The unitary extension principle on locally compact abelian groups

Ole Christensen and Say Song Goh

Abstract

The unitary extension principle (UEP) by Ron and Shen yields conditions for the construction of a multi-generated tight wavelet frame for $L^2(\mathbb{R}^s)$ based on a given refinable function. In this paper we show that the UEP can be generalized to locally compact abelian groups. In the general setting, the resulting frames are generated by modulates of a collection of functions; via the Fourier transform this corresponds to a generalized shift-invariant system. Both the stationary and the nonstationary case are covered. We provide general constructions, based on B-splines on the group itself as well as on characteristic functions on the dual group. Finally, we consider a number of concrete groups and derive explicit constructions of the resulting frames.

1 Introduction

The unitary extension principle (UEP) by Ron and Shen [19] and its many variants ([7, 9], to name a few) are key results in wavelet analysis. They allow construction of tight wavelet frames with compact support, desired smoothness, and good approximation theoretic properties. In this paper we show how the theory can be generalized to the setting of locally compact abelian (LCA) groups. The advantage of this approach is threefold. First, we are able to cover several variants of the unitary extension principle, e.g., the standard case on $\mathbb{R}^s$ and the periodic case corresponding to the group $\mathbb{T}^s$, all at once. Secondly, we can now apply the UEP to a large set of other groups; among these, the group of integers $\mathbb{Z}$ is particularly interesting. Thirdly, the general approach throws new light on the classical UEP: it reveals the structure that is necessary if we want to consider the UEP from a more general point of view – a structure that in the case of the group $\mathbb{R}^s$ turns out to coincide with the classical wavelet structure.

The general approach presented here is able to handle both the stationary and the nonstationary case. In the full generality of LCA groups we will derive explicit conditions for the UEP construction of tight frames, based on either B-splines on the group or characteristic functions on the dual group. Finally, a number of explicit constructions on groups of particular interest are provided.

Denoting the underlying LCA group by G, the construction yields frames for $L^2(\widehat{G})$, where $\widehat{G}$ denotes the dual group. The frames are obtained by letting a class of modulation operators act on a family of functions in $L^2(\widehat{G})$. Via the Fourier transform, this immediately yields a generalized shift-invariant system that forms a frame for $L^2(G)$; see the more technical description right after the definition of the modulation operator in (2.5).

The discussion of the general results is complemented by explicit constructions on the LCA groups $\mathbb{R}^s, \mathbb{T}, \mathbb{Z}$, and $\mathbb{Z}_N$ (the integers modulo N). Some of them are based on a generalization of B-splines to LCA groups that was discovered already in 1994 (independently by Dahlke [8] and Tikhomirov [23]); other constructions are generated by characteristic functions for certain sets in $\widehat{G}$.

Section 2 will provide us with the necessary background on LCA groups; the general version of the UEP and its proof is stated in Section 3. The formulation of the results based on B-splines are presented in Section 4, while the case of characteristic functions on $\widehat{G}$ are in Section 5; in both cases applications to a number of specific LCA groups are given.

Let us end this introduction with a few comments about technicalities. The main difficulty in the extension of the UEP to LCA groups is that there is no scaling operator on LCA groups. The way to overcome this issue turns out to be to consider a collection of modulation operators acting on a family of frame generators, rather than the usual collection of scaled and translated versions of a fixed function. This leads to a different form of the scaling equation. Also, the traditional assumption of the wavelet subspaces being nested has to be replaced by a condition on a nested sequence of lattices in the group. After getting familiar with these new aspects, the reader will observe that several of the technical results follow the same pattern as in the classical proofs of the UEP.

We also note that the approach in the paper heavily uses the LCA group structure. We would like to point the attention of the reader to a different generalization of the UEP, taking place on smooth and compact Riemannian manifolds; see [24]. We also mention that there is a growing literature on wavelet analysis on the p-adic numbers; see [1] and the references therein. The p-adic numbers do not have nontrivial lattices [3], and are not covered by our methods. A more general wavelet theory on LCA groups with a compact open subgroup is developed in [2].

2 Preliminaries on LCA groups

In this section we will give a short introduction to the necessary background on LCA groups; for more information we refer to the books [14, 18, 21].

Let G be an LCA group, with the group composition denoted by the symbol " + " and the neutral element 0. We will assume that G is equipped with a Hausdorff topology, and that G is a countable union of compact sets and metrizable. A *character* on G is a function $\gamma : G \to \mathbb{T} := \{z \in \mathbb{C} \bigm| |z| = 1\}$ that satisfies the condition $\gamma(x + y) = \gamma(x)\gamma(y),\ x, y \in G$. We denote the set of continuous characters by $\widehat{G}$, which also forms an LCA group, the *dual group* of G, when equipped with the composition $(\gamma + \gamma')(x) := \gamma(x)\gamma'(x)$, $\gamma, \gamma' \in \widehat{G}$,

$x \in G$, and an appropriate topology. It is a classical result that the double-dual group $\widehat{\widehat{G}}$ is topologically isomorphic to the group G; usually we can identify the two groups and we will simply write $\widehat{\widehat{G}} = G$. Thus $\gamma(x)$ can be interpreted as either the action of $\gamma \in \widehat{G}$ on $x \in G$, or the action of $x \in \widehat{\widehat{G}} = G$ on $\gamma \in \widehat{G}$; for this reason we will from now on use the notation

$$(x, \gamma) := \gamma(x), \ x \in G, \ \gamma \in \widehat{G}.$$

The LCA group G can be equipped with a Radon measure μ_G that is *translation invariant*, which means that for all continuous functions f on G with compact support,

$$\int_G f(x+y)\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x), \ \forall\, y \in G. \tag{2.1}$$

This measure is unique up to scalar multiplication, and is called the *Haar measure.* We will consider the Haar measure μ_G as fixed throughout the paper. Based on the Haar measure, we define the spaces $L^1(G), L^2(G)$ and $L^\infty(G)$ in the usual way. The space $L^2(G)$ is a Hilbert space, and our assumption of G being a countable union of compact sets and metrizable implies (and is, in fact, equivalent to) $L^2(G)$ being separable.

The *Fourier transform* is defined by

$$\mathcal{F} : L^1(G) \to C_0(\widehat{G}), \quad \mathcal{F}f(\gamma) := \hat{f}(\gamma) := \int_G f(x)(-x, \gamma)\, d\mu_G(x). \tag{2.2}$$

The *inversion theorem* states that with appropriate normalization of the Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$, for $f \in L^1(G)$ such that $\hat{f} \in L^1(\widehat{G})$, it holds that

$$f(x) = \int_{\widehat{G}} \hat{f}(\gamma)\, (x, \gamma) d\mu_{\widehat{G}}(\gamma), \ x \in G. \tag{2.3}$$

We will always normalize the measure on $\widehat{G}$ in such a way that the inversion formula holds. With this choice, the Fourier transform can be extended to a surjective isometry $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$, exactly as in the classical case of $G = \mathbb{R}$. To simplify notations, from now onwards, in all integrals when the context is clear (e.g., (2.1)–(2.3)), we simply write $d\mu_G(x) = dx$ and $d\mu_{\widehat{G}}(\gamma) = d\gamma$.

Among the examples of LCA groups, we find $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}_N$, as well as their higher-dimensional variants and direct products hereof; following [18] and [10] we will call such groups *elementary LCA groups.* As discussed in [10], various typical problems in signal processing can be modeled using elementary LCA groups.

A *lattice* (sometimes called a uniform lattice) in an LCA group G is a discrete subgroup Λ for which G/Λ is compact. Lattices are known explicitly in all the elementary LCA groups and in many other LCA groups; however, there also exist LCA groups without lattices, see, e.g., [16, 3, 15]. The *annihilator* $\Lambda^\perp$ of a lattice Λ is defined by

$$\Lambda^\perp := \{\gamma \in \widehat{G} \mid (x, \gamma) = 1, \ \forall x \in \Lambda\}.$$

It follows from the definition of the topology on $\widehat{G}$ that the annihilator $\Lambda^\perp$ is a closed subgroup of $\widehat{G}$. A lattice in G leads to a splitting of the groups G and $\widehat{G}$ into disjoint cosets, see, e.g., [16]:

Lemma 2.1 *Let G be an LCA group and Λ a lattice in G. Then the following hold:*

(i) *There exists a Borel measurable relatively compact set $Q \subseteq G$ such that*

$$G = \bigcup_{\lambda\in\Lambda} (\lambda + Q), \quad (\lambda + Q) \cap (\lambda' + Q) = \emptyset \text{ for } \lambda \neq \lambda', \ \lambda, \lambda' \in \Lambda. \tag{2.4}$$

(ii) *The set $\Lambda^\perp$ is a lattice in $\widehat{G}$, and there exists a Borel measurable relatively compact set $V \subseteq \widehat{G}$ such that*

$$\widehat{G} = \bigcup_{\omega\in\Lambda^\perp} (\omega + V), \quad (\omega + V) \cap (\omega' + V) = \emptyset \text{ for } \omega \neq \omega', \ \omega, \omega' \in \Lambda^\perp.$$

A set Q as in (2.4) that has the properties in Lemma 2.1(i) is called a *fundamental domain* associated to the lattice Λ. For convenience we will allow sets Q for which the two conditions in (2.4) hold up to a set of measure zero. When we speak about a *periodic function* $f \in L^\infty(Q)$, it is understood that we extend f to a function on G by

$$f(\lambda + x) := f(x), \ \lambda \in \Lambda, \ x \in Q.$$

Given a lattice Λ in G, choose a fundamental domain Q. The *density* of Λ is defined by $s(\Lambda) := \mu_G(Q)$. It is well known that this is independent of the chosen fundamental domain, and that (see [12]) $s(\Lambda)\, s(\Lambda^\perp) = 1$.

Given any $\lambda \in G$, consider the *generalized modulation operator*

$$\mathcal{M}_\lambda : L^2(\widehat{G}) \to L^2(\widehat{G}), \ (\mathcal{M}_\lambda f)(\gamma) := (\lambda, \gamma)\, f(\gamma). \tag{2.5}$$

As for the modulation operator on $\mathbb{R}$, it is easy to see that $\mathcal{M}_\lambda$ is a unitary operator.

The main outcome of the current paper is a method for constructing frames for $L^2(\widehat{G})$ of the form $\{\mathcal{M}_\lambda \Gamma_k\}_{k\in J, \lambda\in\Lambda_k}$, where $\{\Gamma_k\}_{k\in J}$ is a countable collection of functions in $L^2(\widehat{G})$ and $\{\Lambda_k\}_{k\in J}$ a collection of lattices in G. Considering the *translation operator* T_y, $y \in G$, on $L^2(G)$ defined by $T_y : L^2(G) \to L^2(G)$, $T_y f(x) := f(x - y)$, $x \in G$, it is well known that $\mathcal{F}T_y = \mathcal{M}_{-y}\mathcal{F}$; thus, if $\{\mathcal{M}_\lambda \Gamma_k\}_{k\in J, \lambda\in\Lambda_k}$ is a frame for $L^2(\widehat{G})$ it immediately follows that the system $\{\mathcal{F}^{-1}\mathcal{M}_\lambda \Gamma_k\}_{k\in J, \lambda\in\Lambda_k} = \{T_\lambda \mathcal{F}^{-1}\Gamma_k\}_{k\in J, \lambda\in\Lambda_k}$ is a frame for $L^2(G)$. This system is a so-called *generalized shift-invariant system* in the terminology of Ron and Shen [20]. The case where the lattices Λ_k are independent of k correspond to the classical shift-invariant systems; for a detailed analysis of such systems we refer to [4].

We will need the following result, which is a variant of Lemma 3.3 in [17]; the version stated here is proved in [6] and repeated in [5]. For technical reasons in the lemma and subsequently in Section 3, we consider the dense subspace $C_c(\widehat{G})$ of $L^2(\widehat{G})$ defined by

$$C_c(\widehat{G}) := \{f \in L^2(\widehat{G}) \,|\, f \text{ is continuous and compactly supported}\}.$$

Lemma 2.2 *Let Λ be a lattice in G, and let $V \subseteq \widehat{G}$ denote a fundamental domain associated with the lattice $\Lambda^\perp$. Let $F, \Phi \in L^2(\widehat{G})$. Then the function*

$$\alpha : \widehat{G} \to \mathbb{C},\ \alpha(\gamma) := \sum_{\omega \in \Lambda^\perp} F(\omega+\gamma)\overline{\Phi(\omega+\gamma)}$$

is well defined, belongs to $L^1(V)$, and satisfies that

$$\alpha(\gamma+\omega') = \alpha(\gamma),\ \forall\, \gamma \in \widehat{G}, \omega' \in \Lambda^\perp.$$

In addition, if $F \in C_c(\widehat{G})$, then

$$\sum_{\lambda \in \Lambda} |\langle F, \mathcal{M}_\lambda \Phi\rangle|^2 = \mu_{\widehat{G}}(V) \int_V \left| \sum_{\omega \in \Lambda^\perp} F(\omega+\gamma)\overline{\Phi(\omega+\gamma)} \right|^2 d\gamma.$$

One of the central ingredient in our generalization of the UEP is to consider a family of nested lattices $\{\Lambda_k\}_{k \in I}$ in the group G. We will need to consider relations between the lattices and their corresponding fundamental domains; for easy reference we will formulate here the relevant connections between just two lattices.

Lemma 2.3 *Consider two lattices $\Lambda_0 \subset \Lambda_1$ in an LCA group G. Then the following hold:*

(i) *The quotient group Λ_1/Λ_0 is finite, with cardinality $|\Lambda_1/\Lambda_0| = \frac{s(\Lambda_0)}{s(\Lambda_1)}$.*

(ii) *Let $\{\eta_\ell\}_{\ell=1}^d \subset \Lambda_1$ be a fundamental domain for Λ_0, considered as a subgroup of Λ_1, chosen such that $\eta_1 = 0$. Then*

$$\Lambda_1 = \bigcup_{\ell=1}^d (\eta_\ell + \Lambda_0),\ \textit{with } (\eta_\ell+\Lambda_0) \cap (\eta_{\ell'}+\Lambda_0) = \emptyset \textit{ if } \ell \neq \ell'.$$

(iii) *Choose Q_1 as a fundamental domain associated with the lattice Λ_1 in G. Then*

$$(\eta_\ell + Q_1) \cap (\eta_{\ell'} + Q_1) = \emptyset \textit{ if } \ell \neq \ell'. \tag{2.6}$$

Furthermore, the set $Q_0 := \bigcup_{\ell=1}^d (\eta_\ell + Q_1)$ is a fundamental domain associated with the lattice Λ_0 in G.

(iv) *$\Lambda_1^\perp$ is a subgroup of $\Lambda_0^\perp$; the quotient group $\Lambda_0^\perp/\Lambda_1^\perp$ is finite, and $\left|\Lambda_0^\perp/\Lambda_1^\perp\right| = |\Lambda_1/\Lambda_0|$.*

(v) *Let $\{\nu_\ell\}_{\ell=1}^d \subset \Lambda_0^\perp$ be a fundamental domain for $\Lambda_1^\perp$, considered as a subgroup of $\Lambda_0^\perp$, chosen such that $\nu_1 = 0$. Then*

$$\Lambda_0^\perp = \bigcup_{\ell=1}^d (\nu_\ell + \Lambda_1^\perp),\ \textit{with } (\nu_\ell+\Lambda_1^\perp) \cap (\nu_{\ell'}+\Lambda_1^\perp) = \emptyset \textit{ if } \ell \neq \ell'.$$

(vi) *Let $V_0 \subset \widehat{G}$ be a fundamental domain associated with the lattice $\Lambda_0^\perp$ in $\widehat{G}$. Then*

$$(\nu_\ell + V_0) \cap (\nu_{\ell'} + V_0) = \emptyset \text{ if } \ell \neq \ell'.$$

Furthermore, the set $V_1 := \bigcup_{\ell=1}^d (\nu_\ell + V_0)$ is a fundamental domain associated with the lattice $\Lambda_1^\perp$ in $\widehat{G}$.

