On transformations between Gabor frames and wavelet frames

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Abstract—**We describe a procedure that enables us to construct dual pairs of wavelet frames from certain dual pairs of Gabor frames. Applying the construction to Gabor frames generated by appropriate exponential B-splines gives wavelet frames generated by functions whose Fourier transforms are compactly supported splines with geometrically distributed knot sequences. There is also a reverse transform, which yields pairs of dual Gabor frames when applied to certain wavelet frames.**

I. Introduction

In this note we will discuss a procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames, and vice versa. Applying this to Gabor frames generated by exponential B-splines produces a class of attractive dual wavelet frame pairs generated by functions whose Fourier transform are compactly supported splines with geometrically distributed knots. Our main purpose here is to demonstrate the usefulness of the method; the proofs of the theoretical results are given in [2].

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\,||f||^2,\ \forall f \in \mathcal{H}. \qquad \text{(I.1)}$$

The constants A and B are *frame bounds*. The sequence $\{f_i\}_{i\in I}$ is a *Bessel sequence* if at least the upper bound in (I.1) is satisfied. A frame is *tight* if we can choose $A = B$ in (I.1). For any frame $\{f_i\}_{i\in I}$ there exists at least one *dual frame,* i.e., a frame $\{\widetilde{f}_i\}_{i\in I}$ for which

$$f = \sum_{i\in I} \langle f, f_i\rangle \widetilde{f}_i,\ \forall f \in \mathcal{H}.$$

We will consider Gabor frames and wavelet frames in the Hilbert space $L^2(\mathbb{R})$. A *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given function $g \in L^2(\mathbb{R})$. Using the *translation operators* $T_af(x) := f(x-a), a \in \mathbb{R}$, and the *modulation operators* $E_bf(x) := e^{2\pi ibx}f(x), b \in \mathbb{R}$, both acting on $L^2(\mathbb{R})$, we will denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. On the other hand, a *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for some parameters $a > 1, b > 0$ and a given function $\psi \in L^2(\mathbb{R})$. Introducing the *scaling operators* $(D_af)(x) := a^{1/2}f(ax)$, $a > 0$, acting on $L^2(\mathbb{R})$, the wavelet system can be written as $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$.

The duality conditions for a pair of Gabor systems were obtained by Ron & Shen [9], [10]. We state the formulation due to Janssen [8]:

Theorem 1.1: Given $b, \alpha > 0$, two Bessel sequences $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$, where $g, \tilde{g} \in L^2(\mathbb{R})$, form dual Gabor frames for $L^2(\mathbb{R})$ if and only if for all $n \in \mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}} \overline{g(x+j\alpha)}\tilde{g}(x+j\alpha+n/b) = b\delta_{n,0},\ a.e.\ x \in \mathbb{R}.$$

There are also characterizing equations for dual wavelet frames; see [5]. They are formulated in terms of the Fourier transform, for $f \in L^1(\mathbb{R})$ defined by $\widehat{f}(\gamma) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i\gamma x}dx$, and extended to $L^2(\mathbb{R})$ in the usual way.

Theorem 1.2: Given $a > 1$, $b > 0$, two Bessel sequences $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$, where $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$, form dual wavelet frames for $L^2(\mathbb{R})$ if and only if the following two conditions hold:

(i) $\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma) = b$ for a.e. $\gamma \in \mathbb{R}$.

(ii) For any number $\alpha \neq 0$ of the form $\alpha = m/a^j$, $m, j \in \mathbb{Z}$,

$$\sum_{(j,m)\in I_\alpha} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma + m/b) = 0, \ a.e. \ \gamma \in \mathbb{R},$$

where $I_\alpha := \{(j,m) \in \mathbb{Z}^2 \mid \alpha = m/a^j\}$.

For more information on fundamental results of Gabor frames and wavelet frames, see, e.g., [1], [7], and [6].

II. From Gabor frames to wavelet frames

The goal of this section is to show how we can construct dual wavelet frame pairs based on certain dual Gabor frame pairs. The key is the following transform that allows us to move the Gabor structure into the wavelet structure.

