# BAND-LIMITED REFINABLE FUNCTIONS FOR WAVELETS AND FRAMELETS

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**Abstract**

Extending band-limited constructions of orthonormal refinable functions, a special class of periodic functions is used to generate a family of band-limited refinable functions. Characterizations of Riesz bases and frames formed by integer shifts of these refinable functions are obtained. Such families of refinable functions are employed to construct band-limited biorthogonal wavelet bases and biframes with desirable time-frequency localization.

*Keywords:* Band-limited refinable functions; Refinement masks; Wavelet bases; Frames

## 1. Introduction

Signals in practice are often band-limited where their frequency contents are restricted to prescribed bands. For such signals, it is natural to expand them in terms of band-limited functions such as band-limited wavelets. Band-limited refinable functions play a fundamental role in the construction of band-limited wavelets. Well-known examples of band-limited refinable functions and wavelets include the orthonormal Shannon's and Meyer's scaling functions and wavelets (see for instance [5, 16, 20]). In [9], the periodized Meyer's wavelet is applied to deconvolution and image deblurring; and in [18], the Meyer's wavelet is used in a bifiltering algorithm to improve results in image compression. Among others, contributions to the theory of band-limited refinable functions and wavelets are made in [1, 2, 10, 11, 15, 16, 17].

Band-limited orthonormal refinable functions and wavelets are already well studied in the literature (see for instance [2, 10, 15, 16]). Therefore we do not focus on the orthonormal case here. This note instead aims to provide a general family of band-limited refinable functions and illustrate how it can be easily employed to construct band-limited biorthogonal wavelet bases and biframes. These band-limited refinable functions, wavelets and framelets could be designed to have certain desired bandwidth and lie in the Schwartz class of rapidly decaying infinitely differentiable functions.

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Fixing notations, let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ be the usual inner product and norm of the space $L^2(\mathbb{R})$. We denote the Fourier transform of $f \in L^2(\mathbb{R})$ by $\hat{f}$; and for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\hat{f}$ takes the form $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx$, $\xi \in \mathbb{R}$.

Let $I$ be a countable index set. A sequence $\{v_n\}_{n\in I}$ in $L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $A$, $B > 0$ such that

$$A\|f\|_2^2 \le \sum_{n\in I} |\langle f,\, v_n\rangle|^2 \le B\|f\|_2^2, \quad f \in L^2(\mathbb{R}). \tag{1.1}$$

A frame is said to be *tight* if we may take $A = B = 1$ in (1.1). A sequence $\{v_n\}_{n\in I}$ in $L^2(\mathbb{R})$ for which the second inequality in (1.1) holds is known as a *Bessel sequence*. In addition, $\{v_n\}_{n\in I}$ in $L^2(\mathbb{R})$ is a *Riesz basis* for $L^2(\mathbb{R})$ if its linear span is dense in $L^2(\mathbb{R})$ and there exist constants $A$, $B > 0$ such that

$$A\sum_{n\in I} |c_n|^2 \le \Big\| \sum_{n\in I} c_n v_n \Big\|_2^2 \le B\sum_{n\in I} |c_n|^2, \quad \{c_n\}_{n\in I} \in \ell^2(I).$$

If $A = B = 1$, then the Riesz basis $\{v_n\}_{n\in I}$ is an *orthonormal basis* for $L^2(\mathbb{R})$. Similar to these definitions on $L^2(\mathbb{R})$, we have the notions of frames, Riesz bases and orthonormal bases for subspaces of $L^2(\mathbb{R})$.

Here we are interested in frames and Riesz bases that are affine systems of the form

$$X(\psi_1, \ldots, \psi_n) := \{2^{j/2}\psi_\ell(2^j \cdot -k) \,:\, j,\, k \in \mathbb{Z},\, \ell = 1, \ldots, n\},$$

where $\psi_\ell \in L^2(\mathbb{R})$, $\ell = 1, \ldots, n$. The functions $\psi_1, \ldots, \psi_n$ are called *mother wavelets*, or simply, *wavelets*. If $X(\psi_1, \ldots, \psi_n)$ forms a frame for $L^2(\mathbb{R})$, $\psi_1, \ldots, \psi_n$ are also referred to as *framelets*. Wavelets are often constructed from refinable functions. A function $\phi \in L^2(\mathbb{R})$ is called a *refinable function* or *scaling function* if it satisfies the *refinement equation*

$$\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi), \quad \text{a.e. } \xi \in \mathbb{R}, \tag{1.2}$$

where $\hat{a}$ is a $2\pi$-periodic measurable function known as a *refinement mask*. If $\phi \in L^2(\mathbb{R})$ is refinable with $|\hat{\phi}| > 0$ on a neighborhood of 0 and $\hat{a}$ bounded, then the sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by

$$V_j := \overline{\operatorname{span}\{2^{j/2}\phi(2^j \cdot -k) : k \in \mathbb{Z}\}}, \quad j \in \mathbb{Z}, \tag{1.3}$$

satisfies the properties of a *multiresolution analysis* of $L^2(\mathbb{R})$, namely, $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (see for instance [3]).

In Section 2, we consider a special class of refinement masks and its corresponding family of band-limited refinable functions. Such a family of refinable functions, which we denote by $\mathcal{B}_{\delta,\Omega}$ where $0 < \delta \le \Omega \le 2\pi/3$, is a natural extension of the Shannon's and Meyer's scaling functions. For any $\phi \in \mathcal{B}_{\delta,\Omega}$, its Fourier transform is supported on $[-2\Omega,\, 2\Omega]$ and takes the value 1 on $[-2\delta,\, 2\delta]$. We also provide characterizations of whether the integer shifts of $\phi$ form a Riesz basis or a frame for their closed linear span. These characterizations are particularly simple to check. In Section 3, we illustrate how the band-limited refinable functions in $\mathcal{B}_{\delta,\Omega}$ could be readily used to construct band-limited wavelets with desirable time-frequency localization. Two setups are considered, one for constructing biorthogonal Riesz bases and the other for frames of $L^2(\mathbb{R})$. In particular, we illustrate how explicit expressions for the Fourier transforms of the wavelets could be obtained from the class of refinement masks.

