Characterization of Optimal Strategy for Multi-Asset Investment and Consumption with Transaction Costs

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Abstract. We consider the optimal consumption and investment with transaction costs on multiple assets, where the prices of risky assets jointly follow a multi-dimensional geometric Brownian motion. We characterize the optimal investment strategy and in particular prove by rigorous mathematical analysis that the trading region has the shape that is very much needed for well defining the trading strategy, e.g., the no-trading region has distinct corners. In contrast, the existing literature is restricted to either single risky asset or multiple uncorrelated risky assets.

Key words. Portfolio selection, Optimal investment and consumption, Transaction costs, Multiple risky assets, Shape and location of no-trading regions.

AMS subject classifications. 91G10, 93E20

1. Introduction. We consider the optimal investment and consumption decision of a risk-averse investor who has access to multiple risky assets as well as a riskfree asset. Proportional transaction costs are incurred when the investor buys or sells the risky assets whose prices are assumed to follow a multi-dimensional geometric Brownian motion. We aim to provide a theoretical characterization of the optimal strategy.

In the absence of transaction costs, the problem described above has been studied by Merton (1969, 1971). It turns out that the optimal strategy of a constant relative risk aversion (CRRA) investor is to keep a constant fraction of total wealth in each assets and consume at a constant fraction of total wealth. In contrast, the optimal strategy of a constant absolute risk aversion (CARA) investor is to keep a certain fixed amount in each risky asset and a consumption that is affine in the total wealth. Merton’s strategy requires continuous trading in all risky assets and thus must be suboptimal when transaction costs are incurred.

Magill and Constantinides (1976) introduce proportional transaction costs to Merton’s model with single risky asset and a CRRA investor. They provide a fundamental insight that there exists an interval, known as the no-trading region, such that the optimal investment strategy is to keep the fraction of wealth invested in the risky asset within the interval (i.e., no-trading region). Hence, as long as the initial fraction falls within the no-trading region, the future transactions only occurs at the boundary of the region. For a CARA investor, it can be shown that the optimal investment strategy is to keep the dollar amount in the risky asset between two levels [cf. Liu (2004) and Chen et al. (2012)].

Since the seminal work of Magill and Constantinides (1976), portfolio selection with trans-

Most of existing theoretical characterizations of optimal strategy are for the single risky-asset case. In contrast, there is relatively limited literature on the multiple risky-asset case. Assuming that there are multiple uncorrelated risky assets available for investment, Akian et al. (1996) obtain some qualitative results on the optimal strategy of a CRRA investor. Liu (2004) considers a CARA investor who is also restricted to invest in uncorrelated risky assets. He shows that the problem can be reduced, by virtue of the separability of the CARA utility function, to the single risky-asset case. This leads to the separability of the optimal investment strategy which is to keep the dollar amount invested in each asset between two constant levels. Unfortunately, such a reduction does not work when the risky assets are correlated.

The main contribution of this paper is to provide a thorough characterization of the optimal investment strategy for a risk-averse investor who can access multiple correlated risky assets as well as a riskfree asset. We focus on the CARA utility case, and an extension to the CRRA utility case is explained later. To illustrate our results, we take as an example the scenario of two risky assets. We will show that the shape of trading and no-trading regions must be as in Figure 1, where “S1”, “B1”, and “N1” represent selling, buying, and no trading in asset i, respectively. The no-trading region N1 ∩ N2 locates in the center, surrounded by eight trading regions. Moreover, each intersection ∂S1 ∩ ∂S2, ∂S1 ∩ ∂B2, ∂B1 ∩ ∂S2, and ∂B1 ∩ ∂B2 is a singleton. In addition, we show that the boundary of each of corner regions S1 ∩ S2, S1 ∩ B2, B1 ∩ B2, and B1 ∩ S2 consists of one vertical and one horizontal half line, whereas the boundary of each of S1 ∩ N2, N1 ∩ S2, B1 ∩ N2, and N1 ∩ B2 consists of two parallel either vertical or horizontal half lines and a curve in between connecting the end points of the two half lines. These characterizations on the shapes of trading regions are extremely

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1There do exist many papers working on perturbation analysis or numerical solutions for the multiple risky-asset case, e.g. Law et al. (2009), Bichuch and Shreve (2011), Muthuraman and Kumar (2006), Dai and Zhong (2010).
important because they are necessary conditions for the trading strategy to be well-defined.\footnote{It should be pointed out that these results have been conjectured or numerically verified by some researchers [e.g. Liu (2004), Dai and Zhong (2010), Bichuch and Shreve (2011)]. However no theoretical analysis has been given so far.} For example, given an initial portfolio in $S_1 \cap B_2$, the investor should sell asset 1 and buy asset 2 to the unique corner $\partial S_1 \cap \partial B_2$; similarly, given an initial portfolio in $S_1 \cap N_2$, the investor should sell asset 1 and keep asset 2 unchanged to the unique portfolio on $\partial S_1 \cap N_2$.

Recently, Soner and Touzi (2012) and Possamaï et al. (2013) use homogenization techniques to prove rigorous asymptotic expansion for optimal investment and consumption in a multidimensional market with proportional transaction costs in a very general setting. In particular, the model in the papers permits transaction between risky assets (unlike in the present paper) and the shape of the resulting trading/no-trading regions is different from that as described above. It should be pointed out that our method and results cannot be extended to models with such a feature; see Remark 3.1.

We also prove that the no-trading region is contained in a union of uniformly bounded ellipses. Thus in numerical simulation one need only perform computation on the bounded union. Furthermore, we provide a precise characterization of the corners of the no-trading region.

We present the model and collect main theoretical results in Section 2. In section 3, we explain an extension of our results to the CRRA utility. Section 4 is devoted to the proof of our main results regarding shape and location of non-trading zone for the “infinite horizon” problem (Theorems 2.4 and 2.5), with the proof of a technical $C^1$ regularity of viscosity solution (Theorem 2.3) being left to Section 5.

Since negative wealth is permitted with the CARA utility, we need to impose some constraint to prevent the investor from unlimited consumptions. This motivates us to start from a finite horizon problem formulation, where the expected utility is from not only the intermediate consumption but also the terminal wealth (i.e., bequest). We provide some basic

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{shape.png}
\caption{Shape of trading and no-trading regions with CARA utility}
\end{figure}
properties of the value function associated with the finite horizon problem; in particular, we
let the finite horizon $T \to \infty$ to obtain the “infinite horizon” problem addressed in this paper.
These results are presented as preliminaries in Section 2 and their proofs are placed in the
Appendix.

2. Problem Formulation and Main Results. Consider a portfolio consisting of one risk-
free asset (bank account) and $n$ risky assets whose unit share prices are stochastic process
$\{(S^0_t, S^1_t, \cdots, S^n_t)\}$ described by the stochastic differential equations

\[
\frac{dS^0_t}{S^0_t} = r dt, \quad \frac{dS^i_t}{S^i_t} = \alpha_i dt + \sum_{j=1}^n a_{ij} dW^j_t \quad \text{for } i = 1, \cdots, n,
\]

where $\{W^1_t, \cdots, W^n_t\}_{t \geq 0}$ is a standard $n$-dimensional Wiener process, $r > 0$ is the constant
bank rate, $\alpha_i$ is the constant expected return rates of the $i$-th risky asset, and $(a_{ij})_{n \times n}$
is a constant positive definite matrix. We consider optimal strategies of investment and
consumption subject to transaction cost which are proportional to the amount of transactions.

2.1. Investment and Consumption. We introduce a non-negative parameter $\kappa$ where
$\kappa = 0$ corresponds to the no-consumption case. Suppose the terminal time is $T$ and current
time is $t < T$. For $s \in [t, T)$, we denote by $\kappa c_s ds$ the consumption, deducted from the bank
account, during time interval $[s, s + ds)$. Here we assume that $\kappa$ has the same unit as $r$, being
1/year, and that $c_s$ has the unit of dollars.\footnote{The parameters $\kappa$ and $K$ in (2.3) below are both used as the weight between consumption and terminal
wealth. However $K$ is dimensionless while $\kappa$ has the same unit as $r$. It should be pointed out that the condition
$r < 1$ in Chen et al. (2012) indeed means $r < \kappa$.} We denote by $dL^i_s$ the transfer of money from
the bank account to the $i$-th risky assets during $[s, s + ds)$, which incurs purchasing costs $\lambda_i dL^i_s$.
Similarly, we denote by $dM^i_s$ the money transferred from the $i$-th risky asset to the bank
account during $[s, s + ds)$, which incurs selling costs $\mu_i dM^i_s$. Here $\lambda_i \geq 0$ and $\mu_i \in [0, 1)$ are
the constant proportions of transaction costs for purchasing and selling the $i$-th risky asset, respectively.

Let $x_s$ and $y_s = (y^1_s, \cdots, y^n_s)$ be dollar values at time $s \in [t, T]$ invested in the bank
account and risky assets, respectively. Their evolutions are described by

\[
\begin{align*}
\frac{dx_s}{x_s} &= (r x_s - \kappa c_s) ds - \sum_i (1 + \lambda_i) dL^i_s + \sum_i (1 - \mu_i) dM^i_s, \\
\frac{dy^i_s}{y^i_s} &= y^i_s (\alpha_i ds + \sum_j a_{ij} dW^j_s) + dL^i_s - dM^i_s, \quad i = 1, \cdots, n.
\end{align*}
\]

For simplicity, we define an admissible (investment-consumption) strategy as $S = (C, L, M)$
where $C = \{c_s\}_{s \in [t, T]}$, $L = \{L^1_s, \cdots, L^n_s\}_{s \in [t, T]}$, and $M = \{M^1_s, \cdots, M^n_s\}_{s \in [t, T]}$ are adapted
processes satisfying

\[
dL^i_s \geq 0, \quad dM^i_s \geq 0,
\]

and $\{x_s, y_s\}_{t \leq s \leq T}$ is the solution of (2.1) subject to constant initial conditions. We denote by
$\mathcal{A}$ all the admissible strategies.
2.2. The Merton’s Problem. Given concave utilities \( U(x, y) \) and \( V(c) \) for the terminal portfolio and consumption respectively, a constant discount factor \( \beta > 0 \), and positive dimensionless constant weight \( K \), we consider the measure of quality of an investment-consumption strategy \( S \) defined by

\[
J(S, t) := KU(x_T, y_T)e^{-\beta(T-t)} + \int_t^T V(c_s)e^{-\beta(s-t)}Kds,
\]

where \( \{x_s, y_s\}_{s \in [t,T]} \) is the solution of (2.1) with given strategy \( S \in A_t \). The Merton’s problem is to maximize the expected utility:

\[
\Phi(x, y, t) = \sup_{S \in A_t} E_{x,y}^t[J(S, t)] \quad \forall x \in \mathbb{R}, y \in \mathbb{R}^n, t \leq T,
\]

where \( E_{x,y}^t \) is the expectation under the condition \( (x_t, y_t) = (x, y) \). In this paper, we mainly consider the exponential utility

\[
V(c) := -e^{-\gamma c}, \quad U(x, y) := V(x + \ell(y)),
\]

where \( \ell(y) \) is the liquidation value of the holdings in the risky assets:

\[
\ell(y) = \sum_i \ell_i(y_i), \quad \ell_i(y_i) = \begin{cases} (1 - \mu_i)y_i & \text{if } y_i \geq 0, \\ (1 + \lambda_i)y_i & \text{if } y_i < 0. \end{cases}
\]

Notice that \( \ell_i(\cdot) \) is a concave function and

\[
\ell_i(y_i) = \min_{1 - \mu_i \leq k \leq 1 + \lambda_i} \{ky_i\} = \min\{(1 - \mu_i)y_i, (1 + \lambda_i)y_i\} \quad \forall y_i \in \mathbb{R}.
\]