Let $\theta > 1$ be given. Associated with a function $g \in L^2(\mathbb{R})$ for which $g(\log_\theta |\cdot|) \in L^2(\mathbb{R})$, we define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma) = \begin{cases} g(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \quad \text{(II.1)}$$

Note that by (II.1), for any $a > 0, j \in \mathbb{Z}$ and $\gamma \in \mathbb{R}\backslash\{0\}$,

$$\widehat{\psi}(a^j\gamma) = g(j\log_\theta(a) + \log_\theta(|\gamma|)). \quad \text{(II.2)}$$

Also, if $g \in L^2(\mathbb{R})$ is a bounded function with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$, then

$$\operatorname{supp}\widehat{\psi} \subseteq [-\theta^N, -\theta^M] \cup [\theta^M, \theta^N].$$

Note that (II.2) gives a convenient way to obtain functions ψ with the partition of unity property

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(a^j\gamma) = 1, \ \gamma \in \mathbb{R}. \quad \text{(II.3)}$$

Indeed, just take any function g satisfying the partition of unity condition

$$\sum_{j\in\mathbb{Z}} g(x+j) = 1, \ x \in \mathbb{R}, \quad \text{(II.4)}$$

and apply the construction in (II.1) with $\theta := a$. Comparing the corresponding conditions in Theorem 1.2(i) and Theorem 1.1, (II.3) provides a possible starting point for constructing dual wavelet frames, similar to (II.4) for dual Gabor frames, see, e.g., [3].

If g has compact support and is smooth, then the function $\widehat{\psi}$ in (II.1) is also smooth. Thus, by taking smooth functions g we obtain functions ψ with fast decay in the time domain.

A. Construction of dual pairs of wavelet frames

For fixed parameters $b, \alpha > 0$ we will consider two bounded compactly supported functions $g, \tilde{g} \in L^2(\mathbb{R})$ and the associated Gabor systems $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$. For a fixed $\theta > 1$, define the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ by (II.1) from $g, \tilde{g}$ respectively.

Theorem 2.1: Let $b > 0$, $\alpha > 0$, and $\theta > 1$ be given. Assume that $g, \tilde{g} \in L^2(\mathbb{R})$ are bounded functions with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$ and that $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. With $a := \theta^\alpha$, if $b \leq \frac{1}{2\theta^N}$, then $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.

The proof follows from (II.2) and the characterizations of duality for Gabor frames and wavelet frames in Theorem 1.1 and Theorem 1.2.

If $g = \tilde{g}$ in Theorem 2.1, then $\psi = \widetilde{\psi}$, i.e., the result enables a tight wavelet frame to be constructed from a tight Gabor frame.

B. Explicit constructions

Based on Theorem 2.1, the rich theory for construction of dual pairs of Gabor frames enables us to provide explicit constructions of wavelet frame pairs.

Proposition 2.2: Let $g \in L^2(\mathbb{R})$ be a bounded real-valued function with support in the interval $[M, N]$ for some $M, N \in \mathbb{Z}$. Suppose that g satisfies the partition of unity condition (II.4). Let $a > 1$ and $b \in (0, \min(\frac{1}{2(N-M)-1}, 2^{-1}a^{-N})]$ be given, and take any real sequence $\{c_n\}_{n=-N+M+1}^{N-M-1}$ such that

$$c_0 = b, c_n + c_{-n} = 2b, \ n = 1, \ldots, N-M-1.$$

Then the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ defined by (II.1) and

$$\widehat{\widetilde{\psi}}(\gamma) = \sum_{n=-N+M+1}^{N-M-1} c_n g(\log_a(|\gamma|) + n), \ \gamma \neq 0, \quad \text{(II.5)}$$

generate dual wavelet frames $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. It follows from Theorem 3.1 in [3] that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and the Gabor system generated by $\tilde{g}(x) = \sum_{n=-N+M+1}^{N-M-1} c_n g(x+n)$ form dual Gabor frames for $L^2(\mathbb{R})$ (the condition $b \leq \frac{1}{2(N-M)-1}$ is assumed in that result). Now the result follows from Theorem 2.1 with $\theta := a$. □

We will now consider a class of exponential B-splines that yields attractive dual pairs of wavelet frames, for which the Fourier transform of the generators are compactly supported splines with geometrically distributed knots and desired smoothness. These exponential splines are of the form

$$\mathcal{E}_N(\cdot) := e^{\beta_1(\cdot)}\chi_{[0,1]}(\cdot) * \cdots * e^{\beta_N(\cdot)}\chi_{[0,1]}(\cdot),$$

where $\beta_k = (k-1)\beta$, $k = 1, \ldots, N$, for some $\beta > 0$. Similar to the classical B-splines given by the choice $\beta_k = 0, k = 1, \ldots, N$, the exponential B-spline $\mathcal{E}_N$ is $N-2$ times differentiable (for $N \geq 2$) and its support is $[0, N]$. An explicit formula for $\mathcal{E}_N$ is given by Theorem 2.2 in [4] (note that there is a typo in the expression for $\mathcal{E}_N(x)$ for $x \in [k-1, k]$ on page 304 of [4]: the expression $e^{a_{j_1}} + \cdots + e^{a_{j_{k-1}}}$ should be $e^{a_{j_1} + \cdots + a_{j_{k-1}}}$). In Theorem 3.1 in the same paper, it is shown that for $N \geq 2$,

