FROM DUAL PAIRS OF GABOR FRAMES TO DUAL PAIRS OF WAVELET FRAMES AND VICE VERSA

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Abstract. We discuss an elementary procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames and vice versa. The construction preserves tightness of the involved frames. Starting with Gabor frames generated by characteristic functions the construction leads to a class of tight wavelet frames that include the Shannon (orthonormal) wavelet, and applying the construction to Gabor frames generated by certain exponential B-splines yields wavelet frames generated by functions whose Fourier transforms are compactly supported splines with geometrically distributed knot sequences. On the other hand, the pendant of the Meyer wavelet turns out to be a tight Gabor frame generated by a $C^\infty(\mathbb{R})$ function with compact support. As an application of our results we show that for each given pair of bandlimited dual wavelet frames it is possible to construct dual wavelet frames for any desired scaling and translation parameters.

1. Introduction

At the beginning of the "wavelet era", wavelet analysis and Gabor analysis were treated in a parallel way, where results and methods in one of the settings would often have an immediate impact on the other. This is illustrated by the work by Daubechies, Grossmann & Meyer [9], Daubechies [7], [8], and Heil & Walnut [16] that contain sections on wavelet analysis as well as Gabor analysis. Also, the Feichtinger & Gröchenig theory for atomic decomposition [10] lays a common foundation for wavelet expansions and Gabor analysis.

From 1990, wavelet analysis became focused on multiscale construction that has no pendant in Gabor analysis. Also the trends in Gabor analysis went new ways, e.g., into studies of special classes of operators (spreading operators, pseudodifferential operators, among others; see [12], [13] and the references therein), as well as studies of frame properties for Gabor windows versus dual windows [15]. Thus, practically since 1990, wavelet analysis and Gabor analysis have been two separate research areas, with little impact on each other.

The purpose of this paper is to reestablish the connection between the two topics, taking the subsequent developments into account. We will provide a procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames, and vice versa. We begin, in Section 2, with the introduction of a transform that will move the Gabor structure (to be defined below) into the wavelet structure. Based on this we show in

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Section 2.1 how to construct bandlimited wavelet frames (i.e., wavelet frames generated by functions with compactly supported Fourier transform) based on Gabor frames with compactly supported window functions. For example, an application to Gabor frames generated by the characteristic functions $\chi_{[N-1,N]}$, $N \in \mathbb{R}$, leads to a class of tight wavelet frames that includes Shannon's wavelet. Other explicit constructions based on Gabor frames generated by exponential B-splines are provided in Section 2.2; the outcome is a class of attractive dual wavelet frame pairs generated by functions whose Fourier transform are compactly supported splines with geometrically distributed knots. In Section 2.3 we introduce extra degrees of freedom that allow us to change the parameters of the constructed frames.

In Section 3 we consider the analogous problem of constructing Gabor frames based on wavelet frames. We first introduce the transformation that moves the wavelet structure into the Gabor structure. In its basic form, this transform appears in the fundamental paper [9] by Daubechies, Grossmann & Meyer (see Section 3 for a description). In Section 3.1 we present the results about how to construct Gabor frames with compactly supported windows based on bandlimited wavelet frames. The special case of tight frames is considered in Section 3.2. In Section 3.3 we introduce extra degrees of freedom that allow us to change the parameters of the constructed frames. As a result, we can then construct a Gabor frame based on the Meyer wavelet; it turns out to be a tight frame generated by a $C^\infty(\mathbb{R})$ function with compact support. In Section 3.4 we highlight an instance of gaining additional insight to bandlimited wavelet frames by applying our transforms between wavelet frames and Gabor frames. It shows that for such dual wavelet frames, there is an elementary way to construct other wavelet frames with desired (and arbitrary) dilation and translation parameters.

In the rest of this introduction we review some results from frame theory. Let $\mathcal{H}$ denote a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are *frame bounds.* The sequence $\{f_i\}_{i\in I}$ is a *Bessel sequence* if at least the upper bound in (1.1) is satisfied. A frame is *tight* if we can choose $A = B$ in (1.1). For any frame $\{f_i\}_{i\in I}$ there exists at least one *dual frame,* i.e., a frame $\{\widetilde{f}_i\}_{i\in I}$ for which

$$f = \sum_{i\in I} \langle f, f_i\rangle \widetilde{f}_i, \ \forall f \in \mathcal{H}. \tag{1.2}$$

We will consider Gabor frames and wavelet frames, both in the Hilbert space $L^2(\mathbb{R})$. A *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi i m b x} g(x - na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given function $g \in L^2(\mathbb{R})$. Using the *translation operators* $T_a f(x) := f(x - a), a \in \mathbb{R}$, and the *modulation operators* $E_b f(x) := e^{2\pi i b x} f(x), b \in \mathbb{R}$, both acting on $L^2(\mathbb{R})$, we will denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

A *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for some parameters $a>1, b>0$ and a given function $\psi\in L^2(\mathbb{R})$. Introducing the *scaling operators* $(D_af)(x):=a^{1/2}f(ax)$, $a>0$, acting on $L^2(\mathbb{R})$, the wavelet system can be written as $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$.

The key tools in our approach are equations characterizing dual frame pairs in the Gabor setting and the wavelet setting. For the readers' convenience we state both of them here.

The duality conditions for two Gabor systems were found by Ron & Shen [23], [24]. We use the formulation due to Janssen [17]:

Theorem 1.1. *Given $b,\alpha>0$, two Bessel sequences $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$, where $g,\tilde{g}\in L^2(\mathbb{R})$, form dual Gabor frames for $L^2(\mathbb{R})$ if and only if for all $n\in\mathbb{Z}$,*

$$\sum_{j\in\mathbb{Z}}\overline{g(x+j\alpha)}\tilde{g}(x+j\alpha+n/b) \;=\; b\delta_{n,0},\ \textit{a.e. } x\in\mathbb{R}. \tag{1.3}$$

Note that for $n=0$, the condition (1.3) amounts to

$$\sum_{j\in\mathbb{Z}}\overline{g(x+j\alpha)}\tilde{g}(x+j\alpha) \;=\; b,\ \text{a.e. } x\in\mathbb{R}. \tag{1.4}$$

If the functions g and $\tilde{g}$ are compactly supported, the conditions in (1.3) are automatically satisfied for $n\neq 0$ whenever $b>0$ is sufficiently small. Thus, in that case it is enough to check (1.4). For more general information on Gabor analysis, we refer to the monograph [14] by Gröchenig and the compiled volumes [12], [13].

We now state the characterizing equations for dual wavelet frames; see [5]. Here $\widehat{f}$ denotes the Fourier transform for $f\in L^1(\mathbb{R})$ defined by $\widehat{f}(\gamma):=\int_{-\infty}^{\infty}f(x)e^{-2\pi i\gamma x}dx$, and extended to $L^2(\mathbb{R})$ in the usual way.

Theorem 1.2. *Given $a>1$, $b>0$, two Bessel sequences $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$, where $\psi,\widetilde{\psi}\in L^2(\mathbb{R})$, form dual wavelet frames for $L^2(\mathbb{R})$ if and only if the following two conditions are satisfied:*

- (i) $\sum_{j\in\mathbb{Z}}\overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma)=b$ *for a.e.* $\gamma\in\mathbb{R}$.
- (ii) *For any number $\alpha\neq 0$ of the form $\alpha=m/a^j,\ m,j\in\mathbb{Z}$,*

$$\sum_{\{(j,m)\in\mathbb{Z}^2\,|\,\alpha=m/a^j\}}\overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma+m/b)=0,\ \textit{a.e. }\gamma\in\mathbb{R}. \tag{1.5}$$

The condition (1.5) is clearly satisfied if

$$\overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma+q/b)=0,\ \text{a.e. }\gamma\in\mathbb{R},\ \forall\, q\in\mathbb{Z}\setminus\{0\}.$$

Again, this shows that if ψ and $\widetilde{\psi}$ are functions for which the Fourier transform have compact support, then for sufficiently small values of $b>0$ the wavelet systems $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ when the condition (i) in Theorem 1.2 is satisfied.

2. From Gabor frames to wavelet frames

The goal of this section is to show how we can construct dual wavelet frame pairs based on certain dual Gabor frame pairs. The key is the following transform that allows us to move the Gabor structure into the wavelet structure.

Let $\theta > 1$ be given. Associated with a function $g \in L^2(\mathbb{R})$ with the property that $g(\log_\theta |\cdot|) \in L^2(\mathbb{R})$ we define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma) = \begin{cases} g(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \tag{2.1}$$

Note that this function could also be considered for $\theta \in (0,1)$; however, this turns out not to give any essential new information for the considered transform. After acceptance of the current paper, we were informed that Feichtinger, Holighaus & Wiesmeyr, independent of our work, also consider a form of the transform (2.1) in their forthcoming manuscript [11].

All results in this section are based on the following relationships between the functions g and ψ.

Proposition 2.1. *Let $g \in L^2(\mathbb{R})$ be a bounded function with support in the interval $[M,N]$ for some $M, N \in \mathbb{R}$. Let $\theta > 1$ be given, and define the function $\psi \in L^2(\mathbb{R})$ by (2.1). Then the following hold:*

(i) *For any $a > 0, j \in \mathbb{Z}$ and $\gamma \in \mathbb{R} \setminus \{0\}$,*

$$\widehat{\psi}(a^j\gamma) = g(j\log_\theta(a) + \log_\theta(|\gamma|)).$$

(ii) *If g is continuous, then $\widehat{\psi}$ is continuous.*

(iii) $\operatorname{supp}\widehat{\psi} \subseteq [-\theta^N, -\theta^M] \cup [\theta^M, \theta^N]$.