For later use, we define

\[
\sigma_{ij} := \sum_k a_{ik}a_{jk}
\]

and

\[
m_j := \sum_i \sigma^{ji}(\alpha_i - r) \quad \forall j, \quad A_0 := \frac{1}{2} \sum_{i,j} m_i \sigma_{ij} m_j,
\]

where \( (\sigma^{ij})_{n \times n} \) is the inverse matrix of \( (\sigma_{ij})_{n \times n} \). Here \( m := (m_1, \ldots, m_n) \) is the optimal investment strategy for the Merton’s problem without transaction costs, being the solution of the linear system

\[
\sum_j \sigma_{ij} m_j = \alpha_i - r \quad \forall i = 1, \ldots, n.
\]
2.3. Preliminary Results. With the exponential utility, we have the following:

**Theorem 2.1.** There exists a function \( \psi \) defined on \( \mathbb{R}^n \times [0, \infty) \) such that

\[
\Phi(x, y, t) = -e^{-\gamma \xi(t)x + (r - \beta)\beta + Z(r)\ln K - \ln \xi(t) - \psi(z, \tau)},
\]

where \( \tau = T - t, z = \gamma \xi(t)y, \) and \( \xi, Z, b \) are defined by

\[
\begin{align*}
\xi(\tau) &= \frac{re^{\tau}}{r + ke^{\tau} - \kappa}, & Z(\tau) &= \xi(\tau)e^{-\tau}, & b(\tau) &= \frac{\kappa(e^{\tau} - 1 - r\tau) + r^2\tau}{r(r + ke^{\tau} - \kappa)},
\end{align*}
\]

i.e., unique solutions of the following

\[
\begin{align*}
\xi' &= (r - \kappa\xi)\xi \text{ on } [0, \infty), & \xi(0) &= 1; \\
Z' &= -\kappa Z \text{ on } [0, \infty), & Z(0) &= 1; \\
b' &= -\kappa\xi b + 1 \text{ on } [0, \infty), & b(0) &= 0.
\end{align*}
\]

In addition,

1. for each \( \tau \geq 0, \psi(\cdot, \tau) \) is Lipschitz continuous:

\[
\ell(z - z') \leq \psi(z, \tau) - \psi(z, \tau) \leq -\ell(z - z') \quad \forall z, z' \in \mathbb{R}^n;
\]
2. for \( A_0 \) defined in (2.5),

\[
\ell(z) \leq \psi(z, \tau) \leq \ell(z) + A_0 b(\tau) \quad \forall z \in \mathbb{R}^n, \tau \geq 0;
\]
3. for each \( \tau \geq 0, \psi(\cdot, \tau) \) is concave;
4. \( \psi \) is a viscosity solution of

\[
\begin{align*}
\min \left\{ \partial_\tau \psi - A^\tau[\psi], B(\nabla \psi) \right\} &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
\psi(\cdot, 0) &= \ell(\cdot) \quad \text{on } \mathbb{R}^n \times \{0\},
\end{align*}
\]

where \( A^\tau \) and \( B \) are defined by

\[
\begin{align*}
\{a_{ij}[\psi] := \frac{1}{2} \sum_{i,j} \sigma_{ij} z_i z_j (\partial_{z_i} \psi \cdot \partial_{z_j} \psi) + \sum_{i} (\alpha^i - \tau)z_i \partial_{z_i} \psi + \kappa \xi(\tau) \left| z \cdot \nabla \psi - \psi \right|, \\
B(p) := \min \min_{\tau} \left\{ 1 + \lambda_i - p_i, -1 + \mu_i + p_i \right\} \quad \forall p = (p_1, \cdots, p_n) \in \mathbb{R}^n.
\end{align*}
\]

We regard the infinite horizon problem as the limit of finite horizon problem under appropriate scales, as the finite horizon \( T \rightarrow \infty \). For this, we have the following:

**Theorem 2.2.** Suppose \( \kappa > 0 \). Then there exist a function \( u \) and a constant \( M \) such that

\[
|\partial_\tau \psi(z, \tau)| + |\psi(z, \tau) - u(z)| \leq M(1 + r\tau)e^{-rt} \quad \forall \tau > 0, z \in \mathbb{R}^n.
\]

In addition, \( u \) is a Lipschitz continuous concave viscosity solution of the following equation

\[
\min \{-A[u], B(\nabla u)\} = 0, \quad 0 \leq u - \ell \leq \frac{A_0}{r} \quad \text{in } \mathbb{R}^n.
\]
where \( \ell(z) \), \( A_0 \), and \( B \) are defined in (2.4), (2.5), and (2.15) respectively, and

\[
A[u] := \frac{1}{2} \sum_{i,j} \sigma_{ij} z_i z_j (\partial_{z_i} z_j u - \partial_{z_i} u \partial_{z_j} u) + \sum_i \alpha_i z_i \partial_{z_i} u - ru.
\]

The estimate (2.16) implies that \( \psi \) and \( \partial_z \psi \) converge at an exponential rate governed by the interest rate \( r > 0 \), instead of the discount factor \( \beta \). If \( r \) goes to 0, the analytic infinite horizon optimal strategy in the absence of transaction costs indicates that the dollar values invested in risky assets and consumption tend to infinity [see, e.g., Merton (1969) and Chen et al. (2012)].

The proof of the above two theorems is given in the Appendix.

2.4. Main Results. We are mainly interested in the infinite horizon problem. Recall that \( \tau = T - t \) and \( z = \gamma \xi(T)y \). Hence, setting \( \Phi^T(x, y) = \Phi(x, y, 0) \) where \( \Phi \) is the solution of the finite \( T \)-horizon problem, we have, when \( \kappa > 0 \),

\[
\Phi^\infty(x, y) := \lim_{T \to \infty} \Phi^T(x, y) = - \lim_{T \to \infty} e^{(r-\beta)b(T)-\gamma \xi(T)x+z(T)\ln K-\ln \xi(T)-\psi(\gamma \xi(T)y,T)}
\]

\[
= -e^{(r-\beta)b(\infty)-\gamma \xi(\infty)x+Z(\infty)\ln K-\ln \xi(\infty)-u(\gamma \xi(\infty)y)}
\]

\[
= -\frac{K}{r} e^{(r-\beta)/r-\gamma \xi(\infty)x-u(\gamma \xi(\infty)y/\kappa)}.
\]

Hence, the solution \( u \) to the equation (2.17) can be formally regarded as the value function associated with the infinite horizon utility maximization problem:

\[
\Phi^\infty(x, y) = \sup_{S \in A_\infty} \mathbb{E}_0^{x,y} \left[ \int_0^\infty V(c_s) e^{-\beta s} ds \right]
\]

where \( A_\infty \) is certain admissible strategy. Note that without any restriction on \( A_\infty \), the optimal strategy is \( c_s \equiv \infty \). Hence, if one considers directly the infinite horizon problem, some technical conditions should be given [see, e.g., Liu (2004)]. We instead regard the infinite horizon problem as the limit, as \( T \to \infty \), of the finite \( T \)-horizon problem with the addition of a bequest utility which prevents unlimited consumption.

For solution \( u \) of the “infinite horizon” problem (2.17), we have the following regularity result that plays a critical role in the analysis of the shape and location of the trading and no-trading regions:

**Theorem 2.3.** Let \( u \) be the solution of (2.17). Then \( z_i \partial_z u \in C(\mathbb{R}^n) \) for each \( i = 1, \ldots, n \); consequently, \( u \in C^1(\Omega) \) where \( \Omega = \{ (z_1, \ldots, z_n) \in \mathbb{R}^n \mid \prod_i z_i \neq 0 \} \).

The shape of the trading and no-trading regions is characterized by the following:

**Theorem 2.4.** Assume that \(^5\) \( n = 2 \) and for \( i = 1, 2 \) define

\[
B_i := \{ z \mid \partial_z u(z) = 1 + \lambda_i \},
\]

\[
S_i := \{ z \mid \partial_z u(z) = 1 - \mu_i \},
\]

\[
N_i := \{ z \mid 1 - \mu_i < \partial_z u(z) < 1 + \lambda_i \},
\]

\(^4\)It should be admitted that the model with the CARA utility discussed here is less plausible than the one with the CRRA utility.

\(^5\)Here we consider only the two dimensional case. We expect that analogous results remain true for the higher dimensional case.
and denote $SS = S_1 \cap S_2, SN = S_1 \cap N_2, SB = S_1 \cap B_2, NS = N_1 \cap S_2, NT = N_1 \cap N_2, NB = N_1 \cap B_2, BS = B_1 \cap S_2, BN = B_1 \cap N_2$, and $BB = B_1 \cap B_2$. Then

(1) For $i = 1, 2$, there are bounded functions $l^+_i(\cdot)$ (defined on $\mathbb{R}^{n-1}$) such that

$$B_i = \{ (z_1, z_2) \mid z_i \in \mathbb{R}, z_i \leq l^+_i(z_i) \},$$

$$S_i = \{ (z_1, z_2) \mid z_i \in \mathbb{R}, z_i \geq l^+_i(z_i) \},$$

$$N_i = \{ (z_1, z_2) \mid z_i \in \mathbb{R}, l^+_i(z_i) < z_i < l^+_i(z_i) \},$$

where $i = \{1, 2\} \setminus \{i\}$, i.e., $i = 2$ if $i = 1$ and $i = 1$ if $i = 2$.

(2) Each intersection $\partial S_1 \cap \partial S_2, \partial S_1 \cap \partial B_2, \partial B_1 \cap \partial S_2$, and $\partial B_1 \cap \partial B_2$ is a singleton, so the four boundaries $\partial S_1, \partial B_1, \partial S_2, \partial B_2$ divide the plane into nine regions, with open region $NT$ in the center surrounded in clockwise order by closed regions $SS, SN, SB, NB, BB, BN, BS, $ and $NS$.

(3) The boundary of each of corner regions $SS, SB, BB, $ and $BS$ consists of one vertical and one horizontal half line, whereas the boundary of each of $SN, NS, BN, $ and $NB$ consists of two parallel either vertical or horizontal half lines and a curve in between connecting the end points of the two half lines; c.f. Figure 1.

We remark when $n \geq 3, i = \{1, \cdots, n\} \setminus \{i\}$, so $l_i(z_i)$ is indeed a function of $n-1$ variables.

The theorem implies the following: There are intervals $[b_i^+, s_i^+]$ such that

$$SS = [s_1^+, \infty) \times [s_2^+, \infty),$$

$$SN = \{ (z_1, z_2) \mid z_2 \in (b_2^+, s_2^+), z_1 \geq l^+_1(z_2) \},$$

$$SB = [s_1^+, \infty) \times (-\infty, b_2^-),$$

$$NB = \{ (z_1, z_2) \mid z_1 \in (b_1^+, s_1^-), z_2 \leq l^-_2(z_1) \},$$

$$BB = (-\infty, b_1^-] \times (-\infty, b_2^-),$$

$$BN = \{ (z_1, z_2) \mid z_2 \geq (b_2^+, s_2^-), z_1 \leq l^-_1(z_1) \},$$

$$BS = (-\infty, b_1^-] \times [s_2^-, \infty),$$

$$NS = \{ (z_1, z_2) \mid z_1 \in (b_1^+, s_1^-), z_2 \geq l^-_2(z_1) \}.$$

The no-trading region $NT$ is bounded by four curves: $\Gamma_{2+}$ from the right, $\Gamma_{2-}$ from the left, $\Gamma_{1+}$ from the top, and $\Gamma_{1-}$ from the bottom where

$$\Gamma_{2\pm} := \{ (l^+_1(z_2), z_2) \mid z_2 \in (b_2^+, s_2^+) \}, \quad \Gamma_{1\pm} := \{ (z_1, l^+_2(z_1)) \mid z_1 \in (b_1^+, s_1^-) \}.$$

These four curves connect each other only at their tips:

$$l^+_i(b_i^+) = \lim_{z_i \to b_i^+} l^+_i(z_i), \quad l^+_i(s_i^+) = \lim_{z_i \to s_i^+} l^+_i(z_i),$$

$$l^+_i(b_i^-) = (l^+_1(s_2^-), s_2^-), \quad l^+_i(s_i^+) = (s_1^+, l^+_2(s_1^-)),$$

$$(l^-_1(b_2^-), b_2^-) = (b_1^-, l^-_2(b_1^-)), \quad (s_1^-, l^-_2(s_1^-)) = (l^-_1(b_2^+), b_2^+).$$

Each function $l^+_i$ is constant outside $(b_i^+, s_i^+)$:

$$l^+_i(z_i) = l^+_i(b_i^+) \quad \forall z_i \leq b_i^+, \quad l^+_i(z_i) = l^+_i(s_i^+) \quad \forall z_i \geq s_i^+.$$


As emphasized in the introduction, Theorem 2.4 is very much needed for the trading strategy to be well-defined. It is well-known that except at the initial time, transactions occur at the boundary of the no-trading region. When the initial portfolio falls outside the no-trading region, the shape of the trading regions stated in the theorem implies a unique trading strategy to move the portfolio to the boundary of the no-trading region. However, at this stage we cannot prove the smoothness of the curves $l_i^\pm$, so we are unable to prove rigorously the optimal controlled portfolio process as a reflected diffusion as in Davis and Norman (1990) and Shreve and Soner (1994).

