SINGULAR INTEGRALS, SCALE-SPACE AND WAVELET TRANSFORMS

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Abstract. The Gaussian scale-space is a singular integral convolution operator with scaled Gaussian kernel. For a large class of singular integral convolution operators with differentiable kernels, a general method for constructing mother wavelets for continuous wavelet transforms is developed, and Calderón type inversion formulas, in both integral and semi-discrete forms, are derived for functions in L^p spaces. In the case of the Gaussian scale-space, the semi-discrete inversion formula can further be expressed as a sum of wavelet transforms with the even order derivatives of the Gaussian as mother wavelets. Similar results are obtained for B-spline scale-space, in which the high frequency component of a function between two consecutive dyadic scales can be represented as a finite linear combination of wavelet transforms with the derivatives of the B-spline or the spline framelets of Ron and Shen as mother wavelets.

Key words: singular integral operators, scale-space, wavelet transforms, Gaussian scale-space and wavelet transforms, B-spline scale-space, wavelets and framelets.

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1. Introduction

The Fourier transform transforms a signal that encodes information in time into a function that expresses the information in terms of its frequency. The local behavior in time of the signal is completely lost in its frequency representation. The *short-time Fourier transform* (also called *windowed Fourier transform* or *Gabor transform*) is a way to capture the information of a signal in both time and frequency in its frequency representation. It applies the Fourier transform to segments of the signal, i.e. to different time durations of the signal within a certain moving window by shifting a locally supported function, which is often symmetrical about the origin (see [4, 5, 19]). The *continuous wavelet transforms* were developed by Grossmann and Morlet [17] in the early 80's to address some of the deficiencies of short-time Fourier transforms. A continuous wavelet transform is an integral transform similar to the windowed Fourier transform, where modulated window kernels are replaced by the dilations and shifts of a localized oscillating kernel called a *mother wavelet* (see [4, 5, 9, 18, 19]). An important example of a mother wavelet is the *Mexican hat function,* $\omega(t) := \frac{1}{\sqrt{2\pi}}(1-t^2)e^{-t^2/2}$, $t \in \mathbb{R}$, which is actually the negative of the second derivative of the Gaussian function, $G(t) := \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

A mathematical concept similar to time-frequency representation had already been developed before wavelets. Known as *scale-space* or *scale-space representation*, it was inspired by the visual system of higher mammals, that sees different structures in images and objects at different scales (see [30, 41]). The main motivation for this development was the automatic extraction of image structures from image data for the purpose of image processing and computer vision. Scale-space is a method of handling multiscale structures in such a way that fine-scale features can be successively filtered away while the large-scale features emerge (see [25, 26]). The idea of scale-space was first introduced by Iijima in 1959 (see [39]) and rediscovered independently by Witkin [40] in 1983 and Koenderink [22] in 1984. Initially developed as a linear filtering by the Gaussian function for computer vision, scale-space has been extended to linear filtering by a one-parameter family of kernels that includes the Gaussian (see [11, 12, 13, 29, 33] and the references therein), as well as to nonlinear diffusion process, for the purpose of image and signal processing (see [21, 34, 38]). For a given signal f, the standard Gaussian scale-space representation is a family of signals formed by convolving f with the scaled Gaussian $\frac{1}{\sqrt{2r}}G(t/\sqrt{2r})$, where $r > 0$ represents the scale (see [25, 26]). The Gaussian scale-space transform is a singular integral operator of convolution type. The objective of this paper is to study a class of such singular integral operators that approximate the Gaussian scale-space transform and explore their relationships with the continuous and semi-discrete wavelet transforms for representation of functions in frequency bands and for signal recovery.

For a kernel $\phi \in L^1(\mathbb{R})$, with $\int_{-\infty}^{\infty} \phi(t)dt = 1$ and hence $\int_{-\infty}^{\infty} s\phi(st)dt = 1$ for all $s > 0$,

$$S_\phi f(s,x) := \int_{-\infty}^{\infty} s\phi(s(x-t))f(t)dt, \quad x \in \mathbb{R}, \tag{1.1}$$

is called a *singular integral operator* of the convolution type. In order to develop an elegant mathematical theory to connect scale-space with singular integral operators and the continuous wavelet transforms, we define the *Gaussian scale-space transform*, $S_G f(s,x)$, as the convolution of f with the scaled Gaussian $G_s(t) := sG(st) = \frac{1}{\sqrt{2\pi}} s e^{-s^2t^2/2}$, $t \in \mathbb{R}$, $s > 0$, i.e.

$$S_G f(s,x) := G_s * f(x) = \int_{-\infty}^{\infty} sG(s(x-t))f(t)dt, \quad x \in \mathbb{R},\ s > 0. \tag{1.2}$$

In this formulation $s = \frac{1}{\sqrt{2r}}$, where r is the standard representation for scale. The scale-space filtering process (1.2) is also an integral transform, but unlike the continuous wavelet transforms the Gaussian function is not oscillatory and does not satisfy the admissibility condition. Since the integral of G over $\mathbb{R}$ is 1, it is of the form (1.1). It is known in Approximation Theory as the *Gauss-Weierstrass operator* [2]. If $s = 1$, $S_G f(1,x)$ is also called the *Weierstrass transform* of f. If f is uniformly continuous and bounded, $S_G f(s,\cdot)$ converges uniformly to f as $s \to \infty$ (see [2]). On the other hand, the Gaussian function is unimodal and satisfies the *continuous refinement equation* $G(x) = \int_{\mathbb{R}} \alpha G(\alpha x - y)dg(y)$, $x \in \mathbb{R}$, where $\alpha > 1$, and g is the absolutely continuous measure given by $dg(y) = \frac{1}{\sqrt{2\pi(\alpha^2-1)}} e^{-y^2/(2(\alpha^2-1))}dy$. Therefore, the Gaussian scale-space is a smoothing operator as the scale $r = \frac{1}{2s^2}$ increases. To connect scale-space with wavelet transforms, we modify the continuous wavelet transform and define it as

$$W_\omega f(s,x) := \int_{-\infty}^{\infty} s\omega(s(x-t))f(t)dt, \quad x \in \mathbb{R},\ s > 0, \tag{1.3}$$

where ω is an admissible mother wavelet. An important family of mother wavelets are the derivatives of the Gaussian function. Indeed, $-G^{(n)}$, for all positive integers n, form a class of admissible wavelets. Since $G^{(n)}$ is the product of the Hermite polynomial of degree n with the Gaussian, it is well localized and has exactly n oscillations. Therefore, the Gaussian scale-space together with the continuous wavelet transforms $W_{-G^{(n)}}f$ defined by the derivatives of the Gaussian, form a complete set of convolution transforms that decompose a function into different frequency components. For fixed values of n and s, $W_{-G^{(n)}}f(s,\cdot)$ is a multiple of the n^{th} order derivative of the Gaussian scale-space.

Derivatives of the Gaussian scale-space have been widely used to estimate local information of images, such as edges, corners and ridges. In general, linear operators that provide local information of images are referred to as *neighborhood operators* [23]. They are related to the continuous wavelet transforms. In an attempt to find a unified framework for neighborhood operators under the assumptions of scale invariance and that they do not introduce new features as the scale increases, Koenderink and van Doorn [23] have shown that the resulting neighborhood operators are indeed the derivatives of the Gaussian scale-space, which are referred to as the *generic neighborhood operators.* They are used to express geometric features of images at multiple scales for visual operations, such as feature detection, classification and image matching (see [28]).

Now for a continuous function f on $\mathbb{R}$, define $V(f) := \sup V(f(t_1), f(t_2), \ldots, f(t_N))$, where the supremum is taken over all N and all $t_1 < t_2 < \cdots < t_N$ in $\mathbb{R}$, and the quantity $V(f(t_1), f(t_2), \ldots, f(t_N))$ denotes the number of strict sign changes for the sequence $\{f(t_1), f(t_2), \ldots, f(t_N)\}$. Then a kernel ϕ as in (1.1) is said to be *variation diminishing* if for all continuous f, $V(S_\phi f(1, \cdot)) \leq V(f)$. In particular, the Gaussian function is a variation diminishing kernel. We say that the kernel ϕ has the *causality property* if for all continuous f,

$$V(S_\phi f(2^{-(k+1)}, \cdot)) \leq V(S_\phi f(2^{-k}, \cdot)), \quad k \in \mathbb{Z}, \tag{1.4}$$

which is variation diminishing across dyadic scales as the scale $r = 2^{2k-1}$ increases. The variations of a function and its derivatives are associated with features of images, and so the causality property (1.4) can be interpreted as no new features are introduced as the scale increases. It is shown in [1, 22, 27, 42] that ϕ has the causality property if and only if it is a shift and dilation of a Gaussian. The causality property is a stronger condition than variation diminishing, and in [24] (see also [25, 31]), Lindeberg extends the definition of a scale-space kernel to one that is variation diminishing.

It is well known that the standardized uniform B-splines, with mean 0 and variance 1, approximate the Gaussian function (see [8, 36]). Therefore, they can also be used as kernels in place of the Gaussian scale-space kernel in (1.2) to define a linear scale-space, in a generalized sense described below, to approximate the Gaussian scale-space [14, 37]. Also a B-spline scale-space provides a natural multiscale representation of geometric data modeled by B-spline curves and box-spline surfaces with fast and efficient algorithms involving only taking moving averages and repeating control points [20]. However, the B-spline kernels are not variation diminishing in the sense described above and so do not qualify as scale-space kernels in the sense of Lindeberg. A B-spline is an example of a scaling function. A compactly supported *scaling function* ϕ satisfies a *discrete refinement* *equation* of the form

$$\phi(t) = \sum_{j=0}^{N} h_j \phi(2t - j), \quad t \in \mathbb{R}. \tag{1.5}$$

The polynomial $H(z) := \sum_{j=0}^{N} h_j z^j$ is called the *symbol* of ϕ. If the symbol has all roots in the left half plane, then for any $N \geq 1$, real numbers $t_1 < t_2 < \cdots < t_N$ and integers $k_1 < k_2 < \cdots < k_N$,

$$\det(\phi(k_i - t_j))_{i,j=1}^{N} \geq 0, \tag{1.6}$$

and ϕ is called a *ripplet* (see [16]). In particular, B-splines are ripplets. It follows from the results in [3] that standardized ripplets, with mean 0 and variance 1, approximate the Gaussian. A ripplet ϕ satisfies the following weaker form of the *variation diminishing property* [14]: for any $a \in \mathbb{R}$ and continuous f on $\mathbb{R}$,

$$V\left(\{S_\phi f(j + a)\}_{j \in \mathbb{Z}}\right) \leq V(f). \tag{1.7}$$

It also satisfies the following weaker form of the *causality property* [14]: for any $a \in \mathbb{R}$, continuous f on $\mathbb{R}$ and $k \in \mathbb{Z}$,

$$V\left(\{S_\phi f(2^{-(k+1)}, 2^{k+1} j + a)\}_{j \in \mathbb{Z}}\right) \leq V\left(\{S_\phi f(2^{-k}, 2^k j + a)\}_{j \in \mathbb{Z}}\right). \tag{1.8}$$

Extending Lindeberg's definition of scale-space, we shall define a *scale-space kernel* as a function ϕ that satisfies (1.7). In particular, ripplets are scale-space kernels, and ripplet scale-space transforms approximate the Gaussian scale-space transform [14], where the orders of approximation are derived.

