Fourier-like frames on locally compact abelian groups

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Abstract

We consider a class of functions, defined on a locally compact abelian group by letting a class of modulation operators act on a countable collection of functions. We derive sufficient conditions for such a class of functions to form a Bessel sequence or a frame and for two such systems to be dual frames. Explicit constructions are obtained via various generalizations of the classical B-splines to the setting of locally compact abelian groups.

1 Introduction

It is known that several parts of the classical frame theory for structured function systems (e.g., Gabor systems and generalized shift-invariant systems on $\mathbb{R}^s$) have parallel versions on locally compact abelian (LCA) groups. For example, the Ron and Shen theory for shift-invariant spaces were generalized to this setting by Cabrelli and Paternostro [1], while the equations characterizing Parseval frames via generalized shift-invariant systems were obtained in the LCA-group setting by Kutyniok and Labate [15].

In this paper we consider a class of functions, defined on an LCA group by letting a class of modulation operators act on a given countable collection of functions. Via the Fourier transform these systems correspond to generalized shift-invariant systems, and we refer to such systems as Fourier-like systems. In contrast to most contributions in the literature, our focus is on simple sufficient conditions and explicit constructions, rather than characterizations. These constructions show that the theory is applicable in practice on LCA groups, not just on the formal level of deriving the theorems.

The concrete constructions will be based on a generalization of the classical B-splines to the setting of LCA groups that was presented already in 1994, independently by Dahlke [7] and Tikhomirov [20]. We will, however, provide more freedom than in these papers, by allowing certain weights to appear in the definition of the splines; in the concrete setting of $\mathbb{R}$, this yields a class of splines that also contains the exponential B-splines and several other types of splines.

There are many advantages of the LCA-group approach. Besides the impact of the generalization to the LCA-group case itself, it enables us to consider key questions in frame analysis from an abstract angle, and to cover the analysis of several types of systems of functions with a notation that is much simpler than the one applied in the calculations for a concrete case. Another advantage of the group-theoretical approach is that it immediately covers the higher-dimensional case (even the matrix case), without any notational complication compared to the one-dimensional setting.

Often the LCA-group approach is considered as "just" a unified way to the analysis on the four elementary groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_m$ (the finite group of integers modulo m) and their higher dimensional variants. However, the unifying approach has more merits than that. As explained in the book [2] by Cariolaro and the paper [8] by Feichtinger and Kozek, signal processing often involves products of the above four groups; for example, as mentioned in [8], a multichannel video signal can be considered as a function in $\mathbb{Z}^p \times \mathbb{Z}_m$, where p is the number of channels and m is the pixel number of each image. The general LCA-group approach applies to all groups of the form $G = \mathbb{R}^s \times \mathbb{Z}^p \times \mathbb{T}^q \times \mathbb{Z}_m$, with concrete conditions that are easy to verify, while a direct derivation would be rather painful.

The paper is organized as follows. In Section 2 we give a brief introduction to the classical harmonic analysis on LCA groups, with focus on the relationship between the group and its dual group, and the Fourier transform. Section 3 contains the main results and the explicit frame constructions, while Section 4 translates the results into the setting of generalized shift-invariant systems. As standard references to harmonic analysis on LCA groups we refer to [19] by Rudin and [12] by Hewitt and Ross. Within the particular area of frame expansions we mention the papers [1] by Cabrelli and Paternostro, [10] by Gröchenig, and [15] by Kutyniok and Labate.

2 Preliminaries on LCA groups

2.1 LCA groups

Let G denote an LCA group, with the group composition denoted by the symbol " + " and neutral element 0. For technical reasons (see below) we will assume that G is equipped with a Hausdorff topology, and that G is a countable union of compact sets and metrizable. A *character* on G is a function $\gamma : G \to \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, for which $\gamma(x+y) = \gamma(x)\gamma(y),\ \forall x, y \in G$. The set of continuous characters is denoted by $\widehat{G}$, and also forms an LCA group, the *dual group* of G, when equipped with an appropriate topology and the composition

$$(\gamma + \gamma')(x) := \gamma(x)\gamma'(x),\ \ \gamma, \gamma' \in \widehat{G}, x \in G.$$

The assumptions on G imply that also $\widehat{G}$ is a countable union of compact sets and metrizable.

One can prove that $\widehat{\widehat{G}} = G$, so $\gamma(x)$ can either be interpreted as the action of $\gamma \in \widehat{G}$ on $x \in G$, or as the action of $x \in \widehat{\widehat{G}} = G$ on $\gamma \in \widehat{G}$. For this reason we will from now on use the notation

$$(x, \gamma) := \gamma(x),\ \ x \in G,\ \ \gamma \in \widehat{G}.$$

A (uniform) *lattice* in the LCA group G is a discrete subgroup Λ for which G/Λ is compact; the *annihilator* $\Lambda^\perp$ of Λ is defined by

$$\Lambda^\perp := \{\gamma \in \widehat{G} \mid (x,\gamma) = 1, \ \forall x \in \Lambda\}.$$

It follows from the definition of the topology on $\widehat{G}$ that the annihilator $\Lambda^\perp$ is a closed subgroup of $\widehat{G}$. Lattices are known explicitly in most of the classical LCA groups; however, there also exist LCA groups without lattices, see, e.g., [13] and [14].

Let us collect some of the classical results about these concepts. The proofs can be found, e.g., in [12].

Lemma 2.1 *Let G be an LCA group and Λ a lattice in G. Then the following hold:*

(i) *If G is discrete, then $\widehat{G}$ is compact.*

(ii) *If G is compact, then $\widehat{G}$ is discrete.*

(iii) $\widehat{G/\Lambda} = \Lambda^\perp$ (*in the sense of topological group isomorphism*).

(iv) $\widehat{\widehat{G}/\Lambda^\perp} = \Lambda$ (*in the sense of topological group isomorphism*).

A lattice in G leads to a splitting of the group G, as well as the dual group, into disjoint cosets:

Lemma 2.2 *Let G be an LCA group and Λ a lattice in G. Then the following hold:*

(i) *There exists a Borel measurable relatively compact set $Q \subset G$ such that*

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + Q), \quad (\lambda + Q) \cap (\lambda' + Q) = \emptyset \textit{ for } \lambda \neq \lambda', \ \lambda, \lambda' \in \Lambda.$$

(ii) *The set $\Lambda^\perp$ is a lattice in $\widehat{G}$, and there exists a Borel measurable relatively compact set $V \subset \widehat{G}$ such that*

$$\widehat{G} = \bigcup_{\omega \in \Lambda^\perp} (\omega + V), \quad (\omega + V) \cap (\omega' + V) = \emptyset \textit{ for } \omega \neq \omega', \ \omega, \omega' \in \Lambda^\perp. \tag{2.1}$$

Proof. The result in (i) is well known, so we skip the proof. But let us show how the results in Lemma 2.1 can be used to prove (ii). First, since G/Λ is compact by definition, Lemma 2.1(ii)+(iii) imply that $\Lambda^\perp$ is discrete. Also, since Λ is a discrete subgroup of G, its dual group is compact by Lemma 2.1(i). By (iv) in the same lemma, this implies that $\widehat{G}/\Lambda^\perp$ is compact (recall that the double-dual of a group is the group itself). Thus, $\Lambda^\perp$ is a lattice in $\widehat{G}$, and the result in (ii) follows from the first part of the lemma. □

2.2 Fourier analysis on LCA groups

Any LCA group G can be equipped with a Radon measure μ_G which is *translation invariant*, such that for all continuous functions on G with compact support, i.e. $f \in C_c(G)$,

$$\int_G f(x+y)\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x),\ \forall\, y \in G.$$

The measure is unique up to multiplication with a positive scalar, and is called the *Haar measure.* With this measure on hand, we can now define the spaces $L^p(G),\ 1 \leq p < \infty$, in the usual way; we will only need the spaces $L^1(G)$ and $L^2(G)$. The space $L^2(G)$ is a Hilbert space in the obvious way; furthermore, our assumption of G being a countable union of compact sets and metrizable implies (and is, in fact, equivalent to) $L^2(G)$ being separable.

We will now consider the Haar measure μ_G as fixed in the rest of the paper. The *Fourier transform* is defined as the operator

$$\mathcal{F} : L^1(G) \to C_0(\widehat{G}),\ \mathcal{F}f(\gamma) := \int_G f(x)(-x,\gamma)\, d\mu_G(x).$$

We will often use the notation $\hat{f} := \mathcal{F}f$. The *inversion theorem* states that with appropriate normalization of the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$, for $f \in L^1(G)$ such that $\hat{f} \in L^1(\widehat{G})$, it holds that

$$f(x) = \int_{\widehat{G}} \hat{f}(\gamma)\,(x,\gamma) d\mu_{\widehat{G}}(\gamma),\ x \in G. \tag{2.2}$$

As in the classical case of $G = \mathbb{R}$, the Fourier transform can be extended to a surjective isometry $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$. Also, the same normalization of the Haar measure that makes the inversion formula work leads to the Plancherel theorem (see [19]),

$$\int_G f(x)\overline{g(x)}\, d\mu_G(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\, d\mu_{\widehat{G}}(\gamma),\ f, g \in L^2(G).$$

We will always choose the Haar measure on $\widehat{G}$ such that the inversion formula (2.2) holds for the pairs G and $\widehat{G}$.

Our approach relies strongly on the following result.

Lemma 2.3 *Let Λ denote a lattice in the LCA group G, and choose the set $V \subset \widehat{G}$ as in* (2.1)*. Choose the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that the inversion formula* (2.2) *holds. Then, for all $f \in L^2(V)$,*

$$\mu_{\widehat{G}}(V) \int_V |f(\gamma)|^2\, d\mu_{\widehat{G}}(\gamma) = \sum_{\lambda\in\Lambda} |\widehat{f\chi_V}(\lambda)|^2, \tag{2.3}$$

where the hat denotes the Fourier transform on the group $\widehat{G}$.