We can now state our main result regarding the location of no trading region:

**Theorem 2.5.** Assume that $n = 2$. Let $(m_1, m_2)$ and $A_0$ be as given in (2.5), and

$$
\sigma_1 = \sqrt{\sigma_{11}}, \quad \sigma_2 = \sqrt{\sigma_{22}}, \quad \rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}, \quad M(h) := \sqrt{\frac{2(A_0 - rh)}{1 - \rho^2}}.
$$

Also, define the ellipse $C(k_1, k_2, h)$ and constant $c_{ij}$ by

$$
C(k_1, k_2, h) := \left\{ (z_1, z_2) \mid \frac{1}{2} \sum_{i,j} \sigma_{ij}(k_i z_i - m_i)(k_j z_j - m_j) = A_0 - rh \right\},
$$

$$
c_{ij} := \lim_{z_1 \to \infty, z_2 \to \infty} \{ u(z) - \ell(z) \}, \quad i, j = 1, 2.
$$

1. The no-trading region $\mathbf{NT}$ is contained in the set

$$
D := \bigcup_{1 - \mu_i < k_i < 1 + \lambda_i, h \geq u(0)} C(k_1, k_2, h).
$$

2. The corner $(s_1^+, s_2^+)$ of $\mathbf{SS}$ lies on the ellipse $C(1 - \mu_1, 1 - \mu_2, c_{22})$ and on its top-left part in the sense that

$$
-\rho \leq \frac{(1 - \mu_1)s_1^+ - m_1}{M(c_{22})/\sigma_1} \leq 1, \quad -\rho \leq \frac{(1 - \mu_2)s_2^+ - m_2}{M(c_{22})/\sigma_2} \leq 1.
$$

3. The corner $(s_1^-, b_2^+)$ of $\mathbf{SB}$ lies on the bottom-right part of $C(1 - \mu_1, 1 + \lambda_2, c_{21})$:

$$
\rho \leq \frac{(1 - \mu_1)s_1^- - m_1}{M(c_{21})/\sigma_1} \leq 1, \quad -1 \leq \frac{(1 + \lambda_2)b_2^+ - m_2}{M(c_{21})/\sigma_2} \leq -\rho.
$$

4. The corner $(b_1^-, b_2^-)$ of $\mathbf{BB}$ lies on bottom-left part of $C(1 + \lambda_1, 1 + \lambda_2, c_{11})$:

$$
-1 \leq \frac{(1 + \lambda_1)b_1^- - m_1}{M(c_{11})/\sigma_1} \leq \rho, \quad -1 \leq \frac{(1 + \lambda_1)b_2^- - m_2}{M(c_{11})/\sigma_2} \leq \rho.
$$

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The regularity proof of free boundary in high dimension is always challenging. We leave it for future research.
5. The corner \((b^+_1, s^-_2)\) of BS lies on top-left of \(C(1 + \lambda_1, 1 - \mu_2, c_{12})\):
\[
-1 \leq \frac{(1 + \lambda_1)b^+_1 - m_1}{M(c_{12})/\sigma_1} \leq -\rho, \quad \rho \leq \frac{(1 - \mu_2)s^-_2 - m_2}{M(c_{12})/\sigma_2} \leq 1.
\]

We remark that
\[
0 \leq u(0) \leq c_{ij} \leq A_0 \frac{r}{r},
\]
since \(0 \leq u(z) - \ell(z) \leq A_0/r\) and \(u(z) - \ell(z)\) is an increasing concave function in each radial direction.

The theorem shows that the no-trading region is contained in a union of uniformly bounded ellipses. Moreover, the location of the corners of the no-trading region is estimated. Hence we can restrict attention to a bounded domain to study the problem. In particular, this allows us to do computations in a bounded domain.

The proof of Theorems 2.4 and 2.5 will be given in Section 4 and the proof of Theorem 2.3 in Section 5.

3. Extension to the CRRA Utility. Now let us examine the case with the CRRA utility, namely,
\[
V(c) = \frac{1}{\gamma} c^\gamma, \quad \gamma < 1, \quad \gamma \neq 0,
\]
for which we require that the liquidated wealth be non-negative:
\[
x + \sum_i \ell_i(y_i) \geq 0.
\]

Note that we can directly consider the infinite horizon problem for the CRRA utility and the above solvency constraint. Let \(\Phi (x, y_1, y_2)\) be the associated value function which satisfies (cf. Dai and Zhong (2010))
\[
(3.1) \quad \min \left\{ -L\Phi, \min_i [(1 + \lambda_i) \partial_x \Phi - \partial_{y_i} \Phi], \min_i [- (1 - \mu_i) \partial_x \Phi + \partial_{y_i} \Phi] \right\} = 0,
\]
in \(x + \sum_i \ell_i(y_i) > 0\), where \(L\) is as given in (A.6).

For illustration, we still consider the case of two risky assets. The homogeneity of the utility function allows us to make the following transformation:
\[
z_i = \frac{y_i}{x + y_1 + y_2}, \quad i = 1, 2,
\]
\[
\varphi (z_1, z_2) \equiv \Phi (1 - z_1 - z_2, z_1, z_2) = \frac{1}{(x + y_1 + y_2)^\gamma} \Phi (x, y_1, y_2).
\]

Then (3.1) reduces to
\[
\min \left\{ -\tilde{A}\varphi, \min_i \left[ \lambda_i \gamma \varphi - \sum_k (\delta_{ik} + \lambda_i z_k) \partial_{z_k} \varphi \right], \min_i \left[ \mu_i \gamma \varphi - \sum_k (-\delta_{ik} + \mu_i z_k) \partial_{z_k} \varphi \right] \right\} = 0
\]
in $\mathcal{D} = \left\{ (z_1, z_2) : \sum_{i=1}^{2} [z_i - \ell_i(z_i)] < 1 \right\}$, where the expression of $\tilde{A}$ is omitted.

Now we define, for $i = 1, 2$,

$$B_i = \left\{ (z_1, z_2) \in \mathcal{D} : \lambda_i \gamma \varphi - \sum_{k=1}^{2} (\delta_{ik} + \lambda_i z_k) \partial_{z_k} \varphi = 0 \right\},$$

$$S_i = \left\{ (z_1, z_2) \in \mathcal{D} : \mu_i \gamma \varphi - \sum_{k=1}^{2} (-\delta_{ik} + \mu_i z_k) \partial_{z_k} \varphi = 0 \right\},$$

$$N_i = \mathcal{D} \cap B_i^c \cap S_i^c.$$  

We aim to show that $\mathcal{D}$ is partitioned into nine regions as shown in Figure 2. In particular, $N_1 \cap N_2$ has four distinct corners. We point out that the shape of trading/no-trading regions is the same as that postulated by Bichuch and Shreve (2011) who deal with a slightly different setting: the prices of risky assets follow arithmetic Brownian motions.

Let us consider, as an example, the region $S_1 \cap S_2$. It is worthwhile pointing out that regarding the proof for the CARA utility case, two conditions play critical roles: 1) $\varphi$ is concave and is $C^1$ except on the coordinates planes; 2) $\partial_{z_i} \varphi$, $i = 1, 2$ are constants in $S_1 \cap S_2$. The former still holds true with the CRRA utility whereas the latter does not. This motivates us to make a new transformation:

$$\pi_i = \frac{y_i}{x + (1 - \mu_1) y_1 + (1 - \mu_2) y_2}, \quad i = 1, 2,$$

$$\varphi(\pi_1, \pi_2) \equiv \Phi (1 - (1 - \mu_1) \pi_1 - (1 - \mu_2) \pi_2, \pi_1, \pi_2)
= \frac{1}{[x + (1 - \mu_1) y_1 + (1 - \mu_2) y_2]^\gamma} \Phi (x, y_1, y_2).$$

It is easy to verify that

$$-(1 - \mu_i) \partial_x \Phi + \partial_{y_i} \Phi = [x + (1 - \mu_1) y_1 + (1 - \mu_2) y_2]^{\gamma-1} \partial_{\pi_i} \varphi,$$

which implies

$$\partial_{\pi_i} \varphi = 0 \text{ in } S_1 \cap S_2, \quad i = 1, 2.$$  

Since $\varphi$ is also concave and is $C^1$ except on the coordinates planes, we can use the same analysis as in the CARA utility case to obtain the desired result. We point out that most results presented in the CARA utility can be extended to the CRRA utility case.

**Remark 3.1.** If transactions between risky assets are allowed, as modeled in Soner and Touzi (2012) and Possamaï et al. (2013), we are unable to find a transformation that gives equation (3.2), and therefore cannot extend our analysis to the case. Indeed, numerical calculation presented in Possamaï et al. (2013) indicates that there is no analogous conclusion for the shape of the trading and no trading regions.

Now we return to the CARA case.
4. Shape and Location of The Trading/No-Trading Zones. In this section, we investigate the shape and location of the trading/no-trading regions in the two dimensional case for the infinite horizon problem; that is, we prove Theorem 2.4 and Theorem 2.5 using the properties of $u$ established in Theorem 2.2 and Theorem 2.3 whose proof will be given in the Appendix and the next section, respectively.

4.1. The Bound of the No-Trading Region.

Let $u$ be the viscosity solution of (2.17). For each $z^0$, we define the first order approximation of $u$ by

$$
\pi(z^0, z) := u(z^0) + \nabla u(z^0) \cdot (z - z^0).
$$

By concavity, we have

$$
u(z) \leq \pi(z^0, z),
$$

which will be often used later on. In particular,

(4.1) \quad u(0) \leq \pi(z^0, 0) \leq u(z^0) - \nabla u(z^0)z^0.$
We now decompose \( A[u] \) as \( A[u] = \mathcal{L}u - f \) where

\[
\mathcal{L}u(z) = \frac{1}{2} \sum_{i,j} \sigma_{ij} z_i z_j \partial_{z_i z_j} u(z),
\]

\[
f(z) = \frac{1}{2} \sum_{i,j} \sigma_{ij} z_i z_j \partial_{z_i} u(z) \partial_{z_j} u(z) - \sum_i \alpha_i z_i \partial_{z_i} u(z) + ru(z)
\]

where, \( m_j \) and \( A_0 \) are defined in (2.5). Thanks to Theorem 2.3 (to be proven in the next section), \( f \) is continuous. If \( z \in \mathbf{NT} \), then \( A[u](z) = 0 \) so \( \mathcal{L}u(z) = f(z) \). Using concavity of \( u \) we derive

\[
0 \geq \mathcal{L}u(z) = f(z) \geq \frac{1}{2} \sum_{i,j} \sigma_{ij} (z_i \partial_{z_i} u - m_i)(z_j \partial_{z_j} u - m_j) - A_0 + ru(0),
\]

where we use (4.1) in the last inequality. Thus, \( z \) is on or inside the ellipse \( C(\partial_{z_1} u, \partial_{z_2} u, u(0)) \). Since \( \partial_{z_i} u(z) \in (1 - \mu_i, 1 + \lambda_i) \) we see that \( z \in \mathbf{D} \) defined in (2.24). The first assertion of Theorem 2.5 thus follows.

