APPELL SEQUENCES, CONTINUOUS WAVELET TRANSFORMS AND SERIES EXPANSIONS

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Abstract. A series expansion with remainder for functions in a Sobolev space is derived in terms of the classical Bernoulli polynomials, the B-spline scale-space and the continuous wavelet transforms with the derivatives of the standardized B-splines as mother wavelets. In the limit as their orders tend to infinity, the B-splines and their derivatives converge to the Gaussian function and its derivatives respectively, the associated Bernoulli polynomials converge to the Hermite polynomials, and the corresponding series expansion is an expansion in terms of the Hermite polynomials, the Gaussian scale-space and the continuous wavelet transforms with the derivatives of the Gaussian function as mother wavelets. A similar expansion is also derived in terms of continuous wavelet transforms in which the mother wavelets are the spline framelets that approximate the derivatives of the standardized B-splines.

1. Introduction

The object of this paper is to present some new results involving the classical Bernoulli and Hermite polynomials, and to connect scale-space and wavelet transforms with classical series expansions of real-valued functions in terms of these Appell polynomials. A brief review and a general approach to scale-space and continuous wavelet transforms in the context of singular integral operators can be found in [2]. The Bernoulli polynomials are associated with the uniform B-splines in the same way as the Hermite polynomials are associated with the Gaussian function. The main result is Theorem 1.1 below, but in order to give a systematic presentation, we shall begin with the general setting and consider a compactly supported probability measure or probability density function μ, of

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which the standardized uniform B-splines are examples. Then for any integer $m \geq 0$,

$$\langle \mu^{(m)}, e^{(\cdot)z} \rangle = (-1)^m \langle \mu, z^m e^{(\cdot)z} \rangle = (-1)^m z^m \widehat{\mu}(iz)$$

and so

$$\left\langle (-1)^m \mu^{(m)}, \frac{e^{(\cdot)z}}{\widehat{\mu}(iz)} \right\rangle = z^m \tag{1.1}$$

in a neighborhood of 0, where the derivatives $\mu^{(m)}$ are in the sense of distribution, $\langle \cdot, \cdot \rangle$ denotes the action of distributions on test functions, and $\widehat{\mu}$ is the Fourier transform defined by $\widehat{\mu}(u) := \int_{-\infty}^{\infty} e^{-iut} d\mu(t)$. Since μ is compactly supported, $\widehat{\mu}$ is analytic, and so we can define a sequence of Appell polynomials $P_{\mu,m}$ by the generating function

$$\frac{e^{xz}}{\widehat{\mu}(iz)} = \sum_{m=0}^{\infty} \frac{P_{\mu,m}(x)}{m!} z^m,$$

which, together with (1.1), give the biorthogonal relation

$$\left\langle (-1)^n \mu^{(n)}, \frac{P_{\mu,m}}{m!} \right\rangle = \delta_{m,n}. \tag{1.2}$$

The biorthogonal system $\{(-1)^m \mu^{(m)}\}_{m=0}^{\infty}$, $\{P_{\mu,m}/m!\}_{m=0}^{\infty}$ provides a formal biorthogonal expansion for $f \in C^\infty$ of the form

$$f(x) = \sum_{m=0}^{\infty} \langle (-1)^m \mu^{(m)}, f \rangle \frac{P_{\mu,m}(x)}{m!} = \sum_{m=0}^{\infty} \langle \mu, f^{(m)} \rangle \frac{P_{\mu,m}(x)}{m!}. \tag{1.3}$$

Such expansions include a large class of classical series expansions, such as Taylor's series, the Euler-Maclaurin expansion and the Lidstone series [8]. For the case where μ is the Gaussian density function $G(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, which is not compactly supported but the above procedure is applicable, the corresponding Appell sequence comprises the Hermite polynomials, the biorthogonal relation (1.2) is the orthonormal relation with respect to the weighted inner product with Gaussian weight and the expansion (1.3) is the classical Hermite polynomial expansion.

This paper focuses on the case where the probability density functions are the standardized B-splines and derives an expansion of the form (1.3) for functions in a Sobolev space. To this end, we define for $\nu > 1$, the Sobolev space $H^\nu(\mathbb{R})$ that comprises all tempered distributions f for which

$$\|f\|_{H^\nu}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{f}(u)|^2 (1 + |u|^2)^\nu du < \infty.$$

For the forward uniform B-splines M_n of order n, the Fourier transforms are

$$\widehat{M_n}(u) = \left(\frac{1 - e^{-iu}}{iu} \right)^n, \quad u \in \mathbb{R},$$

and so

$$\widehat{M}_n(iz) = \left(\frac{e^z - 1}{z}\right)^n, \quad z \in \mathbb{C}.$$

The corresponding Appell sequences of polynomials, which we denote by $B_{n,m}$, are the Bernoulli polynomials of order n generated by

$$\frac{z^n e^{xz}}{(e^z - 1)^n} = \sum_{m=0}^{\infty} \frac{B_{n,m}(x)}{m!} z^m, \quad |z| < 2\pi.$$

As a probability distribution, the mean and variance of M_n are $\mu_n = \frac{n}{2}$ and $\sigma_n^2 = \frac{n}{12}$ respectively. Let $\widetilde{M}_n(x) := \sigma_n M_n(\sigma_n x + \mu_n)$ be the standardized B-splines with mean 0 and variance 1. Then

$$\widehat{\widetilde{M}}_n(u) = e^{iun/(2\sigma_n)} \left(\frac{1 - e^{-iu/\sigma_n}}{iu/\sigma_n}\right)^n, \quad u \in \mathbb{R}.$$

Further the *standardized Bernoulli polynomials*

$$\widetilde{B}_{n,m}(x) := \frac{1}{\sigma_n^m} B_{n,m}(\sigma_n x + \mu_n),$$

are generated by the generating function

$$\frac{e^{xz}}{\widehat{\widetilde{M}}_n(iz)} = \sum_{m=0}^{\infty} \frac{\widetilde{B}_{n,m}(x)}{m!} z^m, \quad |z| < 2\pi\sigma_n, \tag{1.4}$$

where for every $x \in \mathbb{R}$, the convergence is uniform on $|z| \le \rho$, for any $\rho < 2\pi\sigma_n$.

The main result is

Theorem 1.1. *If $f \in H^\nu(\mathbb{R})$ for some $\nu \ge 3/2$ and is continuous, then for $x, t \in \mathbb{R}$, $s > 0$,*

$$f(x) = S_{\widetilde{M}_n} f(s,t) + \sum_{m=1}^{n-1} \frac{(-1)^m \widetilde{B}_{n,m}(s(t-x))}{m!} W_{\widetilde{M}_n^{(m)}} f(s,t) + R_n f(x,s,t), \tag{1.5}$$

where $R_n f(x,s,t) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$ and $s > 0$ and uniformly for $t \in \mathbb{R}$, and

$$S_{\widetilde{M}_n} f(s,t) := \int_{-\infty}^{\infty} s\widetilde{M}_n(s(t-x)) f(x) dx, \quad t \in \mathbb{R}, s > 0,$$

is the B-spline scale-space transform of f,

$$W_{\widetilde{M}_n^{(m)}} f(s,t) := \int_{-\infty}^{\infty} s\widetilde{M}_n^{(m)}(s(t-x)) f(x) dx, \quad t \in \mathbb{R}, s > 0,$$

are continuous wavelet transforms of f with the derivatives of the standardized B-splines as mother wavelets.

The representation of $f(x)$ in (1.5) holds for any $t \in \mathbb{R}$ and $s > 0$. In particular, if $t = 0$ and $s = 1$, it becomes

$$f(x) = \sum_{m=0}^{n-1} \langle (-1)^m \widetilde{M}_n^{(m)}, f\rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + R_n f(x),$$

where $R_n f(x) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$, which is the expansion of f, with remainder, in terms of the biorthogonal system $\{(-1)^m \widetilde{M}_n^{(m)}\}$, $\{\widetilde{B}_{n,m}/m!\}$. On the other hand, if $t = x$, then (1.5) gives

$$f(x) = S_{\widetilde{M}_n} f(s,x) + \sum_{m=1}^{n-1} \frac{(-1)^m \widetilde{B}_{n,m}(0)}{m!} W_{\widetilde{M}_n^{(m)}} f(s,x) + R_n f(s,x),$$

which is a decomposition of f into its B-spline scale-space and wavelet transforms. Since spline functions were introduced by Schoenberg [4], such an expansion involving spline functions and Bernoulli polynomials will be referred as the *Bernoulli-Schoenberg series.*

To prove Theorem 1.1, we first develop the corresponding expansion where the Gaussian function $G(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ defines the Gaussian scale-space

$$S_G f(s,t) := \int_{-\infty}^{\infty} sG(s(t-x)) f(x)dx, \quad t \in \mathbb{R}, s > 0,$$

and its derivatives $G^{(m)}$, $m = 1, 2, \ldots$, are mother wavelets that define the continuous wavelet transforms

$$W_{G^{(m)}} f(s,t) := \int_{-\infty}^{\infty} sG^{(m)}(s(t-x)) f(x)dx, \quad t \in \mathbb{R}, s > 0.$$

The Gaussian function also defines the Appell sequence of Hermite polynomials H_m, $m = 0, 1, \ldots$, by the generating function

$$\frac{e^{xz}}{\widehat{G}(iz)} = \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} z^m, \tag{1.6}$$

where $\widehat{G}(u) = e^{-u^2/2}$ is the Fourier transform of G, and for every $x \in \mathbb{R}$, the series converges locally uniformly on $\mathbb{C}$. Also $(-1)^m G^{(m)}(t) = H_m(t)G(t)$ and $H_m/m!$ are biorthogonal sequences, which is the well-known fact that the normalized Hermite polynomials $H_m/\sqrt{m!}$, $m = 0, 1, \ldots$, form an orthonormal basis for the Hilbert space $L^2_G(\mathbb{R})$ with inner product $\langle f, g\rangle_G := \int_{-\infty}^{\infty} f(t)g(t)G(t)dt$, and so for every $f \in L^2_G(\mathbb{R})$,

$$f(x) = \sum_{m=0}^{\infty} \langle f, H_m\rangle_G \frac{H_m(x)}{m!},$$

where the convergence is in $L^2_G(\mathbb{R})$. We shall show that if $f \in H^\nu(\mathbb{R})$ for some $\nu > 1$ and is continuous, then

$$f(x) = S_G f(s,t) + \sum_{m=1}^{\infty} \frac{H_m(s(x-t))}{m!} W_{G^{(m)}} f(s,t), \tag{1.7}$$

where the series converges uniformly in $t \in \mathbb{R}$ and locally uniformly in $x \in \mathbb{R}$ and $s > 0$. Note that if $t = 0$ and $s = 1$, the resulting series is the Hermite expansion in Sobolev space.

On the other hand, since

$$H_m(0) = \begin{cases} 0, & m = 2j+1, \\ \frac{(-1)^j (2j)!}{2^j j!}, & m = 2j, \end{cases} \quad j = 0, 1, \ldots,$$

if $t = x$, the expansion becomes

$$f(x) = S_G f(s,x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{2^j j!} W_{G^{(2j)}} f(s,x),$$

which is a decomposition of f into its Gaussian scale-space and wavelet transforms. Therefore, (1.7) provides an expansion that unifies the classical Hermite polynomial expansion on one hand and Gaussian scale-space and wavelet decomposition on the other. Section 2 gives a proof of (1.7).