Scaling functions are fundamental in the construction of dyadic mother wavelets that provide the discrete wavelet transforms (see [5, 7, 9, 32]). Scaling functions are also fundamental in the construction of Ron and Shen's tight wavelet frames (see [35], and also [4, 6, 10]). A set of functions $\{\phi; \psi_1, \ldots, \psi_m\}$ generates a tight wavelet frame in $L^2(\mathbb{R})$ if for $f \in L^2(\mathbb{R})$,

$$\|f\|_2^2 = \sum_{j=-\infty}^{\infty} |\langle f, \phi(\,\cdot - j)\rangle|^2 + \sum_{\ell=1}^{m} \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} |\langle f, 2^{k/2} \psi_\ell(2^k \cdot - j)\rangle|^2.$$

In Ron-Shen's construction [35], ϕ is a scaling function and ψ_ℓ are constructed from ϕ by the unitary extension principle. For convenience, we shall call ψ_ℓ *mother framelets.* In wavelet analysis, a scaling function defines a smoothing process. Smoothing between two scales discards high frequency components of a signal between the scales. Wavelets and wavelet frames provide bases and frame systems for the recovery and representation of these high frequency components as well as recovery of functions from their discrete wavelet transforms. On the other hand, in the smoothing process by the Gaussian scale-space the rate of smoothing with respect to scale is the continuous wavelet transform with the Mexican hat kernel. Therefore, the high frequency component between two sufficiently close scales of the Gaussian scale-space can be approximated by the Mexican hat continuous wavelet transform. It can be approximated to higher orders of accuracy by finite sums of wavelet transforms with derivatives of the Gaussian as mother wavelets. It is shown in Section 2 and Section 3 that the high frequency component of a function between scales α and αs, $\alpha \neq 1$, $s > 0$, can be represented exactly as an infinite series of continuous wavelet transforms with the derivatives of the Gaussian function as the mother wavelets. We shall call such a representation a *semi-discrete wavelet representation.* Various Calderón type reconstruction formulas of functions in terms of their scale-space and continuous and semi-discrete wavelet transforms are obtained in Section 2. A formula connecting Gaussian scale-space and continuous wavelet transforms defined by derivatives of the Gaussian is derived in Section 3. These results further motivate the investigation of the corresponding singular integral operators and scale-space transforms defined by unimodal kernels with oscillating derivatives. We call a kernel *unimodal of order n* if the number of sign changes in its k^{th} order derivative equals k for $k = 0, 1, \ldots, n$. The Gaussian kernel is unimodal of infinite order. Unimodality is closely associated with scale-space kernels. It is shown in [24] that unimodality of order 0 in the spatial and Fourier domains are necessary conditions for a kernel to be a discrete scale-space kernel.

In this paper the relationships between scale-space and continuous wavelet transforms are studied within the general framework of convolution type singular integral operators. We then focus on the useful cases of Gaussian scale-space and B-spline scale-space and the corresponding continuous and semi-discrete wavelet transforms. The relationships between singular integral operators and continuous and semi-discrete wavelet transforms are studied in Section 2. With a mild condition on the kernel, general reconstruction formulas for functions in $L^p(\mathbb{R})$ in terms of the singular integral operator and the corresponding wavelet transforms follow readily from properties of singular integral operators. For the case of Gaussian scale-space, these reconstruction formulas correspond to the recovery of signals in terms of Gaussian scale-space and the high frequency components represented by the continuous wavelet transforms defined by derivatives of the Gaussian. The B-splines approximate the Gaussian function and inherit, at least approximately, many useful properties: variation diminishing (1.7), unimodality of higher order, total positivity (1.6), casuality property (1.8), asymptotic optimality in time-frequency localization [15]. Further in two dimensions, a tensor product of two B-splines of the same order possesses rotational symmetry of order 4 and higher order rotational symmetry can be achieved using box-splines. As in the case of Gaussian scale-space, the high frequency component of a function between two consecutive dyadic scales of a B-spline scale-space can be represented as a finite linear combination of continuous wavelet transforms defined by derivatives of the B-splines (Proposition 4.2). Interestingly, it can also be represented as a finite linear combination of continuous wavelet transforms defined by the spline framelets of Ron and Shen [35]. These are shown in Section 4.

In Section 5, we study the singular integral operators defined by the kernel $f_n(t) := -M_n'(t)/t$, where M_n is the centered uniform B-spline of order n. These are examples of singular integral operators defined by non-refinable functions, which approximate the Gaussian scale-space. Similar to the uniform B-splines, f_n is unimodal of order $n-3$. It is not known whether f_n has any form of variation diminishing or causality property, but their scale-space transforms approximate the Gaussian scale-space transform. Perhaps a more inclusive definition of scale-space kernel is required to include f_n as scale-space kernels, but more study has to be conducted.

2. Scale-space and singular integral operators

We consider a general class of singular integral operators of convolution type, each defined by a bounded integrable kernel $\phi : \mathbb{R} \to \mathbb{R}$ with $\int_{-\infty}^{\infty} \phi(t)dt = 1$. For real-valued f in $L^p(\mathbb{R})$, $1 \le p < \infty$, or $C(\mathbb{R})$, define

$$S_\phi f(s,x) := \int_{-\infty}^{\infty} s\phi(s(x-t))f(t)dt, \quad x \in \mathbb{R},\ s > 0. \tag{2.1}$$

Here $C(\mathbb{R})$ is the space of bounded uniformly continuous real-valued functions f with supremum norm $\|f\|_C := \sup_{t\in\mathbb{R}} |f(t)|$. We shall work with real-valued functions unless otherwise stated.

By Young's inequality for convolution, for $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and $s > 0$, $\|S_\phi f(s,\ \cdot)\|_p \le \|f\|_p \|\phi\|_1$, and so $S_\phi f(s,\ \cdot) \in L^p(\mathbb{R})$. If $f \in C(\mathbb{R})$,

$$|S_\phi f(s,x) - S_\phi f(s,y)| \le \|f(x-\ \cdot) - f(y-\ \cdot)\|_C \, \|\phi\|_1, \quad x,y \in \mathbb{R},\ s > 0,$$

and so $S_\phi f(s,\ \cdot) \in C(\mathbb{R})$. Note that the above inequalities hold for any $\phi \in L^1(\mathbb{R})$.

Lemma 2.1. *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$,*

$$\lim_{s\to 0} S_\phi f(s,x) = 0 \text{ uniformly on } \mathbb{R}, \tag{2.2}$$

and

$$\lim_{s\to 0} \|S_\phi f(s,\ \cdot)\|_p = 0. \tag{2.3}$$

Proof. Since $\phi \in L^1(\mathbb{R})$ and is bounded, it belongs to $L^q(\mathbb{R})$ for $1 \leq q \leq \infty$. If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then for $x \in \mathbb{R}$, $s > 0$,

$$\begin{aligned} |S_\phi f(s,x)| &\leq \int_{-\infty}^{\infty} s|\phi(s(x-t))|\,|f(t)|dt \leq s\|f\|_p \left(\int_{-\infty}^{\infty} |\phi(s(x-t))|^q dt\right)^{1/q} \\ &= s^{1-1/q}\|f\|_p\|\phi\|_q = s^{1/p}\|f\|_p\|\phi\|_q, \end{aligned}$$

where $1/p + 1/q = 1$. Therefore (2.2) holds.

Since $f \in L^p(\mathbb{R})$, by Young's inequality, for any $\epsilon > 0$ there is a number $R > 0$ such that $\int_{|x|>R} |S_\phi f(s,x)|^p dx < \epsilon$ uniformly for $s > 0$. By (2.2),

$$\lim_{s\to 0}\int_{-\infty}^{\infty} |S_\phi f(s,x)|^p dx \leq \lim_{s\to 0}\int_{-R}^{R} |S_\phi f(s,x)|^p dx + \epsilon = \epsilon,$$

and so (2.3) holds. □

On the other hand, as $\phi \in L^1(\mathbb{R})$ is normalized so that $\int_{-\infty}^{\infty} \phi(t)dt = 1$, then it is known that $S_\phi f(s, \cdot)$ converges to f as $s \to \infty$. To be more precise, we summarize the results as a proposition, which can be found in [2] page 121, or deduced from the results therein.