Proof. (i) By definition of a lattice we know that G/Λ_0 is relatively compact; since $\Lambda_1 \subseteq G$ this implies that Λ_1/Λ_0 is also relatively compact. But since Λ_1 is discrete, Λ_1/Λ_0 is also discrete; hence the set must be finite. We postpone the proof of the cardinality by a few lines.

(ii) By definition of the fundamental domain, $\Lambda_1 = \bigcup_{\lambda \in \Lambda_0} (\lambda + \{\eta_\ell\}_{\ell=1}^d) = \bigcup_{\ell=1}^d (\eta_\ell + \Lambda_0)$. By construction the union is disjoint, so (ii) holds.

(iii) As Q_1 is a fundamental domain associated with Λ_1, we know that $(\lambda + Q_1) \cap (\lambda' + Q_1) = \emptyset$ if $\lambda, \lambda' \in \Lambda_1$, $\lambda \neq \lambda'$. Since $\eta_\ell \in \Lambda_1$, (2.6) follows. In order to prove the rest of (iii), using (ii) we see that

$$G = \bigcup_{\lambda \in \Lambda_1} (\lambda + Q_1) = \bigcup_{\lambda \in \Lambda_0} \bigcup_{\ell=1}^{d} (\eta_\ell + \lambda + Q_1) \; = \; \bigcup_{\lambda \in \Lambda_0} \big(\lambda + \bigcup_{\ell=1}^{d} (\eta_\ell + Q_1)\big) = \bigcup_{\lambda \in \Lambda_0} (\lambda + Q_0).$$

Now assume that for some $\lambda, \lambda' \in \Lambda_0$ we have $\mu_G\big((\lambda + Q_0) \cap (\lambda' + Q_0)\big) > 0$. Then for some $\eta_\ell, \eta_{\ell'} \in \Lambda_1$ we have $\mu_G\big((\lambda + \eta_\ell + Q_1) \cap (\lambda' + \eta_{\ell'} + Q_1)\big) > 0$. Since $\lambda + \eta_\ell, \lambda' + \eta_{\ell'} \in \Lambda_1$, we conclude that $\lambda + \eta_\ell = \lambda' + \eta_{\ell'}$; by (ii), this implies that $\eta_\ell = \eta_{\ell'}$ and therefore $\lambda = \lambda'$, i.e., Q_0 is indeed a fundamental domain associated with Λ_0.

Let us now give the proof of the cardinality in (i). Note that the cardinality of the set Λ_1/Λ_0 equals the number d introduced in (ii). Since Q_1 is a fundamental domain associated with Λ_1, by definition we have $s(\Lambda_1) = \mu_G(Q_1)$. It now follows from (iii), the disjointness (up to a set of measure zero) of the sets $\eta_\ell + Q_1$, $\ell = 1, \dots, d$, and the translation invariance of the measure that

$$s(\Lambda_0) = \mu_G(Q_0) = \mu_G\big(\bigcup_{\ell=1}^{d} (\eta_\ell + Q_1)\big) = d\, \mu_G(Q_1) = d\, s(\Lambda_1),$$

as claimed.

(iv) This follows from Proposition 4.2.24 in [18], but let us give a direct proof. By the definition of the annihilator, the assumption $\Lambda_0 \subset \Lambda_1$ implies that $\Lambda_1^\perp \subset \Lambda_0^\perp$; since both are groups it is clear that $\Lambda_1^\perp$ is a subgroup of $\Lambda_0^\perp$. That $\Lambda_0^\perp/\Lambda_1^\perp$ is finite now follows from (i). Using (i), we arrive at $\left|\Lambda_0^\perp/\Lambda_1^\perp\right| = \frac{s(\Lambda_1^\perp)}{s(\Lambda_0^\perp)} = \frac{s(\Lambda_0)}{s(\Lambda_1)} = |\Lambda_1/\Lambda_0|$, which proves (iv).

(v), (vi) Finally, the results in (v) and (vi) follow immediately from (ii) and (iii) applied to the inclusion $\Lambda_1^\perp \subset \Lambda_0^\perp$. □

3 The unitary extension principle

In order to avoid a long list of assumptions in the formulation of the UEP, we will state the standing assumptions for this section in a "General setup". Before we do this, let us mention a few conventions that will help us avoid cumbersome notations.

First, the UEP will be based on a sequence of functions $\{\Phi_k\}_{k\in I}$, indexed by a countable sequence of consecutive integers in $\mathbb{Z}$, i.e., either $I=\{k\}_{k=k_0}^{\infty}$ or $I=\{k\}_{k=k_0}^{k_1}$ for some $k_0,k_1\in\mathbb{Z}$. In what follows we will tacitly assume that $I=\{k\}_{k=k_0}^{\infty}$ and leave the minor modifications in the case $I=\{k\}_{k=k_0}^{k_1}$ to the reader. A typical example: a condition involving Φ_k and Φ_{k+1} makes perfect sense for $k\in I$ if we assume that $I=\{k\}_{k=k_0}^{\infty}$. On the other hand, for the case $I=\{k\}_{k=k_0}^{k_1}$ one would have to assume that $k\in\{k_0,\dots,k_1-1\}$. For the rest of the paper, we will let k_0 denote the starting index of the set I.

The UEP on $\mathbb{R}^s$ by Ron and Shen is formulated in terms of conditions on some filters, which are periodic functions. We will need an analog concept in our setting, and we will use the convention stated right after Lemma 2.1. As we will see, the relevant periodic functions are actually defined on the dual group $\widehat{G}$.

Let us now state the standing assumptions for this section:

General setup: Let I be a sequence of consecutive numbers in $\mathbb{Z}$. Let $\{\Lambda_k\}_{k\in I}$ be a nested sequence of lattices in G, i.e.,

$$\Lambda_{k_0}\subset\Lambda_{k_0+1}\subset\Lambda_{k_0+2}\subset\cdots. \tag{3.1}$$

Let $\{\Phi_k\}_{k\in I}$ be a sequence of functions in $L^2(\widehat{G})$. Furthermore, for each $k\in I$, let V_k denote a fundamental domain associated with the lattice $\Lambda_k^{\perp}$; then, in particular, for each $k\in I$,

$$\widehat{G}=\bigcup_{\omega\in\Lambda_k^{\perp}}(\omega+V_k),\ \ (\omega+V_k)\cap(\omega'+V_k)=\emptyset\text{ for }\omega\neq\omega',\ \omega,\omega'\in\Lambda_k^{\perp}. \tag{3.2}$$

Assume the following conditions:

(i) For every compact set S in $\widehat{G}$, there exists $K_1\in I$ such that

$$\mu_{\widehat{G}}\big((\omega+S)\cap(\omega'+S)\big)=0\text{ for }\omega\neq\omega',\ \omega,\omega'\in\Lambda_{K_1}^{\perp}. \tag{3.3}$$

(ii) For every compact set S in $\widehat{G}$ and any $\epsilon>0$, there exists $K_2\in I$ such that for all $k\geq K_2$, $k\in I$,

$$\big|\mu_{\widehat{G}}(V_k)\,|\Phi_k(\gamma)|^2-1\big|\leq\epsilon,\ \forall\gamma\in S. \tag{3.4}$$

(iii) For all $k\in I$ and some periodic functions $H_{k+1}\in L^{\infty}(V_{k+1})$ (see the convention in Section 2),

$$\Phi_k(\gamma)=H_{k+1}(\gamma)\,\Phi_{k+1}(\gamma),\text{ a.e. }\gamma\in\widehat{G}. \tag{3.5}$$

For $k \in I$, given periodic functions $G_{k+1}^{(m)} \in L^\infty(V_{k+1})$, $m = 1, \ldots, \rho_k$, define the functions $\Psi_k^{(m)} \in L^2(\widehat{G})$, $m = 1, \ldots, \rho_k$, by

$$\Psi_k^{(m)}(\gamma) := G_{k+1}^{(m)}(\gamma)\, \Phi_{k+1}(\gamma),\ \gamma \in \widehat{G}. \tag{3.6}$$

Our goal is to identify conditions on the *filters* H_k and $G_k^{(m)}$ such that the collection of functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0,\, \lambda \in \Lambda_k, m=1,\ldots,\rho_k}$$

forms a tight frame for $L^2(\widehat{G})$ with frame bound 1.

We will see later in Example 3.7 that for the case $G = \mathbb{R}$, the assumptions (3.4) and (3.5) correspond directly to the assumptions and setup used in the classical UEP constructions; on the other hand, the condition (3.3) is automatically satisfied in this case, and does not appear explicitly in the classical UEP.

Starting with a suitable choice of a sequence $\{\nu_{k_0,\ell}\}_{\ell=1,\ldots,d_{k_0}}$ associated with the "lowest" level lattice $\Lambda_{k_0}^\perp$, repeated use of Lemma 2.3(v) shows that for each $k \in I$, we can choose a sequence $\{\nu_{k,\ell}\}_{\ell=1,\ldots,d_k} \subset \widehat{G}$ such that $\nu_{k,1} = 0$ and

$$\Lambda_k^\perp \ = \ \bigcup_{\ell=1}^{d_k} (\nu_{k,\ell} + \Lambda_{k+1}^\perp), \quad (\nu_{k,\ell} + \Lambda_{k+1}^\perp) \cap (\nu_{k,\ell'} + \Lambda_{k+1}^\perp) = \emptyset \text{ for } \ell \neq \ell'. \tag{3.7}$$

For $k \in I$, consider the $(\rho_k + 1) \times d_k$ matrix-valued function P_k defined by

$$P_k(\gamma) := \begin{pmatrix} H_{k+1}(\gamma + \nu_{k,1}) & \cdot\ \cdot\ \cdot & H_{k+1}(\gamma + \nu_{k,d_k}) \\ G_{k+1}^{(1)}(\gamma + \nu_{k,1}) & \cdot\ \cdot\ \cdot & G_{k+1}^{(1)}(\gamma + \nu_{k,d_k}) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ G_{k+1}^{(\rho_k)}(\gamma + \nu_{k,1}) & \cdot\ \cdot\ \cdot & G_{k+1}^{(\rho_k)}(\gamma + \nu_{k,d_k}) \end{pmatrix},\ \gamma \in V_k. \tag{3.8}$$

By Lemma 2.3(vi) applied to the lattices $\Lambda_k^\perp$ and $\Lambda_{k+1}^\perp$, we know that for any fundamental domain V_k associated with $\Lambda_k^\perp$,

$$(\nu_{k,\ell} + V_k) \cap (\nu_{k,\ell'} + V_k) = \emptyset \text{ for } \ell \neq \ell'. \tag{3.9}$$

Lemma 2.3(vi) also shows that the set

$$V'_{k+1} := \bigcup_{\ell=1}^{d_k} (\nu_{k,\ell} + V_k) \tag{3.10}$$

is a fundamental domain associated with the lattice $\Lambda_{k+1}^\perp$. This observation turns out to be important: in fact, some of the analysis to follow applies whenever $\{V_k\}_{k \in I}$ is an arbitrary collection of fundamental domains associated with the lattices $\{\Lambda_k^\perp\}_{k\in I}$, but some of the results require a relationship between fundamental domains "on consecutive levels"; in such cases we will apply (3.10).

Since we will work with different choices of fundamental domains V_k, we need the following elementary result:

Lemma 3.1 *For any two fundamental domains V_k and V_k' associated with $\Lambda_k^\perp$, we have $\mu_{\widehat{G}}(V_k) = \mu_{\widehat{G}}(V_k')$ and $L^\infty(V_k) = L^\infty(V_k')$.*

Lemma 3.1 will help us to simplify the notation: indeed, whenever we need to deal with, e.g., the fundamental domain V_{k+1}' in (3.10), we will always write $\mu_{\widehat{G}}(V_{k+1})$ instead of $\mu_{\widehat{G}}(V_{k+1}')$ and $L^\infty(V_{k+1})$ instead of $L^\infty(V_{k+1}')$.

Let us comment on the fact that P_k for the moment is only considered on V_k. Looking at the entries of the matrix P_k, we see that by considering $\gamma\in V_k$, we exactly use the information about the functions H_{k+1} and $G_{k+1}^{(m)}$ on the fundamental domain V_{k+1}' defined by (3.10); thus, considering P_k only on V_k is consistent with our desire to consider H_{k+1} and $G_{k+1}^{(m)}$ as periodic functions in $L^\infty(V_{k+1})$.

We will see that the essential condition in the UEP is that

$$P_k(\gamma)^* P_k(\gamma) = d_k I_{d_k}, \text{ a.e. } \gamma \in V_k. \tag{3.11}$$

Note that $d_k = \frac{\mu_{\widehat{G}}(V_{k+1})}{\mu_{\widehat{G}}(V_k)}$. We will now show that if (3.11) holds on *any* fundamental domain V_k, then it automatically holds on $\widehat{G}$.

Lemma 3.2 *Let V_k denote any fundamental domain associated with $\Lambda_k^\perp$. If* (3.11) *holds for a.e. $\gamma\in V_k$, then* (3.11) *holds for a.e. $\gamma\in\widehat{G}$.*

Proof. The condition (3.11) means that for a.e. $\gamma\in V_k$ and $\ell,\ell'=1,\dots,d_k$,

$$H_{k+1}(\gamma+\nu_{k,\ell})\overline{H_{k+1}(\gamma+\nu_{k,\ell'})} + \sum_{m=1}^{\rho_k} G_{k+1}^{(m)}(\gamma+\nu_{k,\ell})\overline{G_{k+1}^{(m)}(\gamma+\nu_{k,\ell'})} = d_k\delta_{\ell,\ell'}. \tag{3.12}$$

Consider now any γ' belonging to the fundamental domain V_{k+1}' chosen in (3.10). Such γ' can be written in the form $\gamma' = \gamma + \nu_{k,\tilde{\ell}}$ for some $\gamma\in V_k$ and some $\tilde{\ell}\in\{1,\dots,d_k\}$. Using (3.12), that $\nu_{k,\tilde{\ell}} + \nu_{k,\ell} \in \Lambda_k^\perp = \cup_{q=1}^{d_k}(\nu_{k,q}+\Lambda_{k+1}^\perp)$, and the periodicity of the functions H_{k+1} and $G_{k+1}^{(m)}$ in $L^\infty(V_{k+1})$, it follows that γ' also satisfies (3.12) (we leave the details to the reader). Using now the periodicity of the entries of the matrix P_k, it follows that (3.12) holds for a.e. $\gamma\in\widehat{G}$. $\square$

We now state a lemma which allows us to move around between different levels of Φ_k.