(iv) *In the time-domain, the function ψ is given by*

$$\psi(x) = 2\ln(\theta)\int_M^N g(t)\theta^t\cos(2\pi x\theta^t)dt, \ x \in \mathbb{R}.$$

(v) $||\psi||^2 = 2\ln(\theta)\displaystyle\int_M^N |g(t)|^2\theta^t\,dt.$

Proof. The result in (i) follows by direct computation. The result in (ii) is a consequence of g having compact support and the fact that $\log_a(|\gamma|) \to -\infty$ as $\gamma \to 0$. The result in (iii) follows from a direct computation. In order to prove (iv) we use the inverse Fourier transform. Since $\widehat{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the inversion theorem shows that

$$\psi(x) = \int_{-\infty}^{\infty}\widehat{\psi}(\gamma)e^{2\pi ix\gamma}\,d\gamma = \int_{-\theta^N}^{-\theta^M} g(\log_\theta(-\gamma))e^{2\pi ix\gamma}\,d\gamma + \int_{\theta^M}^{\theta^N} g(\log_\theta(\gamma))e^{2\pi ix\gamma}\,d\gamma.$$

Using the change of variable $t = \log_\theta(-\gamma)$ in the first integral and $t = \log_\theta(\gamma)$ in the second integral we arrive at

$$\begin{aligned}\psi(x) &= \int_N^M g(t)e^{-2\pi i x\theta^t}(-\theta^t \ln(\theta))\,dt + \int_M^N g(t)e^{2\pi i x\theta^t}\theta^t \ln(\theta)\,dt \\ &= \ln(\theta)\int_M^N g(t)\theta^t\left[e^{-2\pi i x\theta^t} + e^{2\pi i x\theta^t}\right]dt = 2\ln(\theta)\int_M^N g(t)\theta^t\cos(2\pi x\theta^t)dt,\end{aligned}$$

as desired. For the proof of (v),

$$||\psi||^2 = ||\widehat{\psi}||^2 = \int_{-\infty}^{\infty} |g(\log_\theta(|\gamma|))|^2\,dt.$$

Using the same calculation as in the proof of (iv) now leads to the result. □

Remark 2.2. Proposition 2.1 yields a convenient way of constructing functions ψ satisfying the partition of unity condition

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(a^j\gamma) = 1,\ \gamma \in \mathbb{R}. \tag{2.2}$$

In fact, just take any function g satisfying the partition of unity condition

$$\sum_{j\in\mathbb{Z}} g(x+j) = 1,\ x \in \mathbb{R}, \tag{2.3}$$

and apply the construction in (2.1) with the choice $\theta := a$. There is a rich theory for functions satisfying (2.3): it could be any B-spline, or any scaling function in the context of multiresolution analysis. The condition (2.2) plays a key role, e.g., in [19].

Remark 2.3. If g has compact support and is smooth, then the function $\widehat{\psi}$ in (2.1) is also smooth. Thus, by taking smooth functions g we obtain functions ψ with fast decay in the time domain.

2.1. **Construction of dual pairs of wavelet frames.** We will now show how the above correspondence between the functions g and ψ can be used to construct wavelet frames based on certain Gabor frames.

For fixed parameters $b, \alpha > 0$ we will consider two bounded compactly supported functions $g, \tilde{g} \in L^2(\mathbb{R})$ and the associated Gabor systems $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$. For a fixed $\theta > 1$, define the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma) = \begin{cases} g(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0, \end{cases} \qquad \widehat{\widetilde{\psi}}(\gamma) = \begin{cases} \tilde{g}(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \tag{2.4}$$

We will now construct dual pairs of wavelet frames $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ for certain $a > 1$. As we will see, there is an interplay between the parameters θ and a: if we have chosen a value of θ, the next result tells us which value we need to take for a, but we can also insist on a certain value for a and take θ accordingly.

Theorem 2.4. *Let $b > 0$, $\alpha > 0$, and $\theta > 1$ be given. Assume that $g, \tilde{g} \in L^2(\mathbb{R})$ are bounded functions with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$ and that $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. Then with $a := \theta^\alpha$ the following hold:*

(i) $\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma) = b$ *for a.e.* $\gamma \in \mathbb{R}$.

(ii) $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ *are Bessel sequences.*

(iii) *If* $b \le \frac{1}{2\theta^N}$, *then* $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ *are dual frames for* $L^2(\mathbb{R})$.

(iv) *If* $b \le \frac{1}{2\theta^N}$ *and* $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ *is a tight frame for* $L^2(\mathbb{R})$ *with frame bound* 1, *then* $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *is also a tight frame for* $L^2(\mathbb{R})$ *with frame bound* 1.

(v) *If* $N = \log_\theta 2 + M$ *and* $b = \frac{1}{\theta^N}$, *then* $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ *are dual frames for* $L^2(\mathbb{R})$.

Proof. We first observe that $a = \theta^\alpha \Leftrightarrow \alpha = \log_\theta(a)$. The result in (i) follows from this: in fact, using Proposition 2.1 and Theorem 1.1,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma) &= \sum_{j\in\mathbb{Z}} \overline{g(j\log_\theta(a) + \log_\theta(|\gamma|))}\tilde{g}(j\log_\theta(a) + \log_\theta(|\gamma|)) \\ &= \sum_{j\in\mathbb{Z}} \overline{g(\log_\theta(|\gamma|) + j\alpha)}\tilde{g}(\log_\theta(|\gamma|) + j\alpha) = b, \text{ a.e. } \gamma \in \mathbb{R}.\end{aligned}$$

In order to prove (ii) we note that Proposition 2.1 shows that the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ have support in $[-\theta^N, -\theta^M] \cup [\theta^M, \theta^N]$. Since $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are bounded functions this implies that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are Bessel sequences, see Theorem 11.2.3 in [1]. Finally, note that the support of $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ is contained in the interval $[-\theta^N, \theta^N]$ of length $2\theta^N$. Thus if $b \le \frac{1}{2\theta^N}$, then

$$\overline{\widehat{\psi}(a^j\gamma)}\widehat{\widetilde{\psi}}(a^j\gamma + q) = 0 \text{ for a.e. } \gamma \in \mathbb{R} \text{ if } q \in b^{-1}\mathbb{Z} \setminus \{0\}. \tag{2.5}$$

By the comment after Theorem 1.2 this implies that (iii) holds.

Part (iv) is an immediate consequence of (iii). In order to prove (v) we notice that under the condition $N = \log_\theta 2 + M$, the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are supported on

$$[-\theta^N, -\theta^M] \cup [\theta^M, \theta^N] = [-2\theta^M, -\theta^M] \cup [\theta^M, 2\theta^M].$$

The intersection of this set and its shifts by $k2\theta^M$, $k \in \mathbb{Z}\setminus\{0\}$, has measure 0. This implies that (2.5) holds if we take $b = (2\theta^M)^{-1} = \theta^{-N}$. As above, this concludes the proof of (v). $\square$

At a first glance Theorem 2.4(v) might look like a very particular case, but in fact it leads to a construction of the Shannon wavelet:

Example 2.5. Consider the function $g = \chi_{[N-1,N]}$ for some $N \in \mathbb{R}$. It is well known that $\{E_m T_n \chi_{[N-1,N]}\}_{m,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$. According to Proposition 2.1 the construction in (2.1) yields the function

$$\psi(x) = 2\ln(\theta) \int_{N-1}^{N} \theta^t \cos(2\pi x\theta^t) dt = \frac{\sin(2\pi x\theta^N) - \sin(2\pi x\theta^{N-1})}{\pi x}.$$

We will now consider the special situation of $\theta = 2$, where the above function becomes

$$\psi(x) = \frac{\sin(2\pi x 2^N) - \sin(2\pi x 2^{N-1})}{\pi x} = \frac{\sin(\pi x 2^N)}{\pi x}(2\cos(\pi x 2^N) - 1).$$

First we look at the cases $N \leq -1$ and $N = 0$. Since $\alpha = b = 1$, these two cases satisfy the conditions in (iv) and (v) of Theorem 2.4 respectively with $M = N - 1$ and $g = \tilde{g}$. Thus, for $N \leq -1$ or $N = 0$, the function ψ generates a tight wavelet frame $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$. When $N = 0$, we obtain the function

$$\psi(x) = \frac{\sin(\pi x)}{\pi x}(2\cos(\pi x) - 1),$$

which is the *Shannon wavelet*. Note that by Proposition 2.1(v), for any given $N \in \mathbb{R}$,

$$||\psi||^2 = 2\ln(2) \int_{M}^{N} |g(t)|^2 2^t \, dt = 2\ln(2) \int_{N-1}^{N} 2^t \, dt = 2^N.$$

Thus, only for $N = 0$, the construction yields an orthonormal basis $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$. As for the remaining cases of $-1 < N < 0$ and $N > 0$, the condition $b \leq \frac{1}{2\theta^N}$ or $b = \frac{1}{\theta^N}$ in Theorem 2.4 is not satisfied when $\theta = 2$ and $b = 1$. These cases can be handled by shifting the function g as in Corollary 2.6 below. □

As we have seen, our procedure of constructing the function ψ based on g immediately leads to a way to derive the condition (i) in Theorem 1.2 from (1.3) with $n = 0$. The remaining conditions in the characterizations of dual wavelet frames and dual Gabor frames, i.e., condition (ii) in Theorem 1.2 and (1.3) with $n \neq 0$, are not related by the transform. For this reason we need the relationship between the parameter b and the support size that is stated in, e.g., Theorem 2.4(iii), which ensures that the "trouble terms" vanish.