Proof. Let $f \in L^2(V)$. Weyl's formula implies that for an appropriate normalization of the measure on $\widehat{G}/\Lambda^\perp$ (see [17] or [15, p. 199]) and with $\dot{g} = g + \Lambda^\perp$,

$$\begin{aligned}\int_V |f(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) &= \int_{\widehat{G}} |f\chi_V(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) = \mu_{\widehat{G}}(V) \int_{\widehat{G}/\Lambda^\perp} \sum_{h\in\Lambda^\perp} |f\chi_V(g+h)|^2 d\mu_{\widehat{G}/\Lambda^\perp}(\dot{g}) \\ &= \mu_{\widehat{G}}(V) \int_{\widehat{G}/\Lambda^\perp} \left| \sum_{h\in\Lambda^\perp} f\chi_V(g+h) \right|^2 d\mu_{\widehat{G}/\Lambda^\perp}(\dot{g}). \qquad (2.4)\end{aligned}$$

Note that with this normalization of the measure on $\widehat{G}/\Lambda^\perp$, we have $\mu_{\widehat{G}/\Lambda^\perp}(\widehat{G}/\Lambda^\perp) = 1$. Thus, using the Plancherel theorem on $\widehat{G}/\Lambda^\perp$, as well as Lemma 2.1(iv), followed by a new application of Weyl's formula and the definition of the Fourier transform,

$$\begin{aligned}\int_{\widehat{G}/\Lambda^\perp} \left| \sum_{h\in\Lambda^\perp} f\chi_V(g+h) \right|^2 d\mu_{\widehat{G}/\Lambda^\perp}(\dot{g}) &= \sum_{\lambda\in\Lambda} \left| \int_{\widehat{G}/\Lambda^\perp} \sum_{h\in\Lambda^\perp} (f\chi_V)(g+h)(-g,\lambda) \, d\mu_{\widehat{G}/\Lambda^\perp}(\dot{g}) \right|^2 \\ &= \sum_{\lambda\in\Lambda} \left| \frac{1}{\mu_{\widehat{G}}(V)} \int_{\widehat{G}} (f\chi_V)(\gamma)(-\gamma,\lambda) d\mu_{\widehat{G}}(\gamma) \right|^2 \\ &= \frac{1}{\mu_{\widehat{G}}(V)^2} \sum_{\lambda\in\Lambda} |\widehat{f\chi_V}(\lambda)|^2.\end{aligned}$$

Inserting this in (2.4) yields the result. □

The following example shows that we can consider Lemma 2.3 as an abstract version of Parseval's theorem:

Example 2.4 Consider the group $G := \mathbb{R}$, with dual group $\widehat{G} = \mathbb{R}$. It is well known that the functions $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis for $L^2[0,1]$. Thus, writing the Fourier coefficients for $f \in L^2[0,1]$ as $c_n = \int_0^1 f(x)\, e^{-2\pi inx}\, dx,\ n \in \mathbb{Z}$, Parseval's theorem shows that $\int_0^1 |f(x)|^2 \, dx = \sum_{n\in\mathbb{Z}} |c_n|^2$. Note that with the Fourier transform defined on $L^1(\mathbb{R})$ (or rather on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and then extended to $L^2(\mathbb{R})$) by $\mathcal{F}f(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi iyx}\, dx$, this can be written as

$$\int_0^1 |f(x)|^2 \, dx = \sum_{n\in\mathbb{Z}} \left| \widehat{f\chi_{[0,1]}}(n) \right|^2. \qquad (2.5)$$

This is a special case of the result in Lemma 2.3. In fact, consider the set $V = [0,1)$ as a subset of $\widehat{G} = \mathbb{R}$. Defining the lattice $\Lambda := \mathbb{Z}$ in $\mathbb{R}$, we have that $\Lambda^\perp = \mathbb{Z}$. Thus, the set $V = [0,1)$ satisfies (2.1), and Lemma 2.3 tells us that (2.3) holds. Inserting the sets V and Λ in (2.3) shows that this is exactly the same as (2.5). □

3 Analysis of systems $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ on LCA groups

In the entire section we will consider an LCA group G, with dual group $\widehat{G}$. We will fix a Haar measure μ_G on G, and normalize the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that the inversion formula and Plancherel formula hold. Compared to Subsection 2.2 we will simplify the notation and for $f \in L^1(G)$ simply write $\int_G f(x)\,dx := \int_G f(x)\,d\mu_G(x)$. Similarly, for $f \in L^1(\widehat{G})$, we write $\int_{\widehat{G}} f(\gamma)\,d\gamma := \int_{\widehat{G}} f(\gamma)\,d\mu_{\widehat{G}}(\gamma)$. Given any $\lambda \in G$, consider the *generalized modulation operator*

$$\mathcal{M}_\lambda : L^2(\widehat{G}) \to L^2(\widehat{G}),\ (\mathcal{M}_\lambda f)(\gamma) := (\lambda, \gamma)\, f(\gamma).$$

Like the modulation operator on $\mathbb{R}$, it is easy to see that $\mathcal{M}_\lambda$ is a unitary operator. Our goal in this section is to consider actions by a class of operators $\mathcal{M}_\lambda$ on a countable collection of functions $\{\Phi_k\}_{k\in I}$ in $L^2(\widehat{G})$, with λ belonging to lattices Λ_k that depend on $k \in I$. The resulting class of functions in $L^2(\widehat{G})$ is then of the form $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$, which we refer to as a *Fourier-like system.* In order to avoid a messy notation in the proofs we will first consider the action of a class of such operators associated with a single lattice, on a single function.

For technical reasons we will often need the dense subspace $B_c(\widehat{G})$ of $L^2(\widehat{G})$ defined by

$$B_c(\widehat{G}) := \{f \in L^2(\widehat{G}) \,\big|\, f \text{ is bounded, measurable and compactly supported}\}.$$

3.1 The case of one generator

In this subsection we consider a lattice Λ in G and analyze the action by the operators $\mathcal{M}_\lambda,\ \lambda \in \Lambda$, on a single function $\Phi \in L^2(\widehat{G})$. We will use the notation from Subsection 2.2 and consequently let V denote a relatively compact set that is chosen as in Lemma 2.2.

Lemma 3.1 *Let Λ be a lattice in G, and choose the relatively compact set V in $\widehat{G}$ as in* (2.1)*. Let $F, \Phi \in L^2(\widehat{G})$. Then the following hold:*

(i) *The function*

$$\alpha : \widehat{G} \to \mathbb{C},\ \alpha(\gamma) := \sum_{\omega\in\Lambda^\perp} F(\omega+\gamma)\overline{\Phi(\omega+\gamma)},$$

is well defined a.e., belongs to $L^1(V)$, and satisfies that

$$\alpha(\gamma+\omega') = \alpha(\gamma),\ \forall\, \gamma \in \widehat{G}, \omega' \in \Lambda^\perp.$$

(ii) *For any $\lambda \in \Lambda$,*

$$\langle F, \mathcal{M}_\lambda\Phi\rangle = \int_V \alpha(\gamma)\overline{(\lambda,\gamma)}\,d\gamma = \widehat{\alpha\chi_V}(\lambda),$$

where the hat denotes the Fourier transform on the group $\widehat{G}$.

Proof. Using Lemma 2.2(ii),

$$\begin{aligned}\int_V \sum_{\omega\in\Lambda^\perp} \left|F(\omega+\gamma)\overline{\Phi(\omega+\gamma)}\right| d\gamma &= \sum_{\omega\in\Lambda^\perp}\int_V |F(\omega+\gamma)\Phi(\omega+\gamma)|\, d\gamma \\ &= \sum_{\omega\in\Lambda^\perp}\int_{\omega+V} |F(\gamma)\Phi(\gamma)|\, d\gamma = \int_{\widehat{G}} |F(\gamma)\Phi(\gamma)|\, d\gamma,\end{aligned}$$

which is finite by the Cauchy-Schwarz inequality. This shows that $\alpha(\gamma)$ is well defined pointwise for almost all $\gamma \in V$, and also implies that $\alpha \in L^1(V)$. Now, any $\gamma \in \widehat{G}$ can be written as $\gamma = \gamma' + \omega'$ for some $\gamma' \in V,\ \omega' \in \Lambda^\perp$. Then

$$\begin{aligned}\sum_{\omega\in\Lambda^\perp} \left|F(\omega+\gamma)\overline{\Phi(\omega+\gamma)}\right| &= \sum_{\omega\in\Lambda^\perp} \left|F(\omega+\gamma'+\omega')\overline{\Phi(\omega+\gamma'+\omega')}\right| \\ &= \sum_{\omega\in\Lambda^\perp} \left|F(\omega+\gamma')\overline{\Phi(\omega+\gamma')}\right|,\end{aligned}$$

where we used the change of summation variable $\omega \to \omega - \omega'$ (which is allowed because $\Lambda^\perp$ is a group). Thus, the series defining $\alpha(\gamma)$ is absolutely convergent for a.e. $\gamma \in \widehat{G}$. The same argument, just without the absolute value, shows that $\alpha(\gamma+\omega') = \alpha(\gamma)$, which proves (i).