Lemma 3.3 *In addition to the general setup, assume that for some $k\in I$, the matrix-valued function P_k satisfies* (3.11). *Then for all $F\in C_c(\widehat{G})$,*

$$\sum_{\lambda\in\Lambda_{k+1}} |\langle F, \mathcal{M}_\lambda \Phi_{k+1}\rangle|^2 = \sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda\Phi_k\rangle|^2 + \sum_{m=1}^{\rho_k}\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Psi_k^{(m)}\rangle|^2. \tag{3.13}$$

Proof. Fix $k \in I$. We first note that by Lemma 3.2 the condition (3.11) holds for a.e. $\gamma \in \widehat{G}$. We will now use Lemma 2.2 to rewrite the three expressions appearing in (3.13). For technical reasons we use the given fundamental domain V_k for the terms on the right-hand side, and the domain V'_{k+1} in (3.10) for the left-hand side:

$$\sum_{\lambda\in\Lambda_{k+1}} |\langle F, \mathcal{M}_\lambda \Phi_{k+1}\rangle|^2 = \mu_{\widehat{G}}(V_{k+1}) \int_{V'_{k+1}} \left| \sum_{\omega\in\Lambda_{k+1}^\perp} F(\omega+\gamma)\overline{\Phi_{k+1}(\omega+\gamma)}\right|^2 d\gamma; \quad (3.14)$$

$$\sum_{\lambda\in\Lambda_{k}} |\langle F, \mathcal{M}_\lambda \Phi_{k}\rangle|^2 = \mu_{\widehat{G}}(V_{k}) \int_{V_{k}} \left| \sum_{\omega\in\Lambda_{k}^\perp} F(\omega+\gamma)\overline{\Phi_{k}(\omega+\gamma)}\right|^2 d\gamma; \quad (3.15)$$

$$\sum_{m=1}^{\rho_k}\sum_{\lambda\in\Lambda_{k}} |\langle F, \mathcal{M}_\lambda \Psi_{k}^{(m)}\rangle|^2 = \sum_{m=1}^{\rho_k}\mu_{\widehat{G}}(V_{k}) \int_{V_{k}} \left| \sum_{\omega\in\Lambda_{k}^\perp} F(\omega+\gamma)\overline{\Psi_{k}^{(m)}(\omega+\gamma)}\right|^2 d\gamma. \quad (3.16)$$

Based on these expressions, a natural approach is to start with (3.14) and apply the disjoint splitting of V'_{k+1} in terms of V_k, see (3.10); this yields

$$\begin{aligned}
\sum_{\lambda\in\Lambda_{k+1}} |\langle F, \mathcal{M}_\lambda \Phi_{k+1}\rangle|^2 &= \mu_{\widehat{G}}(V_{k+1}) \int_{V'_{k+1}} \left| \sum_{\omega\in\Lambda_{k+1}^\perp} F(\omega+\gamma)\overline{\Phi_{k+1}(\omega+\gamma)}\right|^2 d\gamma \\
&= \mu_{\widehat{G}}(V_{k+1}) \sum_{\ell=1}^{d_k} \int_{\nu_{k,\ell}+V_k} \left| \sum_{\omega\in\Lambda_{k+1}^\perp} F(\omega+\gamma)\overline{\Phi_{k+1}(\omega+\gamma)}\right|^2 d\gamma \\
&= \mu_{\widehat{G}}(V_{k+1}) \int_{V_k} \sum_{\ell=1}^{d_k} \left| \sum_{\omega\in\Lambda_{k+1}^\perp} F(\omega+\nu_{k,\ell}+\gamma)\overline{\Phi_{k+1}(\omega+\nu_{k,\ell}+\gamma)}\right|^2 d\gamma. \quad (3.17)
\end{aligned}$$

Define the functions a_k^ℓ, $\ell = 1, \ldots, d_k$, and b_k on V_k by

$$\begin{aligned}
a_k^\ell(\gamma) &:= \sum_{\omega\in\Lambda_{k+1}^\perp} F(\omega+\nu_{k,\ell}+\gamma)\overline{\Phi_{k+1}(\omega+\nu_{k,\ell}+\gamma)}, \ \gamma\in V_k; \\
b_k(\gamma) &:= \sum_{\omega\in\Lambda_{k}^\perp} F(\omega+\gamma)\overline{\Phi_{k}(\omega+\gamma)}, \ \gamma\in V_k.
\end{aligned}$$

By Lemma 2.2, these functions are well defined. We will also consider the vector

$$c_k(\gamma) := \begin{pmatrix} \sum_{\omega\in\Lambda_k^\perp} F(\omega+\gamma)\overline{\Psi_k^{(1)}(\omega+\gamma)} \\ \cdot \\ \cdot \\ \sum_{\omega\in\Lambda_k^\perp} F(\omega+\gamma)\overline{\Psi_k^{(\rho_k)}(\omega+\gamma)} \end{pmatrix}, \ \gamma\in V_k.$$

Using (3.5) and that $H_{k+1}(\gamma+\omega)=H_{k+1}(\gamma)$ for $\gamma\in\widehat{G}$, $\omega\in\Lambda_{k+1}^{\perp}$,

$$\begin{aligned}\sum_{\ell=1}^{d_k} a_k^{\ell}(\gamma)\overline{H_{k+1}(\gamma+\nu_{k,\ell})} &= \sum_{\ell=1}^{d_k}\sum_{\omega\in\Lambda_{k+1}^{\perp}} F(\omega+\nu_{k,\ell}+\gamma)\overline{\Phi_{k+1}(\omega+\nu_{k,\ell}+\gamma)H_{k+1}(\omega+\gamma+\nu_{k,\ell})}\\ &= \sum_{\ell=1}^{d_k}\sum_{\omega\in\Lambda_{k+1}^{\perp}} F(\omega+\nu_{k,\ell}+\gamma)\overline{\Phi_{k}(\omega+\nu_{k,\ell}+\gamma)}.\end{aligned}$$

Via the disjoint splitting of $\Lambda_k^{\perp}$ in (3.7), it follows that

$$\sum_{\ell=1}^{d_k} a_k^{\ell}(\gamma)\overline{H_{k+1}(\gamma+\nu_{k,\ell})} = \sum_{\omega\in\Lambda_k^{\perp}} F(\omega+\gamma)\overline{\Phi_k(\omega+\gamma)} = b_k(\gamma).$$

In the same way it can be proved from (3.6) that

$$\sum_{\ell=1}^{d_k} a_k^{\ell}(\gamma)\begin{pmatrix}\overline{G_{k+1}^{(1)}(\gamma+\nu_{k,\ell})}\\ \cdot\\ \cdot\\ \overline{G_{k+1}^{(\rho_k)}(\gamma+\nu_{k,\ell})}\end{pmatrix} = c_k(\gamma).$$

Defining the column vectors

$$\beta_k(\gamma) := \begin{pmatrix} b_k(\gamma)\\ c_k(\gamma)\end{pmatrix},\quad \alpha_k(\gamma) := \begin{pmatrix} a_k^1(\gamma)\\ \cdot\\ \cdot\\ a_k^{d_k}(\gamma)\end{pmatrix},$$

and using the definition of $P_k(\gamma)$ in (3.8) these calculations can be summarized as

$$\beta_k(\gamma) = \overline{P_k(\gamma)}\,\alpha_k(\gamma). \tag{3.18}$$

Now, in terms of the vector $\alpha_k(\gamma)$, the result in (3.17) means that

$$\sum_{\lambda\in\Lambda_{k+1}} |\langle F,\mathcal{M}_\lambda\Phi_{k+1}\rangle|^2 = \mu_{\widehat{G}}(V_{k+1})\int_{V_k}\alpha_k(\gamma)^*\alpha_k(\gamma)\,d\gamma.$$

Using the assumption (3.11) and (3.18), it follows that

$$\begin{aligned}&\sum_{\lambda\in\Lambda_{k+1}} |\langle F,\mathcal{M}_\lambda\Phi_{k+1}\rangle|^2 = \mu_{\widehat{G}}(V_{k+1})\,\frac{\mu_{\widehat{G}}(V_k)}{\mu_{\widehat{G}}(V_{k+1})}\int_{V_k}\alpha_k(\gamma)^*\overline{P_k(\gamma)^*P_k(\gamma)}\alpha_k(\gamma)\,d\gamma\\ =\;& \mu_{\widehat{G}}(V_k)\int_{V_k}\beta_k(\gamma)^*\beta_k(\gamma)\,d\gamma = \mu_{\widehat{G}}(V_k)\int_{V_k}\left(|b_k(\gamma)|^2+\|c_k(\gamma)\|^2\right)\,d\gamma\\ =\;& \mu_{\widehat{G}}(V_k)\int_{V_k}\left(\Big|\sum_{\omega\in\Lambda_k^{\perp}}F(\omega+\gamma)\overline{\Phi_k(\omega+\gamma)}\Big|^2+\sum_{m=1}^{\rho_k}\Big|\sum_{\omega\in\Lambda_k^{\perp}}F(\omega+\gamma)\overline{\Psi_k^{(m)}(\omega+\gamma)}\Big|^2\right)d\gamma\\ =\;& \sum_{\lambda\in\Lambda_k}|\langle F,\mathcal{M}_\lambda\Phi_k\rangle|^2+\sum_{m=1}^{\rho_k}\sum_{\lambda\in\Lambda_k}|\langle F,\mathcal{M}_\lambda\Psi_k^{(m)}\rangle|^2,\end{aligned}$$

where (3.15) and (3.16) are used in the final step. $\square$

The next lemma is a consequence of the assumptions (i) and (ii) of the general setup.

Lemma 3.4 *For any $F \in C_c(\widehat{G})$ and any $\epsilon > 0$, there is a $K \in I$ such that for $k \geq K$, $k \in I$,*

$$(1-\epsilon)\,\|F\|^2 \leq \sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 \leq (1+\epsilon)\,\|F\|^2.$$

Proof. Given $F \in C_c(\widehat{G})$, put $S := \operatorname{supp} F$. For $k \in I$, $\omega \in \Lambda_k^\perp$, let

$$S_{k,\omega} := \{\gamma \in V_k \,|\, \omega + \gamma \in S\}.$$

Note that by (3.2),

$$\begin{aligned} S &= S \cap \widehat{G} = S \cap \Big[\bigcup_{\omega\in\Lambda_k^\perp} (\omega + V_k)\Big] = \bigcup_{\omega\in\Lambda_k^\perp} [S \cap (\omega + V_k)] \\ &= \bigcup_{\omega\in\Lambda_k^\perp} \{\omega+\gamma \,|\, \gamma \in V_k,\, \omega+\gamma \in S\} = \bigcup_{\omega\in\Lambda_k^\perp} \{\omega+\gamma \,|\, \gamma\in S_{k,\omega}\} = \bigcup_{\omega\in\Lambda_k^\perp} (\omega + S_{k,\omega}). \qquad (3.19)\end{aligned}$$

Since the decomposition in (3.2) is disjoint, (3.19) is clearly a disjoint decomposition of S (up to a set of measure zero for both decompositions). By Lemma 2.2,

$$\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda\Phi_k\rangle|^2 = \mu_{\widehat{G}}(V_k) \int_{V_k} \Big| \sum_{\omega\in\Lambda_k^\perp} F(\omega+\gamma)\; \overline{\Phi_k(\omega+\gamma)} \Big|^2 \, d\gamma.$$

Note that in the integral we only get contributions for the $\gamma \in V_k$ for which there is an $\omega' \in \Lambda_k^\perp$ such that $\omega' + \gamma \in S$, i.e., we only get contributions for $\gamma \in \bigcup_{\omega'\in\Lambda_k^\perp} S_{k,\omega'}$. Thus

$$\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda\Phi_k\rangle|^2 \;=\; \mu_{\widehat{G}}(V_k) \int_{\left[\bigcup_{\omega'\in\Lambda_k^\perp} S_{k,\omega'}\right]} \Big| \sum_{\omega\in\Lambda_k^\perp} F(\omega+\gamma)\; \overline{\Phi_k(\omega+\gamma)} \Big|^2 \, d\gamma.$$

Now, given any $\epsilon > 0$, take $K \in I$ satisfying (i) and (ii) of the general setup. Then, for $k \in I$ with $k \geq K$, the sets $S_{k,\omega}$, $\omega \in \Lambda_k^\perp$, are disjoint (up to a set of measure zero). Indeed, if $\gamma \in S_{k,\omega} \cap S_{k,\omega'}$ for some $\omega, \omega' \in \Lambda_k^\perp$, then $\gamma \in (-\omega + S) \cap (-\omega' + S)$; by the assumption (3.3) and the fact that $\Lambda_k^\perp \subset \Lambda_K^\perp$, this implies that $\mu_{\widehat{G}}(S_{k,\omega} \cap S_{k,\omega'}) = 0$ if $\omega \neq \omega'$. We can therefore continue our calculation, and obtain that for $k \geq K, k \in I$,

$$\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda\Phi_k\rangle|^2 \;=\; \mu_{\widehat{G}}(V_k) \sum_{\omega'\in\Lambda_k^\perp} \int_{S_{k,\omega'}} \Big| \sum_{\omega\in\Lambda_k^\perp} F(\omega+\gamma)\; \overline{\Phi_k(\omega+\gamma)} \Big|^2 \, d\gamma. \qquad (3.20)$$

Note that for a fixed ω' in the "outer sum", we only get a nonzero contribution in the "inner sum" over $\omega \in \Lambda_k^\perp$ for the choice $\omega = \omega'$. In fact, given $\omega' \in \Lambda_k^\perp$, for $\omega \neq \omega'$ any

$\gamma \in S_{k,\omega'}$ will be outside $S_{k,\omega}$, meaning that $\omega + \gamma \notin S$, i.e., $F(\omega+\gamma)=0$. Therefore (3.20) simplifies to

$$\begin{aligned}\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 &= \mu_{\widehat{G}}(V_k) \sum_{\omega'\in\Lambda_k^\perp} \int_{S_{k,\omega'}} \left|F(\omega'+\gamma)\Phi_k(\omega'+\gamma)\right|^2 d\gamma \\ &= \mu_{\widehat{G}}(V_k) \sum_{\omega'\in\Lambda_k^\perp} \int_{\omega'+S_{k,\omega'}} \left|F(\gamma)\Phi_k(\gamma)\right|^2 d\gamma = \mu_{\widehat{G}}(V_k) \int_S \left|F(\gamma)\Phi_k(\gamma)\right|^2 d\gamma,\end{aligned}$$

where the last step again used that the union in (3.19) is disjoint. Our choice of K and the assumption (3.4) now implies that

$$(1-\epsilon) \int_S |F(\gamma)|^2 \, d\gamma \leq \sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 \leq (1+\epsilon) \int_S |F(\gamma)|^2 \, d\gamma;$$

since $\int_S |F(\gamma)|^2 \, d\gamma = \|F\|^2$, this completes the proof. $\square$

We are now ready to state the main result, the unitary extension principle for LCA groups.

Theorem 3.5 *In addition to the assumptions* (3.3)–(3.5) *in the general setup, assume that for $k \in I$, the matrix-valued function P_k in* (3.8) *satisfies* (3.11), *i.e.,*

$$P_k(\gamma)^* P_k(\gamma) = d_k I_{d_k}, \text{ a.e. } \gamma \in V_k.$$

Then with $\Psi_k^{(m)}$, $k \geq k_0$, $m = 1, \dots, \rho_k$, defined as in (3.6), *the collection of functions*

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k\geq k_0, \lambda\in\Lambda_k, m=1,\dots,\rho_k} \tag{3.21}$$

forms a tight frame for $L^2(\widehat{G})$ with frame bound 1.