The condition on b in Theorem 2.4(iii)+(iv) can be avoided if we apply a shift on the frame generators:

Corollary 2.6. *Let $g, \tilde{g} \in L^2(\mathbb{R})$ be bounded functions with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$. Assume that $\{E_{mb} T_{n\alpha} g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb} T_{n\alpha} \tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ for some $\alpha, b > 0$. Given $\theta > 1$, let $a := \theta^\alpha$. For $\lambda \in \mathbb{R}$, define the functions ψ_λ and $\widetilde{\psi_\lambda}$ by*

$$\widehat{\psi_\lambda}(\gamma) = \begin{cases} g(\log_\theta(|\gamma|) - \lambda), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0, \end{cases} \qquad \widehat{\widetilde{\psi_\lambda}}(\gamma) = \begin{cases} \tilde{g}(\log_\theta(|\gamma|) - \lambda), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases} \tag{2.6}$$

Then the following hold:

(i) $||\psi_\lambda||^2 = 2\,\ln(\theta)\,\theta^\lambda \displaystyle\int_M^N |g(t)|^2\theta^t\,dt.$

(ii) *If* $\lambda \le -N - \log_\theta(2b)$, *then* ψ_λ *and* $\widetilde{\psi_\lambda}$ *generate dual wavelet frames* $\{D_{a^j}T_{kb}\psi_\lambda\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{kb}\widetilde{\psi_\lambda}\}_{j,k\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Proof. The proof of (i) is similar to the proof in Proposition 2.1, so we only sketch it. First,

$$\operatorname{supp}\widehat{\psi_\lambda} \subseteq [\theta^{M+\lambda}, \theta^{N+\lambda}] \cup [-\theta^{N+\lambda}, -\theta^{M+\lambda}].$$

Writing

$$\begin{aligned} ||\psi_\lambda||^2 = ||\widehat{\psi_\lambda}||^2 &= \int_{-\infty}^{\infty} |g(\log_\theta(|\gamma|) - \lambda)|^2\,d\gamma \\ &= \int_{\theta^{M+\lambda}}^{\theta^{N+\lambda}} |g(\log_\theta(\gamma) - \lambda)|^2\,d\gamma + \int_{-\theta^{N+\lambda}}^{-\theta^{M+\lambda}} |g(\log_\theta(-\gamma) - \lambda)|^2\,d\gamma, \end{aligned}$$

the changes of variable $t = \log_\theta(\gamma) - \lambda$, respectively $t = \log_\theta(-\gamma) - \lambda$, yield $d\gamma = \theta^{\lambda+t}\ln(\theta)\,dt$, respectively $d\gamma = -\theta^{\lambda+t}\ln(\theta)\,dt$. This explains the extra term θ^λ in (i) compared to Proposition 2.1(v).

We now prove (ii). The duality condition in Theorem 1.1 shows that if $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$, then for any $\lambda \in \mathbb{R}$, the Gabor systems $\{E_{mb}T_{n\alpha}(T_\lambda g)\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}(T_\lambda \tilde{g})\}_{m,n\in\mathbb{Z}}$ also form dual frames. Thus, under the assumptions in Theorem 2.4 we can replace the interval $[M, N]$ by any interval $[M+\lambda, N+\lambda]$, simply by replacing the generators $g, \tilde{g}$ by $T_\lambda g, T_\lambda \tilde{g}$. For any value of $b > 0$ the condition $b \le \frac{1}{2\theta^{N+\lambda}}$ is satisfied if we choose $\lambda \le -N - \log_\theta(2b)$. $\square$

Remark 2.7. The procedure provided in Theorem 2.4, or more generally Corollary 2.6, also generates a wavelet frame from a Gabor frame with the same frame bounds, regardless of any duality consideration. More precisely, let $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ be a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A, B, where $g \in L^2(\mathbb{R})$ is a bounded function with support in $[M, N]$. It is well known (see, e.g., [7], [1]) that

$$bA \le \sum_{j\in\mathbb{Z}} |g(x + j\alpha)|^2 \le bB, \text{ a.e. } x \in \mathbb{R}. \tag{2.7}$$

If $\psi_\lambda \in L^2(\mathbb{R})$ is defined by (2.6), then similar calculations as in the proofs of Theorem 2.4 and Corollary 2.6 show that

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_\lambda}(a^j\gamma)|^2 = \sum_{j\in\mathbb{Z}} |g(\log_\theta(|\gamma|) - \lambda + j\alpha)|^2, \text{ a.e. } \gamma \in \mathbb{R},$$

and

$$\overline{\widehat{\psi_\lambda}(a^j\gamma)}\widehat{\psi_\lambda}(a^j\gamma + q) = 0 \text{ for a.e. } \gamma \in \mathbb{R} \text{ if } q \in b^{-1}\mathbb{Z}\setminus\{0\}.$$

Applying Theorem 11.2.3 in [1], we conclude from (2.7) that $\{D_{a^j}T_{kb}\psi_\lambda\}_{j,k\in\mathbb{Z}}$ is a wavelet frame for $L^2(\mathbb{R})$ with frame bounds A, B.

2.2. **Explicit constructions.** The rich theory for construction of dual pairs of Gabor frames enables us to provide several explicit constructions of wavelet frame pairs based on Theorem 2.4.

Proposition 2.8. *Let $g \in L^2(\mathbb{R})$ be a bounded real-valued function with support in the interval $[M, N]$ for some $M, N \in \mathbb{Z}$. Suppose that g satisfies the partition of unity condition,*

$$\sum_{j\in\mathbb{Z}} g(x+j) = 1,\ x \in \mathbb{R}. \tag{2.8}$$

Let $a > 1$ and $b \in (0, \min(\frac{1}{2(N-M)-1}, 2^{-1}a^{-N})]$ be given, and take any sequence $\{c_n\}_{n=-N+M+1}^{N-M-1}$ such that

$$c_0 = b, c_n + c_{-n} = 2b,\ n = 1, \dots, N-M-1. \tag{2.9}$$

Then the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ defined (almost everywhere) by

$$\widehat{\psi}(\gamma) = g(\log_a(|\gamma|)),\ \gamma \neq 0, \tag{2.10}$$

and

$$\widehat{\widetilde{\psi}}(\gamma) = \sum_{n=-N+M+1}^{N-M-1} c_n g(\log_a(|\gamma|) + n) = \sum_{n=-N+M+1}^{N-M-1} c_n \widehat{\psi}(a^n\gamma),\ \gamma \neq 0, \tag{2.11}$$

generate dual wavelet frames $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. It follows from Theorem 3.1 in [2] that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and the corresponding Gabor system based on $\tilde{g}(x) = \sum_{n=-N+M+1}^{N-M-1} c_n g(x+n)$ form dual Gabor frames for $L^2(\mathbb{R})$ (the extra condition $b \leq \frac{1}{2(N-M)-1}$ comes from that result). Now the result follows from Theorem 2.4 with $\theta := a$. $\square$

Note that Proposition 2.8 could also be derived based on Theorem 2 in [19]. In fact, the crucial condition in that result is that $\sum_{j\in\mathbb{Z}} \widehat{\psi}(a^j\gamma) = 1$ for a.e. $\gamma \in \mathbb{R}$, which is an immediate consequence of (2.8) and the choice of ψ, see Remark 2.2.

Note also that using the second expression for $\widehat{\widetilde{\psi}}$ in (2.11) it follows that the function $\widetilde{\psi}$ is given directly in terms of ψ in (2.10) by

$$\widetilde{\psi}(x) = \sum_{n=-N+M+1}^{N-M-1} c_n a^{-n}\psi(a^{-n}x),\ x \in \mathbb{R}.$$

Thus, an expression for $\widetilde{\psi}$ can be obtained via the formula in Proposition 2.1(iv).

The crucial condition (2.8) is satisfied for any B-spline and any scaling function. Certain exponential B-splines also satisfy the condition. We will now show that an application to exponential B-splines yields a class of attractive wavelet frames, for which the Fourier transform of the generators are compactly supported splines with geometrically distributed knots and desired smoothness.

Definition 2.9. *Consider a finite sequence of scalars $\beta_1, \beta_2, \ldots, \beta_N \in \mathbb{R}$, $N \geq 1$. An exponential B-spline is a function of the form*

$$\mathcal{E}_N := e^{\beta_1(\cdot)}\chi_{[0,1]}(\cdot) * e^{\beta_2(\cdot)}\chi_{[0,1]}(\cdot) * \cdots * e^{\beta_N(\cdot)}\chi_{[0,1]}(\cdot). \tag{2.12}$$

Similar to the classical B-splines given by the choice $\beta_k = 0, k = 1, \ldots, N$, the exponential B-spline $\mathcal{E}_N$ is $N-2$ times differentiable (for $N \geq 2$) and its support is $[0, N]$. For more general information about exponential B-splines we refer to the papers [3], [6], [18], [22], [25].

In Theorem 3.1 in [3] it is shown that the function $\sum_{k\in\mathbb{Z}} \mathcal{E}_N(x-k)$ is constant if and only if $\beta_k = 0$ for at least one value of $k \in \{1, \ldots, N\}$. For this reason we will now consider exponential B-splines of a special form.