The proof of Theorem 1.1 is based on equation (1.7) and requires precise estimates of the rate of convergence of the Fourier transforms of the derivatives of the standardized B-splines to the Fourier transforms of the derivatives of the Gaussian function, as well as on the rate of convergence of the standardized Bernoulli polynomials to the Hermite polynomials as the order of the B-spline tends to infinity. Section 3 studies the rate of convergence of the derivatives of the standardized B-splines, where we also show that for any nonnegative integer m and $0 < \epsilon < 1$,

$$\left\| \widetilde{M}_n^{(m)} - G^{(m)} \right\|_p = O\left(\frac{1}{n^{1-\epsilon/p}} \right) \text{ as } n \to \infty, \quad 1 \le p \le \infty,$$

a result which is of interest in itself. In particular, $\left\| \widetilde{M}_n^{(m)} - G^{(m)} \right\|_\infty = O\left(\frac{1}{n}\right)$. The convergence of the derivatives of B-splines has been established in [1] and pointwise convergence of order $O(1/n)$ is given in [7], where they also deduce L^p-convergence for $p \in [2, \infty)$, but no convergence rate in uniform or L^p-norm is given. The rate of convergence of the standardized Bernoulli polynomials to the Hermite polynomials is given in Section 4 and the proof of Theorem 1.1 in Section 5. In Section 6 we show that when suitably standardized, the spline framelets of Ron and Shen [3] approximate the derivatives of the standardized B-splines and converge to the derivatives of the Gaussian function with the same rate as that of the derivatives of the B-splines. Bernoulli-Schoenberg series expansion formulas of the form (1.5) are then derived with the standardized spline framelets as mother wavelets in place of the derivatives of the B-splines in the wavelet transforms.

2. Hermite polynomial expansion, Gaussian scale-space and wavelet transforms

The Gaussian function $G(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ defines the Gaussian scale-space, and its derivatives $G^{(m)}$, $m = 1, 2, \ldots$, are mother wavelets that define continuous wavelet transforms. It also generates the Appell sequence of Hermite polynomials, H_m, $m = 0, 1, \ldots$, by the generating function (1.6). Let $L^2_G(\mathbb{R})$ be the Hilbert space with inner product $\langle f, g\rangle_G := \int_{-\infty}^{\infty} fgG$. It is well known that $G^{(m)} = (-1)^m H_m G$, $m = 0, 1, \ldots$, and

$$\langle H_j, H_m/m!\rangle_G = \delta_{jm}, \quad j, m = 0, 1, \ldots.$$

Any $f \in L^2_G(\mathbb{R})$ can be represented by

$$f = \sum_{m=0}^{\infty} \langle f, H_m\rangle_G H_m/m! \tag{2.1}$$

in $L^2_G(\mathbb{R})$.

Letting $\langle f, g\rangle$ denote the usual L^2 inner product, for any $f \in L^2(\mathbb{R}) \subset L^2_G(\mathbb{R})$, (2.1) implies that

$$f = \sum_{m=0}^{\infty} \langle (-1)^m G^{(m)}, f\rangle H_m/m! \tag{2.2}$$

in $L^2_G(\mathbb{R})$. For $t, x \in \mathbb{R}$ and $s > 0$, let $f_{s,t}(x) := f(t - x/s)$. Then for $m = 0, 1, \ldots$,

$$\begin{aligned} \langle (-1)^m G^{(m)}, f_{s,t}\rangle &= (-1)^m \int_{-\infty}^{\infty} G^{(m)}(x) f(t - x/s)dx \\ &= (-1)^m \int_{-\infty}^{\infty} sG^{(m)}(s(t - y)) f(y)dy, \end{aligned}$$

and so

$$\langle G, f_{s,t}\rangle = S_G f(s, t) \tag{2.3}$$

and for $m = 1, 2, \ldots$,

$$\langle (-1)^m G^{(m)}, f_{s,t}\rangle = (-1)^m W_{G^{(m)}} f(s, t). \tag{2.4}$$

Therefore, we have by (2.2),

$$f_{s,t} = S_G f(s, t) + \sum_{m=1}^{\infty} (-1)^m W_{G^{(m)}} f(s, t) H_m/m!$$

in $L^2_G(\mathbb{R})$ for any $t \in \mathbb{R}$ and $s > 0$.

To obtain uniform convergence we require f to belong to some Sobolev space. Here we are concerned with $H^\nu(\mathbb{R})$ for some $\nu > 1$, and in this case, $H^\nu(\mathbb{R}) \subset L^2(\mathbb{R})$. We shall prove

Theorem 2.1. *If $f \in H^\nu(\mathbb{R})$ for some $\nu > 1$ and is continuous, then for any $t, x \in \mathbb{R}$ and $s > 0$,*

$$f(t - x/s) = S_G f(s,t) + \sum_{m=1}^{\infty} (-1)^m W_{G^{(m)}} f(s,t) H_m(x)/m!, \tag{2.5}$$

where the convergence is uniform over $t \in \mathbb{R}$ and locally uniform over $s > 0$ and $x \in \mathbb{R}$.

Remark 1. *Since $H_m(t) = (-1)^m H_m(-t)$, a change of variable in (2.5) gives the following equivalent representation*

$$f(x) = S_G f(s,t) + \sum_{m=1}^{\infty} \frac{H_m(s(x-t))}{m!} W_{G^{(m)}} f(s,t).$$

First we show that under the conditions of Theorem 2.1, convergence is locally uniform in (2.1). The method and estimates in the proof of Theorem 2.2 below will also be used elsewhere in the paper.

Theorem 2.2. *If $f \in H^\nu(\mathbb{R})$ for some $\nu > 1$ and is continuous, then*

$$f(x) = \sum_{m=0}^{\infty} \langle (-1)^m G^{(m)}, f \rangle H_m(x)/m!, \tag{2.6}$$

where the convergence is locally uniform over $x \in \mathbb{R}$.

Proof. For $n = 0, 1, \ldots,$

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(f(x)G(x) - \sum_{m=0}^{n} \langle (-1)^m G^{(m)}, f \rangle \frac{H_m(x)}{m!} G(x) \right)^2 dx \\ \leq\ &\int_{-\infty}^{\infty} \left(f(x) - \sum_{m=0}^{n} \langle (-1)^m G^{(m)}, f \rangle \frac{H_m(x)}{m!} \right)^2 G(x) dx \\ =\ &\left\| f - \sum_{m=0}^{n} \langle (-1)^m G^{(m)}, f \rangle H_m/m! \right\|_G^2 . \end{aligned}$$

Now $H_m G \in L^2(\mathbb{R})$, and since $H^\nu(\mathbb{R}) \subset L^2(\mathbb{R})$, fG also belongs to $L^2(\mathbb{R})$, and by (2.2), this gives

$$f(x)G(x) = \sum_{m=0}^{\infty} \langle (-1)^m G^{(m)}, f \rangle H_m(x) G(x)/m! \tag{2.7}$$

in $L^2(\mathbb{R})$. We want to show that the series on the right of (2.7) converges uniformly.

For $m = 0, 1, \ldots,$ $x \in \mathbb{R}$,

$$(-1)^m H_m(x) G(x) = G^{(m)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (iu)^m e^{-u^2/2} e^{iux} du,$$

and so

$$|H_m(x)G(x)| \le \frac{1}{\pi}\int_0^\infty u^m e^{-u^2/2}du = \frac{2^{\frac{m-1}{2}}}{\pi}\Gamma\left((m+1)/2\right). \tag{2.8}$$

By the Stirling formula,

$$\Gamma\left((m+1)/2\right) \sim \frac{\sqrt{2\pi}}{2^{m/2}}\left(\frac{m}{e}\right)^{m/2}, \quad m! = \Gamma(m+1) \sim \sqrt{2\pi m}\left(\frac{m}{e}\right)^m,$$

and so for all sufficiently large m,

$$\frac{|H_m(x)G(x)|}{m!} \le K\frac{1}{\sqrt{m}}\left(\frac{e}{m}\right)^{m/2}. \tag{2.9}$$

Also for $m = 0, 1, \ldots,$

$$\langle(-1)^m G^{(m)}, f\rangle = (-1)^m\int_{-\infty}^\infty f(t)G^{(m)}(t)dt = \frac{(-1)^m}{2\pi}\int_{-\infty}^\infty \widehat{f}(u)(-iu)^m e^{-u^2/2}du$$

and so for some constant $C > 0$,

$$\begin{aligned}
|\langle(-1)^m G^{(m)}, f\rangle| &\le \frac{1}{2\pi}\int_{-\infty}^\infty |\widehat{f}(u)||u|^\nu |u|^{m-\nu}e^{-u^2/2}du \\
&\le C\|f\|_{H^\nu}\left(\int_{-\infty}^\infty |u|^{2m-2\nu}e^{-u^2}du\right)^{1/2} \\
&= C\|f\|_{H^\nu}\left(2\int_0^\infty u^{2m-2\nu}e^{-u^2}du\right)^{1/2} \\
&= C\|f\|_{H^\nu}\,\Gamma\left(m-\nu+1/2\right)^{1/2}.
\end{aligned} \tag{2.10}$$

Again by the Stirling formula,

$$\Gamma\left(m-\nu+1/2\right) \sim \sqrt{2\pi(m-\nu-1/2)}\left(\frac{m-\nu-1/2}{e}\right)^{m-\nu-1/2} \sim \sqrt{2\pi}\,\frac{1}{m^\nu}\left(\frac{m}{e}\right)^m,$$

and so (2.10) gives

$$|\langle(-1)^m G^{(m)}, f\rangle| \le C\frac{\|f\|_{H^\nu}}{m^{\nu/2}}\left(\frac{m}{e}\right)^{m/2}, \tag{2.11}$$

where the constant C is a generic constant independent of m. It follows from (2.9) and (2.11) that

$$\left|\langle(-1)^m G^{(m)}, f\rangle H_m(x)G(x)/m!\right| \le C\frac{\|f\|_{H^\nu}}{m^{(1+\nu)/2}}, \tag{2.12}$$

for all $x \in \mathbb{R}$. Since $\nu > 1$, we conclude that the series in (2.7) converges uniformly on $\mathbb{R}$. It follows that (2.6) holds locally uniformly on $\mathbb{R}$. □

Remark 2. *Although Theorem* 2.1 *involves classical Hermite polynomial expansion, we are unable to find similar results in the literature. More information on local uniform convergence of Hermite polynomial expansions can be found in the classic monograph of Szegö* ([6], *Theorem* 9.1.6).

Proof of Theorem 2.1. For any $t \in \mathbb{R}$ and $s > 0$, let $f_{s,t}(x) := f(t - x/s)$, $x \in \mathbb{R}$. Then $f_{s,t} \in H^\nu(\mathbb{R})$ and its Fourier transform is given by $\widehat{f}_{s,t}(u) = s\widehat{f}(-us)e^{-iust}$, $u \in \mathbb{R}$. By Theorem 2.2, for $t \in \mathbb{R}$ and $s > 0$,

$$f_{s,t}(x) = \sum_{m=0}^{\infty} \langle (-1)^m G^{(m)}, f_{s,t} \rangle H_m(x)/m!, \tag{2.13}$$

where the convergence is locally uniform over $x \in \mathbb{R}$. A calculation similar to (2.10) shows that for some constant $C > 0$,

$$\left| \langle (-1)^m G^{(m)}, f_{s,t} \rangle \right| \le C \frac{1}{s^{\nu - 1/2}} \|f\|_{H^\nu} \Gamma(m - \nu + 1/2)^{1/2}, \quad t \in \mathbb{R},\ s > 0.$$

Proceeding as in the proof of Theorem 2.2, we see that the convergence of the series in (2.13) is uniform over $t \in \mathbb{R}$ and locally uniform over $s > 0$. Combining with (2.3) and (2.4) gives (2.5). □

Corollary 2.3. *If for some $\nu > 1$, $f \in H^\nu(\mathbb{R})$ and is continuous, then*

$$f(t) = S_G f(s,t) + \sum_{j=1}^{\infty} \frac{(-1)^j}{2^j j!} W_{G^{(2j)}} f(s,t), \tag{2.14}$$

where the convergence is uniform over $t \in \mathbb{R}$ and locally uniform over $s > 0$.