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_N(x-k) = \frac{\prod_{m=1}^{N-1}(e^{\beta m}-1)}{\beta^{N-1}(N-1)!}. \qquad \text{(II.6)}$$

For the partition of unity constraint (II.4) to hold, we apply (II.6) and consider the function

$$g(x) := \frac{\beta^{N-1}(N-1)!}{\prod_{m=1}^{N-1}(e^{\beta m}-1)}\, \mathcal{E}_N(x).$$

Furthermore, let $a := e^{\beta}$. For $\gamma \neq 0$, using that $e^{\beta k \log_{e^\beta}(|\gamma|)} = |\gamma|^k$, we obtain from (II.1) an expression that identifies $\widehat{\psi}$ explicitly as a geometric spline, i.e., as a spline with geometrically distributed knots. Now the formula (II.5) yields a dual wavelet frame generator $\widetilde{\psi}$. Note that $\widehat{\widetilde{\psi}}$ is also a geometric spline.

Example 2.3: Consider the exponential B-spline $\mathcal{E}_3$ with $N = 3$ and $\beta = 1$. Then

$$\mathcal{E}_3(x) = \begin{cases} \frac{1-2e^x+e^{2x}}{2}, & x \in [0,1], \\ \frac{-(e+e^2)+2(e^{-1}+e)e^x-(e^{-2}+e^{-1})e^{2x}}{2}, & x \in [1,2], \\ \frac{e^3-2e^x+e^{-3}e^{2x}}{2}, & x \in [2,3], \\ 0, & x \notin [0,3]. \end{cases}$$

By (II.6) we have

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_3(x-k) = \frac{1}{2}(e-1)(e^2-1), \; x \in \mathbb{R},$$

so we consider $g(x) := 2(e-1)^{-1}(e^2-1)^{-1}\mathcal{E}_3(x)$.

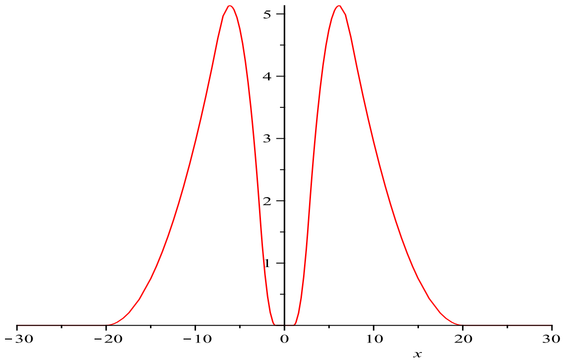

Fig. 1. Plot of the geometric spline $\widehat{\psi}$ in Example 2.3.

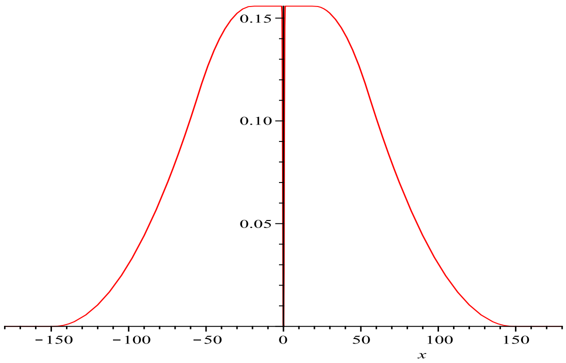

Fig. 2. Plot of the geometric spline $\widehat{\widetilde{\psi}}$ in Example 2.3.

Let $a := e^{\beta} = e$, and define the function ψ by

$$\widehat{\psi}(\gamma) = \begin{cases} \frac{1-2|\gamma|+\gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [1,e], \\ \frac{-(e+e^2)+2(e^{-1}+e)|\gamma|-(e^{-2}+e^{-1})\gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [e,e^2], \\ \frac{e^3-2|\gamma|+e^{-3}\gamma^2}{(e-1)(e^2-1)}, & |\gamma| \in [e^2,e^3], \\ 0, & |\gamma| \notin [1,e^3]. \end{cases}$$

The function $\widehat{\psi}$ is a geometric spline with knots at the points $\pm 1, \pm e, \pm e^2, \pm e^3$.