Proposition 2.2. *Suppose $\phi \in L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} \phi(t)dt = 1$.*

(a) *If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then*

$$\lim_{s\to\infty} \|S_\phi f(s, \cdot) - f\|_p = 0. \tag{2.4}$$

(b) *If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, and is continuous and bounded on $\mathbb{R}$, then*

$$\lim_{s\to\infty} S_\phi f(s,x) = f(x), \quad x \in \mathbb{R}. \tag{2.5}$$

(c) *If $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \leq p < \infty$, then*

$$\lim_{s\to\infty} \|S_\phi f(s, \cdot) - f\|_C = 0. \tag{2.6}$$

If ϕ approximates the Gaussian G, then its singular integral operator approximates the Gaussian scale-space. For easy reference, we shall refer to $S_\phi f$ as the *scale-space defined by ϕ*. The objective is to construct continuous and semi-discrete wavelet transforms from ϕ and derive the corresponding inverse transforms to recover the functions. Now for $\alpha > 1$, $s > 0$, $x \in \mathbb{R}$,

$$\begin{aligned} S_\phi f(\alpha s,x) - S_\phi f(s,x) &= \int_{-\infty}^{\infty} \alpha s\phi(\alpha s(x-t))f(t)dt - \int_{-\infty}^{\infty} s\phi(s(x-t))f(t)dt \\ &= \int_{-\infty}^{\infty} s\Psi_\alpha(s(x-t))f(t)dt, \end{aligned} \tag{2.7}$$

where

$$\Psi_\alpha(t) := \alpha\phi(\alpha t) - \phi(t), \quad t \in \mathbb{R}. \tag{2.8}$$

If ϕ is differentiable, then for $t \in \mathbb{R}$, $\frac{\partial}{\partial \alpha}\Psi_\alpha(t) = \phi(\alpha t) + \alpha t\phi'(\alpha t) = \psi(\alpha t)$, where we define

$$\psi(t) := \phi(t) + t\phi'(t), \quad t \in \mathbb{R}. \tag{2.9}$$

We further assume that $t\phi'(t)$ is integrable and bounded. (Later on in Proposition 2.6, we make the additional assumption that $t\phi(t)$ is integrable.) Then ψ is also integrable and bounded. Since $\Psi_1(t) = 0$, $t \in \mathbb{R}$,

$$\Psi_\alpha(t) = \int_1^\alpha \psi(ut)du, \quad t \in \mathbb{R},\ \alpha > 1,$$

so that by (2.7),

$$S_\phi f(\alpha s, x) - S_\phi f(s, x) = \int_{-\infty}^{\infty} sf(t) \int_1^\alpha \psi(us(x-t))dudt. \tag{2.10}$$

If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then

$$\begin{aligned}\int_1^\alpha \int_{-\infty}^{\infty} s|f(t)||\psi(us(x-t))|dtdu &\le \int_1^\alpha s\|f\|_p \left(\int_{-\infty}^{\infty} |\psi(us(x-t))|^q dt\right)^{1/q} du \\ &\le s^{1/p}p(\alpha^{1/p} - 1)\|f\|_p\|\psi\|_q,\end{aligned}$$

where $1/p + 1/q = 1$. Therefore, we may interchange the order of integration in (2.10) to get for $\alpha > 1$, $s > 0$, $x \in \mathbb{R}$,

$$\begin{aligned}S_\phi f(\alpha s, x) - S_\phi f(s, x) &= \int_1^\alpha \int_{-\infty}^{\infty} s\psi(us(x-t))f(t)dtdu \\ &= \int_s^{\alpha s} \int_{-\infty}^{\infty} \psi(u(x-t))f(t)dtdu. \qquad (2.11)\end{aligned}$$

Now define

$$W_\psi f(s, x) := \int_{-\infty}^{\infty} s\psi(s(x-t))f(t)dt, \quad x \in \mathbb{R},\ s > 0. \tag{2.12}$$

Note that if $t\phi(t)$ is integrable, $\int_{-\infty}^{\infty} \psi(t)dt = 0$, and so the operator $W_\psi f$ in (2.12) is not a singular integral operator. We shall show in Section 2.1 that if $\psi \in L^2(\mathbb{R})$, it is indeed a mother wavelet and $W_\psi f$ is a continuous wavelet transform of f. In the meantime we establish an auxiliary result for recovering f in (2.12).

Note that the representations (2.11) and (2.12) lead to the following result.

Proposition 2.3. *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then for $t > r > 0$ and $x \in \mathbb{R}$,*

$$S_\phi f(t, x) - S_\phi f(r, x) = \int_r^t \frac{1}{s} W_\psi f(s, x)ds. \tag{2.13}$$

Proposition 2.3 leads to the following Calderón type inversion formula.

Proposition 2.4. *Let $\psi(t)$ be defined as in (2.9), where $t\phi'(t)$ is integrable and bounded on $\mathbb{R}$.*

(a) *If* $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, *then*

$$\lim_{\substack{t\to\infty \\ r\to 0}} \left\| f - \int_r^t \frac{1}{s}\, W_\psi f(s,\ \cdot)ds \right\|_p = 0. \tag{2.14}$$

(b) *If* $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, *and is continuous and bounded on* $\mathbb{R}$, *then*

$$f(x) = \int_0^\infty \frac{1}{s}\, W_\psi f(s,x)ds, \quad x \in \mathbb{R}. \tag{2.15}$$

(c) *If* $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \le p < \infty$, *then*

$$\lim_{\substack{t\to\infty \\ r\to 0}} \left\| f - \int_r^t \frac{1}{s}\, W_\psi f(s,\ \cdot)ds \right\|_C = 0.$$

Proof. If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, by (2.13), (2.3) and (2.4),

$$\begin{aligned} \left\| f - \int_r^t \frac{1}{s}\, W_\psi f(s,\ \cdot)ds \right\|_p &= \|f - S_\phi f(t,\ \cdot) + S_\phi f(r,\ \cdot)\|_p \\ &\le \|f - S_\phi f(t,\ \cdot)\|_p + \|S_\phi f(r,\ \cdot)\|_p \to 0, \end{aligned}$$

as $r \to 0,\ t \to \infty$, and so (2.14) holds.

Similarly, assertions (b) and (c) follow from (2.13), (2.2), (2.5) and (2.6). □

Remark 1. *The transform* (2.12) *and its inversion were also considered in* [18], *where they were treated heuristically in the Hilbert space of square integrable functions. In Proposition* 2.4 *we provide the precise conditions on* ϕ *for convergence in* L^p *and uniform norms.*

Remark 2. *Note that if* $\phi = G$, *then from* (2.9), *for* $t \in \mathbb{R}$,

$$\psi(t) = G(t) + tG'(t) = (1-t^2)G(t) = -G''(t),$$

which is the Mexican hat mother wavelet, and W_ψ *in* (2.12) *is the corresponding wavelet transform. If* $\phi(t) = -M_n'(t)/t$, *where* M_n *is the centered uniform B-spline of order* n, *the corresponding mother wavelet* $\psi(t) = -M_n''(t)$.

2.1. **Connection with continuous wavelet transforms.** To connect the inversion formula (2.15) in Proposition 2.4 with an inverse wavelet transform, we introduce the admissibility condition for the continuous wavelet transform (1.3). Thus we say that a function $\omega \in L^2(\mathbb{R})$ satisfies the *admissibility condition* if

$$0 \neq C_\omega := \int_0^\infty \frac{\widehat{\omega}(u)}{u} du = \int_0^\infty \frac{\widehat{\omega}(-u)}{u} du < \infty. \tag{2.16}$$

Here the Fourier transform of $f \in L^1(\mathbb{R})$ is defined by $\widehat{f}(u) := \int_{-\infty}^\infty f(t)e^{-iut}dt$. By (2.16), there is a constant C such that

$$\left| \int_r^\infty \frac{\widehat{\omega}(u)}{u} du \right| \le C, \quad \left| \int_r^\infty \frac{\widehat{\omega}(-u)}{u} du \right| \le C \quad \text{for all } r > 0. \tag{2.17}$$

Proposition 2.5. *Let* $\omega \in L^2(\mathbb{R})$ *satisfy* (2.16). *If* $f \in L^2(\mathbb{R})$ *and* $\widehat{f} \in L^1(\mathbb{R})$,

$$f(x) = \frac{1}{C_\omega} \int_0^\infty \frac{1}{s}\, W_\omega f(s,x) ds, \quad x \in \mathbb{R}. \tag{2.18}$$

Proof. For $x \in \mathbb{R}$, $s > 0$, let $\omega_{s,x}(t) := \omega(s(x-t))$, $t \in \mathbb{R}$. Then $\widehat{\omega}_{s,x}(u) = \frac{1}{s}\, \widehat{\omega}(-u/s) e^{-iux}$, $u \in \mathbb{R}$. Using Parseval's identity and the fact that f is real-valued, it follows from (1.3) that

$$W_\omega f(s,x) = \frac{1}{2\pi} \int_{-\infty}^\infty s\, \widehat{\omega}_{s,x}(u) \overline{\widehat{f}(u)} du = \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{\omega}(u/s) e^{iux} \widehat{f}(u) du.$$

Now for any $R > 0$, by the Cauchy-Schwartz inequality, one has

$$\begin{aligned} \int_0^R \frac{1}{s} \int_{-\infty}^\infty |\widehat{\omega}(u/s) \widehat{f}(u)| du ds &\le \|\widehat{f}\|_2 \int_0^R \frac{1}{s} \left(\int_{-\infty}^\infty |\widehat{\omega}(u/s)|^2 du \right)^{1/2} \\ &= \|\widehat{f}\|_2\, \|\widehat{\omega}\|_2 \int_0^R \frac{1}{\sqrt{s}} < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^R \frac{1}{s} W_\omega f(s,x) ds = \frac{1}{2\pi} \int_{-\infty}^\infty \int_0^R \frac{1}{s} \widehat{\omega}(u/s) \widehat{f}(u) e^{iux} ds du \\ = &\ \frac{1}{2\pi} \int_{-\infty}^0 \int_0^R \frac{1}{s} \widehat{\omega}(u/s) \widehat{f}(u) e^{iux} ds du + \frac{1}{2\pi} \int_0^\infty \int_0^R \frac{1}{s} \widehat{\omega}(u/s) \widehat{f}(u) e^{iux} ds du \\ = &\ \frac{1}{2\pi} \int_{-\infty}^0 \int_{-u/R}^\infty \frac{1}{\tau} \widehat{\omega}(-\tau) \widehat{f}(u) e^{iux} d\tau du + \frac{1}{2\pi} \int_0^\infty \int_{u/R}^\infty \frac{1}{\tau} \widehat{\omega}(\tau) \widehat{f}(u) e^{iux} d\tau du. \end{aligned}$$

By (2.17),

$$\left| \int_{-u/R}^\infty \frac{1}{\tau} \widehat{\omega}(-\tau) \widehat{f}(u) e^{iux} d\tau \right| \le C |\widehat{f}(u)|, \quad u < 0,$$

for all $R > 0$. Since $\widehat{f} \in L^1(\mathbb{R})$, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{2\pi} \int_{-\infty}^0 \int_{-u/R}^\infty \frac{1}{\tau} \widehat{\omega}(-\tau) \widehat{f}(u) e^{iux} d\tau du &= \frac{1}{2\pi} \int_{-\infty}^0 \int_0^\infty \frac{1}{\tau} \widehat{\omega}(-\tau) d\tau \widehat{f}(u) e^{iux} du \\ &= C_\omega \frac{1}{2\pi} \int_{-\infty}^0 \widehat{f}(u) e^{iux} du. \end{aligned}$$