For the proof of (ii), using again Lemma 2.2(ii),

$$\begin{aligned}\langle F, \mathcal{M}_\lambda \Phi\rangle &= \int_{\widehat{G}} F(\gamma)\overline{\Phi(\gamma)}\,\overline{(\lambda,\gamma)}\, d\gamma \\ &= \sum_{\omega\in\Lambda^\perp}\int_{\omega+V} F(\gamma)\overline{\Phi(\gamma)}\,\overline{(\lambda,\gamma)}\, d\gamma = \sum_{\omega\in\Lambda^\perp}\int_V F(\omega+\gamma)\overline{\Phi(\omega+\gamma)}\,\overline{(\lambda,\omega+\gamma)}\, d\gamma.\end{aligned}$$

Note that since $\lambda \in \Lambda$ and $\omega \in \Lambda^\perp$, we have that $(\lambda, \omega+\gamma) = (\lambda,\omega)(\lambda,\gamma) = (\lambda,\gamma)$. Thus the calculation yields that

$$\begin{aligned}\langle F, \mathcal{M}_\lambda \Phi\rangle &= \int_V \sum_{\omega\in\Lambda^\perp} F(\omega+\gamma)\overline{\Phi(\omega+\gamma)}\,\overline{(\lambda,\gamma)}\, d\gamma \\ &= \int_V \alpha(\gamma)\overline{(\lambda,\gamma)}\, d\gamma = \int_V \alpha(\gamma)(-\lambda,\gamma)\, d\gamma = \widehat{\alpha\chi_V}(\lambda),\end{aligned}$$

as desired. □

Lemma 3.2 *Let Λ denote a lattice in the group G, and choose the relatively compact set V in $\widehat{G}$ as in (2.1). Let $\Phi \in L^2(\widehat{G})$. Then the following hold:*

(i) *If $F \in B_c(\widehat{G})$, then*

$$\sum_{\lambda\in\Lambda} |\langle F, \mathcal{M}_\lambda\Phi\rangle|^2 = \mu_{\widehat{G}}(V)\left(\int_{\widehat{G}} |F(\gamma)\Phi(\gamma)|^2\, d\gamma + R(F)\right), \tag{3.1}$$

where

$$|R(F)| \leq \int_{\widehat{G}} |F(\gamma)|^2 \sum_{\omega \in \Lambda^\perp \setminus \{0\}} |\Phi(\gamma)\Phi(\gamma+\omega)| \, d\gamma.$$

(ii) *Assume that*

$$B := \mu_{\widehat{G}}(V) \sup_{\gamma \in \widehat{G}} \sum_{\omega \in \Lambda^\perp} |\Phi(\gamma)\Phi(\gamma+\omega)| < \infty.$$

Then $\{\mathcal{M}_\lambda \Phi\}_{\lambda \in \Lambda}$ *is a Bessel sequence in* $L^2(\widehat{G})$ *with bound* B.

Proof. The assumption $F \in B_c(\widehat{G})$ will justify all interchanges of summations and integrals in the following as $\widehat{G}$ is metrizable and $\Lambda^\perp$ is a discrete subgroup of $\widehat{G}$. In addition, applying the Cauchy-Schwarz inequality followed by Lemma 2.2(ii), it shows that $\alpha \in L^2(V)$. Thus using Lemma 3.1(ii) and Lemma 2.3,

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle F, \mathcal{M}_\lambda \Phi \rangle|^2 &= \sum_{\lambda \in \Lambda} |\widehat{\alpha \chi_V}(\lambda)|^2 \\ &= \mu_{\widehat{G}}(V) \int_V |\alpha(\gamma)|^2 \, d\gamma = \mu_{\widehat{G}}(V) \int_V \alpha(\gamma) \, \overline{\alpha(\gamma)} \, d\gamma. \end{aligned} \tag{3.2}$$

Inserting the expression for $\alpha(\gamma)$ (while keeping the term $\overline{\alpha(\gamma)}$) leads to

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle F, \mathcal{M}_\lambda \Phi \rangle|^2 &= \mu_{\widehat{G}}(V) \int_V \sum_{\omega \in \Lambda^\perp} F(\omega+\gamma) \overline{\Phi(\omega+\gamma)} \, \overline{\alpha(\gamma)} \, d\gamma \\ &= \mu_{\widehat{G}}(V) \sum_{\omega \in \Lambda^\perp} \int_V F(\omega+\gamma) \overline{\Phi(\omega+\gamma)} \, \overline{\alpha(\gamma)} \, d\gamma \\ &= \mu_{\widehat{G}}(V) \sum_{\omega \in \Lambda^\perp} \int_{\omega+V} F(\gamma) \overline{\Phi(\gamma)} \, \overline{\alpha(\gamma-\omega)} \, d\gamma, \end{aligned}$$

where the last step used the translation invariance of the Haar measure. Now, by Lemma 3.1(i) we have that $\alpha(\gamma - \omega) = \alpha(\gamma)$ whenever $\omega \in \Lambda^\perp$. Thus, we arrive at

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle F, \mathcal{M}_\lambda \Phi \rangle|^2 &= \mu_{\widehat{G}}(V) \sum_{\omega \in \Lambda^\perp} \int_{\omega+V} F(\gamma) \overline{\Phi(\gamma)} \, \overline{\alpha(\gamma)} \, d\gamma \\ &= \mu_{\widehat{G}}(V) \int_{\widehat{G}} F(\gamma) \overline{\Phi(\gamma)} \, \overline{\alpha(\gamma)} \, d\gamma, \end{aligned} \tag{3.3}$$

where we used Lemma 2.2(ii) in the last step. Inserting again the expression for $\alpha(\gamma)$ now yields that

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle F, \mathcal{M}_\lambda \Phi \rangle|^2 &= \mu_{\widehat{G}}(V) \int_{\widehat{G}} F(\gamma) \overline{\Phi(\gamma)} \, \overline{\sum_{\omega \in \Lambda^\perp} F(\omega+\gamma) \overline{\Phi(\omega+\gamma)}} \, d\gamma \\ &= \mu_{\widehat{G}}(V) \sum_{\omega \in \Lambda^\perp} \int_{\widehat{G}} F(\gamma) \overline{F(\omega+\gamma)} \, \overline{\Phi(\gamma)} \Phi(\omega+\gamma) \, d\gamma. \end{aligned}$$

Pulling out the term corresponding to $\omega = 0$ gives that

$$\sum_{\lambda\in\Lambda} |\langle F, \mathcal{M}_\lambda \Phi\rangle|^2 = \mu_{\widehat{G}}(V) \left(\int_{\widehat{G}} |F(\gamma)|^2 \, |\Phi(\gamma)|^2 \, d\gamma + R(F) \right),$$

where

$$R(F) := \sum_{\omega\in\Lambda^\perp\setminus\{0\}} \int_{\widehat{G}} F(\gamma)\overline{F(\omega+\gamma)}\,\overline{\Phi(\gamma)}\Phi(\omega+\gamma)\, d\gamma.$$

Clearly,

$$|R(F)| \le \sum_{\omega\in\Lambda^\perp\setminus\{0\}} \int_{\widehat{G}} \left(|F(\gamma)|\,|\Phi(\gamma)\Phi(\omega+\gamma)|^{1/2}\right) \left(|F(\omega+\gamma)|\,|\Phi(\gamma)\Phi(\omega+\gamma)|^{1/2}\right)\, d\gamma;$$

from here, two applications of the Cauchy-Schwarz inequality and a use of the translation invariance of the measure proves (i) in the lemma (the proof is similar to the proof of [3, Theorem 9.1.5]).

For the proof of (ii), combining what we established in (i) shows that for $F \in B_c(\widehat{G})$,

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle F, \mathcal{M}_\lambda\Phi\rangle|^2 &\le \mu_{\widehat{G}}(V) \left(\int_{\widehat{G}} |F(\gamma)\Phi(\gamma)|^2 \, d\gamma + \int_{\widehat{G}} |F(\gamma)|^2 \sum_{\omega\in\Lambda^\perp\setminus\{0\}} |\Phi(\gamma)\Phi(\gamma+\omega)|\, d\gamma \right) \\
&= \mu_{\widehat{G}}(V) \int_{\widehat{G}} |F(\gamma)|^2 \sum_{\omega\in\Lambda^\perp} |\Phi(\gamma)\Phi(\gamma+\omega)|\, d\gamma \\
&\le \mu_{\widehat{G}}(V) \sup_{\gamma\in\widehat{G}} \sum_{\omega\in\Lambda^\perp} |\Phi(\gamma)\Phi(\gamma+\omega)| \int_{\widehat{G}} |F(\gamma)|^2.
\end{aligned}$$

We conclude that the Bessel inequality holds on a dense subset of $L^2(\widehat{G})$. Therefore it holds on $L^2(\widehat{G})$, and we have now proved (ii). □

We now state a consequence of the above results that will be of importance when we consider duality issues in Subsection 3.4.

