar{W}_n(x_1) \leq \bar{W}_n(x_2)$ if $x_1 \leq x_2$,
(c) for each $n \leq N$,

(7.6) $\bar{W}_n(x) = e^x$ if $x \geq (N - n)\sigma \sqrt{\Delta t}$.

Proof. (a) and (b) are obvious. In order to prove (c), we use induction. Suppose (7.6) holds for $n = k + 1$, namely $\bar{W}^{k+1}(x) = e^x$ for $x \geq (N - k - 1)\sigma \sqrt{\Delta t}$. If $x \geq (N - k)\sigma \sqrt{\Delta t}$, then

$$
\bar{W}^k(x) = \max\{e^{-(r + \alpha)\Delta t}[p(x - \sigma \sqrt{\Delta t}) + (1 - p)d(x + \sigma \sqrt{\Delta t})], e^x\}
$$

$$
= \max\{e^{-(r + \alpha)\Delta t}[pue^{-\sigma \sqrt{\Delta t}}(1 - p)de^{-\sigma \sqrt{\Delta t}}], e^x\}
$$

$$
= \max\{e^{-(r + \alpha)\Delta t}e^x, e^x\} = e^x,
$$

which is the desired result.

To simplify notation, (7.5) will also be written as

(7.7) $\bar{W}^n(x) = F(\Delta t)\bar{W}^{n+1}(x)$.

Lemma 7.6. For $\Delta t$ given, there exists unique element $\bar{W}_{\Delta t}(x)$ satisfying $\bar{W}_{\Delta t}(x) = e^x \in \mathcal{L}_\infty(R^+)$ such that

(7.8) $\bar{W}_{\Delta t}(x) = F(\Delta t)\bar{W}_{\Delta t}(x)$.

In addition, $\bar{W}_{\Delta t}(x)$ is a monotone function of $x$ and

(7.9) $\bar{W}^n(x) \leq \bar{W}_{\Delta t}(x)$.

Proof. Let $\tilde{W}^n(x) = \bar{W}^n(x) - e^x$. Then $\tilde{W}^n(x)$ satisfies

$$
\tilde{W}^n(x) = F(\Delta t)(\tilde{W}^{n+1}(x) + e^x) - e^x = G(\Delta t)\tilde{W}^{n+1}(x).
$$

By (7.6), $\tilde{W}^n(x) \in \mathcal{L}_\infty(R^+)$. Hence $G(\Delta t)$ can be regarded as a mapping from $\mathcal{L}_\infty(R^+)$ to $\mathcal{L}_\infty(R^+)$. Next we will show $G(\Delta t)$ is a contraction mapping. Let $U(x)$, $V(x) \in \mathcal{L}_\infty(R^+)$. Then

$$
\|G(\Delta t)U(x) - G(\Delta t)V(x)\|_\infty
$$

$$
= \|F(\Delta t)(U(x) + e^x) - F(\Delta t)(V(x) + e^x)\|_\infty
$$

$$
\leq e^{-(r + \alpha)\Delta t}[p + (1 - p)d]\|U(x) - V(x)\|_\infty
$$

$$
= e^{-\alpha\Delta t}\|U(x) - V(x)\|_\infty.
$$

Therefore, there exists a unique element $\tilde{W}_{\Delta t}(x) \in \mathcal{L}_\infty(R^+)$ such that $\tilde{W}_{\Delta t}(x) = G(\Delta t)\tilde{W}_{\Delta t}(x)$. Owing to Lemma 7.5, $\tilde{W}_{\Delta t}(x)$ is a monotone function of $x$ and

$$
\tilde{W}^n(x) \leq \tilde{W}_{\Delta t}(x).
$$

This completes the proof by denoting $\bar{W}_{\Delta t}(x) = \tilde{W}_{\Delta t}(x) + e^x$. 
Proof of Lemma 7.4. The idea of the proof stems from Dai [10]. Let $\Delta x = \sigma \sqrt{\Delta t}$, $x_j = j \Delta x$, and $u^j = e^{x_j}$, $j = 0, 1, \ldots$. It is not hard to see that $\mathcal{W}_{\Delta t}(x_j)$ satisfies

$$
\begin{align*}
\mathcal{W}_{\Delta t}(x_j) &= \max \left\{ e^{-(r+\alpha)\Delta t} \left[ p\mathcal{W}_{\Delta t}(x_{j-1}) + (1-p)d\mathcal{W}_{\Delta t}(x_{j+1}) \right], u^j \right\}, \\
\mathcal{W}_{\Delta t}(x_0) &= e^{-(r+\alpha)\Delta t} \left[ p\mathcal{W}_{\Delta t}(x_0) + (1-p)d\mathcal{W}_{\Delta t}(x_1) \right],
\end{align*}
$$