Proof. Given any $\epsilon > 0$ (the role hereof will be clear later), choose $K \in I$ such that (3.3) and (3.4) hold. Taking any $F \in C_c(\widehat{G})$ and applying Lemma 3.3 repeatedly, we obtain that for $k \geq K, k \in I$,

$$\sum_{\lambda\in\Lambda_k} |\langle F, \mathcal{M}_\lambda \Phi_k\rangle|^2 = \sum_{\lambda\in\Lambda_{k_0}} |\langle F, \mathcal{M}_\lambda \Phi_{k_0}\rangle|^2 + \sum_{\ell=k_0}^{k-1} \sum_{m=1}^{\rho_\ell} \sum_{\lambda\in\Lambda_\ell} |\langle F, \mathcal{M}_\lambda \Psi_\ell^{(m)}\rangle|^2.$$

Using Lemma 3.4, it follows that

$$(1-\epsilon) \|F\|^2 \leq \sum_{\lambda\in\Lambda_{k_0}} |\langle F, \mathcal{M}_\lambda \Phi_{k_0}\rangle|^2 + \sum_{\ell=k_0}^{k-1} \sum_{m=1}^{\rho_\ell} \sum_{\lambda\in\Lambda_\ell} |\langle F, \mathcal{M}_\lambda \Psi_\ell^{(m)}\rangle|^2 \leq (1+\epsilon) \|F\|^2. \tag{3.22}$$

In the case where $I = \{k\}_{k=k_0}^{\infty}$, we can now let $k \to \infty$, which yields that

$$(1-\epsilon)\,\|F\|^2 \le \sum_{\lambda\in\Lambda_{k_0}} |\langle F, \mathcal{M}_\lambda \Phi_{k_0}\rangle|^2 + \sum_{\ell\in I}\sum_{m=1}^{\rho_\ell}\sum_{\lambda\in\Lambda_\ell} |\langle F, \mathcal{M}_\lambda \Psi_\ell^{(m)}\rangle|^2 \le (1+\epsilon)\,\|F\|^2.$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\|F\|^2 = \sum_{\lambda\in\Lambda_{k_0}} |\langle F, \mathcal{M}_\lambda \Phi_{k_0}\rangle|^2 + \sum_{\ell\in I}\sum_{m=1}^{\rho_\ell}\sum_{\lambda\in\Lambda_\ell} |\langle F, \mathcal{M}_\lambda \Psi_\ell^{(m)}\rangle|^2. \tag{3.23}$$

Due to the fact that F is an arbitrary function in the dense subspace $C_c(\widehat{G})$ of $L^2(\widehat{G})$, we infer that (3.23) holds for all $F \in L^2(\widehat{G})$.

In the case where $I = \{k\}_{k=k_0}^{k_1}$, we simply take $k = k_1$ in (3.22). This eventually leads to the conclusion that the collection $\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k=k_0,\dots,k_1-1,\lambda\in\Lambda_k,m=1,\dots,\rho_k}$ is a tight frame for $L^2(\widehat{G})$ with frame bound 1. $\square$

Applying the inverse Fourier transform to the system in (3.21) immediately leads to a generalized shift-invariant system forming a frame for $L^2(G)$:

Corollary 3.6 *Under the assumptions in Theorem* 3.5*, the generalized shift-invariant system*

$$\{T_\lambda \mathcal{F}^{-1}\Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \bigcup \{T_\lambda \mathcal{F}^{-1}\Psi_k^{(m)}\}_{k\ge k_0,\lambda\in\Lambda_k,m=1,\dots,\rho_k}$$

forms a tight frame for $L^2(G)$ with frame bound 1.

The following example shows how the classical version of the UEP appears via Theorem 3.5. In particular, we will see that the condition of (3.11) holding for all values of k reduces to just one matrix condition in this case. This is not surprising as Theorem 3.5 on LCA groups is established for a much more general nonstationary setting, whereas the classical UEP on the real line deals with stationary tight frames.

Example 3.7 Consider an integer $a > 1$ and the associated *scaling operator* $D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(D_a f)(x) := a^{1/2} f(ax)$, $x \in \mathbb{R}$. Given a function $\phi \in L^2(\mathbb{R})$ satisfying that

$$\lim_{\gamma\to 0} \widehat{\phi}(\gamma) = 1, \tag{3.24}$$

let $\Phi_k := \widehat{D_a^k \phi}$, $k \in \mathbb{Z}$. Also, let $\Lambda_k := a^{-k}\mathbb{Z}$. Then $\Lambda_k^\perp = a^k\mathbb{Z}$, with the fundamental domain $V_k = [0, a^k)$. Notice that for this particular case the disjoint decomposition of $\Lambda_k^\perp$ in terms of $\Lambda_{k+1}^\perp$, see (3.7), takes the form $\Lambda_k^\perp = [0 + \Lambda_{k+1}^\perp] + [a^k + \Lambda_{k+1}^\perp] + \cdots + [(a-1)a^k + \Lambda_{k+1}^\perp]$. That is, $d_k = a$, and $\{\nu_{k,\ell}\}_{\ell=1}^{d_k} = \{(\ell-1)\,a^k\}_{\ell=1}^{a}$.

In the general setup we consider the scaling relation

$$\Phi_{-1}(\gamma) = H_0(\gamma)\Phi_0(\gamma),\ \gamma \in \mathbb{R}, \tag{3.25}$$

for a function $H_0 \in L^\infty(V_0) = L^\infty([0,1))$, extended to a 1-periodic function. The equation means that $\mathcal{F}D_a^{-1}\phi(\gamma) = H_0(\gamma)\mathcal{F}\phi(\gamma)$ or $\sqrt{a}\,\widehat{\phi}(a\gamma) = H_0(\gamma)\widehat{\phi}(\gamma)$; this is the classical scaling equation in wavelet analysis, except that the factor $1/\sqrt{a}$ is usually absorbed in the function H_0. It follows that for any $k \in \mathbb{Z}$, $\sqrt{a}\,\widehat{\phi}(a^k\gamma) = H_0(a^{k-1}\gamma)\widehat{\phi}(a^{k-1}\gamma)$; translated back to the functions Φ_k this means that $\Phi_{-k}(\gamma) = H_0(a^{k-1}\gamma)\Phi_{-k+1}(\gamma)$, or

$$\Phi_k(\gamma) = H_0(a^{-k-1}\gamma)\Phi_{k+1}(\gamma),\ \gamma \in \mathbb{R}.$$

That is, the single scaling equation (3.25) implies that we have the refinement equation $\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma)$ for all levels, with $H_{k+1}(\gamma) = H_0(a^{-k-1}\gamma)$. As for the remaining assumptions (3.3) and (3.4) in the general setup, (3.3) follows from the fact that $\Lambda_k^\perp = a^k\mathbb{Z}$, while (3.4) is a consequence of the condition (3.24) and the calculation

$$\mu_{\mathbb{R}}(V_k)|\Phi_k(\gamma)|^2 = a^k|a^{-k/2}\widehat{\phi}(a^{-k}\gamma)|^2 = |\widehat{\phi}(a^{-k}\gamma)|^2.$$

Assuming now that we have chosen functions $G_0^{(m)} \in L^\infty([0,1))$, $m = 1,\ldots,\rho$, for some integer $\rho \geq a-1$, define the functions $G_{k+1}^{(m)}$ by $G_{k+1}^{(m)}(\gamma) := G_0^{(m)}(a^{-k-1}\gamma)$, $\gamma \in \mathbb{R}$, for $m = 1,\ldots,\rho$, i.e., we take $\rho_k = \rho$ for all $k \in \mathbb{Z}$. Considering the entries in the first row of the $(\rho+1)\times a$ matrix $P_k(\gamma)$ in (3.8), we see that for $\gamma \in V_k = [0,a^k)$,

$$H_{k+1}(\gamma + \nu_{k,\ell}) = H_0(a^{-k-1}(\gamma + (\ell-1)\,a^k)) = H_0(a^{-k-1}\gamma + \nu_{-1,\ell}).$$

A similar calculation works for $G_k^{(m)}$, $m = 1,\ldots,\rho$, so we conclude that for $\gamma \in V_k$, $P_k(\gamma) = P_{-1}(a^{-k-1}\gamma)$. Since $a^{-k-1}\gamma$ runs through $V_{-1} = [0,a^{-1})$ when γ runs through V_k, we conclude that for all $k \in \mathbb{Z}$ the matrix equation (3.11) is equivalent to $P_{-1}(\gamma)^*P_{-1}(\gamma) = a\,I_a$, a.e. $\gamma \in V_{-1}$, which corresponds to the condition in the classical UEP on $\mathbb{R}$. Note that for $\lambda \in \Lambda_k = a^{-k}\mathbb{Z}$, i.e., $\lambda = a^{-k}\ell$ for some $\ell \in \mathbb{Z}$,

$$\mathcal{M}_\lambda\Phi_k(\gamma) = e^{-2\pi i a^{-k}\ell\gamma}\mathcal{F}D_a^k\phi(\gamma) = \mathcal{F}T_{\ell a^{-k}}D_a^k\phi(\gamma),$$

with a similar calculation yielding $\mathcal{M}_\lambda\Psi_k^{(m)}(\gamma) = \mathcal{F}T_{\ell a^{-k}}D_a^k\psi^{(m)}(\gamma)$, $m = 1\ldots,\rho$, where $\widehat{\psi^{(m)}} := \Psi_0^{(m)}$ and $\Psi_k^{(m)} = \widehat{D_a^k\psi^{(m)}}$. If the assumptions in Theorem 3.5 are satisfied, this implies that for any chosen $k_0 \in \mathbb{Z}$ (with $I = \{k\}_{k=k_0}^\infty$), the collection of functions

$$\begin{aligned} & \{\mathcal{M}_\lambda\Phi_{k_0}\}_{\lambda\in\Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda\Psi_k^{(m)}\}_{k\geq k_0,\lambda\in\Lambda_k,m=1,\ldots,\rho} \\ = \ & \{\mathcal{F}T_{\ell a^{-k_0}}D_a^{k_0}\phi\}_{\ell\in\mathbb{Z}} \bigcup \{\mathcal{F}T_{\ell a^{-k}}D_a^k\psi^{(m)}\}_{k\geq k_0,\ell\in\mathbb{Z},m=1,\ldots,\rho} \end{aligned}$$

forms a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Thus the system

$$\begin{aligned} & \{T_{\ell a^{-k_0}}D_a^{k_0}\phi\}_{\ell\in\mathbb{Z}} \bigcup \{T_{\ell a^{-k}}D_a^k\psi^{(m)}\}_{k\geq k_0,\ell\in\mathbb{Z},m=1,\ldots,\rho} \\ = \ & \{a^{k_0/2}\phi(a^{k_0}\cdot -\ell)\}_{\ell\in\mathbb{Z}} \bigcup \{a^{k/2}\psi^{(m)}(a^k\cdot -\ell)\}_{k\geq k_0,\ell\in\mathbb{Z},m=1,\ldots,\rho} \end{aligned}$$

forms a tight frame for $L^2(\mathbb{R})$ with frame bound 1. $\square$

A more recent variant of the UEP, the *oblique extension principle* (OEP), was announced in 2001, independently by two groups of researchers, namely, Chui, He and Stöckler [7], as well as Daubechies, Han, Ron and Shen [9]. The OEP is essentially equivalent to the UEP, but gives easier access to attractive constructions, e.g., constructions with a high number of vanishing moments. Theorem 3.5 can be extended to an OEP version along the lines of the proofs in [7, 9].

Without extra mathematical difficulties (except the ones that come from a more cumbersome notation), Theorem 3.5 can be generalized to the multi-generator case. That is, we can replace each of the functions Φ_k, $k \in I$, in the general setup by a vector of functions $\mathbf{\Phi}_k(\gamma) := (\Phi_k^{(1)}(\gamma), \ldots, \Phi_k^{(r_k)}(\gamma))^T$ and the condition $\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma)$ by a matrix condition $\mathbf{\Phi}_k(\gamma) = \mathbf{H}_{k+1}(\gamma)\mathbf{\Phi}_{k+1}(\gamma)$, where $\mathbf{H}_{k+1}(\gamma)$ is an $r_k \times r_{k+1}$ matrix.

Similarly, assume that for $k \in I$ we are given a $\rho_k \times r_{k+1}$ matrix-valued function $\mathbf{G}_{k+1}$ with entries in $L^\infty(V_{k+1})$, define the functions $\mathbf{\Psi}_k(\gamma) := (\Psi_k^{(1)}(\gamma), \ldots, \Psi_k^{(\rho_k)}(\gamma))^T := \mathbf{G}_{k+1}(\gamma)\mathbf{\Phi}_{k+1}(\gamma)$. The corresponding version of the matrix $P_k(\gamma)$ in (3.8) is

$$\mathbf{P}_k(\gamma) := \begin{pmatrix} \mathbf{H}_{k+1}(\gamma+\nu_{k,1}) & \cdot\ \cdot\ \cdot & \mathbf{H}_{k+1}(\gamma+\nu_{k,d_k}) \\ \mathbf{G}_{k+1}(\gamma+\nu_{k,1}) & \cdot\ \cdot\ \cdot & \mathbf{G}_{k+1}(\gamma+\nu_{k,d_k}) \end{pmatrix},$$

which is now an $(r_k+\rho_k)\times r_{k+1}d_k$ matrix. Finally, while keeping the condition (3.3), replace the technical condition (3.4) by the assumption that for every compact set S in $\widehat{G}$ and any $\epsilon > 0$, there exists $K_2 \in I$ such that for all $k \geq K_2$, $k \in I$,

$$\left| \mu_{\widehat{G}}(V_k) \sum_{m=1}^{r_k} |\Phi_k^{(m)}(\gamma)|^2 - 1 \right| \leq \epsilon, \ \forall \gamma \in S.$$

Such a multi-generator setting is considered in [11] for the UEP on $\mathbb{T}^s$. Following the arguments in [11], the proof of Theorem 3.5 can be easily adapted to give a UEP on LCA groups for the multi-generator setting.

The multi-generator version of Theorem 3.5 now says that if for every $k \in I$,

$$\mathbf{P}_k(\gamma)^*\mathbf{P}_k(\gamma) = d_k I_{r_{k+1}d_k}, \text{ a.e. } \gamma \in V_k,$$

then the collection of functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}^{(m)}\}_{\lambda\in\Lambda_{k_0}, m=1,\ldots,r_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k\geq k_0, \lambda\in\Lambda_k, m=1,\ldots,\rho_k}$$

forms a tight frame for $L^2(\widehat{G})$ with frame bound 1.

4 B-spline generated systems

In this section we will consider certain explicitly given functions on LCA groups and verify some of the UEP conditions without any restriction on the underlying LCA group. The remaining conditions will be verified directly in concrete cases.

The considered systems will be based on a version of the B-splines on LCA groups, as introduced independently by Dahlke [8] and Tikhomirov [23] in 1994. The results generalize the well-known construction by Ron and Shen [19] of a tight frame based on B-splines on $G = \mathbb{R}$.

Consider a nested sequence of lattices in the LCA group G, as in (3.1). We will assume that for each $k \in I$,

$$|\Lambda_{k+1}/\Lambda_k| = 2 \tag{4.1}$$

and apply Lemma 2.3 on each inclusion $\Lambda_k \subset \Lambda_{k+1}$. In particular, there is a $\nu_k \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$ such that we have the disjoint splitting

$$\Lambda_k^\perp = \Lambda_{k+1}^\perp \cup (\nu_k + \Lambda_{k+1}^\perp). \tag{4.2}$$

Similarly, there exists $\eta_k \in \Lambda_{k+1} \setminus \Lambda_k$ such that we have the disjoint splitting

$$\Lambda_{k+1} = \Lambda_k \cup (\eta_k + \Lambda_k). \tag{4.3}$$

In the entire section we let V_k denote an arbitrary fundamental domain associated with the lattice $\Lambda_k^\perp$. Now, fix $k' \geq k_0$ and let $Q_{k'}$ denote a fundamental domain associated with the lattice $\Lambda_{k'}$ in G. It follows from Lemma 2.3 that we successively can construct fundamental domains Q_k for the lattices Λ_k such that for all $k \leq k'$ we have the disjoint splitting

$$Q_k = Q_{k+1} \cup (\eta_k + Q_{k+1}). \tag{4.4}$$

We will assume that (4.4) holds for all $k \in I$ (we will later verify this condition explicitly in concrete cases). On the other hand, we note that it is actually possible to construct "badly chosen" fundamental domains Q_k such that no fundamental domain Q_{k+1} will satisfy (4.4) with a disjoint splitting, regardless of the choice of $\eta_k \in \Lambda_{k+1} \setminus \Lambda_k$. We will verify (4.4) directly for the concrete cases to be considered later.

