MATRIX EXTENSION AND BIORTHOGONAL MULTIWAVELET CONSTRUCTION

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Abstract. Suppose that $P(z)$ and $\tilde{P}(z)$ are two $r \times n$ matrices over the Laurent polynomial ring $\mathcal{R}[z]$, where $r < n$, which satisfy the identity $P(z)\tilde{P}(z)^* = I_r$ on the unit circle $\mathbb{T}$. We develop an algorithm that produces two $n \times n$ matrices $Q(z)$ and $\tilde{Q}(z)$ over $\mathcal{R}[z]$, satisfying the identity $Q(z)\tilde{Q}(z)^* = I_n$ on $\mathbb{T}$, such that the submatrices formed by the first r rows of $Q(z)$ and $\tilde{Q}(z)$ are $P(z)$ and $\tilde{P}(z)$ respectively. Our algorithm is used to construct compactly supported biorthogonal multiwavelets from multiresolutions generated by univariate compactly supported biorthogonal scaling functions with an arbitrary dilation parameter $m \in \mathbb{Z}$, where $m > 1$.

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1. Introduction

This paper presents an algorithmic approach to the construction of compactly supported biorthogonal multiwavelets from multiresolutions generated by univariate compactly supported biorthogonal scaling functions with an arbitrary dilation parameter $m \in \mathbb{Z}$, where $m > 1$. In practice, biorthogonal wavelets and multiwavelets are of interest because they can be real, compactly supported, continuous and symmetric even for dilation as small as $m = 2$. (It is not possible for an orthonormal wavelet with dilation $m = 2$ to have all these properties simultaneously.) It is well known that using a simple manipulation, a pair of biorthogonal wavelets can be obtained from the multiresolutions generated by a pair of biorthogonal scaling functions with dilation $m = 2$. However, no general procedure for obtaining biorthogonal multiwavelets from multiresolutions generated by biorthogonal scaling functions with arbitrary dilation $m > 1$ is available so far.

The construction of biorthogonal multiwavelets from the multiresolutions generated by two sets of compactly supported scaling functions that are biorthogonal can be reduced to the problem of extending two matrices with Laurent polynomial entries. This problem is more complicated than the corresponding matrix extension problems for the construction of orthonormal multiwavelets and prewavelets which only involve the extension of one matrix. The matrix extension problems corresponding to the construction of orthonormal multiwavelets and prewavelets have been well studied. In fact, [2] provides practical algorithms for the solution of these problems. However, it should be mentioned that matrix extension is not the only method of constