Lecture 6&7
Fourier collocation method for differential equations

Katarina Gustavsson

MA5251 Spectral Methods and Applications, 2011

Partial Differential Equations, PDEs

Many different types of partial differential equations:
• Elliptic
• Parabolic
• Hyperbolic
• Time dependent
• Linear
• Non-linear
All of them can be solved by spectral methods, however some are more suitable than others…
Some typical examples of PDEs

- Laplace/Poisson equation: $-u_{xx} = f(x)$
- Helmholtz equation: $-(a(x)u_x)_x + \lambda u = f(x)$
- Linear wave equation: $u_t + au_x = 0$
- Heat equation: $u_t - \epsilon u_{xx} = 0$
- Burgers equation: $u_t + uu_x - \epsilon u_{xx} = 0$
- Korteweg-de Vries equation: $u_t + uu_x + u_{xxx} = 0$

In order to solve any of these equations we need to add appropriate boundary conditions and initial conditions.

Strong and weak form of a PDE

- **Strong form**: the PDE is required to be satisfied at each point in its domain for each time – Collocation methods
- **Weak form**: require that the integral of the PDE against all functions in an appropriate space of test functions to be satisfied - Galerkin methods

**Example**: 

Strong form: $-u_{xx} + \lambda u = f(x)$ \quad $x \in (a,b)$

Weak form: 

Find $u(x) \in V$ such that: $a(u,v) - L(v) = 0 \quad \forall v \in V$

where \(a(u,v) = \left[u_x v_x\right]^b_a + \int_a^b (u_x v + \lambda u v) dx\) and \(L(v) = \int_a^b f v dx\)
Fourier collocation method for PDEs
Linear, time independent problems

Example:
Find the $2\pi$-periodic approximate solution of
\[-u_{xx} + \lambda u = f(x) \quad \lambda > 0, \text{ a constant} \quad (1)\]

- The approximate solution $u^N(x_j)$ is represented as the discrete Fourier series:
  \[u^N(x_j) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx_j}, \quad x_j = \frac{2\pi}{N} j \text{ (collocation points)} \quad (2)\]

- The collocation method requires
  \[-u^N_{xx}(x_j) + \lambda u^N(x_j) - f(x_j) = 0, \quad j = 0, 1, \ldots, N-1 \quad (3)\]

- If we use the expansion, (2) in (3) we obtain for $\tilde{u}_k$
  \[\tilde{u}_k = \frac{\tilde{f}_k}{k^2 + \lambda} \quad (4)\]

- The approximate solution to (1) can be written as
  \[u^N(x_j) = \sum_{k=-N/2}^{N/2-1} \frac{\tilde{f}_k}{k^2 + \lambda} e^{ikx_j} \quad (4)\]

Fourier collocation method for PDEs
Linear, time dependent problems

- We will only use spectral approximation in space
- If $u$ is also a time dependent function, $u(x,t)$, the expansion will be
  \[u^N(x_j,t) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k(t)e^{ikx_j} \quad (4)\]

- Note that the discrete Fourier coefficients now depend on time

Example: The heat equation
\[u_t - \epsilon u_{xx} = 0, \quad u(x,0) = g(x) \]

together with the above expansion will then yield
\[\frac{d}{dt} \tilde{u}_k(t) + \epsilon k^2 \tilde{u}_k(t) = 0, \quad \tilde{u}_k(0) = \tilde{g}_k, \quad k = -N/2, \ldots, N/2 - 1 \]

- So for each Fourier coefficient we obtain an ODE that needs to be solved
- We can time step in Fourier space and transform back to real space only when needed.
Time discretization

\[ \frac{du_j^n}{dt} = \frac{u_j^{n+1} - u_j^n}{\Delta t}, \quad u_j^n = u(x_j,t^n) \]