Lemma 3.3 *Let $\{\mathcal{M}_\lambda\Phi\}_{\lambda\in\Lambda}$ and $\{\mathcal{M}_\lambda\widetilde{\Phi}\}_{\lambda\in\Lambda}$ be Bessel sequences in $L^2(\widehat{G})$. Then for any $F, H \in B_c(\widehat{G})$,*

$$\sum_{\lambda\in\Lambda} \langle F, \mathcal{M}_\lambda\Phi\rangle \overline{\langle H, \mathcal{M}_\lambda\widetilde{\Phi}\rangle} = \mu_{\widehat{G}}(V) \sum_{\omega\in\Lambda^\perp} \int_{\widehat{G}} F(\gamma)\overline{H(\omega+\gamma)}\,\overline{\Phi(\gamma)}\widetilde{\Phi}(\omega+\gamma)\, d\gamma. \tag{3.4}$$

Proof. The proof follows the lines of the proof of Lemma 3.2. First, the Cauchy-Schwarz inequality shows that the sum on the left-hand side of (3.4) is absolutely convergent. As in Lemma 3.1, letting $\beta(\gamma) := \sum_{\omega\in\Lambda^\perp} H(\omega+\gamma)\overline{\widetilde{\Phi}(\omega+\gamma)}$, we can write $\langle H, \mathcal{M}_\lambda \widetilde{\Phi}\rangle = \widehat{\beta\chi_V}(\lambda)$. Thus,

$$\sum_{\lambda\in\Lambda}\langle F, \mathcal{M}_\lambda \Phi\rangle\overline{\langle H, \mathcal{M}_\lambda \widetilde{\Phi}\rangle} = \sum_{\lambda\in\Lambda}\widehat{\alpha\chi_V}(\lambda)\overline{\widehat{\beta\chi_V}(\lambda)} = \mu_{\widehat{G}}(V)\int_V \alpha(\gamma)\overline{\beta(\gamma)}\,d\gamma,$$

where the last step used polarization of the identity in Lemma 2.3. Proceeding exactly as we did in the proof of Lemma 3.2, see (3.2), inserting the expression for $\alpha(\gamma)$ leads to

$$\sum_{\lambda\in\Lambda}\langle F, \mathcal{M}_\lambda \Phi\rangle\overline{\langle H, \mathcal{M}_\lambda \widetilde{\Phi}\rangle} = \mu_{\widehat{G}}(V)\int_{\widehat{G}} F(\gamma)\overline{\Phi(\gamma)}\,\overline{\beta(\gamma)}\,d\gamma,$$

corresponding to (3.3). Inserting the expression for $\beta(\gamma)$ now gives (3.4). $\square$

3.2 The multi-generator case

In this subsection we will use the following

General setup: Let I denote a countable index set, and let $\{\Phi_k\}_{k\in I}$, $\{\widetilde{\Phi_k}\}_{k\in I}$ be two collections of functions in $L^2(\widehat{G})$. Corresponding to a family of lattices $\{\Lambda_k\}_{k\in I}$ in G, we will derive conditions for $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ to be a Bessel sequence or a frame for $L^2(\widehat{G})$, and for $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda\widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ to be dual frames.

Letting $\Lambda_k^\perp$ denote the annihilator of Λ_k, Lemma 2.2(ii) shows that there exist relatively compact sets V_k in $\widehat{G}$ such that for each $k \in I$,

$$\widehat{G} = \bigcup_{\omega\in\Lambda_k^\perp}(\omega + V_k), \quad (\omega+V_k)\cap(\omega'+V_k) = \emptyset \text{ for } \omega\neq\omega',\ \omega,\omega'\in\Lambda_k^\perp. \tag{3.5}$$

Theorem 3.4 *Under the above assumptions, the following hold:*

(i) $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ *is a Bessel sequence in* $L^2(\widehat{G})$ *if*

$$B := \sup_{\gamma\in\widehat{G}}\sum_{k\in I}\mu_{\widehat{G}}(V_k)\sum_{\omega\in\Lambda_k^\perp}|\Phi_k(\gamma)\Phi_k(\gamma+\omega)| < \infty.$$

(ii) *If* (i) *holds, then* $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ *is a frame for* $L^2(\widehat{G})$ *if*

$$A := \inf_{\gamma\in\widehat{G}}\left(\sum_{k\in I}\mu_{\widehat{G}}(V_k)\,|\Phi_k(\gamma)|^2 - \sum_{k\in I}\mu_{\widehat{G}}(V_k)\sum_{\omega\in\Lambda_k^\perp\setminus\{0\}}|\Phi_k(\gamma)\Phi_k(\gamma+\omega)|\right) > 0.$$

Proof. Let $F \in B_c(\widehat{G})$. For each $k \in I$, Lemma 3.2(i) implies that

$$\sum_{\lambda \in \Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k \rangle|^2 \leq \mu_{\widehat{G}}(V_k) \int_{\widehat{G}} |F(\gamma)|^2 \sum_{\omega \in \Lambda_k^\perp} |\Phi_k(\gamma)\Phi_k(\gamma+\omega)| \, d\gamma.$$

Thus,

$$\sum_{k \in I} \sum_{\lambda \in \Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k \rangle|^2 \leq \int_{\widehat{G}} |F(\gamma)|^2 \sum_{k \in I} \mu_{\widehat{G}}(V_k) \sum_{\omega \in \Lambda_k^\perp} |\Phi_k(\gamma)\Phi_k(\gamma+\omega)| \, d\gamma.$$

Under the assumption in (i), this implies that

$$\sum_{k \in I} \sum_{\lambda \in \Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k \rangle|^2 \leq B \int_{\widehat{G}} |F(\gamma)|^2 d\gamma = B \, \|F\|^2.$$

Since this holds on a dense set in $L^2(\widehat{G})$, we conclude that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda \in \Lambda_k, k \in I}$ is a Bessel sequence in $L^2(\widehat{G})$.

Similarly, for each $k \in I$, Lemma 3.2(i) implies that

$$\sum_{\lambda \in \Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k \rangle|^2 \geq \mu_{\widehat{G}}(V_k) \int_{\widehat{G}} |F(\gamma)|^2 \left(|\Phi_k(\gamma)|^2 - \sum_{\omega \in \Lambda_k^\perp \setminus \{0\}} |\Phi_k(\gamma)\Phi_k(\gamma+\omega)| \right) d\gamma.$$

Thus, under the assumptions in (ii),

$$\begin{aligned} &\sum_{k \in I} \sum_{\lambda \in \Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k \rangle|^2 \\ \geq \ & \int_{\widehat{G}} |F(\gamma)|^2 \sum_{k \in I} \mu_{\widehat{G}}(V_k) \left(|\Phi_k(\gamma)|^2 - \sum_{\omega \in \Lambda_k^\perp \setminus \{0\}} |\Phi_k(\gamma)\Phi_k(\gamma+\omega)| \right) d\gamma \\ \geq \ & A \int_{\widehat{G}} |F(\gamma)|^2 = A \, \|F\|^2. \end{aligned}$$

This again implies that the lower frame condition is satisfied. □

Under the assumption that the functions Φ_k have sufficiently small supports (in relation to the given lattices Λ_k), we obtain a characterization of the frame property for $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda \in \Lambda_k, k \in I}$:

Corollary 3.5 *In addition to the general setup, assume that for each $k \in I$, the function Φ_k satisfies that*

$$\operatorname{supp} \Phi_k \cap \operatorname{supp} \Phi_k(\cdot + \omega) = \emptyset, \ \forall \, \omega \in \Lambda_k^\perp \setminus \{0\} \tag{3.6}$$

(*up to a set of measure zero in $\widehat{G}$*). *Then the following hold:*

(i) $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ *is a Bessel sequence in* $L^2(\widehat{G})$ *if and only if*

$$B := \sup_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 < \infty.$$

(ii) *If* (i) *holds, then* $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ *is a frame for* $L^2(\widehat{G})$ *if and only if*

$$A := \inf_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 > 0.$$

Proof. The sufficiency of the conditions in (i) and (ii) follows directly from Theorem 3.4 and the assumption (3.6). Let us show that the condition in (i) is also necessary for $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ to be a Bessel sequence with bound B. First, by (3.1) and the assumption (3.6), for all $F \in B_c(\widehat{G})$ we have

$$\sum_{k\in I}\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 = \sum_{k\in I} \mu_{\widehat{G}}(V_k) \int_{\widehat{G}} |F(\gamma)\Phi_k(\gamma)|^2 \, d\gamma = \int_{\widehat{G}} |F(\gamma)|^2 \sum_{k\in I} \mu_{\widehat{G}}(V_k)\, |\Phi_k(\gamma)|^2 \, d\gamma.$$

Thus, if $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence with bound B,

$$\int_{\widehat{G}} |F(\gamma)|^2 \sum_{k\in I} \mu_{\widehat{G}}(V_k)\, |\Phi_k(\gamma)|^2 \, d\gamma \le B\, \|F\|^2$$

for all $F \in B_c(\widehat{G})$. This implies that $\sum_{k\in I} \mu_{\widehat{G}}(V_k)\, |\Phi_k(\gamma)|^2 \le B$ almost everywhere, as desired: in fact, if $\sum_{k\in I} \mu_{\widehat{G}}(V_k)\, |\Phi_k(\gamma)|^2 > B$ on a set $\mathcal{S}$ of positive measure (we can assume that the measure is finite by switching to a subset, if necessary), taking $F := \chi_{\mathcal{S}}$ would lead to a contradiction. The necessity of the lower bound in (ii) is shown in a similar way. □

3.3 Explicit constructions

The purpose of this subsection is to provide simple and explicit frame constructions based on Theorem 3.4 and Corollary 3.5, in the full generality of LCA groups. The constructions will be based on a generalization of the classical B-splines to the setting of LCA groups, presented independently by Dahlke [7] and Tikhomirov [20] in 1994. We will further extend the definition by allowing certain weight functions to appear, see (3.7) below. This enlarges the class of splines obtained in the same sense as the exponential splines generalize the classical B-splines.

Definition 3.6 *Let* Λ *denote a lattice in the LCA group* G, *with associated fundamental domain* Q. *Let* $r \in \mathbb{N}$. *Given functions* $g_1, \dots, g_r \in L^2(Q)$, *the function on* G *defined by the* r*-fold convolution*

$$W_r := g_1\, \chi_Q * g_2\chi_Q * \cdots * g_r\chi_Q \tag{3.7}$$

is called a weighted B-spline of order r.

Note that since Q is relatively compact, the assumption $g_j \in L^2(Q)$ implies that $g_j \in L^1(Q)$. Therefore the convolution in (3.7) is well defined, and the terms in the convolution can be reordered without changing the function W_r.

Lemma 3.7 *Let Λ denote a lattice in the LCA group G, with associated fundamental domain Q. Given functions $g_1, \dots, g_r \in L^2(Q)$, the weighted B-spline W_r has the following properties:*

(i) *$\{T_\lambda W_r\}_{\lambda\in\Lambda}$ is a Bessel sequence with bound $\prod_{j=1}^r \|g_j\|^2_{L^2(Q)}$.*

(ii) *For $x \in G$, $W_r(x) \neq 0$ only if $x \in rQ := Q + Q + \cdots + Q$; therefore* $\operatorname{supp} W_r \subseteq \overline{rQ}$.