## 2. Band-limited refinable functions

A function $\phi \in L^2(\mathbb{R})$ is said to be *band-limited* if $\operatorname{supp}\hat{\phi} \subseteq [-\Omega,\, \Omega]$ for some $\Omega > 0$. Band-limited orthonormal refinable functions often take the form of bell-shaped functions in the Fourier domain, with their refinement masks of similar shapes (see for instance [2, 16]). Extending from this form, we consider a rather natural class of $2\pi$-periodic functions and a corresponding family of band-limited functions. The former provides a general collection of refinement masks, while the latter contains band-limited refinable functions generated by these masks, which are not necessarily orthonormal.

**Definition 2.1.** *For $0 < \delta \le \Omega \le 2\pi/3$, let $\mathcal{A}_{\delta,\Omega}$ consist of all $2\pi$-periodic functions $\hat{a}$ on $\mathbb{R}$ with the following properties.*

(a) *$\hat{a}$ is nonnegative, even and bounded; it is continuous everywhere on $[-\pi,\, \pi]$ except possibly at the points $\pm\Omega$; and $\lim_{\xi\to\Omega^-} \hat{a}(\xi)$ exists.*

(b) *$\hat{a}$ is positive on $(-\Omega,\, \Omega)$ and vanishes on $[-\pi,\, \pi]\backslash[-\Omega,\, \Omega]$.*

(c) *$\hat{a}(\xi) = 1$ for all $\xi \in [-\delta,\, \delta]$.*

**Definition 2.2.** *For $0 < \delta \le \Omega \le 2\pi/3$, let $\mathcal{B}_{\delta,\Omega}$ consist of all functions $\phi \in L^2(\mathbb{R})$ with the following properties.*

(a) *$\hat{\phi}$ is nonnegative, even and bounded; it is continuous everywhere on $\mathbb{R}$ except possibly at the points $\pm 2\Omega$; and $\lim_{\xi\to(2\Omega)^-} \hat{\phi}(\xi)$ exists.*

(b) *$\hat{\phi}$ is positive on $(-2\Omega,\, 2\Omega)$ and vanishes on $\mathbb{R}\backslash[-2\Omega,\, 2\Omega]$.*

(c) *$\hat{\phi}(\xi) = 1$ for all $\xi \in [-2\delta,\, 2\delta]$.*

Definition 2.2 has as special cases well-known orthonormal refinable functions such as the Shannon's scaling function (where $\delta = \Omega = \pi/2$) and the Meyer's scaling function (where $\delta = \pi/3$, $\Omega = 2\pi/3$). We also note that by part (c) of Definitions 2.1 and 2.2, if $0 < \delta_1 < \delta_2 \le \Omega \le 2\pi/3$, then $\mathcal{A}_{\delta_2,\Omega} \subset \mathcal{A}_{\delta_1,\Omega}$ and $\mathcal{B}_{\delta_2,\Omega} \subset \mathcal{B}_{\delta_1,\Omega}$.

The following theorem shows that a refinable function in the family $\mathcal{B}_{\delta,\Omega}$ has a corresponding refinement mask in the class $\mathcal{A}_{\delta,\Omega}$, and vice versa.

**Theorem 2.1.** *Let $0 < \delta \le \Omega \le 2\pi/3$. For every $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, there exists a unique function $\phi \in \mathcal{B}_{\delta,\Omega}$ (up to a set of measure zero) such that* (1.2) *holds. Conversely, for every $\phi \in \mathcal{B}_{\delta,\Omega}$, there exists a unique function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ (up to a set of measure zero) such that* (1.2) *holds. Furthermore in this correspondence, for $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $\hat{a} \in C^k(\mathbb{R})$ if and only if $\hat{\phi} \in C^k(\mathbb{R})$.*

**Proof**: For $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, we define $\phi \in L^2(\mathbb{R})$ by setting its Fourier transform as

$$\hat{\phi}(\xi) := \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi), \quad \xi \in \mathbb{R}, \tag{2.1}$$

where $\mathbf{1}_{[-2\Omega,\,2\Omega]}$ is the characteristic function over the interval $[-2\Omega,\, 2\Omega]$ and

$$N := \begin{cases} \lceil \log_2(\Omega/\delta) \rceil, & \text{if } \delta < \Omega, \\ 1, & \text{if } \delta = \Omega. \end{cases}$$

(For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.) Using the fact that $\hat{a}(2^{-N}\xi) = 1$ for all $\xi \in [-\Omega,\, \Omega]$ together with $\Omega \le 2\pi/3$, we have

$$\hat{\phi}(2\xi) = \Big[ \prod_{j=1}^{N} \hat{a}(2^{-j}2\xi) \Big] \mathbf{1}_{[-2\Omega,\,2\Omega]}(2\xi) = \hat{a}(\xi) \Big[ \prod_{j=1}^{N-1} \hat{a}(2^{-j}\xi) \Big] \mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \hat{a}(\xi)\hat{\phi}(\xi), \quad \xi \in \mathbb{R},$$

that is, $\phi$ is refinable. (This derivation is similar to an argument in the proof of [2, Lemma 5.14].) In addition, observe that for $j = 1, \dots, N$, $\hat{a}(2^{-j}\cdot)$ is continuous everywhere on $(-2\Omega,\, 2\Omega)$ and $\hat{a}(2^{-j}\xi) = 1$ for all $\xi \in [-2\delta,\, 2\delta]$. Then it follows from (2.1) that $\hat{\phi}$ satisfies all the conditions in Definition 2.2 and so $\phi \in \mathcal{B}_{\delta,\Omega}$. This solution of (1.2) in $\mathcal{B}_{\delta,\Omega}$ is unique up to a set of measure zero. Indeed, if $\varphi$ is another solution in $\mathcal{B}_{\delta,\Omega}$, then by [19, Corollary 3], $\varphi$ and $\phi$ are related by

$$\varphi(x) = \frac{\hat{\varphi}(0)}{\hat{\phi}(0)} \phi(x) = \phi(x), \quad \text{a.e. } x \in \mathbb{R},$$

since $\hat{\varphi}(0) = \hat{\phi}(0) = 1$.