Proof. Since f is continuous, we may set $x = 0$ in (2.5) to give

$$f(t) = S_G f(s,t) + \sum_{m=1}^{\infty} (-1)^m W_{G^{(m)}} f(s,t) H_m(0)/m!, \tag{2.15}$$

where the convergence is uniform over $t \in \mathbb{R}$ and locally uniform over $s > 0$. Now

$$G^{(m)}(0) = (-1)^m H_m(0) G(0) = \frac{1}{\sqrt{2\pi}} (-1)^m H_m(0),$$

and since $G^{(m)}(0) = 0$ if m is odd and $G^{(2j)}(0) = \frac{1}{\sqrt{2\pi}} \frac{(-1)^j (2j)!}{2^j j!}$, (2.14) follows from (2.15). □

3. Convergence of derivatives of B-splines

We shall first prove some lemmas, which will be used to derive the rate of convergence of the derivatives of the standardized B-splines in the frequency domain as well as in the proof of the main theorem.

Lemma 3.1. *For $|u| \le n^\alpha$, $0 < \alpha < 1/2$,*

$$0 \le \widehat{\widetilde{M}}_n(u) \le e^{-u^2/2} \tag{3.1}$$

and for all sufficiently large n,

$$\left| \widehat{\widetilde{M}}_n(u) - e^{-u^2/2} \right| \le K \frac{u^4 e^{-u^2/2}}{n}, \tag{3.2}$$

where K is an absolute constant.

Proof. By Taylor's theorem, for $|u| \le \sqrt{n/3}\,\pi$,

$$\widehat{\widehat{M}}_n(u) = \left(\frac{\sin u\sqrt{3/n}}{u\sqrt{3/n}}\right)^n \le \left(1 - \frac{u^2}{2n} + \frac{3u^4}{40n^2}\right)^n.$$

Now for $0 \le s \le 6/5$,

$$e^{-s} \ge 1 - s + s^2/2 - s^3/6 \ge 1 - s + 3s^2/10,$$

and so for $u^2 \le 12n/5$,

$$1 - \frac{u^2}{2n} + \frac{3u^4}{40n^2} \le e^{-u^2/2n},$$

which gives (3.1).

Also $\widehat{\widehat{M}}_n(u) = e^{-\frac{u^2}{2} - \frac{u^4}{20n} + O\left(\frac{u^6}{n^2}\right)}$, and so

$$\begin{aligned}\left|\widehat{\widehat{M}}_n(u) - e^{-u^2/2}\right| &= e^{-u^2/2}\left|1 - e^{-\frac{u^4}{20n} + O\left(\frac{u^6}{n^2}\right)}\right| \le e^{-u^2/2}\left|\frac{u^4}{20n} + O\left(\frac{u^6}{n^2}\right)\right| \\ &= \frac{e^{-u^2/2}|u|^4}{n}\left|\frac{1}{20} + O\left(\frac{u^2}{n}\right)\right| \le K\frac{|u|^4 e^{-u^2/2}}{n}\end{aligned}$$

for $|u| \le \sqrt{n}$ for all sufficiently large n. □

Lemma 3.2. *Take integers $m > 0$, $n > m$ and $0 < \alpha < 1/2$. Then for all sufficiently large n,*

$$\left|u^m \widehat{\widehat{M}}_n(u)\right| \le n^{m\alpha} e^{-\frac{1}{2}n^{2\alpha}} \quad \text{for} \quad u \ge n^\alpha. \tag{3.3}$$

Proof. Let $K_{n,m} := \sup_{u \ge n^\alpha} \left|u^m \widehat{\widehat{M}}_n(u)\right|$. Then

$$K_{n,m} = \sup_{u \ge n^\alpha}\left|u^m\left(\frac{\sin u\sqrt{3/n}}{u\sqrt{3/n}}\right)^n\right| = \sup_{x \ge \sqrt{3}n^{\alpha-1/2}}\left(\frac{n}{3}\right)^{m/2}\frac{|\sin x|^n}{x^{n(1-m/n)}} = \left(\frac{n}{3}\right)^{m/2} A_{n,m}^n,$$

where

$$A_{n,m} := \sup_{x \ge \sqrt{3}n^{\alpha-1/2}} \frac{|\sin x|}{x^{1-m/n}}.$$

Observe that

$$\sup_{x \ge \pi/2} \frac{|\sin x|}{x^{1-m/n}} = \sup_{\pi/2 \le x \le \pi} \frac{\sin x}{x^{1-m/n}},$$

and so for $\sqrt{3}n^{\alpha-1/2} \le \pi/2$,

$$A_{n,m} = \sup_{\sqrt{3}n^{\alpha-1/2} \le x \le \pi} \frac{\sin x}{x^{1-m/n}}.$$

Consider the function $g(x) := \frac{\sin x}{x^{1-m/n}}$, $0 \le x \le \pi$, where $g(0) = g(\pi) = 0$ and

$$g'(x) = \{x\cos x - (1-m/n)\sin x\}/x^{2-m/n} = \cos x\{x - (1-m/n)\tan x\}/x^{2-m/n}.$$

Then $g'(x) = 0$ only at the unique value $x = \zeta$, $0 < \zeta < \pi/2$, where $\tan\zeta = \zeta/(1 - m/n)$. We want to show that $\zeta < \sqrt{3}n^{\alpha-1/2}$. Consider

$$\frac{1}{x^3}\left\{x - \left(1 - \frac{m}{n}\right)\tan x\right\} = \frac{m}{nx^2} - \left(1 - \frac{m}{n}\right)\frac{\tan x - x}{x^3}.$$

At $x = \sqrt{3}n^{\alpha-1/2}$, as $n \to \infty$, $x \to 0$ and $nx^2 = 3n^{2\alpha} \to \infty$. Thus

$$\lim_{n\to\infty}\frac{1}{x^3}\left\{x - \left(1 - \frac{m}{n}\right)\tan x\right\} = -\lim_{x\to 0}\frac{\tan x - x}{x^3} = -\frac{1}{3}.$$

Therefore for all large enough n, $g'(\sqrt{3}n^{\alpha-1/2}) < 0$. Since $g'(\pi) < 0$, $\zeta < \sqrt{3}n^{\alpha-1/2}$. Hence g is decreasing on $[\sqrt{3}n^{\alpha-1/2}, \pi]$. Thus

$$A_{n,m} = \sup_{\sqrt{3}n^{\alpha-1/2}\le x\le\pi}\frac{\sin x}{x^{1-m/n}} = \frac{\sin(\sqrt{3}n^{\alpha-1/2})}{(\sqrt{3}n^{\alpha-1/2})^{1-m/n}} = 3^{m/2n}n^{m(\alpha-1/2)/n}\frac{\sin(\sqrt{3}n^{\alpha-1/2})}{\sqrt{3}n^{\alpha-1/2}},$$

and so

$$K_{n,m} = \left(\frac{n}{3}\right)^{m/2}A_{n,m}^n = n^{m\alpha}\left(\frac{\sin(n^\alpha\sqrt{3/n})}{n^\alpha\sqrt{3/n}}\right)^n. \tag{3.4}$$

By (3.1) of Lemma 3.1 with $u = n^\alpha$,

$$\left(\frac{\sin(n^\alpha\sqrt{3/n})}{n^\alpha\sqrt{3/n}}\right)^n \le e^{-\frac{1}{2}n^{2\alpha}}.$$

It follows from (3.4) that $K_{n,m} \le n^{m\alpha}e^{-\frac{1}{2}n^{2\alpha}}$ for all large enough n. □

Theorem 3.3. *For any integer $m \ge 0$,*

$$\sup_{u\in\mathbb{R}}\left|u^m\widehat{\widetilde{M}}_n(u) - u^m e^{-u^2/2}\right| = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{3.5}$$

Proof. For any nonnegative integer m, (3.2) gives

$$\left|u^m\widehat{\widetilde{M}}_n(u) - u^m e^{-u^2/2}\right| \le K\frac{|u|^{m+4}e^{-u^2/2}}{n}, \quad |u| < n^\alpha,\ 0 < \alpha < 1/2.$$

The right hand side of the inequality attains its maximum at $|u| = \sqrt{m+4} < n^\alpha$ for all sufficiently large n. It follows that for all $|u| < n^\alpha$,

$$\left|u^m\widehat{\widetilde{M}}_n(u) - u^m e^{-u^2/2}\right| \le \frac{K}{n}\left(\frac{m+4}{e}\right)^{(m+4)/2} = O\left(\frac{1}{n}\right). \tag{3.6}$$

Since $|u|^m e^{-u^2/2}$ is decreasing for all large enough $|u|$, we see that for all large enough n

$$|u^m e^{-u^2/2}| \le \frac{n^{m\alpha}}{e^{n^{2\alpha}/2}} = O\left(\frac{1}{n}\right), \quad |u| \ge n^\alpha. \tag{3.7}$$

Also from (3.3), for all sufficiently large n,

$$\left|u^m\widehat{\widetilde{M}}_n(u)\right| \le \frac{n^{m\alpha}}{e^{n^{2\alpha}/2}} = O\left(\frac{1}{n}\right), \quad |u| \ge n^\alpha.$$

Therefore,

$$\left|u^m\widehat{\widetilde{M}}_n(u) - u^m e^{-u^2/2}\right| \leq \left|u^m\widehat{\widetilde{M}}_n(u)\right| + |u^m e^{-u^2/2}| = O\left(\frac{1}{n}\right), \quad |u| \geq n^\alpha. \tag{3.8}$$

By (3.6) and (3.8), we obtain (3.5). □

Theorem 3.4. *For any integer $m \geq 0$,*

$$\left\|\widetilde{M}_n^{(m)} - G^{(m)}\right\|_\infty = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

Proof. By taking inverse Fourier transforms, for $n \geq 1$ and $x \in \mathbb{R}$,

$$\widetilde{M}_n^{(m)}(x) - G^{(m)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}(iu)^m\left(\widehat{\widetilde{M}}_n(u) - e^{-u^2/2}\right)du$$

and so

$$\left|\widetilde{M}_n^{(m)}(x) - G^{(m)}(x)\right| \leq \frac{1}{\pi}\int_0^\infty u^m\left|\widehat{\widetilde{M}}_n(u) - e^{-u^2/2}\right|du. \tag{3.9}$$

By (3.2), for $0 < \alpha < 1/2$,

$$\begin{aligned}\int_0^{n^\alpha} u^m\left|\widehat{\widetilde{M}}_n(u) - e^{-u^2/2}\right|du &\leq \frac{K}{n}\int_0^{n^\alpha} u^{m+4}e^{-u^2/2}du \\ &\leq \frac{K}{n}\int_0^{\infty} u^{m+4}e^{-u^2/2}du = O\left(\frac{1}{n}\right). \end{aligned} \tag{3.10}$$