In the sequel, we shall use the following important fact:

**Lemma 4.1.** If \( z^0 = (z^0_1, z^0_2) \in \mathbf{S}_1 \) then \([z^0_1, \infty) \times \{z^0_2\} \in \mathbf{S}_1 \) and \( u(z) = \pi(z^0, z) \) and \( \nabla u(z) = \nabla u(z^0) \) for all \( z \in [z^0_1, \infty) \times \{z^0_2\} \).

Analogous assertions hold also for the cases \( z^0 \in \mathbf{B}_1, z^0 \in \mathbf{S}_2, \) and \( z^0 \in \mathbf{B}_2, \) respectively.

**Proof.** Suppose \( z^0 = (z^0_1, z^0_2) \in \mathbf{S}_1, \) i.e., \( \partial_{z_1} u(z^0) = 1 - \mu_1 \). Since \( \partial_{z_1} u \leq 0 \) and \( \partial_{z_2} u \geq 1 - \mu_1 \), we have \( \partial_{z_1} u(z_1, z^0_2) = 1 - \mu_1 \) for all \( z_1 \geq z^0_1 \); that is, \([z^0_1, \infty) \times \{z^0_2\} \in \mathbf{S}_1 \). In addition, for each \( z = (z_1, z_2^0) \) with \( z_1 \geq z^0_1 \), \( u(z) = u(z_0) + (1 - \mu_1)(z_1 - z^0_1) = \pi(z_0, z) \).

Since \( u(z) \leq \pi(z_0, z) \) for all \( z \in \mathbb{R}^2 \), we also have \( \nabla u(z) = \nabla \pi(z^0, z) \) for every \( z \in [z^0_1, \infty) \times \{z^0_2\} \).

The other cases can be similarly proven. This completes the proof.

### 4.2. Limit Profile.

With \( k^+_i := 1 - \mu_i \) and \( k^-_i := 1 + \lambda_i \), we define the limit profile by

\[
v^+_i(z_2) := \lim_{z_1 \to \pm \infty} (u(z_1, z_2) - k^+_1 z_1), \quad v^+_i(z_1) := \lim_{z_2 \to \pm \infty} (u(z_1, z_2) - k^+_2 z_2).
\]

**Lemma 4.2.**

1. With \( k^+_i := 1 - \mu_i \) and \( k^-_i := 1 + \lambda_i \), \( v^+_i \) defined in (4.4) is concave and

\[
u(z_1, z_2) \leq \min \left\{ v^+_2(z_2) + k^+_1 z_1, \ v^+_1(z_1) + k^+_2 z_2 \right\}, \ \forall z \in \mathbb{R}^2.
\]

2. There are intervals \((b^+_i, s^+_i)\) and functions \( l^+_i \) defined on \((b^+_i, s^+_i)\) such that

\[
v^+_i(z_1) \left\{ \begin{array}{ll}
1 - \mu_i & \text{if } z_i \geq s^+_i, \\
(1 - \mu_i, 1 + \lambda_i) & \text{if } z_i \in (b^+_i, s^+_i), \\
1 + \lambda_i & \text{if } z \leq b^+_i,
\end{array} \right.
\]
Define $\partial^-$ and $\partial^+$.

(2) Let $u(z_1, z_2)$.

Proof. By symmetry, we need only consider the function $v := v^+_2$.

(1) Note that $u(z_1, z_2) - (1 - \mu_1)z_1 = \partial_z u - (1 - \mu_1) \geq 0$, and $0 \leq u(z) - \ell(z) \leq A_0/r$. Hence, $v(z_2) := \lim_{z_1 \to \infty} u(z_1, z_2) - (1 - \mu_1)z_1$ exists, $v$ is concave, and $v(z_2) \geq u(z_1, z_2) - (1 - \mu_1)z_1$.

(2) Let $z_2 \in \mathbb{R}$ be a generic point such that $1 - \mu_2 < v'(z_2) < 1 + \lambda_2$. Since $\partial_2 u(z_1, z_2) \to v'(z_2)$ as $z_1 \to \infty$, $\partial_2 u(z_1, z_2) \in (1 - \mu_2, 1 + \lambda_2)$ for all $z_1 \gg 1$. As NT is bounded, we must have $\partial_2 u(z_1, z_2) = 1 - \mu_1$ for all $z_1 \gg 1$. In addition, since $\partial_2 u \to 1 + \lambda_1$ as $z_1 \to -\infty$, we can define

$$l(z_2) := \min\{z_1 \in \mathbb{R} \mid \partial_1 u(z_1, z_2) = 1 - \mu_1\}.$$

Since $u(\cdot, z_2)$ is concave, we must have

$$\partial_{z_1} u(z_1, z_2) < 1 - \mu_1 \forall z_1 < l(z_2), \quad \partial_{z_1} u(z_1, z_2) = 1 - \mu_1 \forall z_1 \geq l(z_2).$$

Denote $z^0 := (l(z_2), z_2)$. Then by Lemma 4.1, $u(z) = \pi(z^0, z) = v(z^0) + (1 - \mu_1)z_1$ and $\nabla u(z) = \nabla u(z^0)$ for each $z \in l(z_2) \in \{z_2\}$. Consequently, $\partial_2 u(z_1, z_2) = v'(z_2) \in (1 - \mu_2, 1 + \lambda_2)$ for all $z_1 \in l(z_2)$. Thus, for small positive $\varepsilon$, $B(\nabla u(z)) > 0$ on $l(z_2) -
\( \varepsilon, l(z_2)) \times \{z_2\} \). Hence, \( l(z_2) - \varepsilon, l(z_2)) \times \{z_2\} \in \text{NT} \) and \( l(z_2), z_2 \in \partial \text{NT} \). Since NT is bounded and \( v \) is concave, there exist bounded \( b \) and \( s \) such that
\[
v' = 1 + \lambda_2 \text{ on } (-\infty, b), \quad 1 + \lambda_2 > v' > 1 - \mu_2 \text{ on } (b, s), \quad v' = 1 - \mu_2 \text{ on } [s, \infty).
\]

(3) Now we define \( l(s) = \lim_{z_2 \to s} l(z_2), l_s(s) = \lim_{z_2 \to s} l(z_2), \) and \( z^* = (l(s), s) \). By continuity, we have \( \partial z_2 u(l(s), s) = 1 - \mu_1 \). This implies that \( \nabla u(z) = (1 - \mu_1, v'(s)) = (1 - \mu_1, 1 - \mu_2) \) for all \( z \in [l(s), \infty) \times \{s\} \). Hence, \( [l(s), \infty) \times \{s\} \in \text{SS} \).

Note that for each \( z_2 < s \) and \( z_1 \in [l(s), \infty) \), we have
\[
u(z_1, s) - u(z_1, z_2) = v(s) - v(z_2) - \int_{z_1}^s [\partial z_1 u(\xi, s) - \partial z_1 u(\xi, z_2)]d\xi
\]
\[
= \int_{z_2}^s v'(y)dy + \int_{z_1}^s [\partial z_1 u(\xi, z_2) - (1 - \mu_1)]d\xi > (1 - \mu_2)(s - z_2).
\]

It follows by concavity that \( \partial z_2 u(z_1, z_2) > 1 - \mu_2 \). Hence,
\[
(4.5)\quad \partial z_2 u > 1 - \mu_2 \text{ on } [l(s), \infty) \times (-\infty, s).
\]

It then follows by the definition of \( l(s) \) and \( l^*(s) \) that \( [l(s), l^*(s)] \times \{s\} \in \partial \text{NT} \).

Similarly we can work on the other functions \( v^+_i \) to complete the proof of Lemma 4.2.

### 4.3. The Intersection of \( \partial \text{NT} \) with \( B_1 \cap B_2, S_1 \cap S_2 \) and \( B_i \cap S_j \)

In this subsection we prove the following:

**Lemma 4.3.** Let \( c_{ij} \) and \( C(k_i, k_2, h_i) \) be defined as in (2.23) and (2.22).

1. The set \( \partial \text{NT} \cap \text{SS} \) is a single point on top-right of the ellipse \( C(1 - \mu_1, 1 - \mu_2, c_{22}) \).

2. The set \( \partial \text{NT} \cap \text{SB} \) is a single point on bottom-right of \( C(1 - \mu_1, 1 + \lambda_2, c_{21}) \).

3. The set \( \partial \text{NT} \cap \text{BB} \) is a single point on bottom-left of \( C(1 + \lambda_1, 1 + \lambda_2, c_{11}) \).

4. The set \( \partial \text{NT} \cap \text{BS} \) is a single point on top-left of \( C(1 + \lambda_1, 1 - \mu_2, c_{12}) \).

**Proof.** (i) Suppose \( z^0 = (z^0_1, z^0_2) \in S_1 \cap S_2 \).

First we show that \( [z^0_1, \infty) \times [z^0_2, \infty) \subset \text{SS} \). Indeed by Lemma 4.1, \( u(z) = \pi(z^0, z) \) on \( ([z^0_1, \infty) \times [z^0_2, \infty)) \cup \{z^0_1\} \times [z^0_2, \infty) \). This implies that \( u(\cdot) \geq \pi(z^0, \cdot) \) on \([z^0_1, \infty) \times [z^0_2, \infty) \) since \( \pi(z^0, \cdot) \) is linear and \( u(\cdot) \) is concave. On the other hand, we have \( u(z) \leq \pi(z^0, z) \) on \( \mathbb{R}^2 \). Thus we must have \( u(z) = \pi(z^0, z) \) and \( \nabla u(z) = \nabla u(z_0) = (1 - \mu_1, 1 - \mu_2) \) so \( [z^0_1, \infty) \times [z^0_2, \infty) \subset \text{SS} \).

In addition, on \( [z^0_1, \infty) \times [z^0_2, \infty) \),
\[
u(z) = u(z^0) + (1 - \mu_1)(z_1 - z^0_1) + (1 - \mu_2)(z_2 - z^0_2) = c_{22} + z \cdot \nabla u(z_0).
\]

Next since \( -A[u] = f(z) - Lu \geq 0 \) on \( \mathbb{R}^2 \), using the linearity of \( u \) on \( [z^0_1, \infty) \times [z^0_2, \infty) \) we obtain
\[
0 = \lim_{s \searrow 0} Lu(z^0 + se_1 + se_2) \leq \lim_{s \searrow 0} f(z^0 + se_1 + se_2) = f(z^0).
\]
Using $u = c_{22} + z \cdot \nabla u(z_0)$ on $[z_0^1, \infty) \times [z_0^2, \infty)$ we find that $f(z) = f_{22}(z) \geq 0$ where

$$f_{22}(z) := \frac{1}{2} \sum_{i,j} \sigma_{ij}(z_i[1 - \mu_i] - m_i)(z_j[1 - \mu_j] - m_j) - A_0 + r c_{22} \quad \forall z \in \mathbb{R}^2.$$ 

This analysis in particular implies that $f_{22}(z) \geq 0$ for every $z \in \mathbf{SS}$.