Example 2.10. Consider an exponential B-spline of the form (2.12) with the assumption that for some $\beta > 0$,

$$\beta_k = (k-1)\beta,\ k = 1, \ldots, N. \tag{2.13}$$

We will use a formula stated in [3], saying that for $N \geq 2$,

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_N(x-k) = \frac{\prod_{m=1}^{N-1}\left(e^{\beta m}-1\right)}{\beta^{N-1}(N-1)!}, \tag{2.14}$$

and

$$\mathcal{E}_N(x) = \begin{cases} \frac{1}{\beta^{N-1}} \displaystyle\sum_{k=0}^{N-1} \frac{1}{\displaystyle\prod_{\substack{j=1\\ j\neq k+1}}^{N}(k+1-j)} e^{\beta k x}, & x \in [0,1], \\[2ex] \frac{(-1)^{\ell-1}}{\beta^{N-1}} \displaystyle\sum_{k=0}^{N-1} \left(\frac{\displaystyle\sum_{\substack{0\leq j_1<\cdots<j_{\ell-1}\leq N-1\\ j_1,\ldots,j_{\ell-1}\neq k}} e^{\beta j_1+\cdots+\beta j_{\ell-1}}}{\displaystyle\prod_{\substack{j=1\\ j\neq k+1}}^{N}(k+1-j)} \right) e^{\beta k(x-\ell+1)}, & x \in [\ell-1,\ell], \\ & \ell = 2, \ldots, N, \\[2ex] 0, & x \notin [0,N]. \end{cases}$$

Note that there is a typo in the expression for $\mathcal{E}_N(x)$ for $x \in [k-1,k]$ on page 304 of [3]: the expression $e^{a_{j_1}} + \cdots + e^{a_{j_{k-1}}}$ should be $e^{a_{j_1}+\cdots+a_{j_{k-1}}}$.

In order for the partition of unity constraint (2.8) to hold, we will now apply (2.14) and consider the function

$$g(x) := \frac{\beta^{N-1}(N-1)!}{\prod_{m=1}^{N-1}\left(e^{\beta m}-1\right)}\ \mathcal{E}_N(x).$$

Furthermore, let $a := e^{\beta}$. For $\gamma \neq 0$, using (2.10) and that $e^{\beta k \log_{e^\beta}(|\gamma|)} = |\gamma|^k$ we arrive at

$$\begin{aligned}
\widehat{\psi}(\gamma) &= \frac{\beta^{N-1}(N-1)!}{\prod_{m=1}^{N-1}\left(e^{\beta m}-1\right)}\ \mathcal{E}_N(\log_a(|\gamma|)) \\
&= \begin{cases}
\frac{(N-1)!}{\prod_{m=1}^{N-1}(a^m-1)} \sum_{k=0}^{N-1} \frac{1}{\prod_{\substack{j=1\\ j\neq k+1}}^{N}(k+1-j)}\, |\gamma|^k, & |\gamma| \in [1,a], \\
\frac{(-1)^{\ell-1}(N-1)!}{\prod_{m=1}^{N-1}(a^m-1)} \sum_{k=0}^{N-1} \left(\frac{\sum_{\substack{0\le j_1<\cdots<j_{\ell-1}\le N-1\\ j_1,\dots,j_{\ell-1}\neq k}} a^{j_1+\cdots+j_{\ell-1}}}{\prod_{\substack{j=1\\ j\neq k+1}}^{N}(k+1-j)} \right) a^{k(-\ell+1)}\, |\gamma|^k, & |\gamma| \in [a^{\ell-1}, a^{\ell}],\ \ell = 2,\dots,N, \\
0, & |\gamma| \notin [1, a^N].
\end{cases}
\end{aligned}$$

This expression identifies $\widehat{\psi}$ explicitly as a geometric spline, i.e., as a spline with geometrically distributed knots. Now the formula (2.11) yields a dual wavelet frame generator $\widetilde{\psi}$, provided that the scalars c_n are chosen such that (2.9) is satisfied. Note that also $\widehat{\widetilde{\psi}}$ is a geometric spline.

As observed in Remark 2.7, the constructed wavelet frame $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ has the same bounds as the given Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$. Since the dual generator $\tilde{g}$ is a linear combination of shifts of g, the frame bounds for $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ (which are also the frame bounds for $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$) can be estimated by standard methods. □

Let us consider a concrete version of Example 2.10.

Example 2.11. Consider the exponential B-spline $\mathcal{E}_2$ in (2.12) with the choice $\beta_1 = 0, \beta_2 = 1$, i.e., $N = 2$ and $\beta = 1$ in (2.13). Then

$$\mathcal{E}_2(x) = \begin{cases} e^x - 1, & \text{if } x \in [0,1], \\ e - e^{-1}e^x, & \text{if } x \in [1,2], \\ 0, & \text{if } x \notin [0,2]. \end{cases}$$

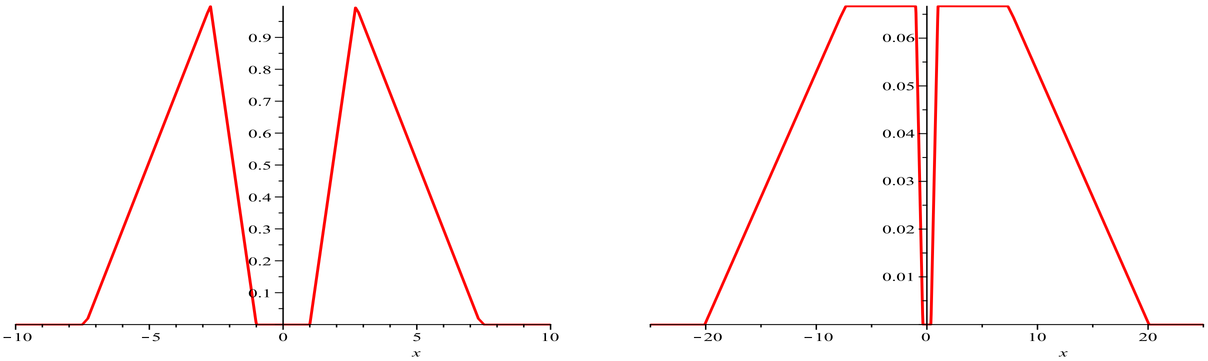

FIGURE 1. Plots of the geometric splines $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ in Example 2.11.

By (2.14) we have

$$\sum_{k\in\mathbb{Z}} \mathcal{E}_2(x-k) = e-1, \ x \in \mathbb{R},$$

so we consider the function $g(x) := (e-1)^{-1}\mathcal{E}_2(x)$.

Let $a := e^{\beta} = e$, and define the function ψ by

$$\widehat{\psi}(\gamma) = \begin{cases} \frac{|\gamma|-1}{e-1}, & \text{if } |\gamma| \in [1,e], \\ \frac{e-e^{-1}|\gamma|}{e-1}, & \text{if } |\gamma| \in [e,e^2], \\ 0, & \text{if } |\gamma| \notin [1,e^2]. \end{cases}$$

The function $\widehat{\psi}$ is a geometric spline with knots at the points $\pm 1, \pm e, \pm e^2$.

The construction in Proposition 2.8 works for $b \leq 2^{-1}e^{-2}$. Taking $b = 15^{-1}$ yields a dual frame generator $\widetilde{\psi}$ given by

$$\widehat{\widetilde{\psi}}(\gamma) = \frac{1}{15}\sum_{n=-1}^{1} \widehat{\psi}(e^n\gamma), \ \gamma \in \mathbb{R}.$$

The function $\widehat{\widetilde{\psi}}$ is a geometric spline with knots at the points $\pm e^{-1}, \pm 1, \pm e^2, \pm e^3$.

Figure 1 shows the graphs of the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$. Note that $\widehat{\widetilde{\psi}}$ is constant on the support of $\widehat{\psi}$ and decays to zero outside this set. Due to (2.14) and the special structure of $\widehat{\widetilde{\psi}}$ in (2.11), the same will occur when the construction is applied to higher order exponential B-splines or any other function whose integer-translates form a partition of unity. One advantage of the construction for higher order splines is that $\widehat{\widetilde{\psi}}$ will have higher order derivatives. $\square$

2.3. **Changing the parameters.** As in Section 2.1, for fixed parameters $b, \alpha > 0$ we will consider two bounded compactly supported functions $g, \tilde{g} \in L^2(\mathbb{R})$ and the associated Gabor systems $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$. Define the functions $\psi, \widetilde{\psi}$ by (2.4). Our aim is to construct dual pairs of wavelet frames with scaling parameter $a > 1$ and translation parameter $\beta > 0$. This yields more flexibility than Theorem 2.4, where the translation parameter in the constructed wavelet frames equals the modulation parameter of the given Gabor frames.

Theorem 2.12. *Let $b > 0$, $\alpha > 0$, and $\theta > 1$ be given. Assume that $g, \tilde{g} \in L^2(\mathbb{R})$ are bounded functions with support in the interval $[M, N]$ for some $M, N \in \mathbb{R}$ and that $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. Given $\beta > 0$, let $\varphi_\beta, \widetilde{\varphi_\beta} \in L^2(\mathbb{R})$ be defined by*

$$\widehat{\varphi_\beta}(\gamma) = \begin{cases} \sqrt{\frac{\beta}{b}}\, g(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0, \end{cases} \qquad \widehat{\widetilde{\varphi_\beta}}(\gamma) = \begin{cases} \sqrt{\frac{\beta}{b}}\, \tilde{g}(\log_\theta(|\gamma|)), & \text{if } \gamma \neq 0, \\ 0, & \text{if } \gamma = 0. \end{cases}$$

Then with $a := \theta^\alpha$ the following hold:

(i) $\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi_\beta}(a^j\gamma)}\widehat{\widetilde{\varphi_\beta}}(a^j\gamma) = \beta$ *for a.e.* $\gamma \in \mathbb{R}$.