Similarly,

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_0^\infty \int_{u/R}^\infty \frac{1}{\tau} \widehat{\omega}(\tau) \widehat{f}(u) e^{iux} d\tau du = C_\omega \frac{1}{2\pi} \int_0^\infty \widehat{f}(u) e^{iux} du,$$

and so

$$\lim_{R\to\infty} \int_0^R \frac{1}{s} W_\omega f(s,x) ds = C_\omega \frac{1}{2\pi} \int_{-\infty}^\infty \widehat{f}(u) e^{iux} du = C_\omega f(x),$$

which gives (2.18). $\square$

Remark 3. *Proposition* 2.4 *and Proposition* 2.5 *give similar results from two different approaches and under two slightly different sets of conditions. Proposition* 2.4 *starts with a bounded integrable function* $\phi : \mathbb{R} \to \mathbb{R}$ *with integral* 1 *such that* $t\phi'(t)$ *is integrable and bounded on* $\mathbb{R}$ *and defines a kernel* $\psi(t) := \phi(t) + t\phi'(t)$ *for a convolution transform* $W_\psi f$ *in* (2.12)*. Then any* $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, *can be recovered from the transform* $W_\psi f$ *in* L^p *norm as well as pointwise (uniformly) if* f *is continuous (uniformly continuous) and bounded. On the other hand, Proposition* 2.5 *starts with a mother wavelet* $\omega \in L^2(\mathbb{R})$ *that satisfies the admissibility condition* (2.16)*. Then any* $f \in L^2(\mathbb{R})$ *with* $\widehat{f} \in L^1(\mathbb{R})$ *can be recovered from its continuous wavelet transform* $W_\omega f$ *by* (2.18)*.*

The following proposition shows the connection between the two approaches.

Proposition 2.6. *Let* $\psi(t)$ *be defined as in* (2.9)*, where* $t\phi(t)$ *and* $t\phi'(t)$ *are integrable on* $\mathbb{R}$*. Then*

$$\left|\int_r^\infty \frac{\widehat{\psi}(u)}{u} du\right| \le \|\phi\|_1, \quad \left|\int_r^\infty \frac{\widehat{\psi}(-u)}{u} du\right| \le \|\phi\|_1 \quad \textit{for all } r > 0,$$

and

$$C_\psi := \int_0^\infty \frac{\widehat{\psi}(u)}{u} du = \int_0^\infty \frac{\widehat{\psi}(-u)}{u} du = 1.$$

Further, if $\psi \in L^2(\mathbb{R})$*, then the conditions of Proposition* 2.5 *are satisfied so that* $f \in L^2(\mathbb{R})$ *with* $\widehat{f} \in L^1(\mathbb{R})$ *can be recovered by* (2.18) *with* $C_\psi = 1$*.*

Proof. Since $\psi(t) = \frac{d}{dt}(t\phi(t))$ and $t\phi(t)$ are integrable,

$$\widehat{\psi}(u) = \int_{-\infty}^\infty \frac{d}{dt}(t\phi(t))\, e^{-iut} dt = iu \int_{-\infty}^\infty t\phi(t) e^{-iut} dt$$

and so

$$\int_r^\infty \frac{\widehat{\psi}(u)}{u} du = \int_r^\infty \int_{-\infty}^\infty it\phi(t) e^{-iut} dt du = -\int_r^\infty \widehat{\phi}\,'(u) du = \widehat{\phi}(r).$$

Therefore,

$$\left|\int_r^\infty \frac{\widehat{\psi}(u)}{u} du\right| = |\widehat{\phi}(r)| \le \|\phi\|_1$$

and

$$\int_0^\infty \frac{\widehat{\psi}(u)}{u} du = \lim_{r \to 0} \widehat{\phi}(r) = \widehat{\phi}(0) = 1.$$

Similarly,

$$\left|\int_r^\infty \frac{\widehat{\psi}(-u)}{u} du\right| = |\widehat{\phi}(-r)| \le \|\phi\|_1$$

and

$$\int_0^\infty \frac{\widehat{\psi}(-u)}{u} du = \lim_{r \to 0} \widehat{\phi}(-r) = 1.$$

□

2.2. **Semi-discrete wavelet representations.** Equation (2.13) provides the representation $S_\phi f(t,x) - S_\phi f(r,x) = \int_r^t \frac{1}{s} W_\psi f(s,x) ds$ for the difference of the scale-space operator $S_\phi f$ at two arbitrary scales $t > r > 0$ in terms of the continuous wavelet transform with mother wavelet ψ. We now consider discrete scales and the wavelet representation for the difference $S_\phi f(\alpha^n, x) - S_\phi f(\alpha^m, x)$, $x \in \mathbb{R}$, of the scale-space operator $S_\phi f$ at two discrete scales α^m and α^n, where $\alpha \neq 1$ is an arbitrary fixed positive number and $m < n$ are integers. We shall call such a representation a *semi-discrete wavelet representation.*

Proposition 2.7. *For positive $\alpha \neq 1$, let $\Psi_\alpha(t)$ be defined as in* (2.8). *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then for $n > m$,*

$$S_\phi f(\alpha^n, x) - S_\phi f(\alpha^m, x) = \sum_{k=m}^{n-1} W_{\Psi_\alpha} f(\alpha^k, x), \quad x \in \mathbb{R}, \tag{2.19}$$

where

$$W_{\Psi_\alpha} f(\alpha^k, x) := \int_{-\infty}^{\infty} \alpha^k \Psi_\alpha(\alpha^k(x-t)) f(t) dt.$$

Proof. We may assume that $\alpha > 1$. By (2.7), for any integer k,

$$S_\phi f(\alpha^{k+1}, x) - S_\phi f(\alpha^k, x) = \int_{-\infty}^{\infty} \alpha^k \Psi_\alpha(\alpha^k(x-t)) f(t) dt,$$

and so

$$S_\phi f(\alpha^n, x) - S_\phi f(\alpha^m, x) = \sum_{k=m}^{n-1} \big(S_\phi f(\alpha^{k+1}, x) - S_\phi f(\alpha^k, x) \big)$$

$$= \sum_{k=m}^{n-1} \int_{-\infty}^{\infty} \alpha^k \Psi_\alpha(\alpha^k(x-t)) f(t) dt = \sum_{k=m}^{n-1} W_{\Psi_\alpha} f(\alpha^k, x), \quad x \in \mathbb{R}.$$

□

Equation (2.19) is a semi-discrete analogue of (2.13). Applying (2.2), (2.3) and Proposition 2.2 to Proposition 2.7 leads to the following semi-discrete analogue of Proposition 2.4.

Corollary 2.8. *Let Ψ_α be defined as in* (2.8).

(a) *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then*

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \left\| f - \sum_{k=m}^{n-1} W_{\Psi_\alpha} f(\alpha^k, \cdot) \right\|_p = 0.$$

(b) *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and is continuous and bounded on $\mathbb{R}$, then*

$$f(x) = \sum_{k=-\infty}^{\infty} W_{\Psi_\alpha} f(\alpha^k, x), \quad x \in \mathbb{R}.$$

(c) *If* $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \le p < \infty$, *then*

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \left\| f - \sum_{k=m}^{n-1} W_{\Psi_\alpha} f(\alpha^k, \cdot) \right\|_C = 0.$$

Semi-discrete wavelet transforms can also be defined directly from kernel functions in $L^2(\mathbb{R})$ that satisfy certain admissibility conditions (see [5]). Here we take a less traditional approach and define semi-discrete wavelet transforms that satisfy another admissibility condition to establish a connection with the inversion formula in Corollary 2.8. We require a function $\omega \in L^2(\mathbb{R})$ to satisfy the admissibility condition

$$\sum_{k=-\infty}^{\infty} \widehat{\omega}(u/\alpha^k) = 1 \quad \text{a. e.}, \tag{2.20}$$

and assume that

$$\left| \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k) \right| \le B, \quad \text{a. e. for any } m < n. \tag{2.21}$$

Define

$$W_\omega f(\alpha^k, x) = \int_{-\infty}^{\infty} \alpha^k \omega(\alpha^k(x-t)) f(t)dt, \quad k \in \mathbb{Z},\ x \in \mathbb{R},$$

for any arbitrary fixed positive number $\alpha \ne 1$. We shall show that

$$f(x) = \sum_{k=-\infty}^{\infty} W_\omega f(\alpha^k, x). \tag{2.22}$$

To be precise, we shall prove the following results.

Proposition 2.9. *Let* $\omega \in L^2(\mathbb{R})$ *satisfy* (2.20) *and* (2.21).

(a) *Equation* (2.22) *holds in* $L^2(\mathbb{R})$ *for any* $f \in L^2(\mathbb{R})$.
(b) *If* $f \in L^2(\mathbb{R})$ *and* $\widehat{f} \in L^1(\mathbb{R})$, *then* (2.22) *holds for all* $x \in \mathbb{R}$.

Proof. Take an $f \in L^2(\mathbb{R})$. For $m < n$, let

$$S_{m,n} f(x) := \sum_{k=m}^{n} W_\omega f(\alpha^k, x) = \sum_{k=m}^{n} \int_{-\infty}^{\infty} \alpha^k \omega(\alpha^k(x-t)) f(t)dt.$$

Then

$$\widehat{S_{m,n} f}(u) = \sum_{k=m}^{n} \widehat{f}(u) \widehat{\omega}(u/\alpha^k), \quad \text{a. e.},$$

and so by (2.20),

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \widehat{S_{m,n} f}(u) = \lim_{\substack{n\to\infty \\ m\to-\infty}} \widehat{f}(u) \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k) = \widehat{f}(u), \quad \text{a. e.}.$$

Also by (2.21),

$$|\widehat{S_{m,n}f}(u)| \leq |\widehat{f}(u)| \left| \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k) \right| \leq B|\widehat{f}(u)| \quad \text{a.e.}\ .$$

By the Dominated Convergence Theorem,

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \int_{-\infty}^{\infty} \widehat{S_{m,n}f}(u)\overline{\widehat{f}(u)}du = \int_{-\infty}^{\infty} |\widehat{f}(u)|^2 du$$

and

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \int_{-\infty}^{\infty} |\widehat{S_{m,n}f}(u)|^2 du = \int_{-\infty}^{\infty} |\widehat{f}(u)|^2 du.$$

So $\lim_{\substack{n\to\infty \\ m\to-\infty}} \widehat{S_{m,n}f} = \widehat{f}$ in $L^2(\mathbb{R})$, and hence $\lim_{\substack{n\to\infty \\ m\to-\infty}} S_{m,n}f = f$ in $L^2(\mathbb{R})$, i.e. (2.22) holds in $L^2(\mathbb{R})$.