(ii) Next, suppose $z^0 \in \partial \mathbf{NT} \cap \mathbf{SS}$. Note that when $z \in \mathbf{NT}$, we have $0 = -\mathcal{A}[u](z) = f(z) - \mathcal{L}u(z)$, i.e. $f(z) = \mathcal{L}[u](z) \leq 0$ (as $u$ is concave). Hence,

$$f(z^0) = \lim_{z \in \mathbf{NT}, z \to z^0} f(z) = \lim_{z \in \mathbf{NT}, z \to z^0} \mathcal{L}u(z) \leq 0.$$ 

Thus, we must have $f(z^0) = 0$, i.e., $z^0 \in C := (1 - \mu_1, 1 - \mu_2, c_{22})$. Moreover, since $f_{22}(z) \geq 0$ for every $z \in [z_0^1, \infty) \times [z_0^2, \infty)$ and $f(z) < 0$ for each $z$ inside the ellipse $C$, we see that $z^0$ lies on the top-right part of the ellipse $C$. Locating the highest and rightmost points of the ellipse $C$, we then derive that

$$f_{22}(z^0) = 0, \quad -\rho \leq \frac{(1 - \mu_1)z_{01}^2 - m_1}{M(c_{22})/\sigma_1} \leq 1, \quad -\rho \leq \frac{(1 - \mu_2)z_{02}^2 - m_2}{M(c_{22})/\sigma_2} \leq 1.$$

(iii) Now we show that $\partial \mathbf{NT} \cap \mathbf{SS}$ is a singleton, by a contradiction argument. Suppose $z^0 = (z_0^1, z_0^2) \in \partial \mathbf{NT} \cap \mathbf{SS}$, $\hat{z}^0 = (\hat{z}_0^1, \hat{z}_0^2) \in \partial \mathbf{NT} \cap \mathbf{SS}$, and $\hat{z}^0 \neq z^0$. Then both $z^0$ and $\hat{z}^0$ lies on the upper-right part of the ellipse $C(1 - \mu_1, 1 - \mu_2, c_{22})$. Exchanging the roles of $z^0$ and $\hat{z}^0$, we can assume that $z_0^2 < \hat{z}_0^2$ and $z_0^1 > \hat{z}_0^1$. Note that $u(z) = \pi(\hat{z}, z)$ for all $z \in [z_0^1, \infty) \times [z_0^2, \infty)$ and $f(z) = \pi(\hat{z}, z)$ for all $z \in [z_0^1, \infty) \cap [z_0^2, \infty)$. Hence, $\pi(z^0, z) = \pi(\hat{z}, z)$ for all $z \in \mathbb{R}^2$. On the other-hand, if $u(z) = \pi(z^0, z)$, then $\nabla u(z) = \nabla \pi(z^0, z) = \nabla u(z_0)$ so $z \in \mathbf{SS}$.

Therefore,

$$\mathbf{SS} := \{z \in \mathbb{R}^2 | \nabla u(z) = (1 - \mu_1, 1 - \mu_1)\} = \{z \in \mathbb{R}^2 | u(z) = \pi(z^0, z)\}.$$ 

Note that $u$ is concave, $\pi(z^0, \cdot)$ is linear, and $u(\cdot) \leq \pi(z^0, \cdot)$ on $\mathbb{R}^n$. We derive that $\mathbf{SS}$ is a convex set. Consequently, it contains $L$, the line segment connecting $z^0$ and $\hat{z}^0$.

Next for each $s \in \mathbb{R}$, denote $z^s = (z_0^1 + s, z_0^2 + s)$. For $s < 0$, $z^s \notin \mathbf{SS}$ since otherwise it would imply $[z_s^1, s, \infty) \times [z_s^2, s, \infty) \in \mathbf{SS}$, contradicting $z^0 \in \partial \mathbf{NT}$. Hence, there exists $s^* > 0$ such that $z^s = (z_0^1 + s^*, z_0^2 + s^*) \in \partial \mathbf{SS}$. The point $z^s$ lies on the line segment $z \to \hat{z}$, so $z^s$ is a convex set. There are two cases: (a) $s^* > 0$, and (b) $s^* = 0$.

Consider case (a) $s^* > 0$. Then for each $s \in [0, s^*)$, $z^s \notin \mathbf{SS}$ since $\mathbf{SS}$ is convex. Also $\partial_1 u(z^s) > 1 - \mu_1$ since by Lemma 4.1, $\partial_1 u(z^s) = 1 - \mu_1$ would imply $\nabla u(z^s) = \nabla u(z) = \nabla u(z_0)$ where $z$ is the intersection of $L$ with the line $z_2 = z_0^2 + s$. Similarly, $\partial_2 u(z^s) > 1 - \mu_2$ for each $z \in [0, s^*)$. Thus, $z^s \in \mathbf{NT}$ for all $s \in [0, s^*)$. This means that $z^s \in \partial \mathbf{NT} \cap \mathbf{SS}$, which is impossible since $z^s \notin \mathbf{C}$.

Consider case (b) $s^* = 0$. First of all for all $\partial_1 u(z_1, z_0^2) > 1 - \mu_1$ for all $z_1 < z_0^1$ since otherwise it would imply $\nabla u(z_1, z_0^2) = \nabla u(z^0)$ and thus $[z_1, \infty) \times [z_0^2, \infty) \in \mathbf{SS}$ contradicting $z^0 \in \partial \mathbf{NT}$. Similarly, $\partial_2 u(z_1, z_0^2) > 1 - \mu_2$ for every $z_2 < z_0^2$. Now for every small positive $\varepsilon$, consider the closed set

$$D_\varepsilon := [z_0^1 - \varepsilon, z_0^1] \times [z_0^2 - \varepsilon, z_0^2] \setminus ([z_0^1 - \varepsilon, z_0^1] \times (z_0^2 - \varepsilon, 2z_0^2) \times (z_0^2 - \varepsilon, 2z_0^2).$$

If $z^s = (z_0^1, z_0^2) \in \mathbf{SS} \cap D_\varepsilon$ we would have $z_0^1 < z^s_0^1$ and $z_0^2 < z^s_0^2$ so $z^0$ is a convex set. Thus, $\mathbf{SS} \cap D_\varepsilon = \emptyset$. Consequently, the closed sets
In this section we prove Theorem 2.17 and 2.20. If

A function thus follow from Lemmas

Therefore $\partial \mathbf{NT} \cap \mathbf{SS}$ is a singleton.

The proof for the singleness of $\partial \mathbf{NT} \cap \mathbf{BS}, \partial \mathbf{NT} \cap \mathbf{SB},$ and $\partial \mathbf{NT} \cap \mathbf{BB}$ is similar. This completes the proof of the Lemma.

4.4. Completion of the Proof of Theorems 2.4, 2.5.

By Lemmas 4.2 and 4.3, we see that the limits in (2.20) exist, and the limits satisfy the matching condition stated in (2.20). We extend $l_1^\pm$ from $(b_1^+, s_1^\pm)$ to $\mathbb{R}$ by (2.21).

By the definition of $l_1^\pm$ on $(b_1^+, s_1^\pm)$ we know that when $z_1 \in (b_1^+, s_1^\pm)$, $\partial \mathbf{z}:u(z_1, z_2) > 1 - \mu_2$ if and only if $z_2 < l_2^+(z_1)$. Also, in view of (4.5) and the matching (2.20), we derive that when $z_1 \in [s_1^+, \infty)$, $\partial \mathbf{z}:u(z_1, z_2) > 1 - \mu_2$ if and only if $z_2 < l_2^+(s_1^\pm) = l_2^+(z_1)$. Similarly, we can show that when $z_1 \in (-\infty, b_1^+)$, $\partial \mathbf{z}:u(z_1, z_2) < 1 - \mu_2$ if and only if $z_2 < l_2^-(b_1^+) = l_2^-(z_1)$.

Thus,

$$S_2 := \{ z \mid \partial \mathbf{z}:u = 1 - \mu_2 \} = \{(z_1, z_2) \mid z_1 \in \mathbb{R}, z_2 \geq l_2^+(z_1) \}.$$ 

Similarly, we can show the other equations in (2.19). The rest assertions of Theorems 2.4 and 2.5 thus follow from Lemmas 4.2 and 4.3. This completes the proof of Theorems 2.4 and 2.5.

5. $C^1$ Regularity. In this section we prove Theorem 2.3; that is, we show that the viscosity solution $u$ of the “infinite horizon” problem (2.17) is $C^1$ except on the coordinates planes where the elliptic operator $\mathcal{A}$ is degenerate. The $C^1$ continuity plays a critical role in the previous section where a key step is to derive the continuity of $f(\cdot)$ defined in (4.3).

We begin with recalling the definition of a viscosity solution:

**Definition 5.1.** A function $u$ defined on $\mathbb{R}^n$ is called a viscosity solution of (2.17) if $u$ is continuous, $u - \ell \in L^\infty(\mathbb{R}^n)$, and the following holds:

1. If $\zeta$ is a $C^2$ function in $B_\varepsilon(z^0) := \{ z \in \mathbb{R}^n \mid |z - z^0| < \varepsilon \}$ for some $z^0 \in \mathbb{R}^n$ and $\varepsilon > 0$, and that $\zeta(z) - u(z) \geq 0 = \zeta(z^0) - u(z^0)$ for every $z \in B_\varepsilon(z^0)$, then

$$\min \left\{ - \mathcal{A}[\zeta](z^0), \ B(\nabla \zeta(z^0)) \right\} \leq 0.$$

2. If $\zeta$ is a $C^2$ function in $B_\varepsilon(z_0) := \{ z \in \mathbb{R}^n \mid |z - z_0| < \varepsilon \}$ for some $z_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, and that $\zeta(z) - u(z) \leq 0 = \zeta(z^0) - u(z^0)$ for each $z \in B_\varepsilon(z^0)$, then

$$\min \left\{ - \mathcal{A}[\zeta](z^0), \ B(\nabla \zeta(z^0)) \right\} \geq 0.$$

One can show that the viscosity solution of (2.17) is unique. Since the solution is the limit of $\psi(\cdot, \tau)$ that is concave, we see that the viscosity solution of (2.17) is concave. We shall use this fact to prove the $C^1$ regularity of $u$.

For any function $f$ defined on $\mathbb{R}^n$, we define its super-differential by

$$\partial f(z) = \{ p \in \mathbb{R}^n : f(\hat{z}) \leq f(z) + p \cdot (\hat{z} - z) \ \forall \, \hat{z} \in \mathbb{R}^n \}.$$ 

We shall use the following fact.

**Lemma 5.2.** Suppose $f$ is a concave function on $\mathbb{R}^n$. Define its super-differential by (5.1). Then the following holds:
1. The set \( \{(z, p) \mid z \in \mathbb{R}^n, p \in \partial f(z)\} \) is closed; i.e., if \( p^k \in \partial f(z^k) \) for all \( k \geq 1 \) and 
\[ \lim_{k \to \infty} (p_k, z_k) = (p, z), \] 
then \( p \in \partial f(z) \).
2. For each \( z \in \mathbb{R}^n, \partial f(z) \) is a non-empty, convex and compact set.
3. If \( \partial f(z) = \{ p \} \) is a singleton, then \( f \) is differentiable at \( z \) and \( p = \nabla f(z) \).
4. If \( \partial f(z) \) is singleton for every \( z \) in an open neighborhood of \( z^0 \in \mathbb{R}^n \), then \( f \) is \( C^1 \) in an open neighborhood of \( z^0 \).
5. For each \( i = 1, \ldots, n \) and fixed \( z \in \mathbb{R}^n \) define
\[ \partial_i f(z) = \{ e_i \cdot p \mid p \in \partial f(z) \}, \quad g(t) = f(z + t e_i). \]

Then
\[ \partial_i f(z) = \partial g(0) = \left[ \lim_{h \searrow 0} \frac{g(h) - g(0)}{h}, \lim_{h \searrow 0} \frac{g(0) - g(-h)}{h} \right]. \]

The conclusion of the lemma is well-known; see Crandall et al. (1992) and references therein.