(ii) $\{D_{a^j}T_{k\beta}\varphi_\beta\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{k\beta}\widetilde{\varphi_\beta}\}_{j,k\in\mathbb{Z}}$ *are Bessel sequences.*

(iii) *If* $\beta \leq \frac{1}{2\theta^N}$, *then* $\{D_{a^j}T_{k\beta}\varphi_\beta\}_{j,k\in\mathbb{Z}}$ *and* $\{D_{a^j}T_{k\beta}\widetilde{\varphi_\beta}\}_{j,k\in\mathbb{Z}}$ *are dual frames for* $L^2(\mathbb{R})$.

Proof. Parts (i) and (ii) follow immediately from Theorem 2.4. For (iii), the assumption $\beta \leq \frac{1}{2\theta^N}$ implies that for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\overline{\widehat{\varphi_\beta}(a^j\gamma)}\widehat{\widetilde{\varphi_\beta}}(a^j\gamma + k/\beta) = 0, \text{ a.e. } \gamma \in \mathbb{R}.$$

Thus the result follows from Theorem 1.2. □

Remark 2.13. Comparing to the use of Theorem 2.4(iv) in Example 2.5, it is natural to ask whether Theorem 2.12(iii), with the additional flexibility provided by the parameter β, can be used to construct a wavelet orthonormal basis $\{D_{a^j}T_{k\beta}\varphi_\beta\}_{j,k\in\mathbb{Z}}$ from a Gabor orthonormal basis $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$. The answer still turns out to be no. In fact, for this to happen, we would need that $||\varphi_\beta|| = 1$, which by Proposition 2.1 means that

$$1 = \frac{\beta}{b}\, 2\ln(\theta) \int_M^N |g(t)|^2 \theta^t \, dt. \tag{2.15}$$

Since $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ is a tight frame with frame bound 1, it follows from (2.7) that $\sum_{j\in\mathbb{Z}} |g(t + j\alpha)|^2 = b$, a.e. $t \in \mathbb{R}$, so $|g(t)|^2 \leq b$, a.e. $t \in \mathbb{R}$. Thus,

$$\frac{\beta}{b}\, 2\ln(\theta) \int_M^N |g(t)|^2 \theta^t \, dt \leq \frac{\beta}{b}\, 2\ln(\theta) \int_M^N b\theta^t \, dt = 2\beta(\theta^N - \theta^M) < 2\beta\theta^N.$$

Therefore (2.15) implies that $\beta > \frac{1}{2\theta^N}$, which contradicts the assumption in Theorem 2.12(iii).

We note that the condition on β in Theorem 2.12(iii) can be removed via a translation of the generators $g, \tilde{g}$, similar to the procedure in Corollary 2.6. However, even in that case the arguments of Remark 2.13 will show that the construction cannot yield an orthonormal basis.

3. From wavelet frames to Gabor frames

We will now discuss how to obtain Gabor frames based on certain wavelet frames. Let us motivate the approach via the work we did as we constructed wavelet frames via Gabor frames. In order to gain maximal flexibility we will use the setup in Corollary 2.6. That is, given a function g that generates a Gabor frame and ignoring $\gamma = 0$, we consider the functions ψ_λ, $\lambda \in \mathbb{R}$, given by

$$\widehat{\psi_\lambda}(\gamma) = g(\log_\theta(|\gamma|) - \lambda) = g(\log_\theta(|\gamma|) - \log_\theta(\theta^\lambda)) = g(\log_\theta(\frac{|\gamma|}{\theta^\lambda})). \tag{3.1}$$

Letting $x := \log_\theta(\frac{|\gamma|}{\theta^\lambda})$, we have $\theta^x = \frac{|\gamma|}{\theta^\lambda}$, i.e., $|\gamma| = \theta^x \theta^\lambda = \theta^{x+\lambda}$.

To construct a Gabor frame with window g based on a wavelet frame $\{D_{a^j} T_{kb} \psi\}_{j,k\in\mathbb{Z}}$, the first step is to define the candidates for the function g. The above considerations and replacing λ by c, suggest that we look at the classes of functions g_c, $c \in \mathbb{R}$, given by

$$g_c(x) = \widehat{\psi}(\theta^{x+c}) \quad \text{ or } \quad g_c(x) = \widehat{\psi}(-\theta^{x+c}). \tag{3.2}$$

Note that these choices of g_c discard the negative, respectively, the positive frequencies completely. The theory for these two choices can be developed in a parallel way; we will focus on the positive choice in (3.2) in the sequel.

With these definitions the two ways (from Gabor to wavelet and from wavelet to Gabor) are almost inverse of each other:

Example 3.1. If we start with a sufficiently well behaving function ϕ and use the first approach in (3.2), we obtain the functions $g_c(x) = \widehat{\phi}(\theta^{x+c})$, $c \in \mathbb{R}$. Now, using our method for constructing functions ψ_λ, see (3.1), on the functions g_c, we arrive at

$$\widehat{\psi_\lambda}(\gamma) = g_c(\log_\theta(|\gamma|) - \lambda) = \widehat{\phi}(\theta^{\log_\theta(|\gamma|) - \lambda + c}) = \widehat{\phi}(|\gamma|\, \theta^{-\lambda+c}).$$

Thus, if we take $\lambda = c$ and the function $\widehat{\phi}$ is symmetric, we have that $\psi_\lambda = \phi$.

On the other hand, starting with a function g and using (3.1), we get the functions ϕ_λ, $\lambda \in \mathbb{R}$, given by $\widehat{\phi_\lambda}(\gamma) = g(\log_\theta(|\gamma|) - \lambda)$, $\gamma \neq 0$; applying the approach in (3.2) on ϕ_λ leads to

$$g_c(x) = \widehat{\phi_\lambda}(\theta^{x+c}) = g(\log_\theta(|\theta^{x+c}|) - \lambda) = g(x + c - \lambda).$$

Thus, we get the original function back if we choose $\lambda = c$. $\square$

Note that the first transform in (3.2) appears in [9] for the special case $\theta = e, c = 0$. Compared with [9], the freedom in the choice of the parameters θ and c leads to more flexibility in the parameters associated with the constructed Gabor frames (see Theorem 3.3). Also, our results deal with the setting of dual frame pairs, while [9] uses the frame inequality as starting point to construct frames and obtains tight frames as a particular case.

Lemma 3.2. *Let $\theta > 1$ and $c \in \mathbb{R}$ be given. Assume that $\psi \in L^2(\mathbb{R})$ and that $\widehat{\psi}$ is bounded and has compact support not containing* 0*.*

(i) *If g_c is defined by $g_c(x) = \widehat{\psi}(\theta^{x+c})$, then*

$$||g_c||^2 = \frac{1}{\ln\theta} \int_0^\infty \frac{|\widehat{\psi}(\gamma)|^2}{\gamma}\, d\gamma.$$

(ii) *If g_c is defined by $g_c(x) = \widehat{\psi}(-\theta^{x+c})$, then*

$$||g_c||^2 = \frac{-1}{\ln\theta} \int_{-\infty}^0 \frac{|\widehat{\psi}(\gamma)|^2}{\gamma}\, d\gamma.$$

In both cases, the proof follows from the change of variable $\gamma = \pm\theta^{x+c}$, which yields $d\gamma = \pm\theta^{x+c}\ln\theta\, dx$.

3.1. **Construction of Gabor frames.** In this section we assume the functions $\psi, \widetilde{\psi} \in L^2(\mathbb{R})$ to be given. For a parameter $\theta > 1$ (to be fixed later) and $c \in \mathbb{R}$ we define the functions $g_c, \widetilde{g_c}$ by

$$g_c(x) := \widehat{\psi}(\theta^{x+c}), \quad \widetilde{g_c}(x) := \widehat{\widetilde{\psi}}(\theta^{x+c}),\ x \in \mathbb{R}. \tag{3.3}$$

The conditions below imply that $g_c, \widetilde{g_c} \in L^2(\mathbb{R})$.

Theorem 3.3. *Let $a > 1$ and $b > 0$. Assume that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$ and that the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are supported in $[-L,-K] \cup [K,L]$ for some $K, L > 0$. Take $\theta > 1$ and $\alpha > 0$ such that $a = \theta^\alpha$, and let $c \in \mathbb{R}$. Then the following hold:*

(i) *The functions g_c and $\widetilde{g_c}$ are compactly supported and bounded, and $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\widetilde{g_c}\}_{m,n\in\mathbb{Z}}$ are Bessel sequences.*

(ii) $\sum_{j\in\mathbb{Z}} \overline{g_c(x+j\alpha)}\widetilde{g_c}(x+j\alpha) = b$, *a.e.* $x \in \mathbb{R}$.

(iii) *If $b \le \frac{1}{\log_\theta(L/K)}$, then $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\widetilde{g_c}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$.*

Proof. To prove (i) we note that the frame condition on $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ implies that $\widehat{\psi}$ is bounded (see, e.g., [4]). Together with the fact that g_c is compactly supported (it is supported on $[\log_\theta(K) - c, \log_\theta(L) - c]$) it follows that $g_c \in L^2(\mathbb{R})$ and that $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence, see Theorem 8.4.4 in [1]. The proof for $\widetilde{g_c}$ is the same. The result in (ii) follows directly from the calculation

$$\sum_{j\in\mathbb{Z}} \overline{g_c(x+j\alpha)}\widetilde{g_c}(x+j\alpha) = \sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(\theta^{x+j\alpha+c})}\widehat{\widetilde{\psi}}(\theta^{x+j\alpha+c}) = \sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}(\theta^{x+c}a^j)}\widehat{\widetilde{\psi}}(\theta^{x+c}a^j) = b$$

and Theorem 1.2. The length of the support of g_c and $\widetilde{g_c}$ is at most $\log_\theta(L/K)$. Thus, the duality conditions in Theorem 1.1 are satisfied if $1/b \ge \log_\theta(L/K)$, i.e., if $b \le \frac{1}{\log_\theta(L/K)}$. $\square$

Remark 3.4. Similar to Remark 2.7, the setup in Theorem 3.3 is also applicable to constructing Gabor frames from wavelet frames, with the frame bounds preserved. Indeed, let $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ be a wavelet frame for $L^2(\mathbb{R})$ with frame bounds A, B, and assume that $\widehat{\psi}$ is supported in $[-L,-K] \cup [K,L]$. It is well known (see, e.g., [8]) that

$$bA \le \sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\gamma)|^2 \le bB, \text{ a.e. } \gamma \in \mathbb{R}. \tag{3.4}$$

If $g_c \in L^2(\mathbb{R})$ is defined by (3.3), then the derivations in the proof of Theorem 3.3 give

$$\sum_{j\in\mathbb{Z}} |g_c(x+j\alpha)|^2 = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(\theta^{x+c}a^j)|^2, \text{ a.e. } x \in \mathbb{R},$$

and

$$\overline{g_c(x+j\alpha)}g_c(x+j\alpha+n/b) = 0 \text{ for a.e. } x\in\mathbb{R} \text{ if } n \in \mathbb{Z}\setminus\{0\}.$$

Using (3.4) and Theorem 8.4.4 in [1], it follows that $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A, B.