To prove (b), consider $f \in L^2(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$. We note that for $x \in \mathbb{R}$, Parseval's identity gives

$$S_{m,n}f(x) = \sum_{k=m}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\omega}(u/\alpha^k)\widehat{f}(u)e^{iux}du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(u)e^{iux} \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k)du.$$

By (2.20),

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \widehat{f}(u)e^{iux} \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k) = \widehat{f}(u)e^{iux} \quad \text{a.e.}\ ,$$

and by (2.21),

$$\left| \widehat{f}(u)e^{iux} \sum_{k=m}^{n} \widehat{\omega}(u/\alpha^k) \right| \leq B|\widehat{f}(u)| \quad \text{a.e.}\ .$$

So by the Dominated Convergence Theorem,

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} S_{m,n}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(u)e^{iux}du = f(x).$$

Since $f \in L^2(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, f is continuous on $\mathbb{R}$ and so (2.22) holds for all $x \in \mathbb{R}$. □

Proposition 2.10. *The mother wavelet* Ψ_α *defined as in* (2.8) *satisfies the conditions* (2.20) *and* (2.21).

Proof. Since ϕ is integrable, $\widehat{\phi}$ is continuous and bounded. The Fourier transform $\widehat{\Psi}_\alpha(u) = \widehat{\phi}(u/\alpha) - \widehat{\phi}(u)$, so that

$$\sum_{k=-\infty}^{\infty} \widehat{\Psi}_\alpha(u/\alpha^k) = \lim_{\substack{n\to\infty \\ m\to-\infty}} \sum_{k=m}^{n} \widehat{\Psi}_\alpha(u/\alpha^k) = \lim_{n\to\infty} \widehat{\phi}(u/\alpha^{n+1}) - \lim_{m\to-\infty} \widehat{\phi}(u/\alpha^m) = \widehat{\phi}(0) = 1.$$

Thus Ψ_α satisfies (2.20). Also

$$\left| \sum_{k=m}^{n} \widehat{\Psi}_\alpha(u/\alpha^k) \right| = |\widehat{\phi}(u/\alpha^{n+1}) - \widehat{\phi}(u/\alpha^m)| \leq 2\|\widehat{\phi}\|_\infty$$

and so Ψ_α satisfies (2.21). □

3. Gaussian scale-space and wavelet representations

The Gaussian function $G(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ defines the Gaussian scale-space $S_G f(s,x) = G_s * f(x)$, where $G_s(t) := sG(st)$. The Fourier transforms of G and G_s are given by $\widehat{G}(u) = \int_{-\infty}^{\infty} G(t) e^{-iut} dt = e^{-u^2/2}$ and $\widehat{G}_s(u) = \widehat{G}(u/s)$. Therefore, in the frequency domain,

$$\widehat{S_G f}(s,u) = \widehat{G}(u/s)\widehat{f}(u) = e^{-u^2/2s^2}\widehat{f}(u), \quad u \in \mathbb{R},\ s > 0.$$

Similarly, for the wavelet transforms $W_{G^{(n)}} f(s,x)$, $n = 1, 2, \ldots$,

$$\widehat{W_{G^{(n)}} f}(s,u) = \widehat{G^{(n)}}(u/s)\widehat{f}(u), \quad u \in \mathbb{R},\ s > 0.$$

The Gaussian kernel G satisfies all the conditions for ϕ in Section 2. The corresponding wavelet $\psi(t) = -G''(t)$ is the Mexican hat function, for which Proposition 2.4 holds.

Similarly, with positive $\alpha \neq 1$, Corollary 2.8 holds for $\Psi_\alpha(t) = \alpha G(\alpha t) - G(t)$. Further, in this case, Ψ_α admits the following expansion:

Lemma 3.1. *For* $0 < \alpha < \sqrt{2}$ *and* $\alpha \neq 1$,

$$\Psi_\alpha(t) = -\sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{\alpha^2 - 1}{2}\right)^j \alpha G^{(2j)}(\alpha t), \quad t \in \mathbb{R}. \tag{3.1}$$

Proof. The Fourier transform of Ψ_α is

$$\begin{aligned} \widehat{\Psi}_\alpha(u) &= \widehat{G}(u/\alpha) - \widehat{G}(u) = \widehat{G}(u/\alpha)\left(1 - e^{-(\alpha^2-1)u^2/2\alpha^2}\right) \\ &= -\sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j (-1)^j (u/\alpha)^{2j} \widehat{G}(u/\alpha) \\ &= -\sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j \widehat{G^{(2j)}}(u/\alpha), \quad u \in \mathbb{R}. \end{aligned} \tag{3.2}$$

For $0 < \alpha < \sqrt{2}$ and $\alpha \neq 1$, the series on the right converges uniformly on $\mathbb{R}$. To see this, observe that for all sufficiently large j,

$$\begin{aligned} \left| \frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j (-1)^j (u/\alpha)^{2j} e^{-u^2/2\alpha^2} \right| &= \frac{1}{j!} |\alpha^2 - 1|^j (u^2/2\alpha^2)^j e^{-u^2/2\alpha^2} \\ &\leq \frac{1}{j!} |\alpha^2 - 1|^j j^j e^{-j} < |\alpha^2 - 1|^j, \end{aligned}$$

where Stirling's formula is employed in the last inequality. Since $0 < \alpha < \sqrt{2}$ and $\alpha \neq 1$, we have $0 < |\alpha^2 - 1| < 1$ and so the series in (3.2) converges uniformly. Taking the inverse Fourier transforms of the functions in (3.2) gives (3.1). □

The series expansion in (3.1) is given in terms of derivatives of the Gaussian function, which provide a family of mother wavelets for continuous wavelet transforms. Further, the derivatives of the Gaussian function are products of Hermite polynomials with the Gaussian, i.e. $G^{(n)}(t) = (-1)^n H_n(t)G(t)$, $n = 0, 1, 2, \dots$, where H_n is the Hermite polynomial of degree n. The Hermite polynomials are also generated by the generating function

$$e^{tz-z^2/2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} z^n, \quad t \in \mathbb{R}. \tag{3.3}$$

The following estimates for Hermite polynomials will be useful later.

Lemma 3.2. *For all sufficiently large integer n and $t \in \mathbb{R}$,*

$$|H_n(t)| \leq K e^{t^2/4} \sqrt{(n+1)!}\ , \tag{3.4}$$

where K is an absolute constant independent of n and t.

Proof. By Cauchy's integral formula, the Hermite polynomials in the expansion (3.3) are given by the contour integrals

$$\frac{H_n(t)}{n!} = \frac{1}{2\pi i} \oint_C \frac{e^{tz-z^2/2}}{z^{n+1}} dz,$$

where C is any circle centered at the origin. Let C_n be the circle of radius $\sqrt{n}$. Then with $z = x + iy$, $x, y \in \mathbb{R}$,

$$\begin{aligned} \frac{|H_n(t)|}{n!} &\leq \frac{1}{2\pi} \oint_{C_n} \frac{|e^{tz}||e^{-z^2/2}|}{|z|^{n+1}} |dz| = \frac{1}{2\pi} \oint_{C_n} \frac{e^{tx-x^2/2+y^2/2}}{|z|^{n+1}} |dz| \\ &= \frac{e^{t^2/4}}{2\pi} \oint_{C_n} \frac{e^{-(x-t/2)^2+x^2/2+y^2/2}}{n^{(n+1)/2}} |dz| \leq \frac{e^{t^2/4}}{2\pi} \oint_{C_n} \frac{e^{x^2/2+y^2/2}}{n^{(n+1)/2}} |dz| = e^{t^2/4} \left(\frac{e}{n}\right)^{n/2}, \end{aligned}$$

and so by Stirling's formula, for all sufficiently large n,

$$|H_n(t)| \leq e^{t^2/4} n! \sqrt{\left(\frac{e}{n}\right)^n} \leq K e^{t^2/4} n! \sqrt{\frac{(2\pi n)^{1/2}}{n!}} \leq K e^{t^2/4} \sqrt{(n+1)!}\ ,$$

where K is a generic constant that does not depend on n and t. □

Lemma 3.3. *Let $0 < \alpha < \sqrt{2}$, $\alpha \neq 1$. For any nonnegative integer k, the series*

$$\sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{\alpha^2 - 1}{2}\right)^j t^k \alpha G^{(2j)}(\alpha t) \tag{3.5}$$

converges uniformly on $\mathbb{R}$.

Proof. By (3.4),

$$\begin{aligned} \left|t^k G^{(2j)}(\alpha t)\right| &= \left|t^k H_{2j}(\alpha t) G(\alpha t)\right| \leq K e^{\alpha^2 t^2/4} \sqrt{(2j+1)!}\ |t|^k e^{-\alpha^2 t^2/2} \\ &= K e^{-\alpha^2 t^2/4}\ (t^2)^{k/2} \sqrt{(2j+1)!} \end{aligned}$$

for some constant $K > 0$. Since the maximum of $e^{-\alpha^2 t^2/4}\,(t^2)^{k/2}$ is attained at $t^2 = 2k/\alpha^2$, we have $e^{-\alpha^2 t^2/4}\,(t^2)^{k/2} \le e^{-k/2}\left(\frac{2k}{\alpha^2}\right)^{k/2}$, and so $\left|t^k G^{(2j)}(\alpha t)\right| \le K'\sqrt{(2j+1)!}$, where K' is a constant that depends only on α, k. Hence

$$\left|\frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j t^k G^{(2j)}(\alpha t)\right| \le K'\frac{1}{j!}\left|\frac{\alpha^2-1}{2}\right|^j \sqrt{(2j+1)!} =: M_j.$$

Now

$$\begin{aligned}\frac{M_{j+1}}{M_j} &= \frac{1}{(j+1)}\left|\frac{\alpha^2-1}{2}\right|\sqrt{(2j+3)(2j+2)} \\ &= \left|\frac{\alpha^2-1}{2}\right|\sqrt{\frac{(2j+3)(2j+2)}{(j+1)^2}} \to |\alpha^2-1| \;\text{ as } j\to\infty.\end{aligned}$$

However, if $0 < \alpha < \sqrt{2}$, the number series $\sum_{j=1}^{\infty} M_j$ converges by the ratio test. Hence the series (3.5) converges uniformly on $\mathbb{R}$. □

Proposition 3.4. *Let $0 < \alpha < \sqrt{2}$ and $\alpha \ne 1$. If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then for any $k \in \mathbb{Z}$,*

$$W_{\Psi_\alpha} f(\alpha^k, x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}\left(\frac{1-\alpha^2}{2}\right)^j W_{G^{(2j)}} f(\alpha^{k+1}, x), \quad \textit{uniformly on } \mathbb{R}. \tag{3.6}$$

The convergence also holds in $L^p(\mathbb{R})$.