which is equivalent to a free boundary problem of a difference equation as follows:

$$
\begin{align*}
\mathcal{W}_{\Delta t}(x_j) &= e^{-(r+\alpha)\Delta t} \left[ p\mathcal{W}_{\Delta t}(x_{j-1}) + (1-p)d\mathcal{W}_{\Delta t}(x_{j+1}) \right] \text{ for } 1 \leq j < j_{\infty}, \\
\mathcal{W}_{\Delta t}(x_0) &= e^{-(r+\alpha)\Delta t} \left[ p\mathcal{W}_{\Delta t}(x_0) + (1-p)d\mathcal{W}_{\Delta t}(x_1) \right], \\
\mathcal{W}_{\Delta t}(x_{j_{\infty}}) &= u^{j_{\infty}}, \\
\mathcal{W}_{\Delta t}(x_{j_{\infty}+1}) &= u^{j_{\infty}+1}.
\end{align*}
$$

Here $j_{\infty}$ is the point of free boundary to be determined. We claim

$$
\mathcal{W}_{\Delta t}(x_j) = C_1 \xi_1^j + C_2 \xi_2^j \text{ for } 0 \leq j \leq j_{\infty} + 1,
$$

where $\xi_1, \xi_2$ are two real roots of the equation $\xi = e^{-(r+\alpha)\Delta t} (pu + (1-p)d\xi^2)$, namely,

$$
\xi_{1,2} = \frac{e^{(r+\alpha)\Delta t} \pm \sqrt{e^{2(r+\alpha)\Delta t} - 4p(1-p)}}{2(1-p)d}.
$$

To determine constants $C_1, C_2$, and $j_{\infty}$, we make use of boundary condition (7.10) and free boundary condition (7.11); we have

$$
\begin{align*}
C_1 &= \frac{(e^{(r+\alpha)\Delta t} - pu) - (1-p)d\xi_2}{(1-p)d\xi_1 - (e^{(r+\alpha)\Delta t} - pu)}, \\
\mathcal{W}_{\Delta t}(x_{j_{\infty}}) &= C_1 \xi_1^{j_{\infty}} + C_2 \xi_2^{j_{\infty}} = u^{j_{\infty}}, \\
\mathcal{W}_{\Delta t}(x_{j_{\infty}+1}) &= C_1 \xi_1^{j_{\infty}+1} + C_2 \xi_2^{j_{\infty}+1} = u^{j_{\infty}+1}.
\end{align*}
$$

By solving (7.14)–(7.16), we get

$$
\begin{align*}
C_1 &= \frac{\xi_2 u^{j_{\infty}} - u^{j_{\infty}+1}}{\xi_1^{j_{\infty}} (\xi_2 - \xi_1)}, \quad C_2 = \frac{\xi_1 u^{j_{\infty}} - u^{j_{\infty}+1}}{\xi_2^{j_{\infty}} (\xi_1 - \xi_2)}, \\
\text{and}
\end{align*}
$$

$$
\begin{align*}
\hat{j}_{\infty} &= \frac{1}{\ln \xi_2 - \ln \xi_1} \ln \left( \frac{\xi_1 u^{j_{\infty}} - u^{j_{\infty}+1} - (e^{(r+\alpha)\Delta t} - pu) \xi_2 - u}{(1-p)d\xi_1 - (e^{(r+\alpha)\Delta t} - pu) \xi_2 - u} \right).
\end{align*}
$$

Noticing that $\mathcal{W}_{\Delta t}(x)$ is monotone with respect to $x$ and combining with (7.12), we have

$$
\mathcal{W}_{\Delta t}(x) \leq \max(e^{x + \Delta x}, e^{\hat{j}_{\infty} \Delta x}).
$$

By symbol operation, one gets

$$
\lim_{\Delta t \to 0} \hat{j}_{\infty} \Delta x = \frac{1}{\lambda_- - \lambda_+} \ln \frac{\lambda_- (\lambda_- - 1)}{\lambda_+ (\lambda_- - 1)} < \infty,
$$

where $\lambda_{\pm}$ are given by (7.3). Then for sufficiently small $\Delta t$,

$$
\mathcal{W}_{\Delta t}(x) \leq \max \left( e^{x}, \left( \frac{\lambda_- (\lambda_- - 1)}{\lambda_+ (\lambda_- - 1)} \right)^{1/(\lambda_- - \lambda_+)} \right) + 1.
$$
Lemma 7.4. Together with (7.4) and (7.9), this implies (7.2), which completes the proof of Lemma 7.4.