If \( f \) is concave, then \( f \) is locally Lipschitz continuous and \( \partial f \) is non-empty and almost everywhere singleton, and coincides with the Sobolev gradient. For convenience, we identify the set \( \partial f(z) \) as a generic vector \( p \) in \( \partial f(z) \).

**Proof of Theorem 2.3.** For illustration, we consider only the two space dimensional case.

First we show that \( \partial_1 u(z_1, z_2) \) is a singleton if \( z_1 \neq 0 \). We use a contradiction argument. Suppose the assertion is not true. Then there exist \( z^0 = (z^0_1, z^0_2) \) with \( z^0_1 \neq 0 \) and \( p = (p_1, p_2) \) and \( q = (q_1, q_2) \) with \( 1 - \mu_1 \leq p_1 < q_1 \leq 1 + \lambda_1 \) such that \( p, q \in \partial u(z^0) \).

By the definition of super-differential, we have
\[ u(z) \leq \min\{u(z^0) + p \cdot (z - z^0), \ u(z^0) + p \cdot (z - z^0)\} \quad \forall z \in \mathbb{R}^2. \]

Now for any \( 0 < \varepsilon \ll 1 \), consider the quadratic concave function
\[ \zeta(z) = u(z^0) + \frac{p + q}{2} \cdot (z - z^0) - \frac{[(q - p) \cdot (z - z^0)]^2}{4\varepsilon} \]
in \( Q_\varepsilon := \{ z \mid |(q-p) \cdot (z - z^0)| \leq \varepsilon \} \). By considering separately the cases \( 0 \leq (q-p) \cdot (z - z^0) \leq \varepsilon \) and \( -\varepsilon \leq (q-p) \cdot (z - z^0) < 0 \), we find that
\[ \zeta(z) - \frac{[(q - p) \cdot (z - z^0)]^2}{4\varepsilon} = u(z^0) + \frac{p + q}{2} \cdot (z - z^0) - \frac{[(q - p) \cdot (z - z^0)]^2}{2\varepsilon} \]
\[ \geq u(z^0) + \min\{p \cdot (z - z^0), q \cdot (z - z^0)\} \geq u(z) \]
for every \( z \in Q_\varepsilon \). Thus, \( \zeta(z) - u(z) \geq 0 = \zeta(z^0) - u(z^0) \) for every \( z \in Q_\varepsilon \). Consequently, by the definition of viscosity solution,
\[ \min\{-A[\zeta](z_0), \ B(\nabla \zeta(z_0))\} \leq 0. \]

It is easy to calculate,
\[ \nabla \zeta(z_0) = \frac{1}{2} (p + q), \quad D^2 \zeta(z_0) = -\frac{1}{2\varepsilon} (q - p) \otimes (q - p). \]
Since \( z_1^0 \neq 0 \) and \( q_1 - p_1 > 0 \), when \( \varepsilon \) is sufficiently small, we have \( -A[\zeta](z_0) > 0 \). Thus, we must have \( B(\frac{1}{2}(p+q)) \leq 0 \). Since \( 1-\mu_1 \leq p_1 < q_1 \leq 1+\lambda_1 \) we have \( 1-\mu_1 < \frac{1}{2}(p_1+q_1) < 1+\lambda_1 \). Hence, we must have one of the following:

(i) \( p_2 = q_2 = 1 - \mu_2 \),

(ii) \( p_2 = q_2 = 1 + \lambda_2 \).

Let’s first consider the case (i) \( p_2 = q_2 = 1 - \mu_2 \). Note that \( u(z_1^0, \cdot) \) is a concave function with super-differential bounded by \( 1 - \mu_2 \) and \( 1 + \lambda_2 \). Since \( p_2 = q_2 = 1 - \mu_2 \), we see that \( \partial_2 u(z_1^0, z_2) = 1 - \mu_2 \) for all \( z_2 > z_2^0 \). We define

\[
\tilde{z}_2^0 = \inf\{ z_2 \leq z_2^0 | u(z_1^0, z_2) = u(z_0^0) + (1 - \mu_2)(z_2 - z_2^0) \}.
\]

Since \( u(z_1^0, z_2) \leq O(1) + (1 + \lambda_2)z_2 \) for \( z_2 \leq 0 \), we see that \( \tilde{z}_2^0 > -\infty \). In addition, set \( \tilde{z}_0^0 = (z_1^0, \tilde{z}_2^0) \) we have

\[
\begin{align*}
\text{(5.4)} \quad & \left\{ \begin{array}{ll}
u(z_1^0, z_2) = u(\tilde{z}_0^0) + (1 - \mu_2)(z_2 - \tilde{z}_2^0) & \forall z_2 \geq \tilde{z}_2^0 \\
u(z_1^0, z_2) < u(\tilde{z}_0^0) + (1 - \mu_2)(z_2 - \tilde{z}_2^0) & \forall z_2 < \tilde{z}_2^0.
\end{array} \right.
\end{align*}
\]

From the definition of \( \zeta \) and the fact that \( p_2 = q_2 = 1 - \mu_2 \), we obtain from (5.3) that, setting \( \beta = q_1 - p_1 \),

\[
\zeta(z) = u(z) + \frac{[\beta(z_1^0 - z_1^0)]^2}{4\varepsilon} \quad \forall z \in [z_1^0 - \varepsilon \beta^{-1}, z_1^0 + \varepsilon \beta^{-1}] \times \mathbb{R},
\]

\[
\zeta(z) = u(z) \quad \forall z \in \{z_1^0\} \times [\tilde{z}_2^0, \infty),
\]

\[
\zeta(z) > u(z) \quad \forall z \in \{z_1^0\} \times (-\infty, \tilde{z}_2^0).
\]

Using \( \zeta(z_1^0, \tilde{z}_2^0 - \varepsilon) > u(z_1^0, \tilde{z}_2^0 - \varepsilon) \) and continuity, we can find \( \eta \in (0, \varepsilon \beta^{-1}) \) such that \( \zeta(z_1^0, \tilde{z}_2^0 - \eta) > u(z_1^0, \tilde{z}_2^0 - \varepsilon) + \eta \) for all \( z_1 \in [z_1^0 - \eta, z_1^0 + \eta] \). Hence,

\[
\zeta(z_1, z_2) - u(z_1, z_2) \geq \begin{cases}
\frac{(\beta \eta)^2}{4\varepsilon} & \text{if } |z_1 - z_1^0| = \eta, \quad z_2 \in \mathbb{R} \\
\eta & \text{if } z_2 = \tilde{z}_2^0 - \varepsilon, |z_1 - z_1^0| \leq \eta.
\end{cases}
\]

Finally, set \( \tilde{\eta} = \min\{(\lambda_2 + \mu_2)/2, \eta/(2\varepsilon) \}, (\beta \eta)^2/(8\varepsilon^2) \} \) and consider the function

\[
\zeta(z) = \zeta(z) + (z_2 - \tilde{z}_2^0)\tilde{\eta} \quad \text{in } D := (z_1^0 - \eta, z_1^0 + \eta) \times (\tilde{z}_2^0 - \varepsilon, \tilde{z}_2^0 + \varepsilon).
\]

Note that \( \zeta(z) > u(z) \) on the boundary \( \partial D \) of \( D \). In addition, \( \zeta(\tilde{z}) = u(\tilde{z}) \). Now set \( m := \max D(u(z) - \zeta(z)) \). Then \( m \geq 0 \). Let \( \tilde{z} \) be the point such that \( u(\tilde{z}) - \zeta(\tilde{z}) = m \), then \( \tilde{z} \in D \) since \( u < \zeta \) on \( \partial D \). Thus, \( [\zeta(z) + m] - u(z) \geq [\zeta(\tilde{z}) + m] - u(\tilde{z}) = 0 \) for every \( z \in D \). Hence, by definition of viscosity solution, we have

\[
\min\{-A[\zeta + m](\tilde{z}), \quad B(\nabla \zeta(\tilde{z})) \} \leq 0.
\]

However, for every \( z \in D \),

\[
p_1 < \partial_1 \zeta(z) < q_1, \quad \partial_2 \zeta(z) = 1 - \mu_2 + \tilde{\eta} \in (1 - \mu_2, 1 - \lambda_2), \quad D^2 \zeta(z) = \frac{\beta^2}{2\varepsilon} e_1 \otimes e_1.
\]
This implies that $B(\nabla \tilde{\zeta}(\tilde{z})) > 0$. Hence, we must have $-A[\tilde{\zeta}(\tilde{z}) + m(\tilde{z})] \leq 0$. However, since $z_1^0 \neq 0$, we have $-A[\tilde{\zeta}(\tilde{z}) + m(\tilde{z})] \to \infty$ as $\varepsilon \searrow 0$; this contradicts $-A[\tilde{\zeta}(\tilde{z}) + m(\tilde{z})] \leq 0$.

Similarly, we can derive a contradiction in the second case (ii) $p_2 = q_2 = 1 + \lambda_2$.

The contradiction shows that $\partial_z u(z_1, z_2)$ is singleton if $z_1 \neq 0$. Now for each fixed $z_2 \in \mathbb{R}$, consider the one dimensional function $g(t) = u(t, z_2)$. By Lemma 5.2 (4), $\partial g(t) = \partial_z u(t, z_2)$ is singleton if $t \neq 0$. Hence, $g \in C^1(\mathbb{R} \setminus \{0\})$, so the classical partial derivative $\frac{\partial g}{\partial z}(z_1) := g'(z_1)$ exists. Since any limit point of $\partial_z u(\tilde{z})$ as $\tilde{z} \to z$ is in $\partial_z u(z)$, we conclude that $\lim_{\tilde{z} \to z} \frac{\partial g}{\partial z}(\tilde{z}) = \frac{\partial g}{\partial z}(z_1) = 0$ if $z_1 \neq 0$. Hence, $\frac{\partial g}{\partial z}(z_1) \in C(\mathbb{R}^2 \setminus \{\{0\} \times \mathbb{R}\})$. As $u$ is Lipschitz continuous, we also know that $z_1 \partial_z u \in C(\mathbb{R})$. This completes the proof of Theorem 2.3.

#### Appendix A. Basic Theory of The Finite Horizon Problem

In this section, we follow the standard technique connecting the value function $\Phi$ to the HJB equation; that is, we prove Theorem 2.1.

**A.1. The Case of No Risky Assets.**

First we establish a useful lower bound of $\Phi(x, 0, t)$ by considering the category of strategies that do not use risky assets; that is, we consider the strategies where $y_s \equiv 0, L_s \equiv 0, M_s \equiv 0$ for all $s \in [t, T]$. Writing $(c_s, x_s)$ as $(c(s), x(s))$, we have $dx(s) = [r x(s) - \kappa c(s)]ds$. Subject to $x(t) = x$ we obtain

$$
(A.1) \quad x(T) = xe^{r(T-t)} - \int_t^T e^{r(T-s)}c(s)\kappa ds.
$$

Thus, for any consumption strategy $c$, the total utility can be written as

$$
(A.2) \quad J_0^{x,t}[c] := -\int_t^T e^{\beta(s-t)} - \gamma c(s)Kds + Ke^{-\gamma x(T-t)} + \gamma \kappa \int_t^T e^{r(T-s)}c(s)ds - \beta(T-t).
$$

We want to find an optimal consumption strategy that maximizes $J_0^{x,t}$

When $\kappa = 0$, we have $J_0^{x,t}[c] = -Ke^{-\gamma x(T-t)} - \beta(T-t)$.