The construction in Proposition 2.2 works for $b \leq 2^{-1}e^{-3}$. Taking $b = 41^{-1}$ and $c_n = 41^{-1}$ for $n = -2, \ldots, 2$, it follows from (II.2) and (II.5) that the resulting dual frame generator $\widetilde{\psi}$ satisfies

$$\widehat{\widetilde{\psi}}(\gamma) = \frac{1}{41}\sum_{n=-2}^{2} \widehat{\psi}(e^n\gamma), \; \gamma \in \mathbb{R}.$$

The function $\widehat{\widetilde{\psi}}$ is a geometric spline with knots at the points $\pm e^{-2}, \pm e^{-1}, \pm 1, \pm e^3, \pm e^4, \pm e^5$.

Figures 1–3 show the graphs of the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$, where Figure 3 re-plots part of the graph in Figure

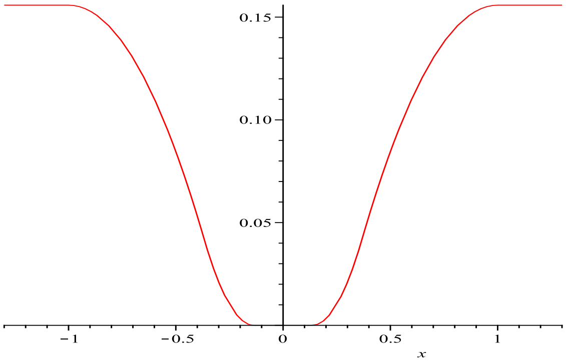

Fig. 3. Plot of the geometric spline $\widehat{\widehat{\psi}}$ in Example 2.3 on the interval $[-1.3, 1.3]$.

2 on a smaller interval to better depict the behavior of $\widehat{\widehat{\psi}}$ around 0. Note that $\widehat{\widehat{\psi}}$ is constant on the support of $\widehat{\psi}$ and decays to zero outside this set. This is due to (II.6) and the special structure of $\widehat{\widehat{\psi}}$ in (II.5). In fact, the same will occur when the construction is applied to any function whose integer-translates form a partition of unity. If higher order smoothness in $\widehat{\psi}$ and $\widehat{\widehat{\psi}}$ is desired, this can be achieved if we use higher order exponential B-splines in the construction. □

III. From wavelet frames to Gabor frames

It is possible to reverse the process discussed so far, and obtain a way to obtain Gabor frames based on certain wavelet frames. Assume the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ to be given. For a parameter $\theta > 1$ we define the functions $g, \tilde{g}$ by

$$g(x) := \widehat{\psi}(\theta^x), \quad \tilde{g}(x) := \widehat{\widetilde{\psi}}(\theta^x), \ x \in \mathbb{R}. \qquad \text{(III.1)}$$

The conditions below imply that $g, \tilde{g} \in L^2(\mathbb{R})$.

Theorem 3.1: Let $a > 1$ and $b > 0$. Assume that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$ and that the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are supported in $[-L, -K] \cup [K, L]$ for some $K, L > 0$. Take $\theta > 1$ and $\alpha > 0$ such that $a = \theta^\alpha$. If $b \le \frac{1}{\log_\theta(L/K)}$, then $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$.

Theorem 3.1 is proved using the characterizations of dual pairs of Gabor frames and wavelet frames in Theorem 1.1 and Theorem 1.2. Again, the result has an immediate consequence for construction of tight Gabor frames via tight wavelet frames.

The result can, e.g., be applied to the Meyer wavelet, which yields a construction of a tight Gabor frame generated by a $C^\infty(\mathbb{R})$, compactly supported function. Details of this are provided in [2].

Let us end this note with a short explanation of why we speak about (III.1) being a reverse transform of (II.1). If we start with a sufficiently well behaving function ψ and use the transform (III.1), we obtain the function $g(x) = \widehat{\psi}(\theta^x)$. Going "back" with the procedure in (II.1) applied on the function g, we arrive at the function

$$\widehat{\phi}(\gamma) = g(\log_\theta(|\gamma|)) = \widehat{\psi}(\theta^{\log_\theta(|\gamma|)}) = \widehat{\psi}(|\gamma|), \ \gamma \neq 0.$$

So, if the function $\widehat{\psi}$ is symmetric, we have that $\phi = \psi$.

On the other hand, starting with a function g and using (II.1), we obtain the function ψ, given by $\widehat{\psi}(\gamma) = g(\log_\theta(|\gamma|))$, $\gamma \neq 0$; applying the approach in (III.1) on $\widehat{\psi}$ leads to the function

$$h(x) = \widehat{\psi}(\theta^x) = g(\log_\theta(|\theta^x|)) = g(x), \ x \in \mathbb{R}.$$

Thus, we get the original function back.

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