We now prove Theorem 7.3. The idea is based on [4] and [14].

Proof of Theorem 7.3. Since $V^*$ and $V_*$ are well defined, it is obvious that $V^* \in USC$ and $V_* \in LSC$, and $V_*(S, A, t) \leq V^*(S, A, t)$. If we show that $V^*$ and $V_*$ are the viscosity subsolution and supersolution of (3.6), respectively, then in terms of the comparison principle (Lemma 7.2), we deduce $V^*(S, A, t) \leq V_*(S, A, t)$ and thus $V^*(S, A, t) = V_*(S, A, t) = V(S, A, t)$, which is the desired conclusion.

We need only to show that $V^*$ is a subsolution of (3.6), (3.2) (and (3.3) for lookback options). It can be shown that

\[
0. \quad \text{By the definition of } \hat{\phi},
\]

Indeed, suppose $(S_{k_i}, A_{k_i}, \hat{t}_{k_i})$ is a global maximum point of $V_{\Delta t_{k_i}} - \phi$ on $B_r$, we can deduce that there is a subsequence $V_{\Delta t_{k_i}}(S_{k_i}, A_{k_i}, \hat{t}_{k_i})$ such that

\[
\Delta t_{k_i} \to 0, \quad (S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \to (S_0, A_0, t_0),
\]

\[
(V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \to (V^* - \phi)(S_0, A_0, t_0)
\]

as $k_i \to \infty$.

Indeed, suppose $(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \to (S, \hat{A}, \hat{t})$; then

\[
(V^* - \phi)(S_0, A_0, t_0) = \lim_{k_i \to \infty} (V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, t_{k_i})
\]

\[
\leq \lim_{k_i \to \infty} (V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \leq (V^* - \phi)(S, \hat{A}, \hat{t}),
\]

which forces $(S, \hat{A}, \hat{t}) = (S_0, A_0, t_0)$ since $(S_0, A_0, t_0)$ is a local strict maximum point of $V^* - \phi$. Therefore

\[
(V_{\Delta t_{k_i}} - \phi)(\cdot, \cdot, \hat{t}_{k_i} + \Delta t_{k_i}) \leq (V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \quad \text{in } B_r;
\]

that is,

\[
V_{\Delta t_{k_i}}(\cdot, \cdot, \hat{t}_{k_i} + \Delta t_{k_i}) \leq \phi(\cdot, \cdot, \hat{t}_{k_i} + \Delta t_{k_i}) + (V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) \quad \text{in } B_r.
\]

Then

\[
V_{\Delta t_{k_i}}(S_{k_i}, A_{k_i}, \hat{t}_{k_i})
\]

\[
= \max\{F_{\Delta t_{k_i}} V_{\Delta t_{k_i}}(S_{k_i}, A_{k_i}, \hat{t}_{k_i}), \Lambda(S_{k_i}, A_{k_i})\}
\]

\[
\leq \max\{F_{\Delta t_{k_i}} \phi(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) + e^{-r \Delta t_{k_i}} (V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}), \Lambda(S_{k_i}, A_{k_i})\},
\]

namely,

\[
\min\{\phi - F_{\Delta t_{k_i}} \phi(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) + (1 - e^{-r \Delta t_{k_i}})(V_{\Delta t_{k_i}} - \phi)(S_{k_i}, A_{k_i}, \hat{t}_{k_i}),
\]

\[
V(S_{k_i}, A_{k_i}, \hat{t}_{k_i}) - \Lambda(S_{k_i}, A_{k_i}) \leq 0.
\]

Here the operator $F_{\Delta t_{k_i}}$ is given by (3.7). Dividing the first argument in the min by
\( \Delta t e^{-r\Delta t_{k_i}} \), letting \( k_i \to \infty \), and noticing that
\[
\frac{1 - e^{-r\Delta t_{k_i}}}{\Delta t e^{-r\Delta t_{k_i}}} (V_{\Delta t_{k_i}} - \phi)(\hat{S}_{k_i}, \hat{A}_{k_i}, \hat{t}_{k_i}) \to (V^* - \phi)(S_0, A_0, t_0) = 0,
\]
we get by consistency and (7.19) that
\[
\min \left\{ -\frac{\partial \phi}{\partial t} - \mathcal{L} \phi, V^* - \Lambda \right\}_{(S_0, A_0, t_0)} \leq 0,
\]
which yields the desired result because of \( V^*(S_0, A_0, t_0) = \phi(S_0, A_0, t_0) \).