(iii) *If $r \geq 2$, then $W_r \in C_c(G)$; in particular, $W_r \in L^p(G)$ for all $p \geq 1$.*

(iv) *If $g_j > 0$ on Q for $j = 1, \dots, r$ and $g_j = C$ for at least one index j, then W_r is nonnegative on G and satisfies the partition of unity condition up to a constant, i.e.,*

$$\sum_{\lambda\in\Lambda} W_r(x-\lambda) = \frac{1}{\mu_G(Q)} \prod_{j=1}^r \int_Q g_j(y)\,dy, \ x \in G.$$

Proof. (i) Given just one function $g \in L^2(Q)$ the system $\{T_\lambda(g\chi_Q)\}_{\lambda\in\Lambda}$ is an orthogonal system (the orthogonality follows from Lemma 2.2) and therefore a Bessel sequence, with Bessel bound $\|g\|^2_{L^2(Q)}$. We will now use a result by Cabrelli and Paternostro [1], which states that a system $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is a Bessel sequence with bound B if and only if $\sum_{\omega\in\Lambda^\perp} |\widehat{\phi}(\gamma+\omega)|^2 \leq B$, a.e. $\gamma \in V$, where V is a relatively compact set in $\widehat{G}$ as in (2.1). Applied to $W_1 := g\chi_Q$, this shows that

$$\sum_{\omega\in\Lambda^\perp} |\widehat{g\chi_Q}(\gamma+\omega)|^2 \leq \|g\|^2_{L^2(Q)}. \tag{3.8}$$

Consider now any weighted B-spline W_r. It follows from (3.8) that all function values of $\widehat{g_j\chi_Q}$, $j = 1, \dots, r-1$, are bounded by $\|g_j\|_{L^2(Q)}$. Then (3.8) applied to g_r implies that

$$\begin{aligned} \sum_{\omega\in\Lambda^\perp} |\widehat{W_r}(\gamma+\omega)|^2 &= \sum_{\omega\in\Lambda^\perp} \prod_{j=1}^r |\widehat{g_j\chi_Q}(\gamma+\omega)|^2 \\ &\leq \prod_{j=1}^{r-1} \|g_j\|^2_{L^2(Q)} \sum_{\omega\in\Lambda^\perp} |\widehat{g_r\chi_Q}(\gamma+\omega)|^2 \leq \prod_{j=1}^r \|g_j\|^2_{L^2(Q)}. \end{aligned}$$

This shows that $\{T_\lambda W_r\}_{\lambda\in\Lambda}$ is a Bessel sequence with the claimed bound.

(ii) This is an immediate consequence of the definition of the convolution.

(iii) Since $g_1, g_2 \in L^2(Q)$, it follows from [19, pp. 4–5] that $W_2 := g_1\chi_Q * g_2\chi_Q \in C_c(G)$, implying that $W_2 \in L^p(G)$ for all $p \geq 1$. Iterating the argument leads to the result.

(iv) First, let f_1 be any nonnegative compactly supported function on G for which there is a constant C_1 such that $\sum_{\lambda\in\Lambda} f_1(x-\lambda) = C_1$, $x \in G$. Then, using an argument from [5], if $f_2 \in L^1(G)$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda} f_1 * f_2(x-\lambda) &= \sum_{\lambda\in\Lambda}\int_G f_1(x-y-\lambda)f_2(y)\,dy \\ &= \int_G \sum_{\lambda\in\Lambda} f_1(x-y-\lambda)f_2(y)\,dy = C_1\int_G f_2(y)\,dy. \qquad (3.9)\end{aligned}$$

This eventually shows that a convolution has the partition of unity property (up to a constant) if at least one of the factors has the property. Indeed, if $g_j = C$ for at least one $j \in \{1,\ldots,r\}$, let us reorder the terms and assume that $g_1 = C$. Then $W_1 := g_1\chi_Q = C\chi_Q$ satisfies that

$$\sum_{\lambda\in\Lambda} W_1(x-\lambda) = C = \frac{1}{\mu_G(Q)}\int_Q g_1(y)\,dy,\ x \in G.$$

The general result now follows by induction based on (3.9). □

We will now show that the weighted B-splines provide natural applications of the results in Section 3.2. For any LCA group G, define the *translation operator* $T_y,\ y \in G$, on $L^2(G)$ by $T_yf(x) = f(x-y),\ y \in G$, and note that

$$\mathcal{F}T_y = \mathcal{M}_{-y}\mathcal{F},\quad \mathcal{F}^{-1}\mathcal{M}_y = T_{-y}\mathcal{F}^{-1}.$$

Using the general setup described in Subsection 3.2 we will construct concrete Gabor-type frames for $L^2(\widehat{G})$ of the form $\{\mathcal{M}_\lambda T_k\Phi\}_{\lambda\in\Lambda,k\in\Gamma}$, where Γ is chosen as a lattice in $\widehat{G}$ and $\Phi \in L^2(\widehat{G})$. The construction is based on the splines as in Definition 3.6, but defined on the group $\widehat{G}$.

Theorem 3.8 *Given a lattice Γ in $\widehat{G}$, let $\Omega \subset \widehat{G}$ denote a fundamental domain. For a fixed $r \in \mathbb{N}$, consider the function $W_r := g_1\,\chi_\Omega * g_2\chi_\Omega * \cdots * g_r\chi_\Omega$, where $g_1,\ldots,g_r \in L^2(\Omega)$, with the assumption that $g_j > 0$ on Ω for $j = 1,\ldots,r$ and $g_j = C$ for at least one index j. Given a lattice Λ in G, assume that the fundamental domain V associated with $\Lambda^\perp$ satisfies that $r\Omega \subseteq V$. Then $\{\mathcal{M}_\lambda T_kW_r\}_{\lambda\in\Lambda,k\in\Gamma}$ is a frame for $L^2(\widehat{G})$.*

Proof. Note that the function W_r is bounded: in fact,

$$0 \le W_r(\gamma) \le C_r := \frac{1}{\mu_{\widehat{G}}(\Omega)}\prod_{j=1}^{r}\int_\Omega g_j(\eta)\,d\eta,\ \gamma \in \widehat{G}. \qquad (3.10)$$

This follows from the assumption that $g_j > 0$ on Ω for $j = 1,\ldots,r$ and $g_j = C$ for at least one index j, which implies via Lemma 3.7(iv) the partition of unity condition (up to a constant):

$$\sum_{k\in\Gamma} W_r(\gamma-k) = C_r,\ \gamma \in \widehat{G}. \qquad (3.11)$$

Without loss of generality we can assume that $g_1 = C$.

Since $V \cap (V + \Lambda^\perp \setminus \{0\})) = \emptyset$ and $r\Omega \subseteq V$, we have $r\Omega \cap (r\Omega + (\Lambda^\perp \setminus \{0\})) = \emptyset$; it then follows from Lemma 3.7(ii) that

$$\operatorname{supp} W_r \cap \operatorname{supp} W_r(\cdot + \omega) = \emptyset, \ \forall \omega \in \Lambda^\perp \setminus \{0\}. \tag{3.12}$$

We will now apply Corollary 3.5 with $\Phi_k := T_k W_r$, i.e., we will estimate the supremum and infimum of

$$\sum_{k\in\Gamma} \mu_{\widehat{G}}(V)|W_r(\gamma - k)|^2 = \mu_{\widehat{G}}(V) \sum_{k\in\Gamma} |W_r(\gamma - k)|^2.$$

Note that (3.12) yields (3.6) in Corollary 3.5 with $\Lambda_k = \Lambda$ for all $k \in \Gamma$.

First, by (3.10) and (3.11), we see that for any $\gamma \in \widehat{G}$,

$$\sum_{k\in\Gamma} |W_r(\gamma - k)|^2 \le C_r \sum_{k\in\Gamma} |W_r(\gamma - k)| = C_r \sum_{k\in\Gamma} W_r(\gamma - k) = C_r^2.$$

We will now show that the term $\sum_{k\in\Gamma} |W_r(\gamma-k)|^2$ also has a strictly positive lower bound. To this end, we notice that

$$\inf_{\gamma\in\widehat{G}} \sum_{k\in\Gamma} |W_r(\gamma - k)|^2 = \inf_{\gamma\in\Omega} \sum_{k\in\Gamma} |W_r(\gamma - k)|^2. \tag{3.13}$$

The inequality $\le$ is obvious. In order to show the opposite inequality, we use that any $\gamma \in \widehat{G}$ can be written in a unique way as $\gamma = \gamma' + k'$ with $k' \in \Gamma, \gamma' \in \Omega$. Thus

$$\sum_{k\in\Gamma} |W_r(\gamma - k)|^2 = \sum_{k\in\Gamma} |W_r(\gamma' + k' - k)|^2;$$

making the change of variable $\ell = k - k'$, this shows that

$$\sum_{k\in\Gamma} |W_r(\gamma - k)|^2 = \sum_{\ell\in\Gamma} |W_r(\gamma' - \ell)|^2 \ge \inf_{\zeta\in\Omega} \sum_{k\in\Gamma} |W_r(\zeta - k)|^2,$$

and (3.13) follows.