Proof. From Lemma 3.3, we know that the series in (3.1) converges uniformly. Now we show that it also converges in $L^p(\mathbb{R})$, $1 \le p < \infty$. Let $S_N(t)$ be the N^{th} partial sum of the series in (3.1). Then

$$\begin{aligned}\|\Psi_\alpha - S_N\|_p &= \left\{\int_{-\infty}^{\infty} |\Psi_\alpha(t) - S_N(t)|^p\, dt\right\}^{1/p} \\ &= \left\{\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^p}\left|(1+t^2)(\Psi_\alpha(t) - S_N(t))\right|^p dt\right\}^{1/p} \\ &\le \sup_{t\in\mathbb{R}}\left|(1+t^2)(\Psi_\alpha(t) - S_N(t))\right| \left\{\int_{-\infty}^{\infty}\frac{1}{(1+t^2)^p} dt\right\}^{1/p}.\end{aligned}$$

By Lemma 3.3, $\sup_{t\in\mathbb{R}} |(1+t^2)(\Psi_\alpha(t) - S_N(t))| \to 0$ as $N \to \infty$, and so

$$\lim_{N\to\infty} \|\Psi_\alpha - S_N\|_p = 0.$$

Now let $\mathcal{S}_N(f, x)$ be the N^{th} partial sum of the series in (3.6). Then

$$W_{\Psi_\alpha} f(\alpha^k, x) - \mathcal{S}_N(f, x) = \alpha^k\left(\Psi_\alpha(\alpha^k\,\cdot) - S_N(\alpha^k\,\cdot)\right) * f(x).$$

By Hölder's inequality,

$$\begin{aligned}\left|W_{\Psi_\alpha} f(\alpha^k, x) - \mathcal{S}_N(f,x)\right| &\leq \alpha^k \left\|\Psi_\alpha(\alpha^k\,\cdot) - S_N(\alpha^k\,\cdot)\right\|_q \|f\|_p \\ &= \alpha^{k/p}\|\Psi_\alpha - S_N\|_q\|f\|_p \quad \text{for all } x \in \mathbb{R},\end{aligned}$$

where $1/p + 1/q = 1$. Hence

$$\left\|W_{\Psi_\alpha} f(\alpha^k,\ \cdot) - \mathcal{S}_N(f,\ \cdot)\right\|_C \leq \alpha^{k/p}\|\Psi_\alpha - S_N\|_q\|f\|_p \to 0 \quad \text{as } N \to \infty.$$

To show the convergence in $L^p(\mathbb{R})$ we use Young's inequality instead of Hölder's inequality, which gives

$$\begin{aligned}\left\|W_{\Psi_\alpha} f(\alpha^k,\ \cdot) - \mathcal{S}_N(f,\ \cdot)\right\|_p &\leq \alpha^k \left\|\Psi_\alpha(\alpha^k\,\cdot) - S_N(\alpha^k\,\cdot)\right\|_1 \|f\|_p \\ &= \|\Psi_\alpha - S_N\|_1\ \|f\|_p \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

□

Combining Corollary 2.8 and Proposition 3.4 gives

Theorem 3.5. *Let $0 < \alpha < \sqrt{2}$ and $\alpha \neq 1$.*

(a) *If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then*

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \left\| f - \sum_{k=m}^{n-1}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}\left(\frac{1-\alpha^2}{2}\right)^j W_{G^{(2j)}} f(\alpha^k,\ \cdot)\right\|_p = 0.$$

(b) *If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, and is continuous and bounded on $\mathbb{R}$, then*

$$f(x) = \sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}\left(\frac{1-\alpha^2}{2}\right)^j W_{G^{(2j)}} f(\alpha^k, x), \quad x \in \mathbb{R}.$$

(c) *If $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \leq p < \infty$, then*

$$\lim_{\substack{n\to\infty \\ m\to-\infty}} \left\| f - \sum_{k=m}^{n-1}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}\left(\frac{1-\alpha^2}{2}\right)^j W_{G^{(2j)}} f(\alpha^k,\ \cdot)\right\|_C = 0.$$

Equation (3.2) gives that for any $s > 0$, $0 < \alpha < \sqrt{2}$ and $\alpha \neq 1$,

$$\widehat{G}(\alpha u/s) = \widehat{G}(u/s) + \sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j \widehat{G^{(2j)}}(u/s), \quad u \in \mathbb{R}.$$

The series on the right converges uniformly on $\mathbb{R}$ by a similar argument as in the proof of Lemma 3.1. So for any $f \in L^2(\mathbb{R})$,

$$\widehat{G}(\alpha u/s)\widehat{f}(u) = \widehat{G}(u/s)\widehat{f}(u) + \sum_{j=1}^{\infty}\frac{1}{j!}\left(\frac{\alpha^2-1}{2}\right)^j \widehat{G^{(2j)}}(u/s)\widehat{f}(u),$$

and letting $\alpha \to 0$ gives

$$\widehat{f}(u) = \widehat{G}(u/s)\widehat{f}(u) + \sum_{j=1}^{\infty} \frac{(-1)^j}{2^j j!} \widehat{G^{(2j)}}(u/s)\widehat{f}(u). \tag{3.7}$$

Since $\widehat{G}(u/s)\widehat{f}(u)$ and $\widehat{G^{(2j)}}(u/s)\widehat{f}(u)$ are Fourier transforms of $S_G f(s,x)$ and $W_{G^{(2j)}} f(s,x)$ respectively, (3.7) suggests the following representation for f:

$$f(x) = S_G f(s,x) + \sum_{j=1}^{\infty} \frac{(-1)^j}{2^j j!} W_{G^{(2j)}} f(s,x). \tag{3.8}$$

Equation (3.8) decomposes a signal f into the sum of a smooth version represented by the Gaussian scale-space $S_G f(s,x)$ and the high frequency components at various frequency levels represented by the wavelet transforms $W_{G^{(2j)}} f(s,x)$. To be more precise, we shall prove the following theorem.

Theorem 3.6. *If $f \in L^2(\mathbb{R})$ then for $s > 0$, (3.8) holds in $L^2(\mathbb{R})$.*

Proof. For $f \in L^2(\mathbb{R})$, we want to show that

$$f = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j j!} \int_{-\infty}^{\infty} sG^{(2j)}(s(\,\cdot - t))f(t)dt$$

in $L^2(\mathbb{R})$. Let the partial sum of the series on the right be

$$P_N f(x) := \sum_{j=0}^{N} \frac{(-1)^j}{2^j j!} \int_{-\infty}^{\infty} sG^{(2j)}(s(x-t))f(t)dt, \quad x \in \mathbb{R}.$$

Its Fourier transform is

$$\widehat{P_N f}(u) = \sum_{j=0}^{N} \frac{(-1)^j}{2^j j!} \left(\frac{iu}{s}\right)^{2j} e^{-u^2/2s^2}\widehat{f}(u) = \sum_{j=0}^{N} \frac{1}{j!} \left(\frac{u^2}{2s^2}\right)^{j} e^{-u^2/2s^2}\widehat{f}(u), \quad u \in \mathbb{R}.$$

Then for $u \in \mathbb{R}$,

$$\lim_{N\to\infty} \widehat{P_N f}(u) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{u^2}{2s^2}\right)^{j} e^{-u^2/2s^2}\widehat{f}(u) = e^{u^2/2s^2}e^{-u^2/2s^2}\widehat{f}(u) = \widehat{f}(u).$$

Since $|\widehat{P_N f}(u)| \le |\widehat{f}(u)|$, by the Dominated Convergence Theorem, $\lim_{N\to\infty} \widehat{P_N f} = \widehat{f}$ in $L^2(\mathbb{R})$, and so $\lim_{N\to\infty} P_N f = f$ in $L^2(\mathbb{R})$. □

4. B-spline scale-space and wavelet representations

We now consider the centered uniform B-spline M_n of order $n \geq 1$ with support $[-n/2, n/2]$. Its Fourier transform is $\widehat{M_n}(u) = \left(\frac{\sin(u/2)}{u/2}\right)^n$, $u \in \mathbb{R}$. As a probability density function, M_n has mean 0 and variance $\sigma_n^2 = n/12$. It is known that $\sigma_n M_n(\sigma_n t) \to G(t)$, uniformly in t as $n \to \infty$, and the rate of convergence was given in [3]. The B-spline M_n has been used to define a scale-space that approximates the Gaussian scale-space [37]. Here we study the corresponding wavelet representations. In particular, we show that the high frequency component of a signal between two consecutive dyadic scales in the B-spline scale-space can be represented as a finite linear combination of continuous wavelet transforms, where the mother wavelets are the derivatives of B-splines or the framelets of Ron and Shen [35].

4.1. **Continuous wavelet transform from B-spline scale-space.** The B-spline M_n satisfies the conditions on the kernel ϕ for the singular integral operators of Section 2. Therefore,

$$\frac{\partial}{\partial s} S_{M_n} f(s,x) = W_{\psi_n} f(s,x), \quad x \in \mathbb{R}, \ s > 0,$$

where

$$\psi_n(t) := M_n(t) + tM_n'(t), \quad t \in \mathbb{R}, \tag{4.1}$$

and so Proposition 2.4 holds for ψ_n. We see from (4.1) that ψ_n is an even function with the same support as M_n. For $n = 2$, ψ_2 is an even discontinuous function on $\mathbb{R}$ with support $[-1, 1]$ and $\psi_2(t) = 1 - 2t$, $0 \leq t < 1$. We shall now show that ψ_n has the same shape as the Mexican hat function $\omega_2 := -G''$, i.e. the same sign structure as ω_2 and its derivatives. To be more precise, we denote by $V(f, I)$ the number of strict sign changes of the function $f : I \to \mathbb{R}$ on an open interval I, and put $V(f) := V(f, \mathbb{R})$.