Take any integer $\ell > 1$. Then by (3.3) of Lemma 3.2 with m replaced by $m+\ell$, for sufficiently large n,

$$\left|u^m\widehat{\widetilde{M}}_n(u)\right| \leq \frac{n^{(m+\ell)\alpha}}{e^{\frac{1}{2}n^{2\alpha}}u^\ell}, \quad u \geq n^\alpha,\ 0 < \alpha < 1/2.$$

Also, as in the proof of (3.7),

$$|u^m e^{-u^2/2}| \leq \frac{n^{(m+\ell)\alpha}}{e^{\frac{1}{2}n^{2\alpha}}u^\ell}, \quad u \geq n^\alpha,$$

and so

$$\begin{aligned}\int_{n^\alpha}^{\infty} u^m\left|\widehat{\widetilde{M}}_n(u) - e^{-u^2/2}\right|du &\leq \int_{n^\alpha}^{\infty}\left(\left|u^m\widehat{\widetilde{M}}_n(u)\right| + \left|u^m e^{-u^2/2}\right|\right)du \leq 2\int_{n^\alpha}^{\infty}\frac{n^{(m+\ell)\alpha}}{e^{\frac{1}{2}n^{2\alpha}}u^\ell}du \\ &= \frac{2n^{(m+\ell)\alpha}}{(\ell-1)e^{\frac{1}{2}n^{2\alpha}}n^{(\ell-1)\alpha}} = \frac{2n^{(m+1)\alpha}}{(\ell-1)e^{\frac{1}{2}n^{2\alpha}}} = O\left(\frac{1}{n}\right). \end{aligned} \tag{3.11}$$

The result follows from (3.9), (3.10) and (3.11). □

Theorem 3.5. *Take any integer $m \geq 0$. Then for any $0 < \epsilon < 1$ and $1 \leq p < \infty$,*

$$\left\|\widetilde{M}_n^{(m)} - G^{(m)}\right\|_p = O\left(\frac{1}{n^{1-\epsilon/p}}\right) \quad \text{as } n \to \infty. \tag{3.12}$$

Proof. By Theorem 3.4, we have

$$\left\|\widetilde{M}_n^{(m)} - G^{(m)}\right\|_\infty = O\left(\frac{1}{n}\right). \tag{3.13}$$

Then for any positive $\epsilon < 1$ and for all $n \geq 1$,

$$\int_{-n^\epsilon}^{n^\epsilon} \left|\widetilde{M}_n^{(m)}(x) - G^{(m)}(x)\right| dx \leq 2n^\epsilon \left\|\widetilde{M}_n^{(m)} - G^{(m)}\right\|_\infty \leq \frac{C}{n^{1-\epsilon}}, \tag{3.14}$$

where $C > 0$ is a generic constant. Thus for all $n \geq 1$,

$$\left|\int_0^{n^\epsilon} \widetilde{M}_n^{(m)}(x)dx - \int_0^{n^\epsilon} G^{(m)}(x)dx\right| \leq \frac{C}{n^{1-\epsilon}}. \tag{3.15}$$

Now for m even, $\int_0^\infty \widetilde{M}_n^{(m)}(x)dx = \int_0^\infty G^{(m)}(x)dx$, and for m odd, by Theorem 3.4,

$$\int_0^\infty \widetilde{M}_n^{(m)}(x)dx - \int_0^\infty G^{(m)}(x)dx = -\widetilde{M}_n^{(m-1)}(0) + G^{(m-1)}(0) = O\left(\frac{1}{n}\right),$$

and so

$$\left|\int_0^\infty \widetilde{M}_n^{(m)}(x)dx - \int_0^\infty G^{(m)}(x)dx\right| \leq \frac{C}{n}. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\left|\int_{n^\epsilon}^\infty \widetilde{M}_n^{(m)}(x)dx - \int_{n^\epsilon}^\infty G^{(m)}(x)dx\right| = O\left(\frac{1}{n^{1-\epsilon}}\right). \tag{3.17}$$

Now $G^{(m)}$ has exactly m simple zeros, $a_1 < a_2 < \cdots < a_m$, and similarly $\widetilde{M}_n^{(m)}$ has exactly m simple zeros, $a_1^n < a_2^n < \cdots < a_m^n$. Then from (3.13) $\lim_{n\to\infty} a_j^n = a_j$, $j = 1, 2, \ldots, m$. So for all large enough n, $G^{(m)}$ and $\widetilde{M}_n^{(m)}$ have no zero on $[n^\epsilon, \infty)$, and therefore

$$\int_{n^\epsilon}^\infty \left|G^{(m)}(x)\right| dx = \left|\int_{n^\epsilon}^\infty G^{(m)}(x)dx\right| = O\left(\frac{1}{n}\right) \tag{3.18}$$

and

$$\int_{n^\epsilon}^\infty \left|\widetilde{M}_n^{(m)}(x)\right| dx = \left|\int_{n^\epsilon}^\infty \widetilde{M}_n^{(m)}(x)dx\right| = \left|\int_{n^\epsilon}^\infty G^{(m)}(x)dx\right| + O\left(\frac{1}{n^{1-\epsilon}}\right), \tag{3.19}$$

by (3.17). Then (3.14), (3.18) and (3.19) give

$$\left\|\widetilde{M}_n^{(m)} - G^{(m)}\right\|_1 = O\left(\frac{1}{n^{1-\epsilon}}\right).$$

Now, using the inequality

$$\|f\|_p \leq \|f\|_\infty^{1-1/p}\|f\|_1^{1/p}$$

for any $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $1 < p < \infty$, we have (3.12). □

4. Convergence of Bernoulli polynomials to Hermite polynomials

To obtain the rate of convergence of the standardized Bernoulli polynomials $\widetilde{B}_{n,m}$ to the Hermite polynomials H_m as $n \to \infty$, we note from (1.4) and (1.6) that

$$\frac{e^{xz}}{\widehat{\widetilde{M}}_n(iz)} - \frac{e^{xz}}{e^{z^2/2}} = \sum_{m=0}^{\infty} \frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!} z^m,$$

and so by Cauchy's integral formula,

$$\frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!} = \frac{1}{2\pi i} \oint_C e^{xz} \left(\frac{1}{\widehat{\widetilde{M}}_n(iz)} - e^{-z^2/2} \right) \frac{dz}{z^{m+1}}, \tag{4.1}$$

where C is a circle with center at the origin and radius $r < 2\pi\sigma_n = \pi\sqrt{n/3}$.

Lemma 4.1. *For z in a compact subset of $\mathbb{C}$,*

$$\left| \frac{1}{\widehat{\widetilde{M}}_n(iz)} - e^{-z^2/2} \right| = O\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty. \tag{4.2}$$

Proof. Take n large enough so that the open disk with center at the origin and radius $\pi\sqrt{n/3}$ contains the given compact set. Then for z in the compact set, a direct computation gives

$$\operatorname{Log}\left(\frac{1}{\widehat{\widetilde{M}}_n(iz)} \right) + \frac{z^2}{2} = \frac{z^4}{20n} + O\left(\frac{z^6}{n^2}\right),$$

and so

$$\left| \operatorname{Log}\left(\frac{1}{\widehat{\widetilde{M}}_n(iz)} \right) + \frac{z^2}{2} \right| \le \frac{A}{n}.$$

Therefore,

$$\left| \frac{1}{\widehat{\widetilde{M}}_n(iz)} - e^{-z^2/2} \right| = \left| e^{\operatorname{Log}\left(1/\widehat{\widetilde{M}}_n(iz)\right)} - e^{-z^2/2} \right| \le K \left| \operatorname{Log}\left(\frac{1}{\widehat{\widetilde{M}}_n(iz)} \right) + \frac{z^2}{2} \right| \le \frac{KA}{n},$$

where A, K are constants independent of n. □

Proposition 4.2. *For positive integers $m < n$ and $x \in \mathbb{R}$,*

$$\left| \frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!} \right| \le \frac{Ke^{x^2}}{n} \left(\frac{e}{2m}\right)^{m/2}, \tag{4.3}$$

where K is a generic constant that is independent of m, n.

Proof. By (4.1) and (4.2) followed by writing $z = \zeta + i\eta$, $\zeta, \eta \in \mathbb{R}$,

$$\begin{aligned}
\left|\frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!}\right| e^{-x^2} &= \left|\frac{1}{2\pi i}\oint_C e^{xz-x^2}\left(\frac{1}{\widehat{\widetilde{M}}_n(iz)} - e^{-z^2/2}\right)\frac{dz}{z^{m+1}}\right| \\
&\le \frac{K}{2\pi n}\oint_C \left|\frac{e^{xz-x^2}}{z^{m+1}}\right| |dz| = \frac{K}{2\pi n}\oint_C \frac{e^{x\zeta-x^2}}{r^{m+1}}|dz| \\
&\le \frac{K}{2\pi n r^{m+1}}\oint_C e^{-(x-\zeta/2)^2} e^{\zeta^2/4}|dz| \\
&\le \frac{K}{2\pi n r^{m+1}}\oint_C e^{\zeta^2/4}|dz| \le \frac{K e^{r^2/4}}{n r^m}, \qquad (4.4)
\end{aligned}$$

for all positive $r < \pi\sqrt{n/3}$, where r is the radius of the circle C. The minimum of $\frac{e^{r^2/4}}{r^m}$ is attained at $r = \sqrt{2m}$, which is less than $\pi\sqrt{n/3}$, since $m < n$. Therefore, (4.4) holds in particular for $r = \sqrt{2m}$, i.e.

$$\left|\frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!}\right| e^{-x^2} \le \frac{K}{n}\left(\frac{e}{2m}\right)^{m/2},$$

and so the result follows. □

5. Proof of Bernoulli-Schoenberg series formula

We first prove the following

Theorem 5.1. *If $f \in H^\nu(\mathbb{R})$ for some $\nu \ge 3/2$ and is continuous, then*

$$f(x) = \sum_{m=0}^{n-1} \langle (-1)^m \widetilde{M}_n^{(m)}, f\rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + R_n f(x), \qquad (5.1)$$

where $R_n f(x) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$.

Proof. We want to show that $R_n f(x) := f(x) - \sum_{m=0}^{n-1}\langle(-1)^m\widetilde{M}_n^{(m)}, f\rangle \widetilde{B}_{n,m}(x)/m! \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$. By (2.6) of Theorem 2.2,

$$\begin{aligned}
R_n f(x) &= \sum_{m=0}^{\infty}\langle(-1)^m G^{(m)}, f\rangle H_m(x)/m! - \sum_{m=0}^{n-1}\langle(-1)^m \widetilde{M}_n^{(m)}, f\rangle\ \widetilde{B}_{n,m}(x)/m! \\
&= A_n f(x) + B_n f(x) + \sum_{m=n}^{\infty}\langle(-1)^m G^{(m)}, f\rangle H_m(x)/m!, \qquad (5.2)
\end{aligned}$$

where

$$\begin{aligned}
A_n f(x) &:= \sum_{m=0}^{n-1}\Big(\langle(-1)^m G^{(m)}, f\rangle - \langle(-1)^m\widetilde{M}_n^{(m)}, f\rangle\Big) H_m(x)/m!, \qquad (5.3) \\
B_n f(x) &:= \sum_{m=0}^{n-1}\langle(-1)^m\widetilde{M}_n^{(m)}, f\rangle\Big(H_m(x) - \widetilde{B}_{n,m}(x)\Big)/m!. \qquad (5.4)
\end{aligned}$$

For the convergence of $A_n f$ we shall use the estimates on $\widehat{\widetilde{M}_n}$ given in Lemmas 3.1 and 3.2, while the convergence of $B_n f$ requires the estimates on $\widetilde{B}_{n,m}(x) - H_m(x)$ given by Proposition 4.2.