Now, for $r = 1$ the (strictly positive) lower bound of $\sum_{k\in\Gamma} |W_r(\gamma-k)|^2$ is obvious because $W_1 = C\chi_\Omega$ and Ω is the fundamental domain associated with Γ. Therefore we now assume that $r \ge 2$. Given any $\eta \in \overline{\Omega}$, the partition of unity condition (3.11), with the nonnegative nature of W_r, shows that there is a lattice point $k_\eta \in \Gamma$ such that $W_r(\eta-k_\eta) > 0$. Since W_r is continuous, for each $\eta \in \overline{\Omega}$ there is a neighborhood $\mathcal{U}_\eta$ around η such that $W_r(\gamma-k_\eta) > 0$ for all $\gamma \in \mathcal{U}_\eta$. The neighborhoods $\mathcal{U}_\eta$, $\eta \in \overline{\Omega}$, form an open cover of the compact set $\overline{\Omega}$, so we can select a finite collection of distinct points $\eta_1, \dots, \eta_n \in \overline{\Omega}$ such that $\overline{\Omega} \subseteq \mathcal{U}_{\eta_1} \cup \mathcal{U}_{\eta_2} \cup \cdots \cup \mathcal{U}_{\eta_n}$; thus, for any $\gamma \in \overline{\Omega}$, at least one of the terms $W_r(\gamma - k_{\eta_j})$, $j = 1, \dots, n$, is positive, and therefore $\sum_{j=1}^n |W_r(\gamma - k_{\eta_j})|^2 > 0$. Since W_r is continuous and $\overline{\Omega}$ is compact, this implies that $\inf_{\gamma\in\overline{\Omega}} \sum_{j=1}^n |W_r(\gamma - k_{\eta_j})|^2 > 0$. Putting everything together, we conclude that

$$\inf_{\gamma\in\widehat{G}} \sum_{k\in\Gamma} |W_r(\gamma - k)|^2 = \inf_{\gamma\in\Omega} \sum_{k\in\Gamma} |W_r(\gamma - k)|^2 \ge \inf_{\gamma\in\overline{\Omega}} \sum_{j=1}^{n} |W_r(\gamma - k_{\eta_j})|^2 > 0,$$

providing the promised lower bound. □

Note that in the classical case of a Gabor system $\{E_{mb}T_k\phi\}_{k,m\in\mathbb{Z}} = \{e^{2\pi imb\cdot}\phi(\cdot-k)\}_{k,m\in\mathbb{Z}}$ (with translation parameter $a=1$ and modulation parameter $b>0$) on $\mathbb{R}$, the technical condition $r\Omega \subseteq V$ means that $[0,r)\subseteq[0,1/b)$. Thus, in this particular case we obtain the well-known result that the classical B-spline B_r on $\mathbb{R}$ (defined by the r-fold convolution of terms $\chi_{[0,1)}$) generates a Gabor frame $\{E_{mb}T_kB_r\}_{k,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ if $b\le 1/r$. It is easy to follow the same approach and find explicit frame constructions for $L^2(\widehat{G})$ for any group of the form $G=\mathbb{R}^s\times\mathbb{Z}^p\times\mathbb{T}^q\times\mathbb{Z}_m$, as discussed in the introduction; we leave the concrete calculations to the reader.

In the next example we show how Theorem 3.8 can be used to construct frames for $L^2(\mathbb{R}^+)$.

Example 3.9 Consider $\mathbb{R}^+$; this is an LCA group with respect to multiplication, and with Haar measure $d\mu = x^{-1}\,dx$, where dx is the Lebesgue measure.

Consider the group $G=\mathbb{R}$, with addition as group operation. The characters are the mappings $x\to e^{2\pi ix\gamma}, \gamma\in\mathbb{R}$, so usually $\widehat{G}$ is identified with $\mathbb{R}$. However, the characters can also be written as $x\to e^{2\pi ix\ln(\gamma)}, \gamma\in\mathbb{R}^+$; hereby we can identify $\widehat{G}$ with $\mathbb{R}^+$.

A lattice Γ in $\mathbb{R}^+$ has the form $\{a^n\}_{n\in\mathbb{Z}}$ for some $a>1$; as fundamental domain, let us take $\Omega=[1,a)$. In order to be concrete, let us consider the B-spline W_2 on $\widehat{G}$, defined as in (3.7) with $g_1=g_2=1$; simple direct calculations lead to

$$W_2(\gamma)=\chi_\Omega*\chi_\Omega(\gamma)=\begin{cases}\ln(\gamma), & \gamma\in[1,a),\\ 2\ln(a)-\ln(\gamma), & \gamma\in[a,a^2),\\ 0, & \gamma\notin[1,a^2).\end{cases}$$

Now, take a lattice $\Lambda=b\mathbb{Z}$ in $G=\mathbb{R}$, where $b>0$. Then a direct calculation gives $\Lambda^\perp=\{\left(e^{1/b}\right)^n \mid n\in\mathbb{Z}\}$; the associated fundamental domain in $\widehat{G}=\mathbb{R}^+$ is $V=[1,e^{1/b})$. Assuming that $2\,\Omega\subseteq V$, i.e., $a^2<e^{1/b}$, Theorem 3.8 yields that the collection of functions

$$\{\mathcal{M}_\lambda T_k W_r\}_{\lambda\in\Lambda,k\in\Gamma}=\{e^{2\pi imb\ln(\cdot)}W_2(\cdot\,a^{-n})\}_{m,n\in\mathbb{Z}}$$

forms a Gabor-type frame for $L^2(\mathbb{R}^+)$, for $\mathbb{R}^+$ equipped with the Haar measure mentioned.

Note that the above result could be derived from classical Gabor analysis on $\mathbb{R}$; however, such a direct approach would not highlight the underlying group structure, and the reason for the choice of the measure $d\mu=1/x\,dx$ on $\mathbb{R}^+$. □

3.4 Duality

Let us now consider duality issues for two sequences $\{\mathcal{M}_\lambda\Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda\widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$. First, as a direct consequence of Lemma 3.3 we have the following:

Proposition 3.10 *If $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are Bessel sequences in $L^2(\widehat{G})$, then for all $F, H \in B_c(\widehat{G})$,*

$$\sum_{k\in I}\sum_{\lambda\in\Lambda_k}\langle F, \mathcal{M}_\lambda\Phi_k\rangle\overline{\langle H, \mathcal{M}_\lambda\widetilde{\Phi_k}\rangle} = \sum_{k\in I}\mu_{\widehat{G}}(V_k)\sum_{\omega\in\Lambda_k^\perp}\int_{\widehat{G}} F(\gamma)\overline{H(\omega+\gamma)}\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\omega+\gamma)\,d\gamma.$$

Proof. Note that the sum on the left-hand side is convergent by the Cauchy-Schwarz inequality and the Bessel assumption. Now the result follows immediately from Lemma 3.3. □

Theorem 3.11 *In addition to the general setup in Subsection 3.2, assume that for each $k \in I$,*

$$\operatorname{supp}\Phi_k \cap \operatorname{supp}\widetilde{\Phi_k}(\cdot+\omega) = \emptyset,\ \forall\omega\in\Lambda_k^\perp\setminus\{0\} \tag{3.14}$$

(up to a set of measure zero in $\widehat{G}$). If $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are Bessel sequences in $L^2(\widehat{G})$, they are dual frames for $L^2(\widehat{G})$ if and only if

$$\sum_{k\in I}\mu_{\widehat{G}}(V_k)\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma) = 1,\ \ a.e.\ \gamma\in\widehat{G}. \tag{3.15}$$

Proof. If (3.15) holds, then Proposition 3.10 shows that for all $F, H \in B_c(\widehat{G})$,

$$\sum_{k\in I}\sum_{\lambda\in\Lambda_k}\langle F, \mathcal{M}_\lambda\Phi_k\rangle\overline{\langle H, \mathcal{M}_\lambda\widetilde{\Phi_k}\rangle} = \langle F, H\rangle.$$

By continuity of the inner product, the above equation also holds for all $F, H \in L^2(\widehat{G})$. Combining with the assumption that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are Bessel sequences, this proves that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are dual frames for $L^2(\widehat{G})$, see, e.g., [16] or [3, Lemma 5.7.1].

Conversely, assume that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda_k,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda_k,k\in I}$ are dual frames such that (3.14) holds. By Proposition 3.10, for $F = H \in B_c(\widehat{G})$,

$$\int_{\widehat{G}}\sum_{k\in I}\mu_{\widehat{G}}(V_k)\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma)|F(\gamma)|^2 d\gamma = \int_{\widehat{G}}|F(\gamma)|^2 d\gamma.$$

Splitting $\sum_{k\in I}\mu_{\widehat{G}}(V_k)\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma)$ into real part and imaginary part, i.e. $a(\gamma)+ib(\gamma) = \sum_{k\in I}\mu_{\widehat{G}}(V_k)\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma)$, yields that

$$\int_{\widehat{G}} a(\gamma)|F(\gamma)|^2 d\gamma = \int_{\widehat{G}}|F(\gamma)|^2 d\gamma \text{ and } \int_{\widehat{G}} b(\gamma)|F(\gamma)|^2 d\gamma = 0$$

for all $F \in B_c(\widehat{G})$, which implies that $a(\gamma) = 1$ and $b(\gamma) = 0$ for a.e. $\gamma \in \widehat{G}$, by exactly the same argument as we used in the proof of Corollary 3.5. □

Let us return to the setup in Theorem 3.8 and consider a Gabor system of the form $\{\mathcal{M}_\lambda T_k W_r\}_{\lambda\in\Lambda,k\in\Gamma}$, where Γ is chosen as a lattice in $\widehat{G}$, Ω is a corresponding fundamental domain, and W_r is a weighted B-spline with $g_1 = C$ and $g_j > 0, j = 2,\ldots,r$, on Ω. By Theorem 3.8, such a system is a frame for $L^2(\widehat{G})$ if $r\Omega \cap (r\Omega + (\Lambda^\perp \setminus \{0\})) = \emptyset$, in particular, if $r\Omega \subseteq V$. We will now impose a stronger assumption, which implies that we can find an explicitly given dual frame $\{\mathcal{M}_\lambda T_k \widetilde{\Phi}\}_{\lambda\in\Lambda,k\in\Gamma}$.