Conversely, for $\phi \in \mathcal{B}_{\delta,\Omega}$, let $\hat{a}$ be the $2\pi$-periodic extension of the function

$$\alpha(\xi) := \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)} \mathbf{1}_{[-\Omega,\,\Omega]}(\xi), \quad \xi \in [-\pi,\, \pi]. \tag{2.2}$$

Then $\hat{a}$ satisfies (1.2). Since $\hat{\phi}$ is positive, even and continuous on $[-\Omega,\, \Omega]$, so is $1/\hat{\phi}$. In addition, $\hat{\phi}(2\xi) = \hat{\phi}(\xi) = 1$ for all $\xi \in [-\delta,\, \delta]$. Hence the corresponding $\hat{a}$ satisfies all the properties in Definition 2.1 and therefore lies in $\mathcal{A}_{\delta,\Omega}$. As any $2\pi$-periodic solution $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ of (1.2) agrees with the function $\alpha$ in (2.2) almost everywhere on $[-\pi,\, \pi]$, it is uniquely determined up to a set of measure zero.

Finally, for $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, based on $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$, the finite product (2.1) shows that $\hat{\phi} \in C^k(\mathbb{R})$. On the other hand, beginning with $\hat{\phi} \in \mathcal{B}_{\delta,\Omega} \cap C^k(\mathbb{R})$, it follows from the restriction of $\hat{a}$ to $[-\pi,\, \pi]$ in (2.2) that $\hat{a} \in C^k(\mathbb{R})$. ∎

Since $\hat{a}$ in Theorem 2.1 is a $2\pi$-periodic function, it is natural to ask whether the assumption of $\Omega \le 2\pi/3$ could be relaxed, for instance, to $\Omega \le \pi$. This is not possible because if $2\pi/3 < \Omega \le \pi$, we have $2\pi - \Omega < 2\Omega$ and for $2\pi - \Omega < |\xi| < 2\Omega$, the left-hand side of (1.2) equals 0 while the right-hand side is nonzero.

For $0 < \delta \le \Omega \le 2\pi/3$, starting from $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, it follows from Theorem 2.1 that $\phi$ as defined in (2.1) is a refinable function in $L^2(\mathbb{R})$. Since $\hat{\phi}(\xi) = 1$ for all $\xi \in [-2\delta,\, 2\delta]$ and $\hat{a}$ is bounded, the sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ given by (1.3) forms a multiresolution analysis of $L^2(\mathbb{R})$.

**Theorem 2.2.** *For $0 < \delta \le \Omega \le 2\pi/3$, consider $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ and define $\phi \in \mathcal{B}_{\delta,\Omega}$ as in* (2.1)*. Let $V_0$ be the closed linear span of the integer shifts of $\phi$ given by* (1.3)*.*

(a) *The integer shifts of* $\phi$ *form a Riesz basis for* $V_0$ *if and only if either* $\Omega > \pi/2$*, or* $\Omega = \pi/2$ *and* $\lim_{\xi \to (\pi/2)^-} \hat{a}(\xi) > 0$.

(b) *For* $\Omega < \pi/2$*, the integer shifts of* $\phi$ *form a frame for* $V_0$ *if and only if* $\lim_{\xi \to \Omega^-} \hat{a}(\xi) > 0$.

**Proof**: For part (a), it is well known (see for instance [3]) that the integer shifts of $\phi$ form a Riesz basis for $V_0$ if and only if there exist constants $A$, $B > 0$ such that

$$A \leq \Phi(\xi) \leq B, \quad \text{a.e. } \xi \in [-\pi,\, \pi], \tag{2.3}$$

where $\Phi$ is the $2\pi$-periodic function defined by

$$\Phi(\xi) := \sum_{q \in \mathbb{Z}} (\hat{\phi}(\xi + 2\pi q))^2, \quad \xi \in [-\pi,\, \pi]. \tag{2.4}$$

By standard calculation, it follows from (1.2) that

$$\Phi(2\xi) = (\hat{a}(\xi))^2 \Phi(\xi) + (\hat{a}(\xi + \pi))^2 \Phi(\xi + \pi), \quad \text{a.e. } \xi \in [-\pi/2,\, \pi/2].$$

Using (2.3), this leads to

$$\frac{A}{B} \leq (\hat{a}(\xi))^2 + (\hat{a}(\xi + \pi))^2 \leq \frac{B}{A}, \quad \text{a.e. } \xi \in [-\pi/2,\, \pi/2], \tag{2.5}$$

which shows that $\Omega \geq \pi/2$. (Otherwise, $(\hat{a}(\xi))^2 + (\hat{a}(\xi + \pi))^2$ takes the value 0 for $\Omega < |\xi| < \pi/2$ which is not possible.) If $\Omega = \pi/2$, (2.5) implies that $\hat{a}(\xi) \geq \sqrt{A/B}$ for all $\xi \in (-\pi/2,\, \pi/2)$, where we have also used the continuity of $\hat{a}$ on $(-\pi/2,\, \pi/2)$. Thus $\lim_{\xi \to (\pi/2)^-} \hat{a}(\xi) \geq \sqrt{A/B} > 0$.