By Theorem 2.2,

$$\sum_{m=n}^{\infty} \langle (-1)^m G^{(m)}, f \rangle H_m(x)/m! \to 0 \text{ locally uniformly as } n \to \infty,$$

and so we need only to prove that if $f \in H^\nu(\mathbb{R})$ for some $\nu \geq 3/2$, then $A_n f(x)$ and $B_n f(x)$ tend to zero locally uniformly for $x \in \mathbb{R}$ as $n \to \infty$.

Now,

$$\begin{aligned}
\langle (-1)^m G^{(m)}, f \rangle - \langle (-1)^m \widetilde{M}_n^{(m)}, f \rangle &= \int_{-\infty}^{\infty} (-1)^m \left(G^{(m)}(x) - \widetilde{M}_n^{(m)}(x) \right) f(x) dx \\
&= \int_{-\infty}^{\infty} \left(G^{(m)}(-x) - \widetilde{M}_n^{(m)}(-x) \right) f(x) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (iu)^m \left(\widehat{G}(u) - \widehat{\widetilde{M}}_n(u) \right) \overline{\widehat{f}(u)} du,
\end{aligned}$$

which gives

$$\left| \langle (-1)^m G^{(m)}, f \rangle - \langle (-1)^m \widetilde{M}_n^{(m)}, f \rangle \right| \leq \int_{-\infty}^{\infty} |u|^m \left| \widehat{G}(u) - \widehat{\widetilde{M}}_n(u) \right| \left| \widehat{f}(u) \right| du = I_{m,n} + J_{m,n}, \tag{5.5}$$

where

$$\begin{aligned}
I_{m,n} &:= \int_{-n^\alpha}^{n^\alpha} |u|^m \left| \widehat{G}(u) - \widehat{\widetilde{M}}_n(u) \right| \left| \widehat{f}(u) \right| du, \\
J_{m,n} &:= \int_{|u|>n^\alpha} |u|^m \left| \widehat{G}(u) - \widehat{\widetilde{M}}_n(u) \right| \left| \widehat{f}(u) \right| du,
\end{aligned}$$

and $0 < \alpha < 1/4$. It follows from (5.3) and (5.5) that

$$|A_n f(x)| \leq \sum_{m=0}^{n-1} I_{m,n} \frac{|H_m(x)|}{m!} + \sum_{m=0}^{n-1} J_{m,n} \frac{|H_m(x)|}{m!}. \tag{5.6}$$

By Lemma 3.1 and as in the proof of (2.10),

$$\begin{aligned}
I_{m,n} &\leq \frac{K}{n^{1-4\alpha}} \int_{-n^\alpha}^{n^\alpha} |u|^m e^{-u^2/2} |\widehat{f}(u)| du \leq \frac{K \|f\|_{H^\nu}}{n^{1-4\alpha}} \left\{ \int_0^\infty u^{2(m-\nu)} e^{-u^2} du \right\}^{1/2} \\
&\leq \frac{K \|f\|_{H^\nu}}{n^{1-4\alpha}} \Gamma\left(m - \nu + 1/2\right)^{1/2},
\end{aligned} \tag{5.7}$$

for all sufficiently large n, where K is a generic constant independent of m, n. We shall show that

$$\sum_{m=0}^{n-1} I_{m,n} |H_m(x)| G(x)/m! \to 0 \text{ uniformly on } \mathbb{R} \text{ as } n \to \infty.$$

By (2.8),

$$|H_m(x)|G(x) \leq \frac{2^{\frac{m-1}{2}}}{\pi}\Gamma\left((m+1)/2\right).$$

Since

$$\Gamma\left((m+1)/2\right) = \begin{cases} j!, & m = 2j+1, \\ \frac{(2j)!}{2^{2j}j!}\sqrt{\pi}, & m = 2j, \end{cases}$$

$$\begin{aligned}\sum_{m=0}^{n-1} I_{m,n}|H_m(x)|G(x)/m! &\leq \sum_{m=0}^{n-1} I_{m,n}\,\frac{2^{\frac{m-1}{2}}}{\pi m!}\Gamma\left((m+1)/2\right) \\ &= \frac{1}{\sqrt{2\pi}}\sum_{j=0}^{[(n-1)/2]} I_{2j,n}\,\frac{1}{2^j j!} + \frac{1}{\pi}\sum_{j=0}^{[(n-2)/2]} I_{2j+1,n}\,\frac{2^j j!}{(2j+1)!}. \qquad (5.8)\end{aligned}$$

Now, as $H^\nu(\mathbb{R}) \subset H^2(\mathbb{R})$ for $\nu > 2$, we may assume that $3/2 \leq \nu \leq 2$. By the functional relation $\Gamma(x+1) = x\Gamma(x)$,

$$\Gamma\left(m-\nu+1/2\right) = (m-\nu-1/2)(m-\nu-3/2)\cdots(5/2-\nu)\Gamma(5/2-\nu).$$

So using (5.7) and absorbing $\Gamma(5/2-\nu)^{1/2}$ into the generic constant K, for $m = 3, 4, \ldots$,

$$I_{m,n}^2 \leq \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}(m-\nu-1/2)(m-\nu-3/2)\cdots(5/2-\nu).$$

Then for $j = 2, 3, \ldots$,

$$\begin{aligned}\left(I_{2j,n}\,\frac{1}{2^j j!}\right)^2 &\leq \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\prod_{\ell=2}^{j}\frac{2\ell-\nu-1/2}{2\ell}\prod_{\ell=2}^{j}\frac{2\ell-\nu-3/2}{2\ell} \\ &= \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\prod_{\ell=2}^{j}\left(1-\frac{\nu+1/2}{2\ell}\right)\prod_{\ell=2}^{j}\left(1-\frac{\nu+3/2}{2\ell}\right) \\ &\leq \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\prod_{\ell=2}^{j}\exp\left(-\frac{\nu+1/2}{2\ell}\right)\prod_{\ell=2}^{j}\exp\left(-\frac{\nu+3/2}{2\ell}\right) \\ &= \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\exp\left(-\sum_{\ell=2}^{j}\frac{\nu+1}{\ell}\right) \\ &\leq \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\,a^2\exp(-(\nu+1)\ln j) = \frac{K^2a^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}j^{\nu+1}},\end{aligned}$$

for some positive constant a, independent of j. Thus

$$\sum_{j=2}^{[(n-1)/2]} I_{2j,n}\,\frac{1}{2^j j!} \leq \frac{Ka\|f\|_{H^\nu}}{n^{1-4\alpha}}\sum_{j=2}^{[(n-1)/2]}\frac{1}{j^{\frac{\nu+1}{2}}} \leq \frac{K\|f\|_{H^\nu}}{n^{1-4\alpha}}, \qquad (5.9)$$

where K is a generic constant independent of n. Similarly, for $j = 2, 3, \dots,$

$$\begin{aligned}\left(I_{2j+1,n}\,\frac{2^j j!}{(2j+1)!}\right)^2 &\le \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\prod_{\ell=2}^{j}\frac{2\ell+1/2-\nu}{2\ell+1}\prod_{\ell=2}^{j}\frac{2\ell-1/2-\nu}{2\ell+1}\\ &= \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\prod_{\ell=2}^{j}\left(1-\frac{\nu+1/2}{2\ell+1}\right)\prod_{\ell=2}^{j}\left(1-\frac{\nu+3/2}{2\ell+1}\right)\\ &\le \frac{K^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\exp\left(-\sum_{\ell=2}^{j}\frac{2\nu+2}{2\ell+1}\right)\\ &\le \frac{K^2b^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}}\exp(-(\nu+1)\ln j) = \frac{K^2b^2\|f\|_{H^\nu}^2}{n^{2(1-4\alpha)}\,j^{\nu+1}},\end{aligned}$$

for some positive constant b, independent of j. Thus

$$\sum_{j=2}^{[(n-2)/2]} I_{2j+1,n}\,\frac{2^j j!}{(2j+1)!} \le \frac{Kb\|f\|_{H^\nu}}{n^{1-4\alpha}}\sum_{j=2}^{[(n-2)/2]}\frac{1}{j^{\frac{\nu+1}{2}}} \le \frac{K\|f\|_{H^\nu}}{n^{1-4\alpha}}, \tag{5.10}$$

where K is a generic constant independent of n. By (5.8)–(5.10),

$$\sum_{m=0}^{n-1} I_{m,n}|H_m(x)|G(x)/m! \le \frac{K\|f\|_{H^\nu}}{n^{1-4\alpha}}, \quad \text{for all } x\in\mathbb{R}. \tag{5.11}$$

Since $\frac{K\|f\|_{H^\nu}}{n^{1-4\alpha}}$ converges to 0 as $n\to\infty$, the sequence of partial sums on the left converges uniformly on $\mathbb{R}$.

On the other hand,

$$J_{m,n} \le J_{1,m,n} + J_{2,m,n}, \tag{5.12}$$

where

$$J_{1,m,n} := \int_{|u|>n^\alpha}|u|^m e^{-u^2/2}|\widehat{f}(u)|du, \quad J_{2,m,n} := \int_{|u|>n^\alpha}|u|^m\left|\widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right|du.$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned}J_{1,m,n} &\le \left\{\int_{|u|>n^\alpha}|u|^{2(m-\nu)}e^{-u^2}du\right\}^{1/2}\left\{\int_{|u|>n^\alpha}|\widehat{f}(u)|^2|u|^{2\nu}du\right\}^{1/2}\\ &= \left\{\int_{n^{2\alpha}}^{\infty}t^{m-\nu-1/2}e^{-t}dt\right\}^{1/2}\|f\|_{H^\nu_{n^\alpha}} \le \Gamma(m-\nu+1/2)^{1/2}\,\|f\|_{H^\nu_{n^\alpha}},\end{aligned}$$

where $\|f\|_{H^\nu_{n^\alpha}} := \left\{\int_{|u|>n^\alpha}|\widehat{f}(u)|^2|u|^{2\nu}du\right\}^{1/2}$. As in the proof of (5.11),

$$\sum_{m=0}^{n-1} J_{1,m,n}|H_m(x)|G(x)/m! \le K\|f\|_{H^\nu_{n^\alpha}}, \quad \text{for all } x\in\mathbb{R}. \tag{5.13}$$