Lemma 4.1 *Assume that* (4.1) *holds and choose* ν_k, η_k *as in* (4.2) *and* (4.3). *Then* $(\eta_k, \nu_k) = -1$ *for all* $k \in I$.

Proof. Since $\nu_k \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$, we see that $\nu_k + \nu_k \in \Lambda_k^\perp \setminus (\nu_k + \Lambda_{k+1}^\perp)$; using the decomposition (4.2), we conclude that $\nu_k + \nu_k \in \Lambda_{k+1}^\perp$. Since $\eta_k \in \Lambda_{k+1}$, it follows that $1 = (\eta_k, \nu_k + \nu_k) = (\eta_k, \nu_k)^2$, i.e., $(\eta_k, \nu_k) = \pm 1$. But if $(\eta_k, \nu_k) = 1$, then for any $\lambda \in \Lambda_k$, $(\eta_k + \lambda, \nu_k) = 1$ and $(\lambda, \nu_k) = 1$; by the decomposition (4.3), this shows that $(\lambda', \nu_k) = 1$ for all $\lambda' \in \Lambda_{k+1}$, i.e., $\nu_k \in \Lambda_{k+1}^\perp$. This contradicts that $\nu_k \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$, and we conclude that $(\eta_k, \nu_k) = -1$. $\square$

Our constructions will be based on the following definition of B-splines on LCA groups.

Definition 4.2 *Consider a sequence of nested lattices* $\{\Lambda_k\}_{k=k_0}^\infty$ *in* G, *with associated fundamental domains* $Q_k, k \geq k_0$. *For* $N \in \mathbb{N}$, *define the B-spline of* N*th order at level* $k, k \geq k_0$, *by the* N*-fold convolution*

$$\phi_{k,N}(x) := \mu_G(Q_k)^{-N+1/2} \chi_{Q_k} * \cdots * \chi_{Q_k}(x),\ x \in G.$$

We have included the factor $\mu_G(Q_k)^{-N+1/2}$ in the definition of the B-spline in order to avoid a later renormalization. Note also that the index k refers to the level of the sets Q_k within the scale of fundamental domains associated with the lattices Λ_k. We will consider a fixed choice of $N \in \mathbb{N}$ and consider the functions $\Phi_{k,N}$, $k \geq k_0$, defined by

$$\Phi_{k,N}(\gamma) := \widehat{\phi_{k,N}}(\gamma) = \mu_G(Q_k)^{-N+1/2} \left(\int_{Q_k} (-x, \gamma) \, dx \right)^N, \ \gamma \in \widehat{G}, \tag{4.5}$$

where the Fourier transform formula (2.2) is used to obtain the final expression.

We now show that for any given $N \in \mathbb{N}$, the functions $\Phi_{k,N}$ satisfy the refinement equation (3.5); we suppress the dependence of N in the notation for the associated filters H_{k+1}. For the one-dimensional elementary groups this result was also proved in [22].

Lemma 4.3 *Assume that* (4.1) *and* (4.4) *are satisfied. Fix* $N \in \mathbb{N}$. *The functions* $\Phi_{k,N}$, $k \in I$, *defined by* (4.5) *satisfy the refinement equation*

$$\Phi_{k,N}(\gamma) = H_{k+1}(\gamma)\Phi_{k+1,N}(\gamma), \ \gamma \in \widehat{G}, \tag{4.6}$$

where $H_{k+1} \in L^\infty(V_{k+1})$ *is given by*

$$H_{k+1}(\gamma) = \frac{1}{2^{N-1/2}} \left(1 + (-\eta_k, \gamma)\right)^N, \ \gamma \in \widehat{G}. \tag{4.7}$$

Proof. Using the disjoint splitting in (4.4) and a change of variable,

$$\begin{aligned}
\Phi_{k,N}(\gamma) &= \mu_G(Q_k)^{-N+1/2} \left(\int_{Q_{k+1}} (-x, \gamma) \, dx + \int_{\eta_k + Q_{k+1}} (-x, \gamma) \, dx \right)^N \\
&= \mu_G(Q_k)^{-N+1/2} (1 + (-\eta_k, \gamma))^N \left(\int_{Q_{k+1}} (-x, \gamma) \, dx \right)^N \\
&= \left(\frac{\mu_G(Q_k)}{\mu_G(Q_{k+1})} \right)^{-N+1/2} (1 + (-\eta_k, \gamma))^N \mu_G(Q_{k+1})^{-N+1/2} \left(\int_{Q_{k+1}} (-x, \gamma) \, dx \right)^N .
\end{aligned}$$

Note that by Lemma 2.3(i), $2 = |\Lambda_{k+1}/\Lambda_k| = \frac{s(\Lambda_k)}{s(\Lambda_{k+1})} = \frac{\mu_G(Q_k)}{\mu_G(Q_{k+1})}$; thus, we see that (4.6) is satisfied with the function H_{k+1} defined as in (4.7). Clearly H_{k+1} is bounded; that H_{k+1} is periodic follows from the fact that for $\omega \in \Lambda_{k+1}^\perp, \gamma \in \widehat{G}$,

$$H_{k+1}(\omega + \gamma) = \frac{1}{2^{N-1/2}} (1 + (-\eta_k, \omega + \gamma))^N = \frac{1}{2^{N-1/2}} (1 + (-\eta_k, \gamma))^N = H_{k+1}(\gamma). \qquad \square$$

We will now provide a condition under which (3.4) holds.

Lemma 4.4 *Assume that* (4.4) *holds and fix* $N \in \mathbb{N}$. *Then the following hold:*

(i) *Let $\delta \in (0,1)$ be given. Assume that for some $k \in I$ and some $\gamma \in \widehat{G}$,*

$$|(-x,\gamma)-1| \le \delta, \ \forall x \in Q_k. \tag{4.8}$$

Then

$$|\mu_{\widehat{G}}(V_k)\,|\Phi_{k,N}(\gamma)|^2 - 1| \le 1-(1-\delta)^{2N}. \tag{4.9}$$

(ii) *Assume that for every $\delta \in (0,1)$ and any compact set S in $\widehat{G}$, there exists $k \in I$ such that the inequality (4.8) holds for all $\gamma \in S$ and $x \in Q_k$. Then the condition (3.4) is satisfied.*

Proof. First, using (4.5) and that $\mu_{\widehat{G}}(V_k)\mu_G(Q_k)=1$,

$$\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 = \mu_{\widehat{G}}(V_k)\mu_G(Q_k)^{-2N+1}\left|\int_{Q_k}(-x,\gamma)\,dx\right|^{2N} = \left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}(-x,\gamma)\,dx\right|^{2N}.$$

It follows that

$$\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 \le \left(\frac{1}{\mu_G(Q_k)}\int_{Q_k}|(-x,\gamma)|\,dx\right)^{2N} = 1,$$

and therefore

$$|\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 - 1| = 1-\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 = 1-\left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}(-x,\gamma)\,dx\right|^{2N}. \tag{4.10}$$

Using the triangle inequality, the assumption (4.8) implies that

$$\begin{aligned}
\left|\left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}(-x,\gamma)\,dx\right|-1\right| &\le \left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}(-x,\gamma)\,dx-1\right| \\
&= \left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}[(-x,\gamma)-1]\,dx\right| \\
&\le \frac{1}{\mu_G(Q_k)}\int_{Q_k}|(-x,\gamma)-1|\,dx \le \delta,
\end{aligned}$$

and therefore $\left|\frac{1}{\mu_G(Q_k)}\int_{Q_k}(-x,\gamma)\,dx\right| \ge 1-\delta$. Via (4.10) we conclude that $|\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 - 1| \le 1-(1-\delta)^{2N}$, which proves (i).

In order to prove (ii), given $\epsilon > 0$, choose $\delta \in (0,1)$ such that $1-(1-\delta)^{2N} \le \epsilon$. Then by (4.9), $|\mu_{\widehat{G}}(V_k)|\Phi_{k,N}(\gamma)|^2 - 1| \le \epsilon$ for all $\gamma \in S$. Since $Q_{k+1} \subset Q_k$ for all $k \in I$, the same holds with k replaced by any k' for which $k' \ge k$. $\square$

Now, fix $N \in \mathbb{N}$ and consider the functions $\Phi_{k,N}$, $k \in I$. Take the functions H_{k+1}, $k \in I$, as in (4.7); our task is to find functions $G_{k+1}^{(1)}, \dots G_{k+1}^{(\rho_k)} \in L^\infty(V_{k+1})$ such that the matrices

$$P_k(\gamma) := \begin{pmatrix} H_{k+1}(\gamma) & H_{k+1}(\gamma+\nu_k) \\ G_{k+1}^{(1)}(\gamma) & G_{k+1}^{(1)}(\gamma+\nu_k) \\ \cdot & \cdot \\ \cdot & \cdot \\ G_{k+1}^{(\rho_k)}(\gamma) & G_{k+1}^{(\rho_k)}(\gamma+\nu_k) \end{pmatrix}, \ \gamma \in V_k, \tag{4.11}$$

satisfy the matrix condition in the UEP, i.e.,

$$P_k(\gamma)^* P_k(\gamma) = 2I_2, \text{ a.e. } \gamma \in V_k. \tag{4.12}$$

As we will see, the LCA case is technically more involved than the case of the real line. We will consider the cases $N = 1$ and $N = 2M$, $M \in \mathbb{N}$. In both instances, we take ρ_k to be N, and for the case $N = 1$, we write $G_{k+1}^{(1)}$ simply as G_{k+1}.

Proposition 4.5 *Assume that* (4.1) *and* (4.4) *are satisfied. For every* $k \geq k_0$, *consider the following two choices of* Φ_k *and associated filters:*

(i) *Let* $N = 1$ *and consider* $\Phi_k := \widehat{\phi_{k,1}}$, *with the associated filter* H_{k+1} *in* (4.7). *Define the function* $G_{k+1} \in L^\infty(V_{k+1})$ *by*

$$G_{k+1}(\gamma) := \frac{1}{\sqrt{2}}\left(1 - (-\eta_k, \gamma)\right), \; \gamma \in \widehat{G}. \tag{4.13}$$

(ii) *Given* $M \in \mathbb{N}$, *let* $\Phi_k := \widehat{\phi_{k,2M}}$, *with the associated filter* H_{k+1} *in* (4.7). *Define the functions* $G_{k+1}^{(1)}, \ldots G_{k+1}^{(2M)} \in L^\infty(V_{k+1})$ *by*

$$G_{k+1}^{(m)}(\gamma) := \frac{1}{2^{2M-1/2}} \sqrt{\binom{2M}{m}} (1 + (-\eta_k, \gamma))^{2M-m} (1 - (-\eta_k, \gamma))^m, \; \gamma \in \widehat{G}. \tag{4.14}$$

Then, in both cases the matrix $P_k(\gamma)$ *in* (4.11) *satisfies the UEP condition* (4.12).

Proof. For the proof of (i), by Lemma 4.3, the function Φ_k satisfies the refinement equation (4.6) with $H_{k+1}(\gamma) = \frac{1}{\sqrt{2}}(1 + (-\eta_k, \gamma))$, $\gamma \in \widehat{G}$. Now, take G_{k+1} as in (4.13). Then, by direct calculation,

$$\begin{aligned} & |H_{k+1}(\gamma)|^2 + |G_{k+1}(\gamma)|^2 \\ = \; & \frac{1}{2}\Big((1 + (-\eta_k, \gamma))\overline{(1 + (-\eta_k, \gamma))} + (1 - (-\eta_k, \gamma))\overline{(1 - (-\eta_k, \gamma))}\Big) = 2. \end{aligned}$$

Similarly, using that $(\eta_k, \nu_k) = -1$ by Lemma 4.1, we arrive at

$$\begin{aligned} & H_{k+1}(\gamma)\overline{H_{k+1}(\gamma + \nu_k)} + G_{k+1}(\gamma)\overline{G_{k+1}(\gamma + \nu_k)} \\ = \; & \frac{1}{2}\big((1 + (-\eta_k, \gamma))(1 + (\eta_k, \gamma + \nu_k)) + (1 - (-\eta_k, \gamma))(1 - (\eta_k, \gamma + \nu_k))\big) \\ = \; & \frac{1}{2}\big((1 + (-\eta_k, \gamma))(1 - (\eta_k, \gamma)) + (1 - (-\eta_k, \gamma))(1 + (\eta_k, \gamma))\big) = 0. \end{aligned}$$

For the proof of (ii), clearly $G_{k+1}^{(m)} \in L^\infty(V_{k+1})$ for $m = 1, \ldots, 2M$. We first show that

$$|H_{k+1}(\gamma)|^2 + \sum_{m=1}^{2M} |G_{k+1}^{(m)}(\gamma)|^2 = 2, \; \gamma \in \widehat{G}. \tag{4.15}$$

We will now use the elementary identity $|1+z|^2+|1-z|^2=4$, which is valid for all $z \in \mathbb{T}$. Taking $z=(-\eta_k,\gamma)$ and raising both sides to the power of $2M$ yields that

$$\frac{1}{2^{4M-1}}\left(|1+(-\eta_k,\gamma)|^2+|1-(-\eta_k,\gamma)|^2\right)^{2M}=2,$$

which by the binomial theorem means that

$$\frac{1}{2^{4M-1}}\sum_{m=0}^{2M}\binom{2M}{m}|1+(-\eta_k,\gamma)|^{4M-2m}|1-(-\eta_k,\gamma)|^{2m}=2.$$

This immediately yields that (4.15) is satisfied; in fact, using (4.7) and (4.14),

$$|H_{k+1}(\gamma)|^2+\sum_{m=1}^{2M}|G_{k+1}^{(m)}(\gamma)|^2=$$

$$\frac{1}{2^{4M-1}}|(1+(-\eta_k,\gamma))|^{4M}+\frac{1}{2^{4M-1}}\sum_{m=1}^{2M}\binom{2M}{m}|(1+(-\eta_k,\gamma))|^{4M-2m}|(1-(-\eta_k,\gamma))|^{2m}=2.$$

We now have to show that

$$H_{k+1}(\gamma)\overline{H_{k+1}(\gamma+\nu_k)}+\sum_{m=1}^{2M}G_{k+1}^{(m)}(\gamma)\overline{G_{k+1}^{(m)}(\gamma+\nu_k)}=0,\ \gamma\in\widehat{G}. \tag{4.16}$$

Use the identity $(1+z)(1-\overline{z})+(1-z)(1+\overline{z})=0, z\in\mathbb{T}$, with $z=(-\eta_k,\gamma)$; then

$$\Big((1+(-\eta_k,\gamma))(1-\overline{(-\eta_k,\gamma)})+(1-(-\eta_k,\gamma))(1+\overline{(-\eta_k,\gamma)})\Big)^{2M}=0,$$

which yields that

$$\sum_{m=0}^{2M}\binom{2M}{m}(1+(-\eta_k,\gamma))^{2M-m}(1-\overline{(-\eta_k,\gamma)})^{2M-m}(1-(-\eta_k,\gamma))^m(1+\overline{(-\eta_k,\gamma)})^m=0. \tag{4.17}$$

Inserting the expressions for H_{k+1} and $G_{k+1}^{(m)}$ into the left-hand side of (4.16) yields

$$\begin{aligned}
&H_{k+1}(\gamma)\overline{H_{k+1}(\gamma+\nu_k)}+\sum_{m=1}^{2M}G_{k+1}^{(m)}\overline{G_{k+1}^{(m)}(\gamma+\nu_k)}\\
=\;&\frac{1}{2^{4M-1}}(1+(-\eta_k,\gamma))^{2M}(1+\overline{(-\eta_k,\gamma+\nu_k)})^{2M}\\
&+\frac{1}{2^{4M-1}}\sum_{m=1}^{2M}\binom{2M}{m}(1+(-\eta_k,\gamma))^{2M-m}(1-(-\eta_k,\gamma))^m\\
&\qquad\times(1+\overline{(-\eta_k,\gamma+\nu_k)})^{2M-m}(1-\overline{(-\eta_k,\gamma+\nu_k)})^m.
\end{aligned}$$

Using again that by Lemma 4.1 $(\eta_k,\nu_k)=-1$, it follows immediately from (4.17) that this expression equals zero, as desired. □

Applying the UEP to the setting in Proposition 4.5(ii), we obtain the following generalization to LCA groups of the B-spline tight frames on $\mathbb{R}$ by Ron and Shen [19].