Conversely, we shall establish (2.3) under the assumption of either $\Omega > \pi/2$, or $\Omega = \pi/2$ and $\lim_{\xi \to (\pi/2)^-} \hat{a}(\xi) > 0$. Indeed, $\Phi$ is bounded above due to the finite number of bounded terms in (2.4). If $\Omega > \pi/2$, since $\phi \in \mathcal{B}_{\delta,\Omega}$ possesses the property of $\hat{\phi}$ being positive and continuous on $(-2\Omega, 2\Omega)$ which contains $[-\pi,\, \pi]$, there exists $A > 0$ such that $\Phi(\xi) \geq (\hat{\phi}(\xi))^2 \geq A$ for all $\xi \in [-\pi,\, \pi]$. On the other hand, if $\Omega = \pi/2$ and $\lim_{\xi \to (\pi/2)^-} \hat{a}(\xi) > 0$, by redefining the values of $\hat{a}(\pm\pi/2)$ as $\lim_{\xi \to (\pi/2)^-} \hat{a}(\xi)$, we see that on the interval $[-\pi/2,\, \pi/2]$, $\hat{a}$ coincides almost everywhere (except possibly at the endpoints $\pm\pi/2$) to a positive continuous function. As such, on the interval $[-\pi,\, \pi]$, $\hat{\phi}$ given by (2.1) equals almost everywhere a positive continuous function. Therefore there exists $A > 0$ such that $\Phi(\xi) \geq (\hat{\phi}(\xi))^2 \geq A$ for almost all $\xi \in [-\pi,\, \pi]$.

To establish part (b), recall from [1] that the integer shifts of $\phi$ form a frame for $V_0$ if and only if there exist constants $A$, $B > 0$ such that

$$A \leq \Phi(\xi) \leq B, \quad \text{a.e. } \xi \in \sigma(\phi), \tag{2.6}$$

where $\sigma(\phi) := \{\xi \in [-\pi,\, \pi] \,:\, \Phi(\xi) > 0\}$. In this case, since $\Omega < \pi/2$ and $\hat{\phi}$ is positive on $(-2\Omega,\, 2\Omega)$, the closure of $\sigma(\phi)$ equals $[-2\Omega,\, 2\Omega]$ and for $\xi \in [-\pi,\, \pi]$, $\Phi(\xi) = (\hat{\phi}(\xi))^2$. Thus (2.6) is equivalent to

$$A \leq (\hat{\phi}(\xi))^2 \leq B, \quad \text{a.e. } \xi \in [-2\Omega,\, 2\Omega]. \tag{2.7}$$

Using (1.2) as well as (2.7) with the continuity of $\hat{\phi}$ on $(-2\Omega,\, 2\Omega)$, we conclude that $\hat{a}(\xi) \geq \sqrt{A/B}$ for all $\xi \in (-\Omega,\, \Omega)$, which gives $\lim_{\xi \to \Omega^-} \hat{a}(\xi) \geq \sqrt{A/B} > 0$.

To complete the proof, it remains to show that $\lim_{\xi\to\Omega^-}\hat{a}(\xi) > 0$ implies (2.7). The upper bound in (2.7) exists as $\hat{\phi}$ is a bounded function. Similar to part (a), by redefining the values of $\hat{a}(\pm\Omega)$ as $\lim_{\xi\to\Omega^-}\hat{a}(\xi)$, on the interval $[-2\Omega, 2\Omega]$, $\hat{\phi}$ given by (2.1) coincides almost everywhere to a positive continuous function. Hence the lower bound in (2.7) also exists. ■

Theorem 2.2 provides easy-to-check characterizations on integer shifts of refinable functions in $\mathcal{B}_{\delta,\Omega}$. For $\Omega \geq \pi/2$, part (a) enables one to verify whether the integer shifts of $\phi \in \mathcal{B}_{\delta,\Omega}$ form a Riesz basis for $V_0$. For $\Omega < \pi/2$, such a collection cannot form a Riesz basis for $V_0$. Instead, part (b) identifies when it forms a frame for $V_0$, which is a fundamental building block of the frame multiresolution analysis in [1].

## 3. Band-limited wavelets and framelets

The family of band-limited refinable functions $\mathcal{B}_{\delta,\Omega}$ arising from the special class of refinement masks $\mathcal{A}_{\delta,\Omega}$ facilitates various constructions of band-limited wavelets and framelets. This section illustrates some of the possible constructions with $\Omega > \pi/2$ for Riesz bases and $\Omega < \pi/2$ for frames.

For $\psi$, $\tilde{\psi} \in L^2(\mathbb{R})$, $(X(\psi), X(\tilde{\psi}))$ is said to be a pair of *biorthogonal wavelet bases* for $L^2(\mathbb{R})$ if each of $X(\psi)$ and $X(\tilde{\psi})$ forms a Riesz basis for $L^2(\mathbb{R})$ and $\langle \psi_{j,k}, \tilde{\psi}_{j',k'}\rangle = \delta_{j,j'}\delta_{k,k'}$ for all $j, k, j', k' \in \mathbb{Z}$, where $\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k)$ and $\tilde{\psi}_{j',k'} := 2^{j'/2}\tilde{\psi}(2^{j'} \cdot -k')$. In [12] and [13], a general approach for constructing biorthogonal wavelet bases is developed. Here, we adapt a certain perspective of [12] and [13] and focus on constructing band-limited wavelets based on pairs of refinement masks from $\mathcal{A}_{\delta,\Omega}$ and $\mathcal{A}_{\delta_0,\Omega_0}$.

**Theorem 3.1.** *For $0 < \delta < \Omega \leq 2\pi/3$, $\Omega > \pi/2$, $0 < \delta_0 < \Omega_0 \leq 2\pi/3$, $\Omega_0 > \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$ and $\hat{a}_0 \in \mathcal{A}_{\delta_0,\Omega_0} \cap C^k(\mathbb{R})$ where $k \in \mathbb{N} \cup \{\infty\}$. Define*

$$\hat{b}(\xi) := e^{-i\xi}\hat{a}_0(\xi+\pi), \qquad \hat{\tilde{a}}(\xi) := \frac{\overline{\hat{b}(\xi+\pi)}}{\overline{\hat{d}(\xi)}}, \qquad \hat{\tilde{b}}(\xi) := -\frac{\hat{a}(\xi+\pi)}{\overline{\hat{d}(\xi)}}, \qquad \xi \in \mathbb{R}, \tag{3.1}$$

*where*

$$\hat{d}(\xi) := \hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{a}(\xi+\pi)\hat{b}(\xi), \qquad \xi \in \mathbb{R}. \tag{3.2}$$