• Often use finite differences in time; Euler, leap-frog, Runge-Kutta…
• In principle, spectral accuracy is sacrificed
• However, time step formulas of order two or higher with a small time step often leave the global accuracy satisfactory
• Better with small time steps then small space steps since it only affects the computational time linearly. Small space steps require a larger storage. Halving the space step typically multiplies the storage by \(2^d\) where \(d\) is the dimension of the problem

Example

• Solution of the linear wave equation with non-constant wave speed
  \[ u_t + c(x)u_x = 0, \quad c(x) = \frac{1}{2} \sin^2(x - 1), \quad 0 \leq x \leq 2\pi \]
  Initial conditions: \( u(x,0) = e^{-100(x-1)^2} \)
  Boundary conditions: \( u(0,t) = u(2\pi,t) \)
• Solved by a Fourier collocation method in space and leap-frog in time
• Numerical parameters: \( N=128, \quad \Delta t = 0.005 \)
Example, cont.

Convergence study:

• For each $N = 16,32,64,…$ form the trigonometric interpolant $I_N u(x) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx}$

• Study the difference between $I_N u(x)$ and $I_{N/2} u(x)$: $\max_{0 \leq x \leq \pi} |I_N u(x) - I_{N/2} u(x)|$ numerically by using a fine mesh (small $h$) for $x$

Spectral accuracy

Example, cont.

Fourier derivative matrix vs. FFT

• For small $N$ the matrix approach faster

• For larger $N$, FFT faster
Fourier derivative matrix vs. FFT

\[ v = v_0 \]

for ii = 1 to number of time steps
  \[ t = t + dt; \]
  \[ v_{\text{hat}} = \text{fft}(v); \]
  \[ w_{\text{hat}} = i \cdot [0:N/2-1 0 -N/2+1:-1] \cdot v_{\text{hat}}; \]
  \[ w = \text{real}(\text{ifft}(w_{\text{hat}})); \]
  \[ v_{\text{new}} = v_{\text{old}} - 2 \cdot dt \cdot c \cdot w; \]
  \[ v_{\text{old}} = v; \]
  \[ v = v_{\text{new}}; \]
end

\[ D = \text{Fouriermatrixt}(N); \]
\[ v = v_0 \]

for ii = 1 to number of time steps
  \[ t = t + dt; \]
  \[ v_{\text{new}} = v_{\text{old}} - 2 \cdot dt \cdot c \cdot (D \cdot v); \]
  \[ v_{\text{old}} = v; \]
  \[ v = v_{\text{new}}; \]
end

Need only to create Fourier derivative matrix once since it only depends on \( N \)

Time step restriction - stability

- If \( \Delta t = 0.007 \) instead of \( \Delta t = 0.005 \), the numerical time-stepping scheme becomes unstable.
- The solution grows unbounded in time even though the exact solutions should stay bounded.
- The numerical time-stepping scheme is unstable for \( \Delta t \geq \) threshold value.
- Threshold value is called the stability limit of the numerical scheme.

- When using leap frog for our advection problem, the stability limit is given by

\[ \Delta t \leq \frac{2}{N \max(c(x_j))} \]
Time-discretization and stability

Rule of thumb*:

The time stepping method is stable if the eigenvalues of the (linearized) spatial discretization operator, $\lambda$, scaled by $\Delta t$ (or $\Delta t^2$ for second order derivatives), lie in the stability region of the time-discretization operator.

Ex: Stability region for leap frog and a first derivative is $-i \leq \Delta t \lambda \leq i$

* L. N. Trefethen

The stability regions for different time-discretization schemes can be found in any of the text books recommended for this course.
Method of integrating factors

When we have a second derivative the time-step restriction can be severe

Ex: \( \frac{d}{dt} \hat{u}_k + \epsilon k^2 \hat{u}_k = \hat{F}_k, \hat{F}_k \) could e.g. the non-linear term in

Burgers equation (later)

Multiply by \( e^{\epsilon k^2 t} \) (integrating factor) and we obtain

\[
e^{\epsilon k^2 t} \frac{d}{dt} \hat{u}_k + \epsilon k^2 e^{\epsilon k^2 t} \hat{u}_k = e^{\epsilon k^2 t} \hat{F}_k \tag{1}
\]

Let \( \tilde{U}_k = e^{\epsilon k^2 t} \hat{u}_k \) so \( \frac{d}{dt} \tilde{U}_k = \epsilon k^2 \tilde{U}_k + e^{\epsilon k^2 t} \frac{d}{dt} \hat{u}_k \)

and (1) can be written as

\[
\frac{d}{dt} \tilde{U}_k = \epsilon k^2 \tilde{U}_k + \epsilon k^2 \tilde{U}_k = e^{\epsilon k^2 t} \hat{F}_k \quad \Rightarrow \quad \frac{d}{dt} \hat{u}_k = e^{\epsilon k^2 t} \hat{F}_k
\]

With e.g. Euler's method we obtain \( \tilde{U}_k^{n+1} = \tilde{U}_k^n + \Delta t e^{\epsilon k^2 (\tilde{U}_k^n)} \) or in terms of \( \tilde{u}_k \)

\[
e^{\epsilon k^2 (\tilde{u}_k^n + \Delta t \epsilon k^2 \tilde{u}_k^n)} = e^{\epsilon k^2 \tilde{u}_k^n + \Delta t e^{\epsilon k^2 \tilde{u}_k^n}} \quad \Rightarrow \quad \tilde{u}_k^{n+1} = e^{-\epsilon k^2 \tilde{u}_k^n} (\tilde{u}_k^n + \Delta t \epsilon k^2 \tilde{u}_k^n)
\]

Fourier collocation method for PDEs
Non-linear problems

- Illustrate by an example; Burgers equation

\[
u_t + u u_x = \varepsilon u_{xx}, \quad u(x,0) = u^0(x)
\]

- Can be written in non-conservative form as above or in

conservative form

\[
u_t + \frac{1}{2} u_x^2 = \varepsilon u_{xx}, \quad u(x,0) = u^0(x)
\]
Example

- Solution to Burgers equation with $\varepsilon = 0.2$
- Solution displayed at $t = 0$ and $t = \pi / 8$
- Solution approximated with $N = 16$, $N = 32$ and $N = 64$
- Discretized in time with 4th order Runge-Kutta with $\Delta t = 0.0001$
- For $N = 16$ and $N = 32$ the transition zone is not resolved and oscillations are displayed

Approximation of non-linear term

- The approximation of the non-linear term
  $u^N(t) = (u^N_0(t), u^N_1(t), \ldots, u^N_{N-1}(t))^T$
  $u^N(t) \otimes u^N(t) = (u^N_0(t) \cdot u^N_0(t), u^N_1(t) \cdot u^N_1(t), \ldots, u^N_{N-1}(t) \cdot u^N_{N-1}(t))^T$
- Conservative form: $\frac{1}{2} \sum_i u_i^2 = \frac{1}{2} \text{D}_N (u^N(t) \otimes u^N(t))$
- Non-conservative form: $u u_t = u^N(t) \otimes \text{D}_N u^N(t)$
Gibbs phenomenon and filtering

- Solution filtering
  \[ u_{N,f}(x_j,t^n) = \sum_{k=-N/2}^{N/2} \sigma_k(t^n)e^{ikx_j} \]
  \( \sigma \) is a filtering function

- Exponential filter
  \[ \sigma_k = \sigma(2\pi k/N) = \sigma(\theta) \]
  \[ \sigma(\theta) = e^{-\alpha \theta}, \alpha > 0 \]

- \( \alpha \) and \( p \) parameters that have to be found by trial-and-error looking for the best trade-off between accuracy and temporal stability

- Accuracy of solution will be of max order \( p \)

Solutions with sharp gradients/discontinuities will suffer from a growth in time of the high frequencies which can lead to a temporal instability.