Proposition 4.6 *Assume that* (3.3), (4.1) *and* (4.4) *are satisfied, and that for each* $\delta \in (0,1)$ *and each compact set* $S \subset \widehat{G}$, *there exists* $k \in I$ *such that the inequality* (4.8) *holds for all* $\gamma \in S$ *and* $x \in Q_k$. *Given* $M \in \mathbb{N}$, *for every* $k \geq k_0$, *let* $\Phi_k := \widehat{\phi_{k,2M}}$, *with the associated filter* H_{k+1} *in* (4.7). *Define the functions* $G_{k+1}^{(m)}, m = 1, \ldots, 2M$, *by* (4.14). *Then the collection of functions*

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,2M}$$

forms a tight frame for $L^2(\widehat{G})$ *with frame bound* 1*; equivalently,*

$$\{T_\lambda \phi_{k_0,2M}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{T_\lambda \psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,2M}$$

forms a tight frame for $L^2(G)$ *with frame bound* 1.

For the first-order B-splines in Proposition 4.5(i), we can of course formulate a UEP-result of exactly the same type as in Proposition 4.6; in this particular case we actually obtain an orthonormal basis. In order to show this, it is sufficient to establish that $\|\Phi_k\| = \|\Psi_k\| = 1$ for all $k \in I$, so let us do that. By the definition of Φ_k and Parseval's identity, we immediately see that $\|\Phi_k\| = 1$ for all $k \in I$. Now, by the definition of Ψ_k in (3.6), using (4.5), (4.13) and then the disjoint splitting (4.4), for $\gamma \in \widehat{G}$,

$$\begin{aligned}
\Psi_k(\gamma) &= \frac{1}{\sqrt{2}}\left(1 - (-\eta_k, \gamma)\right) \frac{1}{\sqrt{\mu_G(Q_{k+1})}} \int_{Q_{k+1}} (-x, \gamma)\, dx \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mu_G(Q_{k+1})}} \left(\int_{Q_{k+1}} (-x, \gamma)\, dx - \int_{Q_{k+1}} (-x - \eta_k, \gamma)\, dx \right) \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mu_G(Q_{k+1})}} \left(\int_{Q_{k+1}} (-x, \gamma)\, dx - \int_{\eta_k + Q_{k+1}} (-x, \gamma)\, dx \right) \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mu_G(Q_{k+1})}} \left(\widehat{\chi_{Q_{k+1}}}(\gamma) - \widehat{\chi_{\eta_k + Q_{k+1}}}(\gamma) \right).
\end{aligned}$$

Thus,

$$\mathcal{F}^{-1}\Psi_k(x) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\mu_G(Q_{k+1})}} \left(\chi_{Q_{k+1}}(x) - \chi_{\eta_k + Q_{k+1}}(x) \right), \, x \in G.$$

Using again the disjoint splitting in (4.4), it now follows that

$$\|\Psi_k\|^2 = \|\mathcal{F}^{-1}\Psi_k\|^2 = \frac{1}{2} \frac{1}{\mu_G(Q_{k+1})} \left(\mu_G(Q_{k+1}) + \mu_G(\eta_k + Q_{k+1}) \right) = 1,$$

as claimed.

We will now provide completely explicit constructions of frames for a number of LCA groups. We will show that all the technical assumptions can be fulfilled in these cases. Then the filters $G_{k+1}^{(m)}$ given in Proposition 4.5 would lead to the desired constructions of $\Psi_k^{(m)}$ via Proposition 4.6 and its above variant.

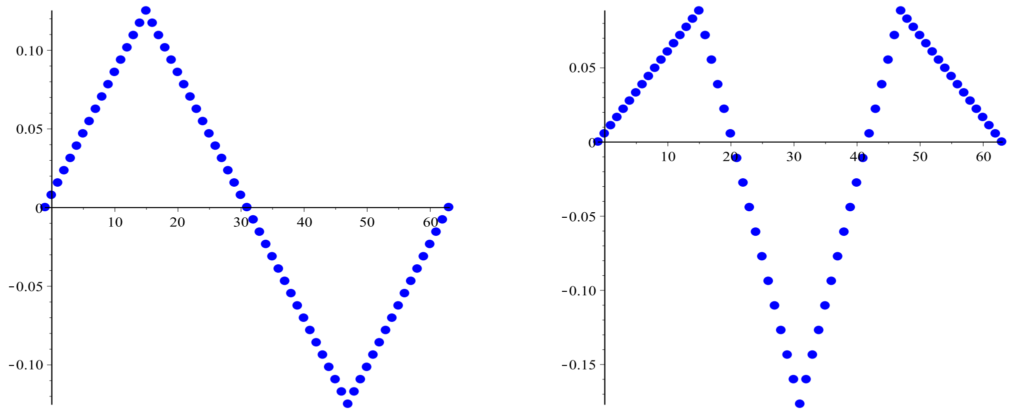

Figure 1: Plots of $\psi_k^{(1)}$ (left) and $\psi_k^{(2)}$ (right) in $\ell^2(\mathbb{Z})$ for $M = 10$ and $k = 5$ in Example 4.7.

Example 4.7 Consider the group $G = \mathbb{Z}$, with dual group $\widehat{G} = \mathbb{T}$. Given $M \in \mathbb{N}$, we will consider the lattices $\Lambda_k := 2^{M-k}\mathbb{Z}$, $k = 0, \ldots, M$; as associated fundamental domains, we take $Q_k = \{0, \ldots, 2^{M-k} - 1\}$. Note that $\Lambda_0 = 2^M\mathbb{Z}$ and $\Lambda_M = \mathbb{Z} = G$. Clearly, for $k = 0, \ldots, M-1$, $\Lambda_k \subset \Lambda_{k+1}$ and $|\Lambda_{k+1}/\Lambda_k| = 2$; furthermore, $\Lambda_{k+1} = \Lambda_k \cup \big(2^{M-k-1} + \Lambda_k\big)$, i.e., (4.3) holds with $\eta_k = 2^{M-k-1}$. Note that

$$Q_k = Q_{k+1} \cup \left(2^{M-k-1} + Q_{k+1}\right),$$

i.e., (4.4) holds for all $k = 0, \ldots, M-1$. Also, for $k = 0, \ldots, M$, $\Lambda_k^\perp = \frac{1}{2^{M-k}}\mathbb{Z}_{2^{M-k}}$; the set $V_k = [0, \frac{1}{2^{M-k}})$ is a fundamental domain associated with the lattice $\Lambda_k^\perp$. Observe that for $k = 0, \ldots, M-1$,

$$\Lambda_k^\perp = \Lambda_{k+1}^\perp + \big(\frac{1}{2^{M-k}} + \Lambda_{k+1}^\perp\big),$$

i.e., (4.2) holds with $\nu_k = \frac{1}{2^{M-k}} \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$.

We now verify the conditions (4.8) and (3.3). To this end, let $0 < \delta < 1$ and consider a compact set S in $\widehat{G} = \mathbb{T}$. Then, taking $k = M$ and noticing that $Q_M = \{0\}$, the condition (4.8) trivially holds. Also, $\Lambda_M^\perp = \{0\}$, so (3.3) clearly holds.

As an illustration, consider the functions $\psi_k^{(1)}, \psi_k^{(2)} \in \ell^2(\mathbb{Z})$, $k = 0, \ldots, M-1$, constructed from the B-splines of second order $\phi_{k+1,2}$, $k = 0, \ldots, M-1$, as in Proposition 4.6. Figure 1 shows the plots of $\psi_k^{(1)}$ and $\psi_k^{(2)}$ for $M = 10$ and $k = 5$. $\square$

Example 4.8 Let $G = \mathbb{R}^2$, with dual group $\widehat{G} = \mathbb{R}^2$. Consider a 2×2 matrix A with integer entries, eigenvalues outside the unit circle and $|\det A| = 2$. We will show that under a certain technical condition (see (4.22)) the conditions (3.3) and (4.4) are satisfied, i.e., the B-spline constructions in Proposition 4.6 can be realized. The condition (4.22) is satisfied, e.g., for the matrices $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$. For $k \geq 0$, consider the lattices $\Lambda_k := (A^k)^\sharp \mathbb{Z}^2 = (A^\sharp)^k \mathbb{Z}^2$ in $G = \mathbb{R}^2$, where $A^\sharp := (A^T)^{-1}$. Observe that as A has integer entries, $A^T \mathbb{Z}^2 \subset \mathbb{Z}^2$; it follows that $\mathbb{Z}^2 \subset A^\sharp \mathbb{Z}^2$, and therefore $\Lambda_k = (A^\sharp)^k \mathbb{Z}^2 \subset (A^\sharp)^{k+1} \mathbb{Z}^2 = \Lambda_{k+1}$, $\forall k \geq 0$, i.e., the sequence of lattices Λ_k, $k \geq 0$, is nested. It is easy to check that $|\Lambda_{k+1}/\Lambda_k| = 2$ for all $k \geq 0$. By general theory, the annihilator of the lattice Λ_k is $\Lambda_k^\perp = \left((A^k)^\sharp\right)^\sharp \mathbb{Z}^2 = A^k \mathbb{Z}^2$, with associated fundamental domain $V_k = A^k[0,1)^2$. For any $k \geq 0$, $\mu_{\widehat{G}}(V_k) = \mu_{\mathbb{R}^2}(A^k[0,1)^2) = |\det A^k| = 2^k$. Let $\{0, \nu\}$ be a full collection of coset representatives of $\mathbb{Z}^2/A\mathbb{Z}^2$. Then

$$\mathbb{Z}^2 = A\mathbb{Z}^2 \cup \left(\nu + A\mathbb{Z}^2\right), \qquad A\mathbb{Z}^2 \cap \left(\nu + A\mathbb{Z}^2\right) = \emptyset,$$

which implies that

$$A^k\mathbb{Z}^2 = A^{k+1}\mathbb{Z}^2 \cup \left(A^k\nu + A^{k+1}\mathbb{Z}^2\right), \qquad A^{k+1}\mathbb{Z}^2 \cap \left(A^k\nu + A^{k+1}\mathbb{Z}^2\right) = \emptyset.$$

Thus,

$$\Lambda_k^\perp = \Lambda_{k+1}^\perp \cup \left(\nu_k + \Lambda_{k+1}^\perp\right), \qquad \Lambda_{k+1}^\perp \cap \left(\nu_k + \Lambda_{k+1}^\perp\right) = \emptyset,$$

where $\nu_k = A^k\nu \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$.

We will now examine the existence of fundamental domains Q_k for the lattices Λ_k such that (4.4) is satisfied. By Theorem 2 in the paper [13] by Gröchenig and Madych, there exists a relatively compact set $Q \subset \mathbb{R}^2$ and $\eta \in \mathbb{R}^2$ such that

$$\begin{aligned} Q \cup (\eta + Q) &= A^T Q, && (4.18)\\ \bigcup_{n \in \mathbb{Z}^2} (n + Q) &= \mathbb{R}^2, && (4.19)\\ Q \cap (\eta + Q) &= \emptyset, && (4.20) \end{aligned}$$

up to a set of measure zero. The set Q is generated iteratively by the algorithm

$$Q^{(r+1)} = A^\sharp Q^{(r)} \cup A^\sharp(\eta + Q^{(r)}),\ r \geq 0.$$

Also, the condition (4.18) is equivalent to

$$Q = A^\sharp Q \cup A^\sharp(\eta + Q), \tag{4.21}$$

i.e., Q is self-similar in the sense of the affine transformation $A^\sharp$.

We will now assume that

$$\mu_{\mathbb{R}^2}(Q) = 1; \tag{4.22}$$

Theorem 3 in [13] provides various equivalent conditions for this condition to hold. Then Lemma 1 or Theorem 3 in [13] implies that $Q \cap (n + Q) = \emptyset$, $\forall n \in \mathbb{Z}^2 \setminus \{0\}$, or, equivalently

$$(n_1 + Q) \cap (n_2 + Q) = \emptyset,\ \forall n_1 \neq n_2,\ n_1, n_2 \in \mathbb{Z}^2, \tag{4.23}$$

again up to a set of measure zero.

Applying $(A^\sharp)^k$ to (4.19), (4.23), (4.21), and $(A^\sharp)^{k+1}$ to (4.20) immediately yields that with $Q_k := (A^\sharp)^k Q$ and $\eta_k := (A^\sharp)^{k+1}\eta$,

$$\begin{aligned}\bigcup_{\omega\in\Lambda_k}(\omega+Q_k) &= \mathbb{R}^2, \quad (\omega_1+Q_k)\cap(\omega_2+Q_k)=\emptyset \text{ for } \omega_1,\omega_2\in\Lambda_k, \omega_1\neq\omega_2, \quad (4.24)\\ Q_k &= Q_{k+1}\cup(\eta_k+Q_{k+1}), \quad Q_{k+1}\cap(\eta_k+Q_{k+1})=\emptyset. \quad (4.25)\end{aligned}$$

By (4.24), the set Q_k is a fundamental domain associated with the lattice Λ_k. Also, (4.25) shows that (4.4) holds.