*Suppose that $\hat{\phi}$, $\hat{\tilde{\phi}}$ are respectively generated by $\hat{a}$ and $\hat{\tilde{a}}$ as in (2.1), and set*

$$\hat{\psi}(\xi) := \hat{b}(\xi/2)\hat{\phi}(\xi/2), \qquad \hat{\tilde{\psi}}(\xi) := \hat{\tilde{b}}(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad \xi \in \mathbb{R}. \tag{3.3}$$

*Then $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\Omega_0} \cap C^k(\mathbb{R})$ where $\tilde{\delta} := \min\{\delta, \delta_0, \pi - \min\{\Omega, \Omega_0\}\}$, $\hat{b}$, $\hat{\tilde{b}} \in C^k(\mathbb{R})$, and $\hat{\phi}$, $\hat{\tilde{\phi}}$, $\hat{\psi}$, $\hat{\tilde{\psi}} \in L^2(\mathbb{R}) \cap C^k(\mathbb{R})$. Furthermore, $(X(\psi), X(\tilde{\psi}))$ forms a pair of band-limited biorthogonal wavelet bases for $L^2(\mathbb{R})$.*

**Proof**: As $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$ and $\hat{a}_0 \in \mathcal{A}_{\delta_0,\Omega_0} \cap C^k(\mathbb{R})$, we observe from (3.1) and (3.2) that for $\xi \in \mathbb{R}$,

$$|\hat{d}(\xi)| = |-e^{i\xi}(\hat{a}(\xi)\hat{a}_0(\xi) + \hat{a}(\xi+\pi)\hat{a}_0(\xi+\pi))| = \hat{a}(\xi)\hat{a}_0(\xi) + \hat{a}(\xi+\pi)\hat{a}_0(\xi+\pi) \geq \hat{a}(\xi)\hat{a}_0(\xi). \tag{3.4}$$

Since $\Omega > \pi/2$ and $\Omega_0 > \pi/2$, on the interval $[-\pi/2, \pi/2]$ the function $\hat{a}\hat{a}_0$ is positive and continuous, and therefore bounded from below. So it follows from (3.4) that the $\pi$-periodic function $|\hat{d}|$ is bounded above and below by positive constants. As a result, $\hat{\tilde{a}}$ in (3.1) is well defined and it simplifies to

$$\hat{\tilde{a}}(\xi) = \frac{\hat{a}_0(\xi)}{\hat{a}(\xi)\hat{a}_0(\xi) + \hat{a}(\xi+\pi)\hat{a}_0(\xi+\pi)}, \quad \xi \in \mathbb{R}.$$

By the properties of $\hat{a}$ and $\hat{a}_0$, this shows that $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\Omega_0}$ where $\tilde{\delta} = \min\{\delta, \delta_0, \pi - \min\{\Omega, \Omega_0\}\}$. In addition, $\hat{\tilde{a}}, \hat{b}, \hat{\tilde{b}} \in C^k(\mathbb{R})$. Applying Theorem 2.1 and (3.3), we see that $\hat{\phi}, \hat{\tilde{\phi}}, \hat{\psi}, \hat{\tilde{\psi}} \in L^2(\mathbb{R}) \cap C^k(\mathbb{R})$.

To establish that $(X(\psi), X(\tilde{\psi}))$ is a pair of biorthogonal wavelet bases for $L^2(\mathbb{R})$, we shall invoke [13, Theorem 3.1]. To this end, we verify easily that $\hat{b}(0) = \hat{\tilde{b}}(0) = 0$ and

$$\begin{pmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi) \\ \hat{b}(\xi) & \hat{b}(\xi+\pi) \end{pmatrix} \begin{pmatrix} \hat{\tilde{a}}(\xi) & \hat{\tilde{a}}(\xi+\pi) \\ \hat{\tilde{b}}(\xi) & \hat{\tilde{b}}(\xi+\pi) \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}.$$

In addition, we use [13, Corollary 2.2] to ensure the positivity of the quantities $\nu_2(\hat{a})$ and $\nu_2(\hat{\tilde{a}})$ required in [13, Theorem 3.1]. This is possible under our setting of $\Omega > \pi/2$ and $\Omega_0 > \pi/2$, because by part (a) of Theorem 2.2, the integer shifts of $\phi$ and the integer shifts of $\tilde{\phi}$ form Riesz bases for their respective closed linear spans. Hence the proof is complete. ∎

**Corollary 3.1.** *For $0 < \delta < \Omega \le 2\pi/3$ and $\Omega > \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$ where $k \in \mathbb{N} \cup \{\infty\}$. Define $\hat{\phi}$ by (2.1), and*

$$\hat{\psi}(\xi) := \hat{b}(\xi/2)\hat{\phi}(\xi/2), \quad \xi \in \mathbb{R}, \tag{3.5}$$

*where*

$$\hat{b}(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}, \quad \xi \in \mathbb{R}. \tag{3.6}$$

*Then $\hat{\psi} \in L^2(\mathbb{R}) \cap C^k(\mathbb{R})$, and $X(\psi)$ forms a Riesz basis for $L^2(\mathbb{R})$.*

**Proof**: The result follows from Theorem 3.1 by choosing $\hat{a}_0 = \hat{a}$ which gives $\hat{b}(\xi) = e^{-i\xi}\hat{a}(\xi+\pi)$, $\hat{\tilde{a}}(\xi) = \hat{a}(\xi)\Big[(\hat{a}(\xi))^2 + (\hat{a}(\xi+\pi))^2\Big]^{-1}$ and $\hat{\tilde{b}}(\xi) = e^{-i\xi}\hat{a}(\xi+\pi)\Big[(\hat{a}(\xi))^2 + (\hat{a}(\xi+\pi))^2\Big]^{-1}$. ∎

It is shown in [8, 14] that beginning with a $B$-spline function, or more generally a pseudo-spline function, the wavelet $\psi$ defined in (3.5) via the alternating flip formula (3.6) generates a Riesz basis $X(\psi)$ for $L^2(\mathbb{R})$. On the other hand, as shown in [12], there exists a compactly supported refinable function which does not give a Riesz basis for $L^2(\mathbb{R})$ through (3.6). Corollary 3.1 examines this possibility for band-limited refinable functions and adds another family of Riesz wavelets created by the alternating flip formula. (In Corollary 3.1, the complex conjugation in (3.6) can be removed as $\hat{a}$ is real-valued.)