By Lemma 3.2,

$$\begin{aligned} J_{2,m,n} &\leq \left\{ \int_{|u|>n^\alpha} \left| u^{(m-\nu)} \widehat{\widetilde{M}}_n(u) \right|^2 du \right\}^{1/2} \|f\|_{H^\nu_{n^\alpha}} \\ &\leq \frac{n^{m\alpha}\|f\|_{H^\nu_{n^\alpha}}}{e^{\frac{1}{2}n^{2\alpha}}} \left\{ \int_{|u|>n^\alpha} \frac{1}{|u|^{2\nu}} du \right\}^{1/2} \leq \frac{K n^{\alpha(m-\nu+1/2)}\|f\|_{H^\nu_{n^\alpha}}}{e^{\frac{1}{2}n^{2\alpha}}}, \end{aligned} \tag{5.14}$$

for all sufficiently large n, where K is a generic constant that is independent of m, n. It follows from (2.8) and (5.14) that

$$\sum_{m=0}^{n-1} J_{2,m,n} |H_m(x)| G(x)/m! \leq \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{\alpha(\nu-1/2)} e^{\frac{1}{2}n^{2\alpha}}} \sum_{m=0}^{n-1} \frac{n^{\alpha m}}{m!} 2^{(m-1)/2} \Gamma\left(\frac{m+1}{2}\right),$$

and since

$$2^{(m-1)/2}\Gamma\left((m+1)/2\right) = \begin{cases} 2^j j!, & m = 2j+1, \\ \frac{(2j)!}{2^j j!}\sqrt{\frac{\pi}{2}}, & m = 2j, \end{cases}$$

this gives

$$\begin{aligned} \sum_{m=0}^{n-1} J_{2,m,n}|H_m(x)|G(x)/m! &\leq \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{\alpha(\nu-1/2)} e^{\frac{1}{2}n^{2\alpha}}} \left(\sum_{j=0}^{[(n-1)/2]} \frac{n^{2j\alpha}}{2^j j!} + \sum_{j=0}^{[(n-2)/2]} \frac{n^{(2j+1)\alpha}\, 2^j j!}{(2j+1)!} \right) \\ &\leq \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{\alpha(\nu-1/2)} e^{\frac{1}{2}n^{2\alpha}}} \left(\sum_{j=0}^{[(n-1)/2]} \frac{n^{2j\alpha}}{2^j j!} + n^\alpha \sum_{j=0}^{[(n-2)/2]} \frac{n^{2j\alpha}}{2^j j!} \right) \\ &\leq \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{\alpha(\nu-1/2)} e^{\frac{1}{2}n^{2\alpha}}} \left(e^{\frac{1}{2}n^{2\alpha}} + n^\alpha e^{\frac{1}{2}n^{2\alpha}} \right). \end{aligned} \tag{5.15}$$

Hence, by (5.12), (5.13) and (5.15), for all $x \in \mathbb{R}$,

$$\sum_{m=0}^{n-1} J_{m,n}|H_m(x)|G(x)/m! \leq 3K\|f\|_{H^\nu_{n^\alpha}}. \tag{5.16}$$

Combining (5.11) and (5.16) gives

$$\sum_{m=0}^{n-1} (I_{m,n} + J_{m,n})|H_m(x)|G(x)/m! \leq \frac{K\|f\|_{H^\nu}}{n^{1-4\alpha}} + 3K\|f\|_{H^\nu_{n^\alpha}},$$

for all $x \in \mathbb{R}$. Since $\alpha < 1/4$, the sequences on the right converge to 0 as $n \to \infty$, and so the sequence of partial sums on the left converges uniformly on $\mathbb{R}$. By (5.6), $A_n f(x) \to 0$ locally uniformly as $n \to \infty$.

Finally we show that $B_n f(x) \to 0$ locally uniformly on $\mathbb{R}$. By (4.3),

$$\left| \sum_{m=0}^{n-1} \langle (-1)^m \widetilde{M}_n^{(m)}, f\rangle \left(\frac{H_m(x) - \widetilde{B}_{n,m}(x)}{m!} \right) e^{-x^2} \right| \leq \frac{K}{n} \sum_{m=0}^{n-1} \left| \langle (-1)^m \widetilde{M}_n^{(m)}, f\rangle \right| \left(\frac{e}{2m} \right)^{m/2},$$

where K is a generic constant that is independent of m, n. Since $\langle(-1)^m \widetilde{M}_n^{(m)}, f\rangle = \int_{-\infty}^{\infty}(iu)^m \widehat{\widetilde{M}}_n(u)\widehat{f}(u)du$,

$$\begin{aligned}\left|\langle(-1)^m \widetilde{M}_n^{(m)}, f\rangle\right| &\leq \int_{-\infty}^{\infty}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du \\ &= \int_{-n^\alpha}^{n^\alpha}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du + \int_{|u|>n^\alpha}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du,\end{aligned}$$

where $0 < \alpha < 1/2$. So

$$\left|\sum_{m=0}^{n-1}\langle(-1)^m \widetilde{M}_n^{(m)}, f\rangle\left(\frac{H_m(x) - \widetilde{B}_{n,m}(x)}{m!}\right)e^{-x^2}\right| \leq T_{n,1} + T_{n,2}, \tag{5.17}$$

where

$$T_{n,1} := \frac{K}{n}\sum_{m=0}^{n-1}\left(\frac{e}{2m}\right)^{m/2}\int_{-n^\alpha}^{n^\alpha}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du, \tag{5.18}$$

$$T_{n,2} := \frac{K}{n}\sum_{m=0}^{n-1}\left(\frac{e}{2m}\right)^{m/2}\int_{|u|>n^\alpha}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du. \tag{5.19}$$

By (3.1) and as in the derivation of (5.7),

$$\int_{-n^\alpha}^{n^\alpha}\left|u^m \widehat{\widetilde{M}}_n(u)\right|\left|\widehat{f}(u)\right| du \leq \int_{-n^\alpha}^{n^\alpha}|u|^m e^{-u^2/2}\left|\widehat{f}(u)\right| du \leq K\|f\|_{H^\nu}\,\Gamma(m-\nu+1/2)^{1/2},$$

and so by (5.18),

$$T_{n,1} \leq \frac{K\|f\|_{H^\nu}}{n}\sum_{m=0}^{n-1}\Gamma(m-\nu+1/2)^{1/2}\left(\frac{e}{2m}\right)^{m/2}.$$

The series on the right converges as $n \to \infty$ by the ratio test, and so

$$T_{n,1} \leq \frac{K\|f\|_{H^\nu}}{n}. \tag{5.20}$$

Also as in the proof of (5.14) and (5.15), (5.19) leads to

$$\begin{aligned}T_{n,2} &\leq \frac{K\,e^{-\frac{1}{2}n^{2\alpha}}\,\|f\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}}\sum_{m=0}^{n-1}n^{m\alpha}\left(\frac{e}{2m}\right)^{m/2} \\ &\leq \frac{K\,e^{-\frac{1}{2}n^{2\alpha}}\,\|f\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}}\left\{\sum_{m=0}^{n-1}\frac{n^{2m\alpha}}{m!}\right\}^{1/2}\left\{\sum_{m=0}^{n-1}m!\left(\frac{e}{2m}\right)^m\right\}^{1/2} \\ &\leq \frac{K\,e^{-\frac{1}{2}n^{2\alpha}}\,\|f\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}}\,e^{\frac{1}{2}n^{2\alpha}}\left\{\sum_{m=0}^{n-1}m!\left(\frac{e}{2m}\right)^m\right\}^{1/2} \leq \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}},\end{aligned} \tag{5.21}$$

since $\sum_{m=0}^{n-1}m!\left(\frac{e}{2m}\right)^m$ converges as $n \to \infty$. Combining (5.17), (5.20) and (5.21) gives

$$\left|\sum_{m=0}^{n-1}\langle(-1)^m \widetilde{M}_n^{(m)}, f\rangle\left(\frac{H_m(x) - \widetilde{B}_{n,m}(x)}{m!}\right)e^{-x^2}\right| \leq \frac{K\|f\|_{H^\nu}}{n} + \frac{K\|f\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}}, \quad \text{for all } x \in \mathbb{R}. \tag{5.22}$$

It follows that the sequence of partial sums on the left of (5.22) converges to 0 uniformly on $\mathbb{R}$ as $n \to \infty$. Hence, $B_n f(x)$ in (5.4) converges locally uniformly on $\mathbb{R}$. □

Proof of Theorem 1.1. For $x, t \in \mathbb{R}$, $s > 0$, let $f_{s,t}(x) := f(t - x/s)$. Then $\widehat{f}_{s,t}(u) = s\widehat{f}(-us)e^{-iust}$, $u \in \mathbb{R}$, and $f_{s,t} \in H^\nu(\mathbb{R})$ if $f \in H^\nu(\mathbb{R})$. As in (5.1)–(5.4),

$$f_{s,t}(x) = \sum_{m=0}^{n-1} \langle (-1)^m \widetilde{M}_n^{(m)}, f_{s,t}\rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + R_n f_{s,t}(x),$$

where

$$\begin{aligned} R_n f_{s,t}(x) &= \sum_{m=0}^{\infty} \langle (-1)^m G^{(m)}, f_{s,t}\rangle H_m(x)/m! - \sum_{m=0}^{n-1} \langle (-1)^m \widetilde{M}_n^{(m)}, f_{s,t}\rangle \; \widetilde{B}_{n,m}(x)/m! \\ &= A_n f_{s,t}(x) + B_n f_{s,t}(x) + \sum_{m=n}^{\infty} \langle (-1)^m G^{(m)}, f_{s,t}\rangle H_m(x)/m!. \end{aligned} \tag{5.23}$$

By (2.12),

$$\left| \sum_{m=n}^{\infty} \langle (-1)^m G^{(m)}, f_{s,t}\rangle H_m(x)/m! \right| \le C\|f_{s,t}\|_{H^\nu} e^{x^2/2} \sum_{m=n}^{\infty} \frac{1}{m^{(1+\nu)/2}}. \tag{5.24}$$

As in the proof of Theorem 5.1, for $\alpha < 1/4$,

$$|A_n f_{s,t}(x)| \le K\, e^{x^2/2} \left\{ \frac{\|f_{s,t}\|_{H^\nu}}{n^{1-4\alpha}} + 3\|f_{s,t}\|_{H^\nu_{n^\alpha}} \right\}, \tag{5.25}$$

and for $\alpha < 1/2$,

$$|B_n f_{s,t}(x)| \le K\, e^{x^2} \left\{ \frac{\|f_{s,t}\|_{H^\nu}}{n} + \frac{\|f_{s,t}\|_{H^\nu_{n^\alpha}}}{n^{1+\alpha(\nu-1/2)}} \right\}. \tag{5.26}$$

Since

$$\|f_{s,t}\|_{H^\nu} \le \begin{cases} \frac{1}{s^{\nu-1/2}}\|f\|_{H^\nu}, & 0 < s < 1, \\ s^{1/2}\|f\|_{H^\nu}, & s \ge 1, \end{cases}$$

and

$$\|f_{s,t}\|_{H^\nu_{n^\alpha}} = \frac{1}{s^{\nu-1/2}}\|f\|_{H^\nu_{sn^\alpha}},$$

it follows from (5.23)–(5.26) that $R_n f_{s,t}(x)$ converges to 0 uniformly for $t \in \mathbb{R}$ and locally uniformly for $x \in \mathbb{R}$ and $s > 0$. □

6. Approximation of derivatives of B-splines by spline framelets

In this section we show that the spline framelets constructed by Ron and Shen [3], when suitably standardized, approximate the derivatives of the standardized B-splines and converge to the derivatives of the Gaussian function as the orders of the spline functions tend to infinity with a convergence rate of $O(\frac{1}{n^{1-\epsilon/p}})$, $0 < \epsilon < 1$, in the L^p-norm, $1 \leq p \leq \infty$. Bernoulli-Schoenberg series expansion formulas of the form (1.5) and (5.1) are then derived with the standardized spline framelets as mother wavelets instead of the derivatives of the B-splines in the wavelet transforms.