We now verify the conditions (4.8) and (3.3). Let $S\subset\mathbb{R}^2$ be compact and consider $0<\delta<1$. Let $\gamma\in S$, and $x\in Q_k$; we can write $x=(A^\sharp)^k q$ for some $q\in Q$. Then

$$|(-x,\gamma)-1| = |e^{-2\pi i\gamma\cdot(A^\sharp)^k q}-1|. \tag{4.26}$$

Now, using that S and Q are (relatively) compact, and thus bounded, there is a constant $C>0$ such that, regardless of the choice of $\gamma\in S$ and $x\in Q_k$,

$$|-2\pi i\gamma\cdot(A^\sharp)^k q| \le 2\pi\|\gamma\|_2\,\|(A^\sharp)^k q\|_2 \le 2\pi\|\gamma\|_2\,\|(A^\sharp)^k\|_2\,\|q\|_2 \le C\,\|(A^\sharp)^k\|_2. \tag{4.27}$$

Since A^T has eigenvalues outside the unit circle, there exists a matrix norm $\|\cdot\|$ such that $\|(A^T)^{-1}\|<1$; since all norms on a finite-dimensional space are equivalent, this implies that there is a constant $D>0$ such that

$$\|(A^\sharp)^k\|_2 = \|(A^T)^{-k}\|_2 \le D\,\|(A^T)^{-k}\| \le D\,\|(A^T)^{-1}\|^k;$$

thus, via (4.27),

$$|-2\pi i\gamma\cdot(A^\sharp)^k q| \le CD\,\|(A^T)^{-1}\|^k, \ \forall\gamma\in S, x\in Q_k.$$

Since $\|(A^T)^{-1}\|^k\to 0$ as $k\to\infty$, it now follows from (4.26) that for any given $\delta\in(0,1)$, there exists $k\in I$ such that the inequality (4.8) holds for all $\gamma\in S$ and $x\in Q_k$.

In order to establish (3.3), for any $k\ge 0$, $n_1,n_2\in\mathbb{Z}^2$, consider $\gamma\in(A^k n_1+S)\cap(A^k n_2+S)$. Writing $\gamma=A^k n_1+s_1$ and $\gamma=A^k n_2+s_2$, where $s_1,s_2\in S$, we have $A^k(n_1-n_2)=s_2-s_1$. Thus, using that S is compact, there is a constant $C_1>0$ (independent of the choice of k) such that $\|A^k(n_1-n_2)\|_2\le C_1$. Then

$$\begin{aligned}\|n_1-n_2\|_2 = \|(A^k)^{-1}A^k(n_1-n_2)\|_2 &\le \|(A^k)^{-1}\|_2\cdot\|A^k(n_1-n_2)\|_2 \le C_1\|(A^k)^{-1}\|_2\\ &\le C_1\|A^{-1}\|_2^k.\end{aligned}$$

But the eigenvalues of A lie outside the unit circle, so $\|A^{-1}\|_2^k\to 0$ as $k\to\infty$; thus for sufficiently large values of k, we have $n_1=n_2$, which finally proves (3.3). $\square$

5 Frames generated by characteristic functions

In the entire section, we will use the setup and notations presented in Section 3. In particular, $\Lambda_k, k \in I$, is a nested sequence of lattices in G. We consider a nested sequence of subsets Ω_k, $k \in I$, of $\widehat{G}$ such that for $k \in I$,

$$\Omega_k \subseteq V_k \tag{5.1}$$

for some fundamental domain V_k associated with $\Lambda_k^{\perp}$. We will further assume that for every compact set S in $\widehat{G}$, there exists $K \in I$ such that $S \subseteq \Omega_K$. Now, for $k \in I$, let

$$\Phi_k(\gamma) := \mu_{\widehat{G}}(V_k)^{-1/2}\chi_{\Omega_k}(\gamma),\ \gamma \in \widehat{G}. \tag{5.2}$$

We will show that with this setup, we can always satisfy the UEP conditions, regardless of the underlying LCA group. As in Section 3, with the set V_k and $\nu_{k,\ell}$ defined as in (3.7), the fundamental domain $V'_{k+1} = \bigcup_{\ell=1}^{d_k}(\nu_{k,\ell} + V_k)$ in (3.10) associated with $\Lambda_{k+1}^{\perp}$ will be useful in our derivations. Note that in contrast with the B-spline generated case in Section 4, we assume neither the condition (4.1) on the lattices, nor the splitting (4.4) of the fundamental domains Q_k associated with the lattices Λ_k.

We will first show that in the above setup, the refinement equation (3.5) and the technical conditions (3.3) and (3.4) are always satisfied.

Lemma 5.1 *Let $\{\Lambda_k\}_{k\in I}$ in G be a nested sequence of lattices in G, and choose the sets V_k and Ω_k as above. Then the functions Φ_k, $k \in I$, in* (5.2) *satisfy the refinement equation* (3.5) *whenever $H_{k+1} \in L^{\infty}(V_{k+1})$ is defined by*

$$H_{k+1}(\gamma) := \begin{cases} \sqrt{d_k}, & \text{if}\, \gamma \in \Omega_k, \\ 0, & \text{if}\, \gamma \in V'_{k+1} \setminus \Omega_k. \end{cases} \tag{5.3}$$

Furthermore, the conditions (3.3) *and* (3.4) *are satisfied.*

Proof. For $k \in I$, if $\gamma \in \Omega_k$, then $\Phi_k(\gamma) = \mu_{\widehat{G}}(V_k)^{-1/2}$ and $\Phi_{k+1}(\gamma) = \mu_{\widehat{G}}(V_{k+1})^{-1/2}$, i.e., (3.5) holds if we let

$$H_{k+1}(\gamma) = \frac{\Phi_k(\gamma)}{\Phi_{k+1}(\gamma)} = \frac{\mu_{\widehat{G}}(V_k)^{-1/2}}{\mu_{\widehat{G}}(V_{k+1})^{-1/2}} = \left(\frac{s(\Lambda_k^{\perp})}{s(\Lambda_{k+1}^{\perp})}\right)^{-1/2} = \left(\frac{s(\Lambda_{k+1})}{s(\Lambda_k)}\right)^{-1/2} = \sqrt{d_k},\ \gamma \in \Omega_k.$$

If $\gamma \in V'_{k+1} \setminus \Omega_k$, then $\Phi_k(\gamma) = 0$ and (3.5) is satisfied when we take $H_{k+1}(\gamma) = 0$, regardless of the value of $\Phi_{k+1}(\gamma)$. We can therefore extend the function H_{k+1} in (5.3) to a periodic function, and (3.5) holds.

We will now check the technical conditions (3.3) and (3.4). First, given a compact set S in $\widehat{G}$, it follows from the inclusions $S \subseteq \Omega_K \subseteq V_K$ that for any $\omega, \omega' \in \Lambda_K^{\perp}$, $\omega \neq \omega'$, $(\omega + S) \cap (\omega' + S) \subseteq (\omega + V_K) \cap (\omega' + V_K)$. Since $\mu_{\widehat{G}}\big((\omega + V_K) \cap (\omega' + V_K)\big) = 0$, this gives (3.3). Next, the definition of Φ_k and the assumptions of the sets Ω_k imply that for $k \geq K$ and $\gamma \in S$, $|\mu_{\widehat{G}}(V_k)|\Phi_k(\gamma)|^2 - 1| = 0$, i.e., (3.4) is satisfied. $\square$

With the considered setup we can apply the UEP as soon as we have defined functions $G_{k+1}^{(m)} \in L^\infty(V_{k+1})$, $m = 1, \ldots, \rho_k$, in such a way that the matrix P_k in (3.8), with ρ_k equals d_k or $d_k - 1$, satisfies (3.11). We note that it is enough to define $G_{k+1}^{(m)}(\gamma + \nu_{k,\ell})$ for $\gamma \in V_k, m = 1, \ldots, \rho_k, \ell = 1, \ldots, d_k$; then, if for $\omega \in \Lambda_{k+1}^\perp$ we define

$$G_{k+1}^{(m)}(\gamma + \nu_{k,\ell} + \omega) := G_{k+1}^{(m)}(\gamma + \nu_{k,\ell}), \tag{5.4}$$

the entries of P_k are defined as periodic functions in $L^\infty(V_{k+1})$.

In order to consider the matrix extension problem (3.11), we need an explicit expression for the first row of the matrix P_k, which follows immediately from Lemma 5.1.

Lemma 5.2 *Let H_{k+1} be defined by* (5.3)*. Then for $\gamma \in V_k$,*

$$(H_{k+1}(\gamma + \nu_{k,1}), \ldots, H_{k+1}(\gamma + \nu_{k,d_k})) = (H_{k+1}(\gamma), 0, \ldots, 0).$$

We will now construct explicitly given tight frames based on the functions Φ_k in (5.2). These tight frames are "bandlimited" in the sense that they are compactly supported on the dual group $\widehat{G}$. We first consider the case where Ω_k is chosen as a proper subset of V_k.

Proposition 5.3 *Let $\{\Lambda_k\}_{k\in I}$ be a nested sequence of lattices in G, and choose the sets V_k and Ω_k as above; assume further that Ω_k is a proper subset of V_k for all $k \in I$. For $k \in I$, consider the function Φ_k in* (5.2) *with the associated filter $H_{k+1} \in L^\infty(V_{k+1})$ in* (5.3)*. Define the functions $G_{k+1}^{(m)}, m = 1, \ldots, d_k - 1$, by*

$$G_{k+1}^{(m)}(\gamma + \nu_{k,\ell}) := \begin{cases} \sqrt{d_k}\delta_{m+1,\ell}, & if \gamma \in \Omega_k, \\ \sqrt{d_k}\delta_{m,\ell}, & if \gamma \in V_k \setminus \Omega_k, \end{cases}$$

and the function $G_{k+1}^{(d_k)}$ by

$$G_{k+1}^{(d_k)}(\gamma + \nu_{k,\ell}) := \begin{cases} 0, & if \gamma \in \Omega_k, \\ \sqrt{d_k}\delta_{d_k,\ell}, & if \gamma \in V_k \setminus \Omega_k, \end{cases}$$

where $\ell = 1, \ldots, d_k$. Then the collection of functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,d_k}$$

forms a tight frame for $L^2(\widehat{G})$ with frame bound 1, and the functions $\Psi_k^{(m)}$ are compactly supported.

Proof. Recall that with H_{k+1} defined by (5.3), Lemma 5.2 determines the first row of the matrix $P_k(\gamma)$ for $\gamma \in V_k$. Then by considering $\gamma \in \Omega_k$, we obtain

$$P_k(\gamma) = \begin{pmatrix} \sqrt{d_k} I_{d_k} \\ \mathbf{0} \end{pmatrix}, \; \gamma \in \Omega_k,$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{C}^{d_k}$. Thus, $P_k(\gamma)^*P_k(\gamma) = d_k I_{d_k}$, $\gamma \in \Omega_k$. Next, by considering $\gamma \in V_k \setminus \Omega_k$, we have

$$P_k(\gamma) = \begin{pmatrix} \mathbf{0} \\ \sqrt{d_k} I_{d_k} \end{pmatrix}, \ \gamma \in V_k \setminus \Omega_k,$$

which implies that $P_k(\gamma)^*P_k(\gamma) = d_k I_{d_k}$, $\gamma \in V_k \setminus \Omega_k$. Altogether, this verifies (3.11). Thus, by Lemma 5.1 and Theorem 3.5, the collection of functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\dots,d_k}$$

forms a tight frame for $L^2(\widehat{G})$ with frame bound 1. Since

$$\Psi_k^{(m)}(\gamma) = G_{k+1}^{(m)}(\gamma)\Phi_{k+1}(\gamma) = \mu_{\widehat{G}}(V_{k+1})^{-1/2} G_{k+1}^{(m)}(\gamma)\chi_{\Omega_{k+1}}(\gamma),$$

we see that $\operatorname{supp} \Psi_k^{(m)} \subseteq \Omega_{k+1} \subset V_{k+1}$, i.e., $\Psi_k^{(m)}$ is compactly supported. □

Note that the construction in Proposition 5.3 forms only one way of extending the first row of the matrix $P_k(\gamma)$, $\gamma \in V_k$ in order to obtain (3.11). Indeed, observe that

$$\sum_{\ell=1}^{d_k} |H_{k+1}(\gamma + \nu_{k,\ell})|^2 = \begin{cases} d_k, & \text{if } \gamma \in \Omega_k, \\ 0, & \text{if } \gamma \in V_k \setminus \Omega_k, \end{cases}$$

i.e., $\sum_{\ell=1}^{d_k} |H_{k+1}(\gamma + \nu_{k,\ell})|^2 \leq d_k$, $\forall \gamma \in V_k$. Thus, for any $\rho_k \geq d_k$, by using the theory of Householder matrices one can construct explicitly $(\rho_k + 1) \times d_k$ matrices $P_k(\gamma)$ that satisfy (3.11). Details can be found in Proposition 4.1 of [11] which deals with the group $\mathbb{T}^s$, but its ideas can be readily adapted to the current setting of general LCA groups.

We will now consider the choice $\Omega_k := V_k$, which leads to an analogue on LCA groups of the Shannon orthonormal basis for $L^2(\mathbb{R})$.

Proposition 5.4 *Given a nested sequence of lattices $\{\Lambda_k\}_{k \in I}$ in G, let V_k be a fundamental domain associated with the lattice $\Lambda_k^\perp$ in $\widehat{G}$, and suppose that the sets V_k, $k \in I$, are nested. Assume that for every compact set S in $\widehat{G}$, there exists $K \in I$ such that $S \subseteq V_K$. For $k \in I$, let*

$$\Phi_k(\gamma) := \mu_{\widehat{G}}(V_k)^{-1/2} \chi_{V_k}(\gamma), \ \gamma \in \widehat{G}, \tag{5.5}$$

with the associated filter H_{k+1} as in (5.3) *(with $\Omega_k = V_k$). For $m = 1, \dots, d_k - 1$, define the function $G_{k+1}^{(m)}$ by*

$$G_{k+1}^{(m)}(\gamma + \nu_{k,\ell}) := \sqrt{d_k}\,\delta_{m+1,\ell}, \ \gamma \in V_k, \tag{5.6}$$

for $\ell = 1, \dots, d_k$. Then the collection of functions

$$\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \bigcup \{\mathcal{M}_\lambda \Psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\dots,d_k-1} \tag{5.7}$$

forms an orthonormal basis for $L^2(\widehat{G})$.

Proof. Using Lemma 5.2, we may simply require that $P_k(\gamma) := \sqrt{d_k} I_{d_k}$, $\gamma \in V_k$, which leads to the choice of $G_{k+1}^{(m)}$ in (5.6). By periodicity as in (5.4), the matrix P_k in (3.8) satisfies (3.11). Thus, by Lemma 5.1 and Theorem 3.5, the functions in (5.7) form a tight frame for $L^2(\widehat{G})$ with frame bound 1. Furthermore, by (5.5), $\|\Phi_{k_0}\| = 1$. Also, for $k \in I, m = 1, \ldots, d_k - 1$, $\Psi_k^{(m)}(\gamma) = G_{k+1}^{(m)}(\gamma)\Phi_{k+1}(\gamma) = \mu_{\widehat{G}}(V_{k+1})^{-1/2} G_{k+1}^{(m)}(\gamma)\chi_{V_{k+1}}(\gamma)$; thus

$$\|\Psi_k^{(m)}\|^2 = \mu_{\widehat{G}}(V_{k+1})^{-1} \int_{V_{k+1}} |G_{k+1}^{(m)}(\gamma)|^2 \, d\gamma.$$

As V_k and V'_{k+1} are fundamental domains associated with $\Lambda_k^{\perp}$ and $\Lambda_{k+1}^{\perp}$ respectively, we may express V_{k+1} as the disjoint union $V_{k+1} = \bigcup_{\omega \in \Lambda_{k+1}^{\perp}} \bigcup_{\ell=1}^{d_k} (\omega + \nu_{k,\ell} + V_k) \cap V_{k+1}$. Using this decomposition of V_{k+1}, it follows that

$$\|\Psi_k^{(m)}\|^2 = \mu_{\widehat{G}}(V_{k+1})^{-1} \sum_{\omega \in \Lambda_{k+1}^{\perp}} \sum_{\ell=1}^{d_k} \int_{(\omega+V_k)\cap(-\nu_{k,\ell}+V_{k+1})} |G_{k+1}^{(m)}(\gamma + \nu_{k,\ell})|^2 \, d\gamma,$$

and therefore by (5.6),

$$\begin{aligned} \|\Psi_k^{(m)}\|^2 &= \mu_{\widehat{G}}(V_{k+1})^{-1} d_k \sum_{\omega \in \Lambda_{k+1}^{\perp}} \mu_{\widehat{G}}((\omega + V_k) \cap (-\nu_{k,m+1} + V_{k+1})) \\ &= \mu_{\widehat{G}}(V_{k+1})^{-1} d_k \sum_{\omega \in \Lambda_{k+1}^{\perp}} \mu_{\widehat{G}}(V_k \cap (-\omega - \nu_{k,m+1} + V_{k+1})) \\ &= \mu_{\widehat{G}}(V_{k+1})^{-1} d_k \, \mu_{\widehat{G}}(V_k) = 1. \end{aligned}$$

Hence, the collection actually forms an orthonormal basis for $L^2(\widehat{G})$, as claimed. □

We will now apply the results in this section to obtain explicit constructions of frames for several concrete LCA groups. In particular we will be able to verify all the technical assumptions made in the section.