**Example 3.1.** For $0 < \delta < \Omega \le 2\pi/3$ and $\Omega > \pi/2$, consider $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ defined by

$$\hat{a}(\xi) := \begin{cases} \sin^m\left(\frac{\pi}{2} g\left(\frac{\xi+\Omega}{\Omega-\delta}\right)\right), & \text{if } \xi \in [-\Omega, -\delta), \\ 1, & \text{if } \xi \in [-\delta, \delta], \\ \cos^m\left(\frac{\pi}{2} g\left(\frac{\xi-\delta}{\Omega-\delta}\right)\right), & \text{if } \xi \in (\delta, \Omega], \\ 0, & \text{if } \xi \in [-\pi, \pi]\backslash[-\Omega, \Omega], \end{cases} \tag{3.7}$$

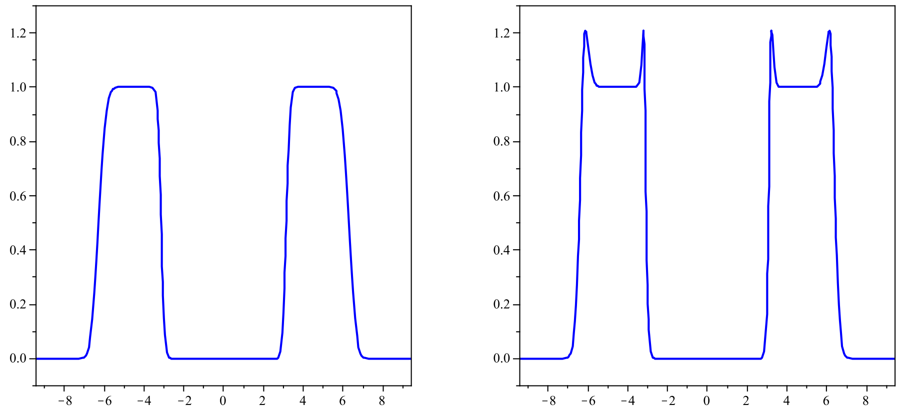

FIGURE 1. Graphs of $|\hat{\psi}|$ (left) and $|\hat{\tilde{\psi}}|$ (right) for a specific construction in Example 3.1.

where $m \in \mathbb{N}$ and $g$ is a $C^k(\mathbb{R})$-function, $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, such that $g(\xi) = 0$ for $\xi < 0$, $g(\xi) = 1$ for $\xi > 1$, and $g(\xi) + g(1-\xi) = 1$ for $0 \le \xi \le 1$. The formula (3.7) is inspired by the refinement mask of the Meyer's scaling function, which is the case of $\delta = \pi/3$, $\Omega = 2\pi/3$ and $m = 1$ (see for example [16, 20]). Since $g \in C^k(\mathbb{R})$, it follows from (3.7) that both $\hat{a}$ and its corresponding $\hat{\phi}$ in (2.1) also lie in $C^k(\mathbb{R})$. With a higher value of $k$, we obtain a refinable function $\phi$ with better decay. In particular, when $k = \infty$, this gives a refinable function $\phi$ in the Schwartz class. There are various possible choices of $g$. For instance, to construct $g$ in $C^\infty(\mathbb{R})$, as in [20], we could define

$$g(\xi) := \frac{\int_{-\infty}^{\xi} u(t)u(1-t)\,dt}{\int_{-\infty}^{\infty} u(t)u(1-t)\,dt}, \quad \xi \in [0,\,1], \tag{3.8}$$

where $u(t) := e^{-1/t^2}$ for $t > 0$ and vanishes elsewhere. On the other hand, for $k \in \mathbb{N}$, if $u$ in (3.8) is replaced by $u(t) := t^k$ for $k \ge 0$ and vanishes elsewhere, then we obtain $g$ in $C^k(\mathbb{R})$ of the form

$$g(\xi) := \frac{\sum_{j=0}^{k} \frac{(-1)^j \xi^{2k-j+1}}{(2k-j+1)j!(k-j)!}}{\sum_{j=0}^{k} \frac{(-1)^j}{(2k-j+1)j!(k-j)!}}, \quad \xi \in [0,\,1]. \tag{3.9}$$

Applying Corollary 3.1 to $\hat{a}$ defined in (3.7), with $\hat{\phi}$ given by (2.1) and $\hat{\psi}$ by (3.6) and (3.5), the affine system $X(\psi)$ forms a Riesz basis for $L^2(\mathbb{R})$. In addition, together with the corresponding $\tilde{\psi}$ generated in Theorem 3.1 when $\hat{a}_0 = \hat{a}$, $(X(\psi), X(\tilde{\psi}))$ forms a pair of biorthogonal wavelet bases for $L^2(\mathbb{R})$.

As an illustration, we choose $\delta = \pi/3$, $\Omega = 2\pi/3$, $m = 2$, and $g$ as in (3.9) with $k = 10$. Then it follows from Theorem 3.1 that $\hat{\psi}$, $\hat{\tilde{\psi}} \in C^{10}(\mathbb{R})$ and $\operatorname{supp}\hat{\psi} = [-8\pi/3,\,-2\pi/3] \cup [2\pi/3,\,8\pi/3] = \operatorname{supp}\hat{\tilde{\psi}}$. Figure 1 displays the plots of $|\hat{\psi}|$ and $|\hat{\tilde{\psi}}|$.