For lookback options (fixed strike call, for example), if \((S_0, A_0, t_0) \in \partial \mathcal{D} \times (0, T) \) and (7.19) holds, we might as well assume either \((\hat{S}_{k_i}, \hat{A}_{k_i}) \in \mathcal{D} \) for all \( k_i \) or \((\hat{S}_{k_i}, \hat{A}_{k_i}) \in \partial \mathcal{D} \) for all \( k_i \). If it is the former, we can use the same argument as before to get \( \min \left\{ -\frac{\partial \phi}{\partial t} - \mathcal{L} \phi, \phi - \Lambda \right\}_{(S_0, A_0, t_0)} \leq 0 \). If \((\hat{S}_{k_i}, \hat{A}_{k_i}) \in \partial \mathcal{D} \), i.e., \( \hat{S}_{k_i} = \hat{A}_{k_i} \), then
\[
(7.22) \quad \hat{A}^u_{k_i} = \hat{S}_{k_i} u \text{ and } \hat{A}^d_{k_i} = \hat{S}_{k_i}.
\]

Using (7.22) and (3.8), we get
\[
(\phi - F_{\Delta t_{k_i}})(\hat{S}_{k_i}, \hat{A}_{k_i}, \hat{t}_{k_i}) = \left( -\frac{\partial \phi}{\partial t} - \mathcal{L} \phi \right) \Delta t_{k_i} - \frac{\sigma \hat{S}_{k_i}}{2} \frac{\partial \phi}{\partial A} \Delta t_{k_i}^{1/2} + O(\Delta t_{k_i}^{1/2}).
\]
Combining with (7.20), which can also be similarly obtained in this case, we deduce
\[
\min \left\{ -\frac{\sigma \hat{S}_{k_i}}{2} \frac{\partial \phi}{\partial A} \Delta t_{k_i}^{1/2} + O(\Delta t_{k_i}) + (1 - e^{-r\Delta t_{k_i}})(V_{\Delta t_{k_i}} - \phi)(\hat{S}_{k_i}, \hat{A}_{k_i}, \hat{t}_{k_i}),
\]
\[
V(\hat{S}_{k_i}, \hat{A}_{k_i}, \hat{t}_{k_i}) - \Lambda(\hat{S}_{k_i}, \hat{A}_{k_i}) \right\} \leq 0.
\]
Dividing the first argument in the min by \( \frac{\sigma \hat{S}_{k_i}}{2} \Delta t_{k_i}^{1/2} e^{-r\Delta t_{k_i}} \), letting \( k_i \to \infty \), and noticing (7.21), we then get by (7.19)
\[
\min \left\{ -\frac{\partial \phi}{\partial A}, \phi - \Lambda \right\}_{(S_0, A_0, t_0)} \leq 0.
\]
Hence, in either case, we have
\[
\min \left\{ \min \left\{ -\frac{\partial \phi}{\partial t} - \mathcal{L} \phi, \phi - \Lambda \right\}, -\frac{\partial \phi}{\partial A} \right\}_{(S_0, A_0, t_0)} \leq 0.
\]
The proof is complete.

Theorem 7.3 indicates that BTMs for American path-dependent options are locally uniformly convergent. It is clear that \( V^* \) and \( V \) are well defined for European path-dependent options because the prices of European options computing by BTMs are always less than those of the corresponding American options. Similar arguments
also give the convergence of BTMs for European path-dependent options. As a result, we assert the following.

**Theorem 7.7.** Binomial tree methods for European/American path-dependent options are uniformly convergent in any bounded closed domain of $D \times (0, T)$.

**Remark 9.** Clearly the convergence proof of BTMs remains valid at $t = 0$ for lookback options. By virtue of Remark 1, the convergence result at $t = 0$ for European Asian options is not too difficult an extension. However, for American Asian options, the convergence at $t = 0$ is currently not available because we cannot prove the strong comparison principle in $D \times [0, T)$ (see also Remark 3).

Due to Lemma 5.2 and Theorems 7.3 and 7.7, we have the following.

**Corollary 7.8.** Let $V_G(S, A, t)$, $V_A(S, A, t)$, and $V_L(S, A, t)$ be the solutions of the continuous models for floating strike geometric average, arithmetic average, and lookback put options (or corresponding fixed strike call options); then we have

$$V_G(S, A, t) \leq V_A(S, A, t) \leq V_L(S, \max(S, A), t).$$

We conclude the paper with the following remark.

**Remark 10.** It is well known that the BTM is not feasible for pricing arithmetic average options because the number of possible arithmetic average values increases exponentially with the number of timesteps. Barraquand and Pudet [5] and Hull and White [13] present modified BTMs that restrict the possible average values to a set of predetermined values. Our technique can also be applied to prove the convergence of their methods. We refer interested readers to [15] for details.

**REFERENCES**