Proposition 4.1. *For $n \geq 1$, let $\psi_n(t) := M_n(t) + tM_n'(t)$, $t \in \mathbb{R}$. Then for $k = 0, 1, \ldots, n-2$, $\psi_n^{(k)}(t) < 0$, $t \in (-n/2, -n/2 + \delta)$, for some $\delta > 0$ and $V(\psi_n^{(k)}) = k + 2$.*

Proof. For $k = 0, 1, \ldots, n-2$, $\psi_n^{(k)}(t) = (k+1)M_n^{(k)}(t) + tM_n^{(k+1)}(t)$, $t \in \mathbb{R} \backslash \mathbb{Z}$. The centered uniform B-spline M_n is a piecewise polynomial of degree $n-1$ with knots at $j - n/2$, $j \in \mathbb{Z}$, and vanishes outside the interval $(-n/2, n/2)$. Its $(n-1)^{\text{th}}$ order derivative is a step function whose values on the subintervals $(j - n/2, j + 1 - n/2)$, $j = 0, 1, \ldots, n-1$, are given by $M_n^{(n-1)}(t) = (-1)^j \binom{n-1}{j}$, and are 0 otherwise. Therefore, for $j = 1, 2, \ldots, n-1$, $M_n^{(n-2)}(j - n/2) = (-1)^{j-1} \binom{n-2}{j-1}$. Then for $j = 0, 1, \ldots, n-1$,

$$\psi_n^{(n-2)}(j - n/2^+) = (n-1)(-1)^{j-1} \binom{n-2}{j-1} + (j - n/2)(-1)^j \binom{n-1}{j},$$

and

$$\psi_n^{(n-2)}(j+1-n/2^-) = (n-1)(-1)^j\binom{n-2}{j} + (j+1-n/2)(-1)^j\binom{n-1}{j},$$

where, as usual, $\binom{n-2}{-1} = \binom{n-2}{n-1} = 0$. Thus

$$(-1)^{j-1}\psi_n^{(n-2)}(j-n/2^+) \;>\; 0, \quad 0 \le j \le n/2, \tag{4.2}$$

$$(-1)^{j-1}\psi_n^{(n-2)}(j-n/2^-) \;>\; 0, \quad n/2 \le j \le n. \tag{4.3}$$

In particular, $\psi_n^{(n-2)}(-n/2^+) < 0$ and so $\psi_n^{(k)}(t) < 0$ for $t \in (-n/2, -n/2+\delta)$ for some $\delta > 0$ and for $k = 0, 1, \ldots, n-3$. Since $\psi_n^{(n-2)}$ is linear on $(j-n/2, j+1-n/2)$, $j = 0, 1, \ldots, n$, from (4.2) and (4.3) we see that $V(\psi_n^{(n-2)}) = n$. By Rolle's theorem,

$$V(\psi_n^{(k)}) \le V(\psi_n^{(k+1)}) - 1, \quad k = 0, 1, \ldots, n-3, \tag{4.4}$$

so that $V(\psi_n^{(k)}) \le k+2, \quad k = 0, 1, \ldots, n-2$. Now $\psi_n(t) < 0$ for $t \in (-n/2, -n/2+\delta) \cup (n/2-\delta, n/2)$ and $\psi_n(0) > 0$. Thus $V(\psi_n) = 2$, and applying (4.4) again gives $V(\psi_n^{(k)}) \ge k+2, \quad k = 0, 1, \ldots, n-2$. □

4.2. **Semi-discrete wavelet representations by derivatives of B-splines and spline framelets.** We now look at the representation for the difference of the B-spline scale-space operator at two consecutive dyadic scales and establish two representations, the first is in terms of derivatives of B-splines and the second in terms of spline framelets of Ron and Shen [35].

Proposition 4.2. *For $f \in L^p(\mathbb{R})$, $1 \le p < \infty$,*

$$S_{M_{2m}}f(s,x) - S_{M_{2m}}f(2s,x) = \sum_{k=1}^{m} \frac{1}{2^{2k}}\binom{m}{k} W_{M_{2m+2k}^{(2k)}} f(2s,x), \quad x \in \mathbb{R},\; s > 0, \tag{4.5}$$

and

$$S_{M_{2m}}f(2s,x) - S_{M_{2m}}f(s,x) = \sum_{k=1}^{m} \binom{m}{k} W_{\psi_{2m,2k}} f(2s,x), \quad x \in \mathbb{R},\; s > 0, \tag{4.6}$$

where $\psi_{2m,\ell}$, $\ell = 1, 2, \ldots, 2m$, are, up to constant factors and a constant scaling, Ron and Shen's spline framelets defined by

$$\widehat{\psi}_{2m,\ell}(u) := \sin^{\ell}(u/2)\cos^{2m-\ell}(u/2)\widehat{M}_{2m}(u), \quad u \in \mathbb{R}. \tag{4.7}$$

Proof. For $u \in \mathbb{R}$,

$$\begin{aligned} \widehat{M}_{2m}(2u) &= \left(\frac{\sin(u/2)}{u/2}\right)^{2m} \cos^{2m}(u/2) = \widehat{M}_{2m}(u)\sum_{k=0}^{m}\binom{m}{k}(-1)^k \sin^{2k}(u/2) \\ &= \widehat{M}_{2m}(u) + \sum_{k=1}^{m}\binom{m}{k}(-1)^k \sin^{2k}(u/2)\widehat{M}_{2m}(u). \end{aligned} \tag{4.8}$$

Now for $\ell = 1, 2, \ldots,$

$$\sin^\ell(u/2)\widehat{M}_{2m}(u) = \left(\frac{u}{2}\right)^\ell \left(\frac{\sin(u/2)}{u/2}\right)^{2m+\ell} = \left(\frac{-i}{2}\right)^\ell (iu)^\ell \widehat{M}_{2m+\ell}(u) = \left(\frac{-i}{2}\right)^\ell \widehat{M^{(\ell)}_{2m+\ell}}(u),$$

and so (4.8) gives

$$\widehat{M}_{2m}(2u) - \widehat{M}_{2m}(u) = \sum_{k=1}^{m} \frac{1}{2^{2k}} \binom{m}{k} \widehat{M^{(2k)}_{2m+2k}}(u).$$

Thus for $s > 0$,

$$\widehat{M}_{2m}(u/s) - \widehat{M}_{2m}(u/2s) = \sum_{k=1}^{m} \frac{1}{2^{2k}} \binom{m}{k} \widehat{M^{(2k)}_{2m+2k}}(u/2s).$$

Since $\widehat{M}_{2m}$ and $\widehat{M^{(2k)}_{2m+2k}}$ are in $L^1(\mathbb{R})$, we may apply the inverse Fourier transform to the functions in the above equation and obtain

$$sM_{2m}(st) - 2sM_{2m}(2st) = \sum_{k=1}^{m} \frac{1}{2^{2k}} \binom{m}{k} 2sM^{(2k)}_{2m+2k}(2st), \quad t \in \mathbb{R}.$$

Then replacing t by $x - t$, multiplying by $f(t)$ and integrating gives (4.5) from (2.1) and (1.3).

To prove (4.6), we write

$$\begin{aligned}
\widehat{M}_{2m}(u) &= \widehat{M}_{2m}(2u)\sec^{2m}(u/2) = \widehat{M}_{2m}(2u)(1+\tan^2(u/2))^m \\
&= \widehat{M}_{2m}(2u)\sum_{k=0}^{m}\binom{m}{k}\tan^{2k}(u/2) = \widehat{M}_{2m}(2u) + \widehat{M}_{2m}(2u)\sum_{k=1}^{m}\binom{m}{k}\tan^{2k}(u/2) \\
&= \widehat{M}_{2m}(2u) + \widehat{M}_{2m}(u)\cos^{2m}(u/2)\sum_{k=1}^{m}\binom{m}{k}\tan^{2k}(u/2) \\
&= \widehat{M}_{2m}(2u) + \sum_{k=1}^{m}\binom{m}{k}\sin^{2k}(u/2)\cos^{2m-2k}(u/2)\widehat{M}_{2m}(u) \\
&= \widehat{M}_{2m}(2u) + \sum_{k=1}^{m}\binom{m}{k}\widehat{\psi}_{2m,2k}(u), \quad u \in \mathbb{R},
\end{aligned}$$

where $\widehat{\psi}_{2m,2k}(u)$ are defined in (4.7). Then a similar argument as above leads to (4.6). ☐

Since M_{2m} satisfies all the conditions for ϕ in Section 2, combining Corollary 2.8, Proposition 2.7 and Proposition 4.2, we have

Corollary 4.3. (a) *If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then*

$$\lim_{\substack{n\to\infty \\ \ell\to-\infty}} \left\| f + \sum_{j=\ell}^{n}\sum_{k=1}^{m} \frac{1}{2^{2k}}\binom{m}{k} W_{M^{(2k)}_{2m+2k}} f(2^j, \cdot)\right\|_p = 0$$

and

$$\lim_{\substack{n\to\infty \\ \ell\to-\infty}} \left\| f - \sum_{j=\ell}^{n} \sum_{k=1}^{m} \binom{m}{k} W_{\psi_{2m,2k}} f(2^j,\ \cdot) \right\|_p = 0.$$

(b) *If* $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, *and is continuous and bounded on* $\mathbb{R}$, *then*

$$f(x) = -\sum_{j=-\infty}^{\infty} \sum_{k=1}^{m} \frac{1}{2^{2k}} \binom{m}{k} W_{M_{2m+2k}^{(2k)}} f(2^j, x), \quad x \in \mathbb{R},$$

and

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{m} \binom{m}{k} W_{\psi_{2m,2k}} f(2^j, x), \quad x \in \mathbb{R}.$$

(c) *If* $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, $1 \le p < \infty$, *the convergence in* (b) *is uniform.*

5. Scale-space defined by $-M_n'(t)/t$

We shall call a function ϕ *unimodal of order* n if it has derivatives up to order n, with at most jump discontinuities, and the number of sign changes of its j^{th} derivatives, $V(\phi^{(j)}) = j$, $j = 0, 1, \ldots, n$. Unimodal functions of some order are said to be of the same shape. The centered uniform B-spline, M_n of order n is unimodal of order $n-1$. The Gaussian function is unimodal of infinite order. Like the Gaussian, unimodal functions of high order are good candidates for scale-space smoothing and their derivatives provide mother wavelets in which the number of oscillations equals the order of the derivative. For the Gaussian, we have $-G'(t)/t = G(t)$. In this section we show that $-M_n'(t)/t$ has the same shape as M_n and the Gaussian. More generally, we shall prove

Proposition 5.1. *Let* ϕ *be a compactly supported symmetric unimodal function of order* $n \ge 2$ *and* $f(t) := \phi'(t)/t$, $t \in \mathbb{R}$, *where* $f(0) := \phi''(0)$. *Then* f *is a symmetric unimodal function of order* $n-2$.