The Fourier transform of the standardized B-splines $\widetilde{M}_n$ is

$$\widehat{\widetilde{M}}_n(u) = \left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^n, \quad u \in \mathbb{R}.$$

For integers $n \geq 1$ $m = 0, 1, \ldots, n$, we define

$$\begin{aligned}\widehat{\widetilde{\psi}}_{n,m}(u) &:= \left(\frac{4n}{3}\right)^{m/2} i^m \left(\sin(u\sqrt{3/4n})\right)^m \left(\cos(u\sqrt{3/4n})\right)^{n-m} \widehat{\widetilde{M}}_n(u/2) \\ &= (iu)^m \left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m} \left(\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}}\right)^{2m}, \quad u \in \mathbb{R}. \end{aligned} \tag{6.1}$$

For even n, $\widetilde{\psi}_{n,m}$ are, up to constant multiples and a constant dilation, the spline framelets of Ron and Shen [3]. They are also, up to constant multiples and a constant dilation, the mother wavelets for the semi-discrete wavelet transforms arising from the B-spline scale-space considered in [2]. We shall call $\widetilde{\psi}_{n,m}$ the *standardized Ron-Shen framelets,* and show that they approximate the derivatives $\widetilde{M}_n^{(m)}$ of the standardized B-splines and also converge to the derivatives of the Gaussian function as $n \to \infty$.

Theorem 6.1. *For any nonnegative integer m,*

$$\sup_{u\in\mathbb{R}} \left|\widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u)\right| = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{6.2}$$

Proof. The Fourier transforms of the derivatives of the standardized B-spline can be expressed as

$$\begin{aligned}\widehat{\widetilde{M}_n^{(m)}}(u) &= (iu)^m \left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^n \\ &= (iu)^m \left(\frac{\sin u\sqrt{3/n}}{u\sqrt{3/n}}\right)^{n-m} \left(\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}}\right)^m \left(\cos(u\sqrt{3/4n})\right)^m . \end{aligned} \tag{6.3}$$

By (6.1) and (6.3) ,

$$\begin{aligned}\left|\widehat{\widehat{\psi}}_{n,m}(u)-\widehat{\widetilde{M}_n^{(m)}}(u)\right| &= \left|u^m\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right|\left|\left(\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}}\right)^m\right| F_{m,n}(u)\\ &\le \left|u^m\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| F_{m,n}(u),\end{aligned}$$

where

$$\begin{aligned}F_{m,n}(u) &:= \left|\left(\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}}\right)^m-\left(\cos(u\sqrt{3/4n})\right)^m\right|\\ &\le m\left|\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}}-\cos(u\sqrt{3/4n})\right|\\ &= m\left|\sum_{j=1}^{\infty}(-1)^{j-1}\frac{2j}{(2j+1)!}\left(u\sqrt{3/4n}\right)^{2j}\right|\le\frac{Kmu^2}{n},\end{aligned}$$

for all positive integers $n \ge m$ and $u \in \mathbb{R}$, where K is a constant independent of m, n and u. Hence

$$\left|\widehat{\widehat{\psi}}_{n,m}(u)-\widehat{\widetilde{M}_n^{(m)}}(u)\right| \le \frac{Km}{n}\left|u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right|, \quad u\in\mathbb{R}. \tag{6.4}$$

For $|u| \le \sqrt{n}$, the proof of (3.1) in Lemma 3.1 gives

$$\left|u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| \le |u|^{m+2}e^{-(n-m)u^2/2n}. \tag{6.5}$$

The maximum of the expression on the right is attained at $u=\sqrt{\frac{n(m+2)}{n-m}}$, which is less than $\sqrt{n}$ for $n > 2(m+1)$, and so

$$\left|u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| \le \left(\frac{m+2}{1-m/n}\right)^{(m+2)/2}e^{-(m+2)/2}. \tag{6.6}$$

For $|u| \ge n^\alpha$, $0 < \alpha < 1/2$, similar arguments as in the proof of Lemma 3.2 show that if $n > 2(m+1)$,

$$\left|u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| \le n^{\alpha(m+2)}e^{-n^{2\alpha}/2}e^{m/2n^{1-2\alpha}} \le Kn^{\alpha(m+2)}e^{-n^{2\alpha}/2} \tag{6.7}$$

for all sufficiently large n, where K is a generic constant independent of n and u. The estimate (6.2) now follows from (6.4), (6.6) and (6.7). □

Corollary 6.2. *For any nonnegative integer m,*

$$\sup_{u\in\mathbb{R}}\left|\widehat{\widehat{\psi}}_{m,n}(u)-(iu)^m e^{-u^2/2}\right| = O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty.$$

Proof. The result follows from Theorems 3.3 and 6.1. □

Theorem 6.3. *For any integer* $m \geq 0$,

$$\left\|\widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)}\right\|_\infty = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{6.8}$$

Proof. By taking inverse Fourier transforms, for any positive integer $n \geq m$ and $x \in \mathbb{R}$,

$$\widetilde{\psi}_{n,m}(x) - \widetilde{M}_n^{(m)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\left(\widehat{\widetilde{\psi}}_{m,n}(u) - \widehat{\widetilde{M}_n^{(m)}}(u)\right) du$$

and so

$$\begin{aligned}\left|\widetilde{\psi}_{n,m}(x) - \widetilde{M}_n^{(m)}(x)\right| &\leq \frac{1}{\pi}\int_0^\infty \left|\widehat{\widetilde{\psi}}_{m,n}(u) - \widehat{\widetilde{M}_n^{(m)}}(u)\right| du \\ &\leq \frac{K}{\pi n}\int_0^\infty \left| u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| du, \end{aligned} \tag{6.9}$$

by (6.4). For a positive $\alpha < 1/2$, a similar argument as in the proof of Theorem 3.4 using (6.5) gives

$$\begin{aligned}\int_0^{n^\alpha} \left| u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| du &\leq \int_0^{n^\alpha} u^{m+2} e^{-(n-m)u^2/2n} du \\ &\leq 2^{m+2}\Gamma((m+3)/2), \quad \text{for } n > 2m, \end{aligned} \tag{6.10}$$

and

$$\int_{n^\alpha}^\infty \left| u^{m+2}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{n-m}\right| du \leq C\, n^{(m+3)\alpha}\, e^{-n^{2\alpha}/2}, \tag{6.11}$$

where C is a constant independent of n. It follows from (6.9)–(6.11) that

$$\left|\widetilde{\psi}_{n,m}(x) - \widetilde{M}_n^{(m)}(x)\right| \leq \frac{K}{\pi n} 2^{m+2}\Gamma((m+3)/2) + \frac{CK}{\pi n} n^{(m+3)\alpha}\, e^{-n^{2\alpha}/2},$$

which gives (6.8). □

Corollary 6.4. *For any integer* $m \geq 0$,

$$\left\|\widetilde{\psi}_{n,m} - G^{(m)}\right\|_\infty = O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{6.12}$$

Proof. The estimate (6.12) follows from Theorems 3.4 and 6.3 . □

Remark 3. *The problem of approximating the spline framelets of Ron and Shen by derivatives of the Gaussian function is also considered recently by Shen and Xu in* [5]. *However, their results are different from ours. They consider the nonstandardized Gaussian function*

$$G_{n,\ell}(t) := \sqrt{\binom{n}{\ell}\frac{12}{\pi(2n-\ell)4^{2\ell}}}\exp\left(-\frac{12t^2}{2n-\ell}\right), \quad t \in \mathbb{R},$$

and show that for large n and $\ell = 1, 2, \ldots, n$, the derivatives $\frac{d^\ell}{dt^\ell} G_{n,\ell}(t - j_n/2)$, where $j_n = 0$ or 1 depending on n is even or odd respectively, approximate the original nonstandardized spline generators $\psi_{n,\ell}$ uniformly. An error estimate is also given.

We now give the rate of convergence in the L^p-norm.

Theorem 6.5. *Take any integer $m \geq 0$ and $1 \leq p < \infty$. Then for any $0 < \epsilon < 1$,*

$$\left\| \widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)} \right\|_p = O\left(\frac{1}{n^{1-\epsilon/p}} \right) \quad \text{as } n \to \infty. \tag{6.13}$$

Proof. By Theorem 6.3, for any positive $\epsilon < 1$ and for all positive integers $n \geq m$,

$$\int_{-n^\epsilon}^{n^\epsilon} \left| \widetilde{\psi}_{n,m}(x) - \widetilde{M}_n^{(m)}(x) \right| dx \leq 2n^\epsilon \left\| \widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)} \right\|_\infty \leq \frac{C}{n^{1-\epsilon}},$$

for some constant $C > 0$. Then for all positive integers $n \geq m$,

$$\left| \int_0^{n^\epsilon} \widetilde{\psi}_{n,m}(x) dx - \int_0^{n^\epsilon} \widetilde{M}_n^{(m)}(x) dx \right| \leq \frac{C}{n^{1-\epsilon}}. \tag{6.14}$$

Now for even m,

$$\int_0^\infty \widetilde{M}_n^{(m)}(x) dx = \int_0^\infty \widetilde{\psi}_{n,m}(x) dx = \begin{cases} \frac{1}{2}, & m = 0, \\ 0, & m > 0. \end{cases}$$

For odd m, defining $g_{n,m-1}$ by

$$\widehat{g}_{n,m-1}(u) := (iu)^{m-1} \left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}} \right)^{n-m} \left(\frac{\sin(u\sqrt{3/4n})}{u\sqrt{3/4n}} \right)^{2m},$$

we have, by (6.1), its derivative $g'_{n,m-1} = \widetilde{\psi}_{n,m}$. Hence

$$\int_0^\infty \widetilde{\psi}_{n,m}(x) dx - \int_0^\infty \widetilde{M}_n^{(m)}(x) dx = -g_{n,m-1}(0) + \widetilde{M}_n^{(m-1)}(0).$$

The proof of Theorem 6.3 gives

$$\left\| g_{n,m-1} - \widetilde{M}_n^{(m-1)} \right\|_\infty = O\left(\frac{1}{n} \right),$$

and so for odd m,

$$\int_0^\infty \widetilde{\psi}_{n,m}(x) dx - \int_0^\infty \widetilde{M}_n^{(m)}(x) dx = O\left(\frac{1}{n} \right).$$

Therefore, for any integer $m \geq 0$,

$$\left| \int_0^\infty \widetilde{\psi}_{n,m}(x) dx - \int_0^\infty \widetilde{M}_n^{(m)}(x) dx \right| \leq \frac{C}{n}. \tag{6.15}$$

It follows from (6.14) and (6.15) that

$$\left| \int_{n^\epsilon}^\infty \widetilde{\psi}_{n,m}(x) dx - \int_{n^\epsilon}^\infty \widetilde{M}_n^{(m)}(x) dx \right| = O\left(\frac{1}{n^{1-\epsilon}} \right).$$

Then similar arguments as in the proof of Theorem 3.5 give

$$\left\| \widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)} \right\|_1 = O\left(\frac{1}{n^{1-\epsilon}} \right),$$

and hence (6.13). □

Corollary 6.6. *Take any integer $m \geq 0$ and $1 \leq p < \infty$. Then for any $0 < \epsilon < 1$,*

$$\left\| \widetilde{\psi}_{n,m} - G^{(m)} \right\|_p = O\left(\frac{1}{n^{1-\epsilon/p}} \right) \quad \text{as } n \to \infty.$$

Proof. The result follows from Theorems 3.5 and 6.5. □

Since the standardized spline framelets $\widetilde{\psi}_{n,m}$ approximate the derivatives $\widetilde{M}_n^{(m)}$ of the standardized B-splines, it is of interest to investigate to what extent the latter can be replaced by the former in (1.5) and (5.1) of Theorems 1.1 and 5.1 respectively. The next theorem gives a result of this form.