Proposition 3.12 *In addition to the setup in Theorem* 3.8, *assume that the set*

$$\Delta := \{k \in \Gamma \bigm| r\Omega \cap (k + r\Omega) \neq \emptyset\} \tag{3.16}$$

satisfies that

$$r\Omega \cap (\Delta + r\Omega + (\Lambda^\perp \setminus \{0\})) = \emptyset. \tag{3.17}$$

Then, with the constant C_r defined as in (3.10), *the function*

$$\widetilde{\Phi}(\gamma) := \frac{1}{\mu_{\widehat{G}}(V)C_r^2} \sum_{k\in\Delta} W_r(\gamma - k), \ \ \gamma \in \widehat{G}, \tag{3.18}$$

generates a dual frame $\{\mathcal{M}_\lambda T_k \widetilde{\Phi}\}_{\lambda\in\Lambda,k\in\Gamma}$ of $\{\mathcal{M}_\lambda T_k W_r\}_{\lambda\in\Lambda,k\in\Gamma}$.

Proof. Note that in the described setup, the condition (3.15) takes the form

$$\sum_{k\in\Gamma} W_r(\gamma - k)\widetilde{\Phi}(\gamma - k) = \frac{1}{\mu_{\widehat{G}}(V)}, \ \text{a.e. } \gamma \in \widehat{G},$$

or,

$$\sum_{k\in\Gamma} (W_r\widetilde{\Phi})(\gamma - k) = \frac{1}{\mu_{\widehat{G}}(V)}, \ \text{a.e. } \gamma \in \widehat{G}, \tag{3.19}$$

Since $\sum_{k\in\Gamma} W_r(\gamma - k) = C_r$ as noted in (3.11), the condition (3.19) is obviously satisfied if we choose the function $\widetilde{\Phi}$ such that $W_r\widetilde{\Phi} = (\mu_{\widehat{G}}(V)C_r)^{-1} W_r$. Thus, it suffices to have that $\widetilde{\Phi}(\gamma) = (\mu_{\widehat{G}}(V)C_r)^{-1}$ for $\gamma \in r\Omega$, a condition that is satisfied if we take $\widetilde{\Phi}$ to be as in (3.18), with the index set Δ defined by (3.16). To see this, note that if $\gamma \in r\Omega$ and $k \in \Gamma \setminus \Delta$, then $\gamma \notin k + r\Omega$, which implies that $W_r(\gamma - k) = 0$. Therefore, for $\gamma \in r\Omega$,

$$\begin{aligned} \widetilde{\Phi}(\gamma) &= \frac{1}{\mu_{\widehat{G}}(V)C_r^2} \sum_{k\in\Delta} W_r(\gamma - k) = \frac{1}{\mu_{\widehat{G}}(V)C_r^2} \left(\sum_{k\in\Delta} W_r(\gamma - k) + \sum_{k\in\Gamma\setminus\Delta} W_r(\gamma - k) \right) \\ &= \frac{1}{\mu_{\widehat{G}}(V)C_r^2} \sum_{k\in\Gamma} W_r(\gamma - k) = \frac{1}{\mu_{\widehat{G}}(V)C_r}, \end{aligned}$$

as desired. With the choice of $\widetilde{\Phi}$ in (3.18), the condition (3.17) ensures that (3.14) holds. Hence, the result follows from Theorem 3.11. □

Let us again relate the result to the classical case of a Gabor system $\{E_{mb}T_k\phi\}_{k,m\in\mathbb{Z}}$ on $\mathbb{R}$. Letting again $\phi := B_r := \chi_{[0,1)} * \chi_{[0,1)} * \cdots * \chi_{[0,1)}$, we see that $\Delta = \{-r+1, \dots, r-1\}$. Thus $\Delta + r\Omega = [-r+1, 2r-1)$; therefore (3.17) is satisfied if $1/b \geq 2r-1$, i.e., if $b \leq \frac{1}{2r-1}$. This is exactly the condition used in order to construct dual pairs in [6]. Similar to the case for Theorem 3.8, it is easy to apply Proposition 3.12 to construct explicit dual pairs of frames for $L^2(\widehat{G})$ for groups of the form $G = \mathbb{R}^s \times \mathbb{Z}^p \times \mathbb{T}^q \times \mathbb{Z}_m$; and, following the approach in Example 3.9, we can use Proposition 3.12 to construct dual pairs of frames for $L^2(\mathbb{R}^+)$. We leave the calculations to the interested reader.

The next result provides a general sufficient condition for dual frames, without explicit assumptions on the supports of Φ_k and $\widetilde{\Phi_k}$, when the lattices Λ_k are independent of $k \in I$.

Proposition 3.13 *Let Λ be a lattice in G, and choose the set V in $\widehat{G}$ as in* (2.1). *Assume that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda,k\in I}$ are Bessel sequences in $L^2(\widehat{G})$. If*

$$\sum_{k\in I} \overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\omega+\gamma) = \frac{1}{\mu_{\widehat{G}}(V)}\,\delta_{\omega,0}, \ \ a.e.\ \gamma\in\widehat{G},\ \ \forall\,\omega\in\Lambda^\perp, \tag{3.20}$$

then $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda,k\in I}$ are dual frames for $L^2(\widehat{G})$.

Proof. Let $F, H \in B_c(\widehat{G})$. Using Proposition 3.10 and pulling out the terms corresponding to $\omega = 0$ yields

$$\begin{aligned}
&\sum_{k\in I}\sum_{\lambda\in\Lambda}\langle F, \mathcal{M}_\lambda\Phi_k\rangle\overline{\langle H, \mathcal{M}_\lambda\widetilde{\Phi_k}\rangle} \\
&= \mu_{\widehat{G}}(V)\sum_{k\in I}\left(\int_{\widehat{G}} F(\gamma)\overline{H(\gamma)}\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma)\,d\gamma + \sum_{\omega\in\Lambda^\perp\setminus\{0\}}\int_{\widehat{G}} F(\gamma)\overline{H(\omega+\gamma)}\,\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\omega+\gamma)\,d\gamma\right) \\
&= \mu_{\widehat{G}}(V)\int_{\widehat{G}} F(\gamma)\overline{H(\gamma)}\sum_{k\in I}\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\gamma)\,d\gamma + \sum_{\omega\in\Lambda^\perp\setminus\{0\}}\int_{\widehat{G}} F(\gamma)\overline{H(\omega+\gamma)}\sum_{k\in I}\overline{\Phi_k(\gamma)}\widetilde{\Phi_k}(\omega+\gamma)\,d\gamma.
\end{aligned}$$

By the assumption (3.20), this implies that

$$\sum_{k\in I}\sum_{\lambda\in\Lambda}\langle F, \mathcal{M}_\lambda\Phi_k\rangle\overline{\langle H, \mathcal{M}_\lambda\widetilde{\Phi_k}\rangle} = \int_{\widehat{G}} F(\gamma)\overline{H(\gamma)}\,d\gamma = \langle F, H\rangle.$$

As in the proof of Theorem 3.11, combining with the Bessel assumption, this implies that $\{\mathcal{M}_\lambda \Phi_k\}_{\lambda\in\Lambda,k\in I}$ and $\{\mathcal{M}_\lambda \widetilde{\Phi_k}\}_{\lambda\in\Lambda,k\in I}$ are dual frames for $L^2(\widehat{G})$. □

4 Generalized shift-invariant systems $\{T_\lambda\phi_k\}_{\lambda\in\Lambda_k,k\in I}$

The results obtained so far have immediate consequences for generalized shift-invariant systems. In fact, let Λ be a lattice in G. Then, for any $\phi \in L^2(G)$ and $\lambda \in \Lambda$,

$$\mathcal{F}^{-1}\mathcal{M}_\lambda\mathcal{F}\phi(x) = T_{-\lambda}\mathcal{F}^{-1}\mathcal{F}\phi(x) = T_{-\lambda}\phi(x) = \phi(x+\lambda).$$

Since the inverse Fourier transform is a unitary operator, it preserves the properties of Bessel sequences, frames and dual frames from $L^2(\widehat{G})$ to $L^2(G)$. By setting $\Phi_k := \widehat{\phi_k}$ and $\widetilde{\Phi_k} := \widehat{\widetilde{\phi_k}}$, we obtain the following immediate consequences of Theorem 3.4, Theorem 3.11 and Proposition 3.13:

Theorem 4.1 *Suppose that $\{\Lambda_k\}_{k\in I}$ is a countable family of lattices in G, and let $\{V_k\}_{k\in I}$ be the corresponding sets in $\widehat{G}$ as in (3.5). Consider two collections of elements $\{\phi_k\}_{k\in I}$, $\{\widetilde{\phi_k}\}_{k\in I}$ in $L^2(G)$. Then the following hold:*

(i) *$\{T_\lambda \phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a Bessel sequence in $L^2(G)$ if*

$$B := \sup_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k) \sum_{\omega\in\Lambda_k^\perp} \left|\widehat{\phi_k}(\gamma)\widehat{\phi_k}(\gamma+\omega)\right| < \infty.$$

(ii) *If* (i) *holds, then $\{T_\lambda \phi_k\}_{\lambda\in\Lambda_k, k\in I}$ is a frame for $L^2(G)$ if*

$$A := \inf_{\gamma\in\widehat{G}} \left(\sum_{k\in I} \mu_{\widehat{G}}(V_k)\, |\widehat{\phi_k}(\gamma)|^2 - \sum_{k\in I} \mu_{\widehat{G}}(V_k) \sum_{\omega\in\Lambda_k^\perp\setminus\{0\}} \left|\widehat{\phi_k}(\gamma)\widehat{\phi_k}(\gamma+\omega)\right| \right) > 0.$$

(iii) *Assume that for each $k\in I$,*

$$\operatorname{supp}\widehat{\phi_k} \cap \operatorname{supp}\widehat{\widetilde{\phi_k}}(\cdot+\omega) = \emptyset,\ \forall\,\omega\in\Lambda_k^\perp\setminus\{0\}$$

(*up to a set of measure zero in $\widehat{G}$*). *If $\{T_\lambda \phi_k\}_{\lambda\in\Lambda_k, k\in I}$ and $\{T_\lambda \widetilde{\phi_k}\}_{\lambda\in\Lambda_k, k\in I}$ are Bessel sequences in $L^2(G)$, they are dual frames for $L^2(G)$ if and only if*

$$\sum_{k\in I} \mu_{\widehat{G}}(V_k)\, \overline{\widehat{\phi_k}(\gamma)}\widehat{\widetilde{\phi_k}}(\gamma) = 1,\ a.e.\,\gamma\in\widehat{G}.$$

(iv) *Assume that $\Lambda_k = \Lambda$ for all $k\in I$, and let V denote the corresponding set in* (2.1). *If $\{T_\lambda \phi_k\}_{\lambda\in\Lambda, k\in I}$ and $\{T_\lambda \phi_k\}_{\lambda\in\Lambda, k\in I}$ are Bessel sequences in $L^2(G)$and*

$$\sum_{k\in I} \overline{\widehat{\phi_k}(\gamma)}\widehat{\widetilde{\phi_k}}(\omega+\gamma) = \frac{1}{\mu_{\widehat{G}}(V)}\,\delta_{\omega,0},\ a.e.\,\gamma\in\widehat{G},\ \forall\,\omega\in\Lambda^\perp,$$

then $\{T_\lambda \phi_k\}_{\lambda\in\Lambda, k\in I}$ and $\{T_\lambda \widetilde{\phi_k}\}_{\lambda\in\Lambda, k\in I}$ are dual frames for $L^2(G)$.

It is well known (see e.g. [18]) that Gabor systems and wavelet systems can be realized as special cases of generalized shift-invariant systems. Translated to the setting of these systems, Theorem 4.1 corresponds to results that are already available in the literature. Let us demonstrate this for the case of a matrix-generated wavelet system in $L^2(\mathbb{R}^s)$.

Example 4.2 For $G = \mathbb{R}^s$, the dual group $\widehat{G}$ can be identified as $\widehat{G} = \mathbb{R}^s$. Define the *scaling operator* associated with an invertible $s \times s$ matrix $\mathcal{A}$ with real entries by $(D_{\mathcal{A}} f)(x) = |\det \mathcal{A}|^{1/2} f(\mathcal{A}x),\ x \in \mathbb{R}^s$. Now, given real invertible $s \times s$ matrices $\mathcal{A}_k$ and $\mathcal{B}_k,\ k \in I$, consider a (nonstationary) wavelet system

$$\{D_{\mathcal{A}_k} T_{\mathcal{B}_k j} \phi\}_{k \in I, j \in \mathbb{Z}^s} = \{|\det \mathcal{A}_k|^{1/2}\, \phi(\mathcal{A}_k \cdot - \mathcal{B}_k j)\}_{k \in I, j \in \mathbb{Z}^s},$$

where $\phi \in L^2(\mathbb{R}^s)$. Note that this general setup contains the classical wavelet systems as well as, e.g., the composite wavelets in [11] as special cases; in particular, when $I = \mathbb{Z}^s$ and $\mathcal{B}_k = \mathcal{B}$ for all $k \in I$. Letting $\phi_k(x) := D_{\mathcal{A}_k} \phi(x) = |\det \mathcal{A}_k|^{1/2} \phi(\mathcal{A}_k x),\ k \in I, x \in \mathbb{R}^s$, we have that

$$T_\lambda \phi_k(x) = \phi_k(x - \lambda) = |\det \mathcal{A}_k|^{1/2} \phi(\mathcal{A}_k x - \mathcal{A}_k \lambda).$$

Thus, taking $\Lambda_k := \mathcal{A}_k^{-1} \mathcal{B}_k \mathbb{Z}^s$, the system $\{T_\lambda \phi_k\}_{k \in I, \lambda \in \Lambda_k}$ is exactly the wavelet system $\{D_{\mathcal{A}_k} T_{\mathcal{B}_k j} \phi\}_{k \in I, j \in \mathbb{Z}^s}$. Since $\Lambda_k^\perp = ((\mathcal{A}_k^{-1} \mathcal{B}_k)^T)^{-1} \mathbb{Z}^s = (\mathcal{A}_k^{-1} \mathcal{B}_k)^\sharp \mathbb{Z}^s = \mathcal{A}_k^T \mathcal{B}_k^\sharp \mathbb{Z}^s$ and $\mathbb{R}^s = \bigcup_{n \in \mathbb{Z}^s} (n + [0,1)^s)$, we can take $V_k = \mathcal{A}_k^T \mathcal{B}_k^\sharp [0,1)^s$ in (3.5). Now,

$$\widehat{\phi_k}(\gamma) = \mathcal{F} D_{\mathcal{A}_k} \phi(\gamma) = D_{\mathcal{A}_k^\sharp} \widehat{\phi}(\gamma),$$

so the condition in Theorem 4.1(i) amounts to

$$B := \sup_{\gamma \in \mathbb{R}^s} \sum_{k \in I} |\det(\mathcal{A}_k^T \mathcal{B}_k^\sharp)| \sum_{\omega \in \Lambda_k^\perp} |\det \mathcal{A}_k^\sharp|\, |\widehat{\phi}(\mathcal{A}_k^\sharp \gamma) \widehat{\phi}(\mathcal{A}_k^\sharp \gamma + \mathcal{A}_k^\sharp \omega)| < \infty,$$

or,

$$B = \sup_{\gamma \in \mathbb{R}^s} \sum_{k \in I} \frac{1}{|\det \mathcal{B}_k|} \sum_{n \in \mathbb{Z}^s} |\widehat{\phi}(\mathcal{A}_k^\sharp \gamma) \widehat{\phi}(\mathcal{A}_k^\sharp \gamma + \mathcal{B}_k^\sharp n)| < \infty.$$

This is a generalized s-dimensional version, which includes the nonstationary case, of the well-known sufficient condition in [3, Theorem 11.7.3]. □

Let us also look at an application to the case of periodic frames in $L^2([0,1]^s)$.

Example 4.3 Let $\mathbb{T}$ denote the torus, i.e., the additive group of all real numbers modulo 1. For $s \in \mathbb{N}$ we consider the group $G = \mathbb{T}^s$. Then $L^2(G)$ is the set of functions on $\mathbb{R}^s$ that are 1-periodic in each variable, and are square-integrable over $[0,1]^s$. It is well known that the dual group of G is the set of functions $\gamma : \mathbb{T}^s \to \mathbb{C}$ that have the form $\gamma(x) = e^{2\pi i n \cdot x}$ for some $n \in \mathbb{Z}^s$. Thus, we use the standard identification $\widehat{G} = \mathbb{Z}^s$. As Haar measure on G, let us take the standard Lebesgue measure on $\mathbb{R}^s$, and we equip $\widehat{G}$ with the counting measure. The Fourier transform becomes $\widehat{f} : \mathbb{Z}^s \to \mathbb{C}$, $\widehat{f}(n) = \int_{[0,1]^s} f(x)\, e^{-2\pi i n \cdot x}\, dx$.

Now, consider a countable sequence $\{\mathcal{N}_k\}_{k \in I}$ of invertible $s \times s$ matrices with integer-entries. For $k \in I$, let L_k denote a full collection of coset representatives of $\mathbb{Z}^s / \mathcal{N}_k \mathbb{Z}^s$. Then

$$\mathbb{Z}^s = \bigcup_{\ell \in L_k} (\ell + \mathcal{N}_k \mathbb{Z}^s) = \{\ell + \omega : \ell \in L_k, \omega \in \mathcal{N}_k \mathbb{Z}^s\} = \bigcup_{\omega \in \mathcal{N}_k \mathbb{Z}^s} (\omega + L_k).$$

Thus, (3.5) holds with $V_k = L_k$. For the lattice $\Lambda_k = \mathcal{N}_k^{-1} L_k$ in G, the annihilator is $\Lambda_k^\perp = \mathcal{N}_k^T \mathbb{Z}^s$. It is well known that the number of elements in L_k is $|\det(\mathcal{N}_k)|$. Thus, $\mu_{\widehat{G}}(V_k) = |\det(\mathcal{N}_k)|$. Consider a collection of functions $\{\phi_k\}_{k\in I}$ in $L^2(G)$. Then, by Theorem 4.1(i), $\{T_\lambda \phi_k\}_{k\in I, \lambda\in\Lambda_k} = \{\phi_k(\cdot - \mathcal{N}_k^{-1}\ell)\}_{\ell\in L_k, k\in I}$ is a Bessel sequence with bound B if

$$B := \sup_{\gamma\in\widehat{G}} \sum_{k\in I} \mu_{\widehat{G}}(V_k) \sum_{\omega\in\Lambda_k^\perp} \left| \widehat{\phi_k}(n)\widehat{\phi_k}(n+\omega) \right| < \infty,$$

i.e., if

$$B := \sup_{n\in\mathbb{Z}^s} \sum_{k\in I} |\det(\mathcal{N}_k)| \sum_{q\in\mathbb{Z}^s} \left| \widehat{\phi_k}(n)\widehat{\phi_k}(n+\mathcal{N}_k^T q) \right| < \infty.$$

This is exactly the s-dimensional version of the condition derived in [4] for the one-dimensional case. □

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