Note that the setup in Theorem 3.3 forces a certain relationship among the parameters a, L, K to hold:

Lemma 3.5. *Given $a > 1, b > 0$ and assume that the function $\widehat{\psi} \in L^2(\mathbb{R})$ has support in a set of the form $[-L,-K] \cup [K,L]$, $K, L > 0$, and that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then*

$$a \le \frac{L}{K}.$$

Proof. If A, B are frame bounds for $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$, then (3.4) holds. For $j \in \mathbb{Z}$, the function $\widehat{\psi}(a^j\cdot)$ has support in the set $[-L(a^{-1})^j, -K(a^{-1})^j] \cup [K(a^{-1})^j, L(a^{-1})^j]$. By (3.4) these sets have to cover $\mathbb{R}$ modulo a set of measure zero. Hence, $L/K \ge a$. □

3.2. **Redundancy and the tight case.** It is interesting to know whether the procedure of constructing a Gabor system based on a wavelet system changes the redundancy. More specifically, if $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis (without any redundant element), will the constructed Gabor system $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ still be a basis, or an overcomplete frame?

It turns out that the construction preserves tightness, but redundancy only remains unchanged in very special cases.

Corollary 3.6. *Let $a > 1$ and $b > 0$. Assume that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1 and that the function $\widehat{\psi}$ is supported in $[-L,-K] \cup [K,L]$ for some $K, L > 0$. Take $\theta > 1$ and $\alpha > 0$ such that $a = \theta^\alpha$, and let $c \in \mathbb{R}$. Assume that*

$$b \le \frac{1}{\log_\theta(L/K)}. \tag{3.5}$$

Then the following hold:

(i) *$\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1.*

(ii) *If $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, the parameters a, L, K satisfy the relation*

$$a = \frac{L}{K}.$$

Proof. Part (i) is an immediate consequence of Theorem 3.3. In order to show (ii), note that if $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, then $\alpha b = 1$ (see, e.g., [1]). Therefore $\theta = a^{1/\alpha} = a^b$, and the condition $b \le \frac{1}{\log_\theta(L/K)}$ means that

$$b \le \frac{\ln\theta}{\ln(L/K)} = b\,\frac{\ln a}{\ln(L/K)}.$$

Thus, $\frac{\ln a}{\ln(L/K)} \ge 1$, which gives the conclusion via Lemma 3.5. □

For certain values of α, e.g., $\alpha = 1$, the conditions in Corollary 3.6 are satisfied for the Shannon wavelet. In this case $b = 1, a = 2, L/K = 2$. The condition (3.5) is satisfied if $\theta \ge 2$, i.e., for $\alpha \le 1$. Thus, the tight frame $\{E_{mb}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ in (ii) is only an orthonormal basis in the special case $\alpha = 1$.

Due to the restriction on the parameter b, Corollary 3.6 cannot be applied to the Meyer wavelet for an arbitrarily given value of α. This is the motivation for the results in the next section, where we show that for *any* $\alpha > 0$, Gabor frames $\{E_{m\beta}T_{n\alpha}g_c\}_{m,n\in\mathbb{Z}}$ can be constructed for sufficiently small values of $\beta > 0$, without the condition (3.5) being satisfied.

3.3. **Changing the parameters and an application to Meyer's wavelet.** In this section we will follow the approach from Section 3.1, but gain extra flexibility in the choice of the translation parameter of the constructed Gabor frames. We will again base the approach on the functions in (3.3), but restrict to the case $c = 0$.

Theorem 3.7. *Let $a > 1$ and $b > 0$. Assume that $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$ and that the functions $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are supported in $[-L,-K] \cup [K,L]$ for some $K, L > 0$. Take $\theta > 1$ and $\alpha > 0$ such that $a = \theta^\alpha$. Furthermore, given any $\beta > 0$, let*

$$h_\beta(x) := \sqrt{\frac{\beta}{b}}\,\widehat{\psi}(\theta^x), \quad \widetilde{h_\beta}(x) := \sqrt{\frac{\beta}{b}}\,\widehat{\widetilde{\psi}}(\theta^x), \ x \in \mathbb{R}. \tag{3.6}$$

Then the following hold:

(i) *If $\beta \le \frac{1}{\log_\theta(L/K)}$, then $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ and $\{E_{m\beta}T_{n\alpha}\widetilde{h_\beta}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$.*

(ii) *If $\beta \le \frac{1}{\log_\theta(L/K)}$ and $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1, then $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ is also a tight frame for $L^2(\mathbb{R})$ with frame bound 1.*

(iii) *If $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1 and $a = L/K$, an orthonormal basis $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ can be obtained by taking $\beta = \alpha^{-1}$.*

Proof. Given any $\beta > 0$, Theorem 3.3(ii) implies that

$$\sum_{j\in\mathbb{Z}} \overline{h_\beta(x+j\alpha)}\widetilde{h_\beta}(x+j\alpha) = \beta, \text{ a.e. } x \in \mathbb{R}.$$

Furthermore, as the length of the support of h_β and $\widetilde{h_\beta}$ is at most $\log_\theta(L/K)$, the condition $\beta \le \frac{1}{\log_\theta(L/K)}$ implies that for all $n \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \overline{h_\beta(x+j\alpha)}\widetilde{h_\beta}(x+j\alpha+n/\beta) = 0, \text{ a.e. } x \in \mathbb{R}.$$

Parts (i) and (ii) now follow from Theorem 1.1. In order to prove (iii), we note that the construction in (ii) yields an orthonormal basis $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ if β is chosen such that $||h_\beta|| = 1$ and $\beta \le \frac{1}{\log_\theta(L/K)}$. By Lemma 3.2(i) with $g_0(x) = \widehat{\psi}(\theta^x)$,

$$||h_\beta||^2 = \frac{\beta}{b}||g_0||^2 = \frac{\beta}{b}\frac{1}{\ln\theta}\int_0^\infty \frac{|\widehat{\psi}(\gamma)|^2}{\gamma}\,d\gamma.$$

Now, by Theorem 3.3.1 in [8],

$$\int_0^\infty \frac{|\widehat{\psi}(\gamma)|^2}{\gamma}\,d\gamma = b\ln a,$$

so we obtain that

$$||h_\beta||^2 = \frac{\beta}{b}\frac{1}{\ln(a^{1/\alpha})}\,b\ln a = \alpha\beta. \tag{3.7}$$

Thus, in order to have $||h_\beta|| = 1$ we must take $\beta = \alpha^{-1}$, and so this leads to the requirement that $1/\alpha \le \frac{1}{\log_\theta(L/K)}$, or $1 \le \frac{\ln a}{\ln(L/K)}$. This condition is satisfied if $a \ge L/K$. However, by Lemma 3.5 we know that a wavelet frame with the properties in Theorem 3.7 exists only if $a \le L/K$. This concludes the proof. $\square$

In Theorem 3.7(iii) one may wonder whether $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ being a tight frame with frame bound 1 and satisfying the given constraints, i.e., the support condition on $\widehat{\psi}$ and $a = L/K$, actually forces $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ to be an orthonormal basis. In fact, it does not. On the other hand, $\widehat{\psi}$ is forced to have a certain form:

Lemma 3.8. *For $L > K > 0$, consider a function $\psi \in L^2(\mathbb{R})$ such that*

$$\operatorname{supp}\widehat{\psi} = [-L,-K] \,\cup\, [K,L],$$

and let $a = L/K$. Then the following hold:

(i) *Let $0 < b \le (2L)^{-1}$. Then $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 1 if and only if $\widehat{\psi}$ has the form*

$$\widehat{\psi}(\gamma) = \begin{cases} \sqrt{b}\,e^{2\pi i\mu(\gamma)}, & \text{if } |\gamma| \in [K,aK], \\ 0, & \text{if } |\gamma| \notin [K,aK], \end{cases} \tag{3.8}$$

(almost everywhere) for some measurable function $\mu : [K, aK] \to \mathbb{R}$.

(ii) *Assume that for some* $b > 0$, $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *is a tight frame for* $L^2(\mathbb{R})$ *with frame bound* 1. *Then* $\widehat{\psi}$ *must be of the form* (3.8). *Furthermore,* $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ *is an orthonormal basis if and only if* $2b(a-1)K = 1$.