We first derive a formula to help prove Proposition 5.1 .

Lemma 5.2. *If a symmetric function* ϕ *has derivatives up to order* $n \ge 2$ *and* $f(t) = \phi'(t)/t$, *then for* $k = 0, 1, \ldots, n-2$,

$$x^{k+1} f^{(k)}(x) = \int_0^x t^k \phi^{(k+2)}(t)dt. \tag{5.1}$$

Proof. Since $\phi'(0) = 0$, the relation (5.1) holds for $k = 0$. If it holds for $k < n-2$, then

$$\begin{aligned} \int_0^x t^{k+1}\phi^{(k+3)}(t)dt &= x^{k+1}\phi^{(k+2)}(x) - (k+1)\int_0^x t^k \phi^{(k+2)}(t)dt \\ &= x^{k+1}\phi^{(k+2)}(x) - (k+1)x^{k+1} f^{(k)}(x). \end{aligned} \tag{5.2}$$

Differentiating $\phi'(x) = xf(x)$ gives

$$\phi^{(k+2)}(x) = (k+1)f^{(k)}(x) + xf^{(k+1)}(x), \quad x \in \mathbb{R},$$

which together with (5.2) lead to

$$\int_0^x t^{k+1}\phi^{(k+3)}(t)dt = x^{k+2}f^{(k+1)}(x).$$

The relation (5.1) follows by induction for $k = 0, 1, \ldots, n-2$. □

Proof of Proposition 5.1. Suppose that $V(\phi^{(k)}) = k$ for $k = 0, 1, \ldots, n$. Clearly $V(f) = 0$. We use induction to show $V(f^{(k)}) = k$, $k = 0, 1, \ldots, n-2$. Suppose $V(f^{(k-1)}) = k-1$. Let $g(x) := \int_0^x t^k\phi^{(k+2)}(t)dt$. Then $g'(x) = x^k\phi^{(k+2)}(x)$ and $V(g', \mathbb{R}_+) = V(\phi^{(k+2)}, \mathbb{R}_+) = [k/2]+1$, where $[x]$ denotes the largest integer $\le x$. Since $g(0) = 0$ and Lemma 5.2 implies that $g(x) = x^{k+1}f^{(k)}(x) \to 0$ as $x \to \infty$, this gives $V(f^{(k)}, \mathbb{R}_+) = V(g, \mathbb{R}_+) \le [k/2]$. Therefore $V(f^{(k)}) \le k$. But $V(f^{(k-1)}) = k-1$ and $f^{(k-1)}(x) \to 0$ as $|x| \to \infty$, which imply $V(f^{(k)}) \ge k$. Hence $V(f^{(k)}) = k$. □

Corollary 5.3. *If* $n \ge 3$, *then the function*

$$f_n(t) := -\frac{1}{t}M_n'(t), \quad t \in \mathbb{R},$$

is unimodal of order $n-3$.

We have

$$f_n(0) = -\lim_{t\to 0}\frac{1}{t}M_n'(t) = -M_n''(0)$$

and

$$\int_{-\infty}^{\infty} t^2 f_n(t)dt = -\int_{-\infty}^{\infty} tM_n'(t)dt = \int_{-\infty}^{\infty} M_n(t)dt = 1. \tag{5.3}$$

For $n = 3, 4, \ldots$, let

$$a_n := \int_{-\infty}^{\infty} f_n(t)dt,$$

and write

$$\widetilde{f}_n(t) := a_n^{-3/2} f_n(a_n^{-1/2}t), \quad t \in \mathbb{R}.$$

Then

$$\int_{-\infty}^{\infty} \widetilde{f}_n(t)dt = a_n^{-3/2}\int_{-\infty}^{\infty} f_n(a_n^{-1/2}t)dt = a_n^{-1}\int_{-\infty}^{\infty} f_n(y)dy = 1,$$

and

$$\int_{-\infty}^{\infty} t^2\widetilde{f}_n(t)dt = \int_{-\infty}^{\infty} a_n^{-3/2}t^2 f_n(a_n^{-1/2}t)dt = \int_{-\infty}^{\infty} y^2 f_n(y)dy = 1,$$

by (5.3), and so $\widetilde{f}_n$ is the standardized form of f_n.

The function

$$\widetilde{f}_n(t) = a_n^{-3/2} f_n(a_n^{-1/2} t) = -\frac{1}{a_n t} M_n'(a_n^{-1/2} t), \quad t \in \mathbb{R},$$

satisfies the conditions for ϕ in Section 2. The corresponding mother wavelet,

$$\widetilde{g}_n(t) := \frac{d}{dt}(t\widetilde{f}_n(t)) = -a_n^{-3/2} M_n''(a_n^{-1/2} t), \quad t \in \mathbb{R},$$

has the same shape as the Mexican hat function $-G''$ in the sense that $V(\widetilde{g}_n^{(k)}) = V(G^{(k+2)}) = k+2$, for $k = 0, 1, \ldots, n-3$. If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, Proposition 2.4 gives the recovery formula,

$$\lim_{\substack{t\to\infty \\ r\to 0}} \left\| f + \frac{1}{a_n} \int_r^t \frac{1}{u} W_{M_n''} f(u, \cdot) du \right\|_p = 0,$$

and if in addition f is continuous and bounded on $\mathbb{R}$,

$$f(x) = -\frac{1}{a_n} \int_0^\infty \frac{1}{u} W_{M_n''} f(u, x) du, \quad x \in \mathbb{R},$$

where the convergence is uniform if $f \in C(\mathbb{R})$.

To compute a_n, we define

$$A_n(j) := \int_{-\infty}^{\infty} \frac{M_n(t - j/2) - M_n(t + j/2)}{t} dt, \tag{5.4}$$

for $n, j = 1, 2, \ldots$, $(n, j) \neq (1, 1)$, and set

$$A_n(0) = 0, \quad n = 1, 2, \ldots. \tag{5.5}$$

From (5.4),

$$\begin{aligned} A_1(j) &= \int_{-\infty}^{\infty} \frac{M_1(t - j/2) - M_1(t + j/2)}{t} dt = 2 \int_{-\infty}^{\infty} \frac{M_1(t - j/2)}{t} dt \\ &= 2 \int_{(j-1)/2}^{(j+1)/2} \frac{1}{t} dt = 2 \ln\left(\frac{j+1}{j-1}\right), \quad j = 2, 3 \ldots. \end{aligned} \tag{5.6}$$

Proposition 5.4. *For $n, j = 1, 2, \ldots$, $(n, j) \neq (1, 1)$,*

$$A_{n+1}(j) = \left(\frac{n+1+j}{2n}\right) A_n(j+1) + \left(\frac{n+1-j}{2n}\right) A_n(j-1), \tag{5.7}$$

with $A_n(0)$, $n = 1, 2, \ldots$, $A_1(j)$, $j = 2, 3, \ldots$, as in (5.5) *and* (5.6) *and arbitrary $A_1(1)$.*

Proof. The centered uniform B-spline of order $n+1$ can be expressed in terms of those of order n by the de Boor-Cox algorithm:

$$M_{n+1}(t) = \left(\frac{n+1-2t}{2n}\right) M_n(t - 1/2) + \left(\frac{n+1+2t}{2n}\right) M_n(t + 1/2), \quad t \in \mathbb{R}. \tag{5.8}$$

By (5.4) and (5.8),

$$\begin{aligned}
A_{n+1}(j) &= \int_{-\infty}^{\infty} \frac{1}{t}\left\{\left(\frac{n+1-2t+j}{2n}\right) M_n(t-\tfrac{j+1}{2}) + \left(\frac{n+1+2t-j}{2n}\right) M_n(t-\tfrac{j-1}{2})\right.\\
&\quad \left. -\left(\frac{n+1-2t-j}{2n}\right) M_n(t+\tfrac{j-1}{2}) - \left(\frac{n+1+2t+j}{2n}\right) M_n(t+\tfrac{j+1}{2})\right\} dt\\
&= \left(\frac{n+1+j}{2n}\right)\int_{-\infty}^{\infty} \frac{M_n(t-\frac{j+1}{2}) - M_n(t+\frac{j+1}{2})}{t} dt\\
&\quad + \left(\frac{n+1-j}{2n}\right)\int_{-\infty}^{\infty} \frac{M_n(t-\frac{j-1}{2}) - M_n(t+\frac{j-1}{2})}{t} dt\\
&\quad + \frac{1}{n}\int_{-\infty}^{\infty} \left\{-M_n(t-\tfrac{j+1}{2}) + M_n(t-\tfrac{j-1}{2}) + M_n(t+\tfrac{j-1}{2}) - M_n(t+\tfrac{j+1}{2})\right\} dt,\\
&\qquad\qquad j = 2, 3, \ldots.
\end{aligned}$$

Since $\int_{-\infty}^{\infty} M_n(t)dt = 1$, (5.7) follows from (5.4). □

Now for $n \geq 3$,

$$A_{n-1}(1) = \int_{-\infty}^{\infty} \frac{M_{n-1}(t-1/2) - M_{n-1}(t+1/2)}{t} dt = -\int_{-\infty}^{\infty} \frac{M_n'(t)}{t} dt = a_n.$$

The expressions for $a_n = A_{n-1}(1)$ can be computed using (5.7). Explicitly, they are given by

$$a_n = \begin{cases} \frac{1}{2^{2m-2}(2m-1)!}\sum_{j=1}^{m}(-1)^{m-j}\binom{2m+1}{m-j}(2j+1)^{2m-1}\ln(2j+1), & n = 2m+1,\\ \frac{2}{(2m)!}\sum_{j=1}^{m}(-1)^{m-j}\binom{2m+2}{m-j}(j+1)^{2m}\ln(j+1), & n = 2m+2,\end{cases}$$

for $m = 1, 2, \ldots$.

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