Next, we turn to constructing wavelets based on refinable functions in $\mathcal{B}_{\delta,\Omega}$ when $\Omega < \pi/2$. This leads to band-limited wavelets of narrow bandwidths. For $\psi_\ell$, $\tilde{\psi}_\ell \in L^2(\mathbb{R})$ where $\ell = 1, \dots, n$, $(X(\psi_1, \dots, \psi_n), X(\tilde{\psi}_1, \dots, \tilde{\psi}_n))$ is said to be a pair of *biframes* for $L^2(\mathbb{R})$ if each of $X(\psi_1, \dots, \psi_n)$

and $X(\tilde{\psi}_1, \dots, \tilde{\psi}_n)$ forms a frame for $L^2(\mathbb{R})$, and

$$f = \sum_{\ell=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f,\, \tilde{\psi}_{\ell,j,k}\rangle \psi_{\ell,j,k} = \sum_{\ell=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f,\, \psi_{\ell,j,k}\rangle \tilde{\psi}_{\ell,j,k}, \quad f \in L^2(\mathbb{R}),$$

where $\psi_{\ell,j,k} := 2^{j/2}\psi_\ell(2^j \cdot -k)$ and $\tilde{\psi}_{\ell,j,k} := 2^{j/2}\tilde{\psi}_\ell(2^j \cdot -k)$. We shall employ the mixed oblique extension principle (see [7] and also [3, 4]) to obtain band-limited biframes for $L^2(\mathbb{R})$. The mixed oblique extension principle does not require the integer shifts of the underlying refinable functions to form frames for their respective closed linear spans. This is fortunate because otherwise by part (b) of Theorem 2.2, for $0 < \delta \le \Omega < \pi/2$, a function $\phi \in \mathcal{B}_{\delta,\Omega}$ must necessarily have its Fourier transform $\hat{\phi}$ discontinuous, which results in slower decay of $\phi$.

**Theorem 3.2.** *For $0 < \delta < \Omega < \pi/2$ and $0 < \tilde{\delta} < \tilde{\Omega} < \pi/2$, consider $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ and $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ such that $\hat{a}(\xi)\hat{\tilde{a}}(\xi) \le 1$ for all $\xi \in \mathbb{R}$. Let $\phi$ and $\tilde{\phi}$ be the respective refinable functions as in (2.1) for the masks $\hat{a}$ and $\hat{\tilde{a}}$. Define*

$$\hat{\psi}_\ell(\xi) := \hat{b}_\ell(\xi/2)\hat{\phi}(\xi/2), \qquad \hat{\tilde{\psi}}_\ell(\xi) := \hat{\tilde{b}}_\ell(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad \xi \in \mathbb{R},\ \ell = 1, 2, \tag{3.10}$$

*where*

$$\hat{b}_1(\xi) := (P(2\xi))^{\epsilon}\, \hat{a}(\xi), \qquad \hat{\tilde{b}}_1(\xi) := (P(2\xi))^{1-\epsilon}\, \hat{\tilde{a}}(\xi), \qquad \xi \in \mathbb{R}, \tag{3.11}$$

$$\hat{b}_2(\xi) := e^{-i\xi}\hat{\tilde{a}}(\xi + \pi), \qquad \hat{\tilde{b}}_2(\xi) := e^{-i\xi}\hat{a}(\xi + \pi), \qquad \xi \in \mathbb{R}, \tag{3.12}$$

*$P$ is a $\pi$-periodic function given by*

$$P(\xi) := 1 - \Big(\hat{a}(\xi)\hat{\tilde{a}}(\xi) + \hat{a}(\xi+\pi)\hat{\tilde{a}}(\xi+\pi)\Big), \quad \xi \in [-\pi/2, \pi/2], \tag{3.13}$$

*and $0 < \epsilon < 1$. Then $(X(\psi_1, \psi_2), X(\tilde{\psi}_1, \tilde{\psi}_2))$ forms a pair of biframes for $L^2(\mathbb{R})$. Furthermore,*

$$\begin{aligned} \operatorname{supp} \hat{\psi}_1 &\subseteq [-\min\{\pi - \delta',\, 2\Omega\},\, -\delta'] \cup [\delta',\, \min\{\pi - \delta',\, 2\Omega\}], \\ \operatorname{supp} \hat{\tilde{\psi}}_1 &\subseteq [-\min\{\pi - \delta',\, 2\tilde{\Omega}\},\, -\delta'] \cup [\delta',\, \min\{\pi - \delta',\, 2\tilde{\Omega}\}], \end{aligned}$$

*where $\delta' := \min\{\delta, \tilde{\delta}\}$. In addition, $\psi_2 \equiv 0$ if and only if $2\Omega \le \pi - \tilde{\Omega}$; and*

$$\operatorname{supp} \hat{\psi}_2 = [-4\Omega,\, -2\pi + 2\tilde{\Omega}] \cup [2\pi - 2\tilde{\Omega},\, 4\Omega] \quad \text{if } 2\Omega > \pi - \tilde{\Omega}.$$

*Likewise, $\tilde{\psi}_2 \equiv 0$ if and only if $2\tilde{\Omega} \le \pi - \Omega$; and*

$$\operatorname{supp} \hat{\tilde{\psi}}_2 = [-4\tilde{\Omega},\, -2\pi + 2\Omega] \cup [2\pi - 2\Omega,\, 4\tilde{\Omega}] \quad \text{if } 2\tilde{\Omega} > \pi - \Omega.$$

**Proof**: Since $\Omega < \pi/2$ and $\hat{a}(\xi)\hat{\tilde{a}}(\xi) \le 1$ for all $\xi \in \mathbb{R}$, it follows from (3.13) that $P(\xi) \ge 0$ for all $\xi \in \mathbb{R}$. Thus the expressions in (3.11) and (3.12) are well defined, and they satisfy the formulation of the mixed oblique extension principle [7, Corollary 5.3] with fundamental function $\theta(\xi) = 1 - P(\xi)$. As $\hat{\phi}$ and $\hat{\tilde{\phi}}$ are compactly supported and bounded, the underlying decay condition in the extension principle holds. The conclusions on $\operatorname{supp} \hat{\psi}_\ell$ and $\operatorname{supp} \hat{\tilde{\psi}}_\ell$, $\ell = 1, 2$, follow easily by taking appropriate intersections of $\operatorname{supp} \hat{\phi}$, $\operatorname{supp} \hat{\tilde{\phi}}$, and the supports of corresponding $\hat{b}_\ell$ and $\hat{\tilde{b}}_\ell$. By [11, Proposition 2.6], both the band-limited families $X(\psi_1, \psi_2)$ and $X(\tilde{\psi}_1, \tilde{\psi}_2)$ are Bessel sequences in $L^2(\mathbb{R})$. Consequently, [7, Corollary 5.3] implies that $(X(\psi_1, \psi_2), X(\tilde{\psi}_1, \tilde{\psi}_2))$ forms a pair of biframes for $L^2(\mathbb{R})$. ■

By setting $\delta = \tilde{\delta}$, $\Omega = \tilde{\Omega}$, $\hat{a} = \hat{\tilde{a}}$ and $\epsilon = 1/2$ in Theorem 3.2, we obtain a result on tight frames.