Example 5.5 Let $M \in \mathbb{N}$, and consider $G = \mathbb{Z}_{2^M} = \{0, \ldots, 2^M - 1\}$. Then $\widehat{G} = G = \mathbb{Z}_{2^M}$, and $L^2(G) = L^2(\widehat{G}) = \mathcal{S}(2^M)$, the space of complex sequences indexed by $\mathbb{Z}$ and with period 2^M. For $k = 0, \ldots, M$, consider the lattice $\Lambda_k := 2^{M-k}\mathbb{Z}_{2^k}$ in G; we note that $\Lambda_0 = \{0\}$ and $\Lambda_M = G$. Using that $2\mathbb{Z}_{2^k} = \{0, 2, \ldots, 2^{k+1} - 2\} \subset \{0, 1, \ldots, 2^{k+1} - 1\} = \mathbb{Z}_{2^{k+1}}$, $\Lambda_k = 2^{M-k-1} 2\mathbb{Z}_{2^k} \subset 2^{M-k-1}\mathbb{Z}_{2^{k+1}} = \Lambda_{k+1}$, i.e., (3.1) holds. It is well known that $\Lambda_k^{\perp} = 2^k \mathbb{Z}_{2^{M-k}}$, $k = 0, \ldots, M$. Direct calculation verifies that for $k = 0, \ldots, M - 1$, $\Lambda_k^{\perp} = \Lambda_{k+1}^{\perp} \cup (2^k + \Lambda_{k+1}^{\perp})$ and $\Lambda_{k+1}^{\perp} \cap (2^k + \Lambda_{k+1}^{\perp}) = \emptyset$, i.e., (3.7) holds with $\nu_{k,0} = 0, \nu_{k,1} = 2^k$. Also, $V_0 = \{0\}$ is a fundamental domain associated with the lattice $\Lambda_0^{\perp}$, and for $k = 0, \ldots, M-1$, the set $V_{k+1} := \mathbb{Z}_{2^{k+1}}$ is a fundamental domain for $\Lambda_{k+1}^{\perp}$.

In order to define appropriate sets $\Omega_k, k = 0, \ldots, M$, let $\{L_k\}_{k=0,\ldots,M}$ be an increasing sequence of nonnegative integers satisfying that

$$L_M = 2^M - 1, \quad L_k \leq 2^k - 1, \; k = 0, \ldots, M - 1.$$

Now, for $k = 0, \ldots, M$, let

$$\Omega_k := \{0, \ldots, L_k\}.$$

Since $L_k \leq 2^k - 1$ and $V_k = \mathbb{Z}_{2^k}$, it follows that $\Omega_k \subseteq V_k$, i.e., (5.1) holds. Also, since $\{L_k\}_{k=0,\ldots,M}$ is increasing, we clearly have $\Omega_k \subseteq \Omega_{k+1}$. By the choice of $L_M = 2^M - 1$, for any compact set S in $\widehat{G}$ we have $S \subseteq \Omega_M = \widehat{G}$. Thus, we have verified all the assumptions for application of the UEP. $\square$

Example 5.6 Let $G = \mathbb{T}$; then $\widehat{G} = \mathbb{Z}$. Let $\{M_k\}_{k\geq 0}$ denote a sequence of integers $M_k \geq 2$, and assume for convenience that M_0 is an even number. Define $N_k := \prod_{\ell=0}^{k} M_\ell$, $k \geq 0$, which are even integers greater than or equal to 2. For $k \geq 0$, consider the lattice $\Lambda_k := \frac{1}{N_k}\mathbb{Z}_{N_k}$ in G; then $\Lambda_k = \frac{1}{N_k M_{k+1}} M_{k+1}\mathbb{Z}_{N_k} \subset \frac{1}{N_k M_{k+1}} \mathbb{Z}_{N_k M_{k+1}} = \Lambda_{k+1}$, i.e., the lattices are nested. Also, for $k \geq 0$, $\Lambda_k^\perp = N_k\mathbb{Z}$, and direct verification yields that $\Lambda_k^\perp = \bigcup_{\ell=1}^{M_{k+1}} \left((\ell-1)N_k + \Lambda_{k+1}^\perp\right)$ and

$$\left((\ell-1)N_k + \Lambda_{k+1}^\perp\right) \cap \left((\ell'-1)N_k + \Lambda_{k+1}^\perp\right) = \emptyset,\ \ell \neq \ell',\ \ell, \ell' = 1, \ldots, M_{k+1},$$

where $(\ell-1)N_k \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$, $\ell = 2, \ldots, M_{k+1}$.

For $k \geq 0$, the set $V_k := \{-\frac{N_k}{2}, -\frac{N_k}{2}+1, \ldots, \frac{N_k}{2}-1\}$ is a fundamental domain associated with $\Lambda_k^\perp$. Now, let $\{L_k\}_{k\geq 0}$ be a (strictly) increasing sequence of nonnegative integers such that $L_k \leq \frac{N_k}{2} - 1$, $k \geq 0$, and define

$$\Omega_k := \{-L_k, \ldots, L_k\},\ k \geq 0.$$

Then $\Omega_k \subset V_k$, and also $\Omega_k \subset \Omega_{k+1}$. Finally, if $S \subset \widehat{G} = \mathbb{Z}$ is compact, i.e., $S \subset \{-R, \ldots, R\}$ for some $R > 0$, taking $K \geq 0$ such that $L_K > R$ implies that $S \subset \Omega_K$; thus, all the assumptions for application of the UEP are verified. $\square$

In our final example, we consider $s \times s$ diagonal scaling matrices and construct nonstationary wavelet frames for $L^2(\mathbb{R}^s)$ (separable as well as nonseparable).

Example 5.7 Let $G = \mathbb{R}^s$; then $\widehat{G} = \mathbb{R}^s$. For $r = 1, \ldots, s$, let $\{M_{k,r}\}_{k\geq 0}$ be a sequence of integers $M_{k,r} \geq 2$. For $k \geq 0$, let $N_{k,r} := \prod_{\ell=0}^{k} M_{\ell,r}$, and consider the $s \times s$ matrix $A_k := \mathrm{diag}\,(N_{k,1}, \ldots, N_{k,s})$. We will now consider the lattice $\Lambda_k := A_k^\sharp \mathbb{Z}^s = \frac{1}{N_{k,1}}\mathbb{Z} \times \cdots \times \frac{1}{N_{k,s}}\mathbb{Z}$; introducing the short notation $\prod_{r=1}^{s} B_r := B_1 \times \cdots \times B_s$ for the cartesian product of s sets, the lattice can be written as

$$\Lambda_k = \prod_{r=1}^{s} \left(\frac{1}{N_{k,r}}\mathbb{Z}\right).$$

Now,

$$\Lambda_k = \prod_{r=1}^{s}\left(\frac{1}{N_{k,r}M_{k+1,r}} M_{k+1,r}\mathbb{Z}\right) = \prod_{r=1}^{s}\left(\frac{1}{N_{k+1,r}} M_{k+1,r}\mathbb{Z}\right) \subset \prod_{r=1}^{s}\left(\frac{1}{N_{k+1,r}}\mathbb{Z}\right) = \Lambda_{k+1},$$

i.e., the sets Λ_k, $k \geq 0$, are nested. Also, for $k \geq 0$, $\Lambda_k^\perp = \prod_{r=1}^s (N_{k,r}\,\mathbb{Z})$. Direct verification yields that $\Lambda_k^\perp = \bigcup_{\ell=1}^{d_k} \left(\nu_{k,\ell} + \Lambda_{k+1}^\perp\right)$, where $d_k = \prod_{r=1}^s M_{k+1,r}$, and

$$\{\nu_{k,\ell}\}_{\ell=1}^{d_k} = \prod_{r=1}^{s} \left(N_{k,r}\{0, 1, \ldots, M_{k+1,r} - 1\}\right)$$

and $\nu_{k,\ell} \in \Lambda_k^\perp \setminus \Lambda_{k+1}^\perp$, $\ell = 2, \ldots, d_k$; finally,

$$\left(\nu_{k,\ell} + \Lambda_{k+1}^\perp\right) \cap \left(\nu_{k,\ell'} + \Lambda_{k+1}^\perp\right) = \emptyset,\ \ell \neq \ell',\ \ell, \ell' = 1, \ldots, d_k.$$

For $k \geq 0$, the set $V_k := \prod_{r=1}^s \left[-\frac{N_{k,r}}{2}, \frac{N_{k,r}}{2}\right)$ is a fundamental domain associated with $\Lambda_k^\perp$.

We shall now identify sets Ω_k, $k \geq 0$, satisfying our technical conditions. For $r = 1, \ldots, s$, let $\{L_{k,r}\}_{k\geq 0}$ be an increasing sequence of nonnegative numbers (not necessarily integers) satisfying that $L_{k,r} \leq \frac{N_{k,r}}{2}$, $\forall k \geq 0$, and $\lim_{k\to\infty} L_{k,r} = \infty$. Then, for $k \geq 0$, we can take $\Omega_k := \prod_{r=1}^s [-L_{k,r}, L_{k,r})$; we leave the easy proof to the reader. This yields separable wavelet frames.

For the construction of nonseparable wavelet frames, let $\{L_k\}_{k\geq 0}$ be an increasing sequence of nonnegative numbers (not necessarily integers) satisfying that

$$L_k < \frac{1}{2}\min(N_{k,1}, \ldots N_{k,s}),\ \forall k \geq 0, \ \text{ and } \ \lim_{k\to\infty} L_k = \infty.$$

For $k \geq 0$, let

$$\Omega_k := \{\gamma \in \mathbb{R}^s \,\big|\, \|\gamma\|_2 \leq L_k\}.$$

For $\gamma = (\gamma_1, \ldots, \gamma_s) \in \Omega_k$, each coordinate γ_r satisfies that $|\gamma_r| \leq L_k < \frac{N_{k,r}}{2}$, i.e., $\gamma \in V_k$; thus $\Omega_k \subset V_k$, i.e., (5.1) holds. It is also clear that $\Omega_k \subseteq \Omega_{k+1}$ for all $k \geq 0$. Finally, let $S \subset \widehat{G} = \mathbb{R}^s$ be a compact set. Then, choosing $R > 0$ such that S is contained in the ball around 0 with radius R and choosing $K \geq 0$ such that $L_K > R$, we clearly have that $S \subset \Omega_K$; thus, again, all the technical assumptions for application of the UEP are satisfied. $\square$

Acknowledgment: Ole Christensen would like to thank the National University of Singapore for its warm hospitality during one month stays in December 2013 – January 2014, December 2014 – January 2015, and December 2015 – January 2016.

References

[1] S. Albeveiro, S. Evdokimov, M. Skopina, p-adic multiresolution analysis and wavelet frames, J. Fourier Anal. Appl. 16 (2010) 693–714.

[2] J. Benedetto, R.L. Benedetto, A wavelet theory for local fields and related groups, J. Geom. Anal. 14 (2004) 423–456.

[3] M. Bownik, K.A. Ross, The structure of translation-invariant spaces on locally compact abelian groups, J. Fourier Anal. Appl. 21 (2015) 849–884.

[4] C. Cabrelli, V. Paternostro, Shift-invariant spaces on LCA groups, J. Funct. Anal. 258 (2010) 2034–2059.

[5] O. Christensen, An Introduction to Frames and Riesz Bases, 2nd expanded ed., Birkhäuser, 2016.

[6] O. Christensen, S.S. Goh, Fourier-like frames on locally compact abelian groups, J. Approx. Theory 192 (2015) 82–101.

[7] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.

[8] S. Dahlke, Multiresolution analysis and wavelets on locally compact abelian groups, in: P.-J. Laurent, A. Le Méhauté, L. Schumaker (Eds.), Wavelets, Images, and Surface Fittings, AK Peters, 1994, pp. 141–156.

[9] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46.

[10] H.G. Feichtinger, W. Kozek, Quantization of TF lattice-invariant operators on elementary LCA groups, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, 1998, pp. 233–266.

[11] S.S. Goh, K.M. Teo, Extension principles for tight wavelet frames of periodic functions, Appl. Comput. Harmon. Anal. 25 (2008) 168–186.

[12] K. Gröchenig, Aspects of Gabor analysis on locally compact abelian groups, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, 1998, pp. 211–231.

[13] K. Gröchenig, W.R. Madych, Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbb{R}^n$, IEEE Trans. Inform. Theory 38 (1992) 556–568.

[14] E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, vol. 1 and 2, Springer, 1963.

[15] M.S. Jakobsen, J. Lemvig, Reproducing formulas for generalized translation invariant systems on locally compact groups, Trans. Amer. Math. Soc. 368 (2016) 8447–8480.

[16] E. Kaniuth, G. Kutyniok, Zeroes of the Zak transform on locally compact abelian groups, Proc. Amer. Math. Soc. 126 (1998) 3561–3569.

[17] G. Kutyniok, D. Labate, Theory of reproducing systems on locally compact abelian group, Colloq. Math. 106 (2006) 197–220.

[18] H. Reiter, J.D. Stegeman, Classical Harmonic Analysis and Locally Compact Groups, 2nd ed., Oxford University Press, 2000.

[19] A. Ron, Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal. 148 (1997) 408–447.

[20] A. Ron, Z. Shen, Generalized shift-invariant systems, Constr. Approx. 22 (2005) 1–45.

[21] W. Rudin, Fourier Analysis on Groups. Interscience Publishers, 1962.

[22] G. Steidl, Wavelets over $\mathbb{R}, \mathbb{Z}, \mathbb{R}/N\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$, in: C.K. Chui, L. Montefusco, L. Puccio (Eds.), Wavelets: Theory, Algorithms and Applications, Academic Press, 1994, pp. 155–179.

[23] V.M. Tikhomirov, Harmonic tools for approximation and splines on locally compact abelian groups, Uspekhi Mat. Nauk 49 (1994), 193–194; translated in Russian Math. Surveys 49 (1994), 200–201.

[24] Y.G. Wang, X. Zhuang, Tight framelets and fast framelet transforms on manifolds, preprint, 2016.

Ole Christensen
Technical University of Denmark
DTU Compute
Building 303, 2800 Lyngby
Denmark
Email: ochr@dtu.dk

Say Song Goh
Department of Mathematics
National University of Singapore
10 Kent Ridge Crescent
Singapore 119260, Republic of Singapore
Email: matgohss@nus.edu.sg