Theorem 6.7. *If $f \in H^\nu(\mathbb{R})$ for some $\nu \geq 3/2$ and is continuous, then*

$$f(x) = \sum_{0 \leq m < \sqrt{n}} \langle (-1)^m \widetilde{\psi}_{n,m}, f \rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + \widetilde{R}_n f(x),$$

where $\widetilde{R}_n f(x) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$.

Proof. The proof uses Theorem 5.1 and similar techniques as in its proof as well as the results on the approximation of the derivatives of the standardized B-splines by the standardized spline framelets. If $f \in H^\nu(\mathbb{R})$, $\nu \geq 3/2$, and is continuous, by the proof of Theorem 5.1 (with the sum over $0 \leq m < \sqrt{n}$ replacing the sum over $0 \leq m \leq n-1$),

$$f(x) = \sum_{0 \leq m < \sqrt{n}} \langle (-1)^m \widetilde{M}_n^{(m)}, f \rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + R_n f(x),$$

where $R_n f(x) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$. Recalling that $\widetilde{\psi}_{n,0} = \widetilde{M}_n$, this gives

$$\begin{aligned} f(x) &= \sum_{0 \leq m < \sqrt{n}} \langle (-1)^m \widetilde{\psi}_{n,m}, f \rangle \frac{\widetilde{B}_{n,m}(x)}{m!} \\ &\quad - \sum_{1 \leq m < \sqrt{n}} (-1)^m \langle \widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)}, f \rangle \frac{\widetilde{B}_{n,m}(x)}{m!} + R_n f(x). \end{aligned}$$

Let

$$S_n(x) := \sum_{1 \leq m < \sqrt{n}} (-1)^{m+1} \langle \widetilde{\psi}_{n,m} - \widetilde{M}_n^{(m)}, f \rangle \frac{\widetilde{B}_{n,m}(x)}{m!}, \quad \widetilde{R}_n f(x) := S_n(x) + R_n f(x).$$

We want to show that $S_n(x) \to 0$ locally uniformly for $x \in \mathbb{R}$, as $n \to \infty$. Applying Parseval's identity,

$$\begin{aligned} |S_n(x)| &\leq \sum_{1\leq m<\sqrt{n}} \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right) \overline{\widehat{f}(u)} du \right| \frac{|\widetilde{B}_{n,m}(x)|}{m!} \\ &\leq \frac{1}{2\pi} \sum_{1\leq m<\sqrt{n}} \left\{ \int_{-n^\alpha}^{n^\alpha} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right| \left| \widehat{f}(u) \right| du \right. \\ &\quad \left. + \int_{|u|>n^\alpha} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right| \left| \widehat{f}(u) \right| du \right\} \frac{|\widetilde{B}_{n,m}(x)|}{m!} \\ &= \sum_{1\leq m<\sqrt{n}} (I_{m,n,1} + I_{m,n,2}) \frac{|\widetilde{B}_{n,m}(x)|}{m!}, \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} I_{m,n,1} &:= \frac{1}{2\pi} \int_{-n^\alpha}^{n^\alpha} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right| \left| \widehat{f}(u) \right| du, \\ I_{m,n,2} &:= \frac{1}{2\pi} \int_{|u|>n^\alpha} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right| \left| \widehat{f}(u) \right| du, \end{aligned}$$

and $0 < \alpha < 1/4$. For $1 \leq m < \sqrt{n}$, by Proposition 4.2,

$$\left| \frac{\widetilde{B}_{n,m}(x) - H_m(x)}{m!} \right| \leq \frac{K}{n} \left(\frac{e}{2m} \right)^{m/2} e^{x^2}$$

where K is a generic constant independent of m, n, and so this together with (2.9) give

$$\begin{aligned} \frac{|\widetilde{B}_{n,m}(x)|}{m!} &\leq \frac{K}{n} \left(\frac{e}{2m} \right)^{m/2} e^{x^2} + \frac{|H_m(x)|}{m!} \\ &\leq \frac{K}{n} \left(\frac{e}{2m} \right)^{m/2} e^{x^2} + \frac{K}{\sqrt{m}} \left(\frac{e}{m} \right)^{m/2} e^{x^2/2} \\ &\leq \frac{K}{\sqrt{m}} \left(\frac{e}{m} \right)^{m/2} e^{x^2}. \end{aligned} \tag{6.17}$$

Now,

$$\begin{aligned} I_{m,n,1} &= \frac{1}{2\pi} \int_{-n^\alpha}^{n^\alpha} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right| \left| \widehat{f}(u) \right| du \\ &\leq \frac{\|f\|_{H^\nu}}{\sqrt{2\pi}} \left\{ \int_{-n^\alpha}^{n^\alpha} (1+|u|^2)^{-\nu} \left| \widehat{\widetilde{\psi}}_{n,m}(u) - \widehat{\widetilde{M}_n^{(m)}}(u) \right|^2 du \right\}^{1/2}. \end{aligned}$$

By (6.4) and using the same method as in the proof of (6.10),

$$\begin{aligned} I_{m,n,1} &\le \frac{Km\|f\|_{H^\nu}}{n}\left\{\int_{-n^\alpha}^{n^\alpha}|u|^{2m+4-2\nu}\left(\frac{\sin(u\sqrt{3/n})}{u\sqrt{3/n}}\right)^{2(n-m)}du\right\}^{1/2} \\ &\le \frac{Km\|f\|_{H^\nu}}{n}\left\{\int_0^{n^\alpha}u^{2m+4-2\nu}e^{-\frac{(n-m)}{n}u^2}du\right\}^{1/2} \\ &\le \frac{Km\|f\|_{H^\nu}}{n}\left(\frac{n}{n-m}\right)^{(2m+5-2\nu)/4}\Gamma(m+5/2-\nu)^{1/2}. \end{aligned}$$

For $m<\sqrt{n}$, the second factor in the last expression is bounded, and so

$$I_{m,n,1}\le\frac{Km\|f\|_{H^\nu}}{n}\Gamma(m+5/2-\nu)^{1/2}. \tag{6.18}$$

Also by (6.4) and (6.7), for $|u|>n^\alpha$, $m<\sqrt{n}$ and $\alpha<1/4$,

$$\begin{aligned} \left|\widehat{\widetilde{\psi}}_{n,m}(u)-\widehat{\widetilde{M}_n^{(m)}}(u)\right| &\le \frac{Km}{n}n^{\alpha(m+2)}e^{-n^{2\alpha}/2}e^{m/2n^{1-2\alpha}} \\ &\le \frac{K}{n^{(1-4\alpha)/2}}n^{m\alpha}e^{-n^{2\alpha}/2}e^{1/2n^{(1-4\alpha)/2}}\le\frac{K}{n^{(1-4\alpha)/2}}n^{m\alpha}e^{-n^{2\alpha}/2}, \end{aligned}$$

and so

$$\begin{aligned} I_{m,n,2} &\le \frac{\|f\|_{H^\nu}}{\sqrt{2\pi}}\left\{\int_{|u|>n^\alpha}|u|^{-2\nu}\left|\widehat{\widetilde{\psi}}_{n,m}(u)-\widehat{\widetilde{M}_n^{(m)}}(u)\right|^2du\right\}^{1/2} \\ &\le \frac{K\|f\|_{H^\nu}}{n^{(1-4\alpha)/2}}n^{m\alpha}e^{-n^{2\alpha}/2}\left\{\int_{|u|>n^\alpha}|u|^{-2\nu}du\right\}^{1/2} \\ &\le \frac{K\|f\|_{H^\nu}}{n^{(1-4\alpha)/2}}n^{\alpha(m-\nu+1/2)}e^{-n^{2\alpha}/2}. \end{aligned} \tag{6.19}$$

It follows from (6.16)–(6.19) that

$$\begin{aligned} |S_n(x)| &\le \frac{K\|f\|_{H^\nu}}{n}\sum_{1\le m<\sqrt{n}}m\Gamma(m+5/2-\nu)^{1/2}\frac{1}{\sqrt{m}}\left(\frac{e}{m}\right)^{m/2}e^{x^2} \\ &+\frac{K\|f\|_{H^\nu}}{n^{(1-4\alpha)/2}}\sum_{1\le m<\sqrt{n}}n^{\alpha(m-\nu+1/2)}e^{-n^{2\alpha}/2}\frac{1}{\sqrt{m}}\left(\frac{e}{m}\right)^{m/2}e^{x^2}=:S_{n,1}(x)+S_{n,2}(x). \end{aligned}$$

Noting that for $\nu\ge 3/2$, $\Gamma(m+5/2-\nu)\le\Gamma(m+1)$, and applying the Stirling formula,

$$\begin{aligned} S_{n,1}(x) &\le \frac{K\|f\|_{H^\nu}}{n}\sum_{1\le m<\sqrt{n}}m^{3/4}\left(\frac{m}{e}\right)^{m/2}\left(\frac{e}{m}\right)^{m/2}e^{x^2} \\ &= \frac{K\|f\|_{H^\nu}}{n}\sum_{1\le m<\sqrt{n}}\frac{m^{15/8}}{m^{9/8}}e^{x^2}\le\frac{K\|f\|_{H^\nu}}{n^{1/16}}\sum_{1\le m<\sqrt{n}}\frac{1}{m^{9/8}}e^{x^2} \\ &\le \frac{K\|f\|_{H^\nu}}{n^{1/16}}e^{x^2}\to 0\ \text{ locally uniformly for } x\in\mathbb{R}. \end{aligned}$$

Applying exactly the arguments of the proof of (5.15) for the second sum,

$$\begin{aligned} |S_{n,2}(x)| &\leq \frac{K\|f\|_{H^\nu}}{n^{(1-4\alpha)/2}} \frac{1+n^\alpha}{n^{\alpha(\nu-1/2)}} e^{x^2} \\ &\leq \frac{K\|f\|_{H^\nu}}{n^{(1-4\alpha)/2}} e^{x^2}, \end{aligned}$$

since $\nu \geq 3/2$. Therefore $S_{n,2}(x)$ and hence $S_n(x)$ and $\widetilde{R}_n f(x)$ converge locally uniformly to 0 as $n \to \infty$. □

We conclude with the following theorem, which can be derived from the proof of Theorem 6.7 in the same way as Theorem 1.1 from the proof of Theorem 5.1.

Theorem 6.8. *If $f \in H^\nu(\mathbb{R})$ for some $\nu \geq 3/2$ and is continuous, then for $x, t \in \mathbb{R}$, $s > 0$,*

$$f(x) = S_{\widetilde{M}_n} f(s,t) + \sum_{1 \leq m < \sqrt{n}} \frac{(-1)^m \widetilde{B}_{n,m}(s(t-x))}{m!} W_{\widetilde{\psi}_{n,m}} f(s,t) + \widetilde{R}_n f(x,s,t),$$

where $\widetilde{R}_n f(x,s,t) \to 0$ as $n \to \infty$ locally uniformly for $x \in \mathbb{R}$ and $s > 0$ and uniformly for $t \in \mathbb{R}$, and $S_{\widetilde{M}_n} f(s,t)$ is the B-spline scale-space transform of f, $W_{\widetilde{\psi}_{n,m}} f(s,t)$ are continuous wavelet transforms of f with the standardized spline framelets as mother wavelets.

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