Next consider the case $\kappa > 0$. The first variation of $J_0^{x,t}$ can be calculated by

$$
\left< \frac{\delta J_0^{x,t}[c]}{\delta c}, \zeta \right> := \lim_{h \to 0} \frac{J_0^{x,t}[c + h\zeta] - J_0^{x,t}[c]}{h} = \kappa \int_t^T e^{-\beta(s-t)}\zeta(s) \left( e^{-\gamma c(s)} - K e^{-\gamma x(T-t)} - \beta(T-t) \right) ds,
$$

where $x(T)$ is as given in (A.1). Hence, if $c^*$ is a critical point of $J_0^{x,t}$, i.e., $\delta J_0^{x,t}[c^*]/\delta c = 0$, then $e^{-\gamma c^*(s)} - Ke^{-\gamma x^*(T-t)} + \beta(T-t) = 0$, where $x^*(T)$ is as in (A.1) with $c$ replaced by $c^*$. Thus $c^*(s) = x^*(T) - [(r - \beta)(T-t) + \ln K]/\gamma$. Using the definition of $x^*(T)$ we then obtain

$$
(A.3) \quad c^*(s) = x\xi(\tau) + \frac{(r - \beta)(s - t - b(\tau)) - Z(\tau)\ln K}{\gamma} \quad \forall s \in [t, T],
$$

where $\tau = T - t$ and $\xi, Z, b$ are defined in (2.7) which satisfy (2.8)-(2.10). Note that $J_0^{x,t}$ is a concave functional, so $c^*$ is the global maximizer.
We have proved the following result: 

**Lemma A.1.** For each \( x \in \mathbb{R} \), the linear function \( c^* \) defined in (A.3), where \( \tau = T - t \) and \( \xi, Z \), and \( b \) are as in (2.7), is the global maximizer of \( J_{0,t}^{x,t} \) defined in (A.2):

\[
J_{0,t}^{x,t}[c] \leq J_{0,t}^{x,t}[c^*].
\]

The strategy that liquidates all risky assets at time \( t \) gives the estimate

\[
\Phi(x, y, t) \geq \Phi(x + \ell(y), 0, t) \geq J_{0,t}^{x+\ell(y),t}[c^*] = -e^{-\gamma \xi (\tau)}|x + \ell(y)| + (r - \beta)b(\tau) - \ln(\xi(\tau) + Z(\tau)) \ln K.
\]

Then we have the following corollary:

**Corollary A.2.** There is the following lower bound for \( \Phi \):

\[
(A.4) \quad \Phi(x, y, t) \geq -e^{-\gamma \xi (\tau)}|x + \ell(y)| + (r - \beta)b(\tau) - \ln(\xi(\tau) + Z(\tau)) \ln K.
\]

### A.2. Separation of Investment and Consumption.

The following result of separation of variables is similar to the one obtained by Davis et al. (1993):

**Lemma A.3.** Let \( \tau = T - t \) and \( \xi \) be as defined in (2.7). Then

\[
(A.5) \quad \Phi(x, y, t) = e^{-\gamma \xi(\tau)} \Phi(0, y, t) \quad \forall (x, y, t) \in \mathbb{R} \times \mathbb{R}^n \times (-\infty, T].
\]

**Proof.** Let \( \mathcal{S} = (C, L, M) \) be an investment-consumption strategy for initial position \((x_t, y_t) = (0, y)\), resulting in the subsequent portfolio \(\{(x_s, y_s)\}_{s \in [t,T]}\). For another initial position \((x, y)\) at time \(t\), we consider the investment-consumption strategy \(\tilde{\mathcal{S}} = (\tilde{C}, \tilde{L}, \tilde{M})\) defined by

\[
(L, M) \equiv (L, M), \quad \tilde{c}_s = c_s + \xi(\tau)x, \quad \tau := T - t.
\]

Denote the corresponding portfolio (starting from \((x, y)\) at time \(t\)) by \(\{\tilde{x}_s, \tilde{y}_s\}_{s \in [t,T]}\). Then \(\tilde{y}_s = y_s\) and \(\tilde{x}_s = x_s + \tilde{x}(s)\) where \(\tilde{x}(s)\) is the solution of \(\frac{d\tilde{x}(s)}{ds} = [r \tilde{x}(s) - \kappa \xi(\tau)x]ds\) subject to \(\tilde{x}(t) = x\). Solving this initial value problem for \(\tilde{x}\) gives

\[
\tilde{x}(T) = \frac{\kappa \xi(\tau)x}{r} + \left( x - \frac{\kappa \xi(\tau)x}{r} \right) e^{rt} = \xi(\tau)x
\]

by the definition of \(\xi(\tau)\) in (2.7). It then follows that

\[
J(\tilde{S}, t) = -Ke^{-\gamma \tilde{x}(T)}U(x_T, y_T) e^{-\beta(T-t)} - \int_t^T e^{-\gamma \xi(\tau)x} e^{-\gamma c_s - \beta(T-s)} \kappa ds = e^{-\gamma \xi(\tau)x} J(S, t).
\]

The relation between \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) is 1-1 and onto, so taking the supremum yields (A.5).


\[ \varphi(x_t, y_t, t) = \varphi(x_T, y_T, T)e^{-\beta(T-t)} - \int_t^T d\left(\varphi_x, y_s, s\right)e^{-\beta(s-t)} \]

\[ = \varphi(x_T, y_T, T)e^{-\beta(T-t)} + \int_t^T V(c_s)e^{-\beta(s-t)}(\kappa ds) - \int_t^T e^{-\beta(s-t)}\sum_{i,j} a_{ij}y_i\partial_{y_j}\varphi dB^i_s \]

\[ + \int_t^T e^{-\beta(s-t)}[(1 + \lambda_i)\partial_x\varphi - \partial_{y_i}\varphi] dB^c_s \]

\[ - \int_t^T e^{-\beta(s-t)}[\partial_t\varphi + L\varphi] ds + \int_t^T e^{-\beta(s-t)}\left[V^*(\partial_x\varphi) - V(c_s) - c_s\partial_x\varphi\right] \kappa ds \]

\[ + \sum_{t \leq s \leq T} e^{-\beta(s-t)}[\varphi(x_{s-}, y_{s-}, s-) - \varphi(x_s, y_s, s)], \]

where \( L^c \) and \( M^c \) are the continuous part of \( L^i \) and \( M^i \) respectively,

\[ V^*(q) := \max_{c \in \mathbb{R}} \left\{ V(c) - cq \right\} \quad \forall q > 0, \]

(A.6) \[ \mathcal{L} \varphi := \frac{1}{2} \sum_{i,j} \sigma_{ij}y_iy_j\partial_{y_iy_j}\varphi + \sum_i \alpha_iy_i\partial_{y_i}\varphi + rx\partial_x\varphi - \beta \varphi + \kappa V^*(\partial_x\varphi) \]

Now suppose \( \varphi \) satisfies \( \varphi(\cdot, T) \geq KU(\cdot) \) and

(A.7) \[ -\partial_t\varphi - \mathcal{L}\varphi \geq 0, \quad (1 + \lambda_i)\partial_x\varphi - \partial_{y_i}\varphi \geq 0, \quad (-1 + \mu_i)\partial_x\varphi + \partial_{y_i}\varphi \geq 0. \]

Combination of (A.7) and \( \partial_x\varphi > 0 \) leads to

\[ \varphi(x_{s-}, y_{s-}, s-) - \varphi(x_s, y_s, s) \geq 0. \]

Now assume \( S = (C, L, M) \) is an admissible strategy. Taking the expectation we obtain

\[ \varphi(x, y, t) \geq E^x_y \left[ KU(x, y, T)e^{-\beta(T-t)} + \int_t^T V(c_s)e^{-\beta(s-t)}(\kappa ds) \right]. \]

Taking the supreme we obtain the following:

**Lemma A.4.** Suppose \( \varphi \) is a smooth function on \( \mathbb{R} \times \mathbb{R}^n \times (-\infty, T] \) satisfying \( \varphi_x > 0, \varphi(\cdot, T) \geq KU(\cdot), \) and (A.7). Then \( \Phi(x, y, t) \leq \varphi(x, y, t) \) for all \( (x, y, t) \in \mathbb{R} \times \mathbb{R}^n \times [0, T] \).

We call such \( \varphi \) a super-solution. Due to the separation property (A.5), we seek a super-solution of the form

\[ \varphi(x, y, t) = -e^{-\gamma(t)x + (r - \beta)b(t) - \ln \xi(t) + Z(t)\ln K - \phi(z, \tau)}, \quad \tau = T - t, \quad z = \gamma(t)y, \]
where $\xi, Z$ and $b$ are defined in (2.7). We can compute

$$-\partial_t \varphi - L \varphi = |\varphi| \left\{ \gamma x (\xi' - r \xi + \kappa \xi^2) + \left[ \frac{\xi'}{\xi} + \kappa \xi - \beta - (r - \beta)(b' + \kappa \xi b) \right] ight. \\
- (Z' + \kappa \xi Z) \ln K + \frac{\xi'}{\xi} \sum_i z_i \partial z_i \phi \\
+ \partial_r \phi - \frac{1}{2} \sum_{i,j} \sigma_{ij} z_i z_j (\partial z_i z_j \phi - \partial z_i \phi \partial z_j \phi) - \sum_i \alpha_i z_i \partial z_i \phi + \kappa \xi \phi \right\}$$

$$- \left( \sum_i \ell_i (z_i) \right)$$

where we have used (2.8)-(2.10) in the last equality and the definition of $A^\tau$ in (2.14). Thus, (A.7) can be written as

$$\min \left\{ \partial_r \phi - A^\tau [\phi], B(\nabla \phi) \right\} \geq 0 \text{ in } \mathbb{R}^n \times (0, \infty).$$

Since $\xi(0) = 1 = Z(0)$ and $b(0) = 0$, the condition $\varphi(\cdot, T) \geq KU(\cdot)$ can be written as

$$\phi(z, 0) \geq \ell(z) = \sum_i \ell_i(z_i),$$

where $\ell$ and $\ell_i$ are as defined in (2.4).

**A.4. Upper Bound.**

Let $k = (k_0, k_1, \cdots, k_n)$ be a constant vector satisfying

$$k_0 \geq 0, \quad 1 - \mu_i \leq k_i \leq 1 + \lambda_i \quad \forall i.$$

Consider the function

$$\bar{\phi}(k; z, \tau) := k_0 b(\tau) + \sum_i k_i z_i.$$

It is easy to see that $\bar{\phi}(k; z, 0) = \sum_i k_i z_i \geq \ell(z), \quad 1 + \lambda_i - \bar{\partial}_z \phi = 1 + \lambda_i - k_i \geq 0$ and $-1 + \mu_i + \bar{\partial}_z \phi = -1 + \mu_i + k_i \geq 0$ for each $i$. Also,

$$\bar{\phi}_r - A^\tau [\bar{\phi}] = (b' + k \xi b)k_0 + \frac{1}{2} \sum_{i,j} (z_i k_i) \sigma_{ij} (z_j k_j) - \sum_i (\alpha_i - r)(z_i k_i)$$

$$= k_0 + \frac{1}{2} \sum_{i,j} (z_i k_i - m_i) \sigma_{ij} (z_j k_j - m_j) - A_0,$$

where $\{m_j\}, A_0$ are as in (2.5). Since $(\sigma_{ij})_{n \times n}$ is positive-definite, taking $k_0 = A_0$ we have $\bar{\phi} - A^\tau \bar{\phi} \geq 0$. Hence, by Lemma A.4,

$$\Phi(x, y, t) \leq -e^{-\gamma \xi(\tau)} x + (r - \beta)(\Theta(\tau) + Z(\tau)) \ln K - \ln \xi(\tau) - \bar{\phi}(k, z, \tau).$$

(A.8)

We are now ready to complete the proof of Theorem 2.1.
A.5. Proof of Theorem 2.1. By (A.4), \( \Phi(x, y, t) > -\infty \). Also, from (A.8) we see that \( \Phi(x, y, t) < 0 \) for every \((x, y, t) \in \mathbb{R} \times \mathbb{R}^n \times (-\infty, T]\). Hence, we can define

\[
\psi(z, \tau) = (r - \beta)b(\tau) + Z(\tau) \ln K - \ln \xi(\tau) - \ln \left| \Phi(0, [\gamma \xi(\tau)]^{-1} z, T - \tau) \right|.
\]

Then by Lemma A.3, we obtain (2.6). Now we establish properties of \( \psi \).