Proof. For (i), since $\operatorname{supp}\widehat{\psi}$ lies inside $[-L, L]$ and $b \le (2L)^{-1}$, it follows from Theorem 1.2 that $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is a tight frame with frame bound 1 if and only if

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\gamma)|^2 = b, \text{ a.e. } \gamma \in \mathbb{R}. \tag{3.9}$$

Due to the condition $\operatorname{supp}\widehat{\psi} = [-aK, -K] \cup [K, aK]$ this is equivalent to

$$|\widehat{\psi}(\gamma)|^2 = \begin{cases} b, & \text{if } |\gamma| \in [K, aK], \\ 0, & \text{if } |\gamma| \notin [K, aK], \end{cases}$$

almost everywhere. This proves (i).

As for (ii), again applying Theorem 1.2, $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ being a tight frame with frame bound 1 for some $b > 0$ gives (3.9) and then (3.8). Thus $||\psi||^2 = ||\widehat{\psi}||^2 = 2b(a-1)K$, and (ii) follows from the observation that $\{D_{a^j} T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis if and only if $||\psi|| = 1$. □

We note that the tight frames characterized by Lemma 3.8(i) cannot form orthonormal bases. This is because for such tight frames, $b \le (2L)^{-1} = (2aK)^{-1}$ which cannot satisfy the condition $2b(a-1)K = 1$ required by Lemma 3.8(ii). On the other hand, the well-known Shannon wavelet satisfies the conditions in Lemma 3.8(ii) with $K = 1/2$, $a = 2$ and $b = 1$.

Now we apply Theorem 3.7(ii) to the Meyer wavelet to obtain $C^\infty(\mathbb{R})$ tight Gabor frames for any translation parameter $\alpha > 0$. Technically this is done by appropriate choices of the modulation parameter β.

Example 3.9. The *Meyer wavelet* is the function $\psi \in L^2(\mathbb{R})$ defined via

$$\widehat{\psi}(\gamma) = \begin{cases} e^{i\pi\gamma}\sin(\frac{\pi}{2}(\nu(3|\gamma|-1))), & \text{if } 1/3 \le |\gamma| \le 2/3, \\ e^{i\pi\gamma}\cos(\frac{\pi}{2}(\nu(3|\gamma|/2-1))), & \text{if } 2/3 \le |\gamma| \le 4/3, \\ 0, & \text{if } |\gamma| \notin [1/3, 4/3], \end{cases}$$

where $\nu : \mathbb{R} \to \mathbb{R}$ is any continuous function for which

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x \ge 1, \end{cases} \tag{3.10}$$

and

$$\nu(x) + \nu(1-x) = 1, \ x \in \mathbb{R}. \tag{3.11}$$

It is known (see, e.g., [21], [8]) that $\{D_{2^j}T_k\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, and clearly

$$\operatorname{supp}\widehat{\psi} = [-4/3,-1/3] \ \cup \ [1/3,4/3].$$

In this case, we have $K = 1/3$, $L = 4/3$, $a = 2$ and $b = 1$, so the condition in Theorem 3.3(iii) does not hold for $\alpha > 1/2$. However, we can apply Theorem 3.7(ii). For an arbitrarily fixed $\alpha > 0$, take any $\beta \le \frac{1}{\log_\theta(L/K)}$ where $\theta = 2^{1/\alpha}$. This means that

$$\beta \le \frac{1}{\alpha}\frac{\ln 2}{\ln(L/K)} = \frac{1}{\alpha}\frac{\ln 2}{\ln 4} = \frac{1}{2\alpha}. \tag{3.12}$$

Then the function h_β defined via (3.6) generates a tight Gabor frame $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ with frame bound 1. The tight frame $\{E_{m\beta}T_{n\alpha}h_\beta\}_{m,n\in\mathbb{Z}}$ is not an orthonormal basis because by (3.7) and (3.12), we have that $||h_\beta||^2 = \alpha\beta \le 1/2 < 1$. As a concrete example of h_β, take $\theta = 2$, $\alpha = 1$ and $\beta = 1/2$. Then

$$h_{1/2}(x) = \begin{cases} \frac{1}{\sqrt{2}}e^{i\pi 2^x}\sin(\frac{\pi}{2}(\nu(3\cdot 2^x - 1))), & \text{if } -\frac{\ln 3}{\ln 2} \le x \le 1 - \frac{\ln 3}{\ln 2}, \\ \frac{1}{\sqrt{2}}e^{i\pi 2^x}\cos(\frac{\pi}{2}(\nu(\frac{3}{2}\cdot 2^x - 1))), & \text{if } 1 - \frac{\ln 3}{\ln 2} \le x \le 2 - \frac{\ln 3}{\ln 2}, \\ 0, & \text{if } x \notin [-\frac{\ln 3}{\ln 2}, 2 - \frac{\ln 3}{\ln 2}]. \end{cases}$$

Note that by taking ν to be a $C^\infty(\mathbb{R})$ function, we hereby obtain a construction of a tight Gabor frame generated by a $C^\infty(\mathbb{R})$, compactly supported function. As an example of an appropriate $C^\infty(\mathbb{R})$ function ν, one can take (see [20, p.36])

$$\nu_0(x) = \begin{cases} \exp\left[-\left\{\exp\left[x/(1-x)\right] - 1\right\}^{-1}\right], & \text{if } 0 < x < 1, \\ 0, & \text{if } x \le 0, \\ 1, & \text{if } x \ge 1, \end{cases} \tag{3.13}$$

and then set

$$\nu(x) := \frac{1}{2}(\nu_0(x) - \nu_0(1-x) + 1), \ x \in \mathbb{R}. \tag{3.14}$$

It is easily verified that ν satisfies (3.10) and (3.11).

We also note that the exponential factors in the expression for $h_{1/2}$ are irrelevant for the Gabor frame property. Thus, our conclusion is that with ν defined by (3.14) and (3.13), the function

$$\tau(x) := \begin{cases} \frac{1}{\sqrt{2}}\sin(\frac{\pi}{2}(\nu(3\cdot 2^x - 1))), & \text{if } -\frac{\ln 3}{\ln 2} \le x \le 1 - \frac{\ln 3}{\ln 2}, \\ \frac{1}{\sqrt{2}}\cos(\frac{\pi}{2}(\nu(\frac{3}{2}\cdot 2^x - 1))), & \text{if } 1 - \frac{\ln 3}{\ln 2} \le x \le 2 - \frac{\ln 3}{\ln 2}, \\ 0, & \text{if } x \notin [-\frac{\ln 3}{\ln 2}, 2 - \frac{\ln 3}{\ln 2}], \end{cases} \tag{3.15}$$

is real-valued, has compact support, belongs to $C^\infty(\mathbb{R})$, and generates a tight Gabor frame $\{E_{m/2}T_n\tau\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ with frame bound 1. The function τ is plotted in Figure 2. □

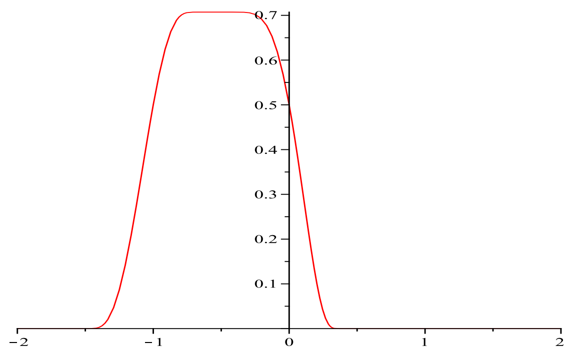

FIGURE 2. The function τ in (3.15).

3.4. **Varying the parameters of wavelet frames.** As an application of our results we now show that for each pair of bandlimited dual wavelet frames it is possible to construct dual wavelet frames with arbitrary scaling parameter $a > 1$ and arbitrary translation parameter $b > 0$.

First, let $\sigma > 1$ and $\rho > 0$. Suppose that $\{D_{\sigma^j} T_{k\rho}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{\sigma^j} T_{k\rho}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, the *continuous wavelet transform* of f with respect to ψ is defined by

$$(W_\psi f)(t,s) := \int_{-\infty}^{\infty} f(x) s^{-1/2} \overline{\psi\left(\frac{x-t}{s}\right)} dx,\ t \in \mathbb{R},\ s > 0.$$

Due to the relationship

$$\langle f, D_{\sigma^j} T_{k\rho}\psi\rangle = (W_\psi f)(k\rho\sigma^{-j}, \sigma^{-j}),\ j,k \in \mathbb{Z},$$

the dual frame condition (1.2) enables f to be reconstructed from sampled values of a continuous wavelet transform; see, e.g., [8]. As the choice of the wavelet parameters σ and ρ determines how the continuous wavelet transforms are sampled, flexibility in the choice of these parameters is desirable.

We remark that for a given wavelet frame, it is possible to construct a wavelet frame with any translation parameter $b > 0$ by a simple scaling. More precisely, suppose that $\{D_{\sigma^j} T_{k\rho}\psi\}_{j,k\in\mathbb{Z}}$ is a wavelet frame for $L^2(\mathbb{R})$ for some $\sigma > 1$ and $\rho > 0$. Then for any $b > 0$, taking $\varphi := D_{\rho b^{-1}}\psi$ leads to $\{D_{b\rho^{-1}\sigma^j} T_{kb}\varphi\}_{j,k\in\mathbb{Z}} = \{D_{\sigma^j} T_{k\rho}\psi\}_{j,k\in\mathbb{Z}}$. However, we see that the flexibility gained by this simple scaling with respect to the translation parameter forces a specific choice of the scaling operators in the resulting system.