**Corollary 3.2.** *For $0 < \delta < \Omega < \pi/2$, suppose that $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ satisfies $\hat{a}(\xi) \le 1$ for all $\xi \in \mathbb{R}$. Let $\phi$, $\psi_1$, $\psi_2$ be defined as in* (2.1) *and* (3.10)*, where*

$$\hat{b}_1(\xi) := \Big(1 - (\hat{a}(2\xi))^2 - (\hat{a}(2\xi+\pi))^2\Big)^{1/2} \hat{a}(\xi), \qquad \hat{b}_2(\xi) := e^{-i\xi}\hat{a}(\xi+\pi), \qquad \xi \in \mathbb{R}. \tag{3.14}$$

*Then $X(\psi_1, \psi_2)$ forms a tight frame for $L^2(\mathbb{R})$. In addition,*

$$\operatorname{supp}\hat{\psi}_1 \subseteq [-\min\{\pi-\delta,\, 2\Omega\},\, -\delta] \cup [\delta,\, \min\{\pi-\delta,\, 2\Omega\}];$$

*$\psi_2 \equiv 0$ if and only if $\Omega \le \pi/3$; and*

$$\operatorname{supp}\hat{\psi}_2 = [-4\Omega,\, -2\pi+2\Omega] \cup [2\pi-2\Omega,\, 4\Omega] \quad \text{if } \Omega > \pi/3.$$

While Theorem 3.2 and Corollary 3.2 give biframes and tight frames for $L^2(\mathbb{R})$, further care is needed if we would like the Fourier transforms of the resulting framelets to lie in $C^k(\mathbb{R})$ for some $k \in \mathbb{N} \cup \{\infty\}$. This is because for $P$ in (3.13) with $\hat{a}$, $\hat{\tilde{a}} \in C^k(\mathbb{R})$, the functions $P^\epsilon$ and $P^{1-\epsilon}$, $0 < \epsilon < 1$, utilized in (3.11) may not lie in $C^k(\mathbb{R})$. The situation in (3.14) is more manageable, which we demonstrate in the example below.

It should also be mentioned that by adapting the ideas and constructions in [6], one could find other setups of $\hat{b}_1$, $\hat{\tilde{b}}_1$, $\hat{b}_2$, $\hat{\tilde{b}}_2$, instead of (3.11) and (3.12), in which the mixed oblique extension principle gives band-limited framelets with Fourier transforms in $C^k(\mathbb{R})$. These setups involve more general fundamental functions, in contrast to the specific fundamental function $\theta(\xi) = 1 - P(\xi)$ in the proof of Theorem 3.2.

**Example 3.2.** For $0 < \delta < \Omega < \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ be defined by (3.7) with $m = 1$, where $g$ is as in Example 3.1. Then

$$\Big(1 - (\hat{a}(\xi))^2 - (\hat{a}(\xi+\pi))^2\Big)^{1/2} = \begin{cases} \cos\left(\frac{\pi}{2} g\left(\frac{\xi+\Omega}{\Omega-\delta}\right)\right), & \text{if } \xi \in [-\Omega,\, -\delta), \\ 0, & \text{if } \xi \in [-\delta,\, \delta], \\ \sin\left(\frac{\pi}{2} g\left(\frac{\xi-\delta}{\Omega-\delta}\right)\right), & \text{if } \xi \in (\delta,\, \Omega], \\ 1, & \text{if } \xi \in [-\pi,\, \pi]\backslash[-\Omega,\, \Omega]. \end{cases}$$

Hence, for $k \in \mathbb{N} \cup \{\infty\}$, applying (2.1), (3.10) and (3.14), $\hat{\psi}_1$, $\hat{\psi}_2 \in C^k(\mathbb{R})$ whenever $g \in C^k(\mathbb{R})$. As noted in Example 3.1, $g$ could be given by (3.9) for $k \in \mathbb{N}$, and by (3.8) for $k = \infty$. Corollary 3.2 shows that the former gives tight framelets with reasonably good decay properties and the latter leads to tight framelets in the Schwartz class.

To illustrate this construction, we choose $\delta = \pi/6$, $\Omega = \pi/3$, and $g$ as in (3.9) with $k = 10$. By Corollary 3.2, $\psi_2 \equiv 0$ and we obtain a tight frame $X(\psi_1)$ for $L^2(\mathbb{R})$, which is generated by a single band-limited framelet $\psi_1$ with $\hat{\psi}_1 \in C^{10}(\mathbb{R})$ and $\operatorname{supp}\hat{\psi}_1 = [-2\pi/3,\, -\pi/6] \cup [\pi/6,\, 2\pi/3]$. Figure 2 shows the graph of $\hat{\psi}_1$.

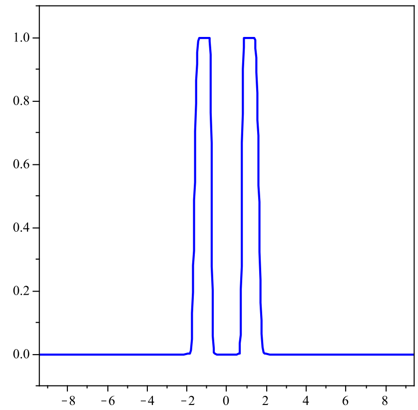


FIGURE 2. Graph of $\hat{\psi}_1$ for a specific construction in Example 3.2.

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