1. Let \( x \in \mathbb{R}, y \in \mathbb{R}^n \) and \( \hat{y} \in \mathbb{R}^n \). Stating at position \((x, \hat{y})\) at time \( t \), one can immediately liquidate \( \hat{y} - y \) amount of risky asset holding to reach the position \((x + \ell(\hat{y} - y), y)\) at time \( t^+\). Hence, we have

\[
\Phi(x, \hat{y}, t) \geq \Phi(x + \ell(\hat{y} - y), y, t) = e^{-\ell(\gamma \xi(\tau)(\hat{y} - y))} \Phi(x, y, t).
\]

In terms of (2.6), this implies that \( \psi(z, \tau) \geq \psi(z, \tau) + \ell(z - z) \). Thus, we obtain (2.11).

2. Combination of the estimate (A.4) (with \( y = z/[\gamma \xi(\tau)] \)) and (2.6) yields the left hand side inequality of (2.12). To show the right hand side inequality, we use (A.8) with \( k_0 := A_0 \)

and (2.6) to derive that \( \psi(z, \tau) \leq \psi(k, z, \tau) = A_0 b(\tau) + \sum_i k_i z_i \) whenever \( k_i \in [1 - \mu_i, 1 + \lambda_i] \) for all \( i \). Hence,

\[
\psi(z, \tau) \leq \min_{1 - \mu_j \leq k_j \leq 1 + \lambda_j} \left( A_0 b(\tau) + \sum_i k_i z_i \right) = A_0 b(\tau) + \ell(z).
\]

3. Thanks to the linearity of transaction costs and the concavity of the exponential utility function, we immediately obtain the concaveness of \( \Phi(\cdot, \cdot, t) \). The concaveness of \( \psi(\cdot, \tau) \) follows by noting (A.9) and the fact that the function \( \ln(\cdot) \) is concave and increasing.

4. Once we know the basic regularity (e.g. \( \Phi \) is Lipschitz continuous), we can use the dynamic programming principle [Pham (2009)] and follow the routine technique detailed by Davis et al. (1993) and Shreve and Soner (1994), to show that \( \psi \) is a viscosity solution of (2.13). Here we do not need the Standing Assumption 2.3 in Shreve and Soner (1994) since we are working on finite horizon problem and we have already shown the (local) boundedness of \( \Phi \). This completes the proof of Theorem 2.1.

Appendix B. The Asymptotic Behavior as \( T \to \infty \).

This section is devoted to the proof of Theorem 2.2. We always assume \( \kappa > 0 \).

We begin with the following estimate:

Lemma B.1. For any \( z \in \mathbb{R}^n \) and \( \tau \geq 0 \),

\[
0 \leq \psi(z, \tau) - z \cdot \partial \psi(z, \tau) \leq A_0 b(\tau).
\]

Proof. The assertion holds for \( z = 0 \) by (2.12).

Fix \( z \in \mathbb{R}^n \setminus \{0\} \). Consider the function

\[
f(s) := \psi(sz, \tau), \quad s \in [0, \infty),
\]

This is a concave and Lipschitz continuous function in one space dimension. Hence, \( f'(s) \) is a decreasing function. Set

\[
p_\infty = \lim_{s \to \infty} \frac{f(s) - f(0)}{s}.
\]
In view of (2.11) and the homogeneity $\ell(sz) = s\ell(z)$ for $s > 0$, we find that $p_\infty \geq \ell(z)$. As $f'(s)$ is a decreasing function, by L'Hôpital's rule, $f'(s) \searrow p_\infty$ as $s \to \infty$. Consequently, for any $s > 0$,

$$0 \leq f(0) \leq f(s) + f'(s)(0 - s) \leq f(s) - p_\infty s \leq A_0 b(\tau) + \ell(sz) - p_\infty s = A_0 b(\tau),$$

by (2.12) and the definition $f(s) = \psi(sz, \tau)$. Note that as super-differential, $z \partial \psi(z, \tau) \subset f'(1)$. Hence, setting $s = 1$ we obtain the assertion of the lemma.

**Proof of Theorem 2.2.** In what follows, we construct viscosity sub and supersolutions to bound the viscosity solution.

For any constant $h > 0$ and smooth function $W(\cdot)$, consider the function $\hat{\psi}(z, \tau) := \psi(z, \tau + h) + W(\tau)$. One can calculate

$$\psi_r(z, \tau + h) - A_r^+ h \psi(z, \tau + h)$$

$$= \hat{\psi}_r - A_r^+ h \hat{\psi} - W'(\tau) - \kappa \xi(\tau + h) W(\tau)$$

$$= \hat{\psi}_r - A_r^+ \hat{\psi} + [\kappa \xi(\tau) - \kappa \xi(\tau + h)](z \cdot \partial \hat{\psi} - \hat{\psi}) - W'(\tau) - \kappa \xi(\tau + h) W(\tau)$$

where

$$f^W(z, \tau) := -[W'(\tau) + \kappa \xi(\tau) W(\tau)]$$

$$+ \kappa [\xi(\tau) - \xi(\tau + h)](z \cdot \partial \psi(z, \tau + h) - \psi(z, \tau + h)).$$

Hence, equation (2.13) with $(\tau, \psi)$ replaced by $(\tau + h, \psi(\cdot, \tau + h))$ can be written as

$$(B.2) \quad \min \{ \hat{\psi}_r - A_r^+ \hat{\psi} + f^W, B(\nabla \hat{\psi}(z, \tau)) \} = 0.$$  

One can show by the definition of viscosity solution that if $\psi$ is a viscosity solution of (2.13), then $\hat{\psi}$ is a viscosity solution of (B.2).

(i) Suppose $0 < r \leq \kappa$. Then $\xi'(\tau) = (r - \kappa)\xi^2 e^{-rt} \leq 0$.

1. Setting $W = 0$ and using the first inequality of (B.1) we find that

$$f^W(z, \tau) = \kappa [\xi(\tau) - \xi(\tau + h)](z \cdot \partial \psi(z, \tau + h) - \psi(z, \tau + h)) \leq 0.$$ 

This implies from (B.2) that $\hat{\psi}(z, \tau) := \psi(z, \tau + h)$ satisfies

$$\min \{ \hat{\psi}_r - A_r^+ \hat{\psi}, B(\nabla \hat{\psi}(z, \tau)) \} \geq 0.$$ 

Also $\hat{\psi}(z, 0) = \psi(z, h) \geq \ell(z) = \psi(z, 0)$. Hence, $\hat{\psi}(z, \tau) := \psi(z, \tau + h)$ is a (viscosity) super-solution, so by comparison principle we have $\psi(z, \tau) \leq \psi(z, \tau + h)$. 

2. Setting \( W = -A_0[b(\tau + h) - b(\tau)] \) and using \( b' + \kappa \xi b = 1 \) and the second inequality in (B.1), we obtain,
\[
f_W(z, \tau) = \kappa [\xi(\tau) - \xi(\tau + h)] [z \cdot \partial \psi(z, \tau + h) - \psi(z, \tau + h) + A_0b(\tau + h)] \geq 0.
\]
Thus, \( \hat{\psi}(z, \tau) := \psi(z, \tau + h) - A_0[b(\tau + h) - b(\tau)] \) satisfies
\[
\min \{ \hat{\psi}_\tau - A^\tau \hat{\psi}, B(\nabla \hat{\psi}(z, \tau)) \} \leq 0.
\]
Also, by (2.12), \( \hat{\psi}(z, 0) = \psi(z, h) - A_0b(h) \leq \ell(z) \). Hence, \( \hat{\psi} \) is a viscosity subsolution, so \( \psi(z, \tau) \geq \psi(z, \tau + h) - A_0[b(\tau + h) - b(\tau)] \).

In conclusion, for every \( h > 0 \),
\[
(B.3) \quad 0 \leq \psi(z, \tau + h) - \psi(z, \tau) \leq A_0[b(\tau + h) - b(\tau)].
\]
 Sending \( h \downarrow 0 \) we obtain
\[
(B.4) \quad 0 \leq \psi_\tau(z, \tau) \leq A_0b'(\tau) = O(1)(1 + \tau)e^{-\tau\tau}.
\]
Thus, \( u(z) := \lim_{\tau \to \infty} \psi(z, \tau) \) exists, and by (2.12), \( 0 \leq u - \ell = A_0b(\infty) = A_0/r \). Finally, sending \( h \to \infty \) we obtain from (B.3) that
\[
0 \leq u(z) - \psi(z, \tau) \leq A_0[b(\infty) - b(\tau)] = O(1)(1 + \tau)e^{-\tau\tau}.
\]
This proves (2.16). Sending \( \tau \to \infty \) in (2.13) and using (B.4), we find that \( u \) is a Lipschitz continuous and concave viscosity solution of (2.17).

(ii) Suppose \( r \geq \kappa \). Then \( \xi' = (r - \kappa)\xi^2e^{-\tau\tau} \geq 0 \).

1. Setting \( W = -A_0b(h)Z(\tau) \) and using \( Z'(\tau) + \kappa \xi(\tau)Z(\tau) = 0 \) we derive that
\[
f_W(z, \tau) = [\kappa \xi(\tau) - \kappa \xi(\tau + h)] [z \cdot \partial \psi(z, \tau + h) - \psi(z, \tau + h) + A_0b(\tau + h)] \geq 0.
\]
Also, \( \hat{\psi}(z, 0) = \psi(z, h) - A_0b(h) \leq \ell(z) \). Hence, \( \hat{\psi} = \psi(z, \tau) - A_0b(h)Z(\tau) \) is a viscosity subsolution, so \( \psi(z, \tau + h) - A_0b(h)Z(\tau) \leq \psi(z, \tau) \).

2. Set \( W = A_0[b(h)Z(\tau) - b(\tau + h) + b(\tau)] \). We derive that
\[
f_W(z, \tau) = [\kappa \xi(\tau) - \kappa \xi(\tau + h)] [z \cdot \partial \psi(z, \tau + h) - \psi(z, \tau + h) + A_0b(\tau + h)] \leq 0.
\]
Also, \( \hat{\psi}(z, 0) = \psi(z, h) \geq \psi(z, 0) \) (by (2.12)). Hence, \( \hat{\psi} \) is a viscosity super-solution, and \( \psi(z, \tau) \leq \psi(z, \tau + h) + A_0[b(h)Z(\tau) - b(\tau + h) + b(\tau)] \).

In summary, we have
\[
(B.5) \quad A_0[b(\tau + h) - b(\tau) - b(h)Z(\tau)] \leq \psi(z, \tau + h) - \psi(z, \tau) \leq A_0b(h)Z(\tau).
\]
Sending \( h \downarrow 0 \) we obtain
\[
A_0[b'(\tau) - Z(\tau)] \leq \psi_\tau(z, \tau) \leq A_0Z(\tau).
\]
In particular, this implies that $|\psi_\tau| = O(1)(1 + \tau)e^{-\tau\tau}$. Consequently, $u = \lim_{\tau \to \infty} \psi(z, \tau)$ exists. Also, sending $h \to \infty$ we obtain from (B.5) that

$$A_0 \left[ \frac{1}{r} - b(\tau) - \frac{1}{r} Z(\tau) \right] \leq u(z) - \psi(z, \tau) \leq \frac{A_0}{r} Z(\tau).$$

This implies that $|\psi(z, \tau) - u(z)| = O(1)(1 + \tau)e^{-\tau\tau}$. Finally, sending $\tau \to \infty$ in (2.13), we find that $u$ is a viscosity solution of (2.17). This completes the proof of Theorem 2.2.

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