We will now prove that for *any* desired wavelet parameters $a > 1$ and $b > 0$, there is an elementary procedure to modify given bandlimited wavelets ψ and $\widetilde{\psi}$ generating dual frames into φ and $\widetilde{\varphi}$ so that every $f \in L^2(\mathbb{R})$ can be recovered from the sampled values $(W_\varphi f)(kba^{-j}, a^{-j})$, $j,k \in \mathbb{Z}$, or $(W_{\widetilde{\varphi}} f)(kba^{-j}, a^{-j})$, $j,k \in \mathbb{Z}$. Our tools to prove this are Theorems 3.7 and 2.12.

Theorem 3.10. *Let $\sigma > 1$ and $\rho > 0$. Assume that $\{D_{\sigma^j}T_{k\rho}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{\sigma^j}T_{k\rho}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$ and that $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are supported in $[-L,-K] \cup [K,L]$ for some $K,L>0$. Given any $a>1$ and $b>0$, fix $r\geq 0$. Define $\varphi_b,\widetilde{\varphi}_b \in L^2(\mathbb{R})$ by*

$$\widehat{\varphi_b}(\gamma) = \sqrt{\frac{b}{\rho}}\widehat{\psi}\left(L\sigma^r(2b|\gamma|)^{\log_a(\sigma)}\right), \quad \widehat{\widetilde{\varphi}_b}(\gamma) = \sqrt{\frac{b}{\rho}}\widehat{\widetilde{\psi}}\left(L\sigma^r(2b|\gamma|)^{\log_a(\sigma)}\right), \ \gamma\in\mathbb{R}. \tag{3.16}$$

Then $\{D_{a^j}T_{kb}\varphi_b\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\varphi}_b\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$, and $\widehat{\varphi_b}$ and $\widehat{\widetilde{\varphi}_b}$ are symmetric and supported in $\big[-\frac{1}{2a^rb}, -\frac{(K/L)^{\log_\sigma(a)}}{2a^rb}\big] \cup \big[\frac{(K/L)^{\log_\sigma(a)}}{2a^rb}, \frac{1}{2a^rb}\big]$.

Proof. First we apply Theorem 3.7 to the dual wavelet frames $\{D_{\sigma^j}T_{k\rho}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{\sigma^j}T_{k\rho}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ by taking $\theta=\sigma$, $\alpha=1$ and $\beta=\frac{1}{\log_\theta(L/K)}$. Then the functions

$$h_\beta(x) = \sqrt{\frac{\beta}{\rho}}\widehat{\psi}(\sigma^x), \quad \widetilde{h_\beta}(x) = \sqrt{\frac{\beta}{\rho}}\widehat{\widetilde{\psi}}(\sigma^x), \ x\in\mathbb{R}, \tag{3.17}$$

generate dual frames $\{E_{m\beta}T_nh_\beta\}_{m,n\in\mathbb{Z}}$ and $\{E_{m\beta}T_n\widetilde{h_\beta}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, and they are bounded and supported in $[\log_\sigma(K),\log_\sigma(L)]$ (see the proof of Theorem 3.3).

For $r\geq 0$, consider

$$\lambda = -\log_\sigma(L) - \log_a(2b) - r. \tag{3.18}$$

Note that $\{E_{m\beta}T_nh_\beta(\cdot-\lambda)\}_{m,n\in\mathbb{Z}}$ and $\{E_{m\beta}T_n\widetilde{h_\beta}(\cdot-\lambda)\}_{m,n\in\mathbb{Z}}$ are also dual frames for $L^2(\mathbb{R})$, and $h_\beta(\cdot-\lambda)$ and $\widetilde{h_\beta}(\cdot-\lambda)$ are supported in $[\log_\sigma(K)+\lambda,\log_\sigma(L)+\lambda]$. Then applying Theorem 2.12 to this pair of Gabor frames with $g=h_\beta(\cdot-\lambda)$ and $\tilde{g}=\widetilde{h_\beta}(\cdot-\lambda)$, we conclude that the functions φ_b and $\widetilde{\varphi}_b$ defined by

$$\widehat{\varphi_b}(\gamma) = \begin{cases} \sqrt{\frac{b}{\beta}}\, h_\beta(\log_a(|\gamma|)-\lambda), & \text{if } \gamma\neq 0, \\ 0, & \text{if } \gamma = 0, \end{cases} \qquad \widehat{\widetilde{\varphi}_b}(\gamma) = \begin{cases} \sqrt{\frac{b}{\beta}}\, \widetilde{h_\beta}(\log_a(|\gamma|)-\lambda), & \text{if } \gamma\neq 0, \\ 0, & \text{if } \gamma = 0, \end{cases}$$

generate dual wavelet frames $\{D_{a^j}T_{kb}\varphi_b\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\varphi}_b\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. This is because the condition in Theorem 2.12(iii) amounts to $b\leq \frac{1}{2a^{\log_\sigma(L)+\lambda}}$, which is equivalent to $r\geq 0$ via a straightforward calculation employing (3.18).

By further calculations, it follows from (3.17) and (3.18) that the expressions for $\widehat{\varphi_b}$ and $\widehat{\widetilde{\varphi}_b}$ reduce to those in (3.16). Indeed, for $\gamma\neq 0$,

$$\sqrt{\frac{b}{\beta}}h_\beta(\log_a(|\gamma|)-\lambda) = \sqrt{\frac{b}{\beta}}\sqrt{\frac{\beta}{\rho}}\widehat{\psi}\left(\sigma^{\log_a(|\gamma|)+\log_\sigma(L)+\log_a(2b)+r}\right) = \sqrt{\frac{b}{\rho}}\widehat{\psi}\left(L\sigma^r(2b|\gamma|)^{\log_a(\sigma)}\right),$$

and similarly for the other expression. For the supports of $\widehat{\varphi_b}$ and $\widehat{\widetilde{\varphi}_b}$, observe that if $\widehat{\varphi_b}(\gamma)$ or $\widehat{\widetilde{\varphi}_b}(\gamma)$ is nonzero, then $K\leq L\sigma^r(2b|\gamma|)^{\log_a(\sigma)}\leq L$, which is the same as $\frac{(K/L)^{\log_\sigma(a)}}{2a^rb}\leq|\gamma|\leq \frac{1}{2a^rb}$. This completes the proof. $\square$

Of course, if the formulas in (3.16) are already available, one could also establish Theorem 3.10 by directly verifying the characterization in Theorem 1.2. However, these formulas do not seem intuitively obvious. By employing our transforms from wavelet frames to Gabor frames and vice versa, we obtain a rather natural way of arriving at the formulas in (3.16) as well as Theorem 3.10. The supports of $\widehat{\varphi_b}$ and $\widehat{\widetilde{\varphi}_b}$ can be made arbitrarily small by increasing the parameter r. On the other hand, they cannot be arbitrarily large; in fact, for all values of $r \geq 0$, they are symmetric subsets of the bounded interval $\left[-\frac{1}{2b}, \frac{1}{2b}\right]$.

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References

[1] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[2] O. Christensen, R.Y. Kim, On dual Gabor frame pairs generated by polynomials, J. Fourier Anal. Appl. 16 (2010) 1–16.

[3] O. Christensen, P. Massopust, Exponential B-splines and the partition of unity property, Adv. Comput. Math. 37 (2012) 301–318.

[4] C.K. Chui, X. Shi, Inequalities of Littlewood-Paley type for frames and wavelets, SIAM J. Math. Anal. 24 (1993) 263–277.

[5] C.K. Chui, X. Shi, Orthonormal wavelets and tight frames with arbitrary real dilations, Appl. Comput. Harmon. Anal. 9 (2000) 243–264.

[6] W. Dahmen, C.A. Micchelli, On multivariate E-splines, Adv. Math. 76 (1989) 33–93.

[7] I. Daubechies, The wavelet transformation, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory 36 (1990) 961–1005.

[8] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[9] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.

[10] H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decomposition I, J. Funct. Anal. 86 (1989) 307–340.

[11] H.G. Feichtinger, N. Holighaus, C. Wiesmeyr, Adaptive continuous and sampled time-frequency representations via warping, in progress.

[12] H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, Boston, 1998.

[13] H.G. Feichtinger, T. Strohmer (Eds.), Advances in Gabor Analysis, Birkhäuser, Boston, 2002.

[14] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2000.

[15] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2004) 105–132.

[16] C. Heil, D. Walnut, Continuous and discrete wavelet transforms, SIAM Rev. 31 (1989) 628–666.

[17] A.J.E.M. Janssen, The duality condition for Weyl-Heisenberg frames, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, Boston, 1998, pp. 33–84.

[18] Y.J. Lee, J. Yoon, Analysis of compactly supported non-stationary biorthogonal wavelet systems based on exponential B-splines, Abst. Appl. Anal. 2011 (2011) Art. ID 593436, 17 pp.

[19] J. Lemvig, Constructing pairs of dual bandlimited framelets in $L^2(\mathbb{R})$, Adv. Comput. Math. 30 (2009) 263–272.

[20] E. Lieb, M. Loss, Analysis, second ed., Graduate Studies in Mathematics, 14, American Mathematical Society, Providence, 2001.

[21] Y. Meyer, Wavelets and Operators, Cambridge Studies in Advanced Mathematics, 37, Cambridge Univesity Press, Cambridge, 1992.

[22] A. Ron, Exponential box splines, Constr. Approx. 4 (1988) 357–378.

[23] A. Ron, Z. Shen, Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$, Can. J. Math. 47 (1995) 1051–1094.

[24] A. Ron, Z. Shen, Weyl-Heisenberg frames and Riesz bases in $L^2(\mathbb{R}^d)$, Duke Math. J. 89 (1997) 237–282.

[25] M. Unser, T. Blu, Cardinal exponential B-splines: Part I – Theory and filtering algorithms, IEEE Trans. Signal Process. 53 (2005) 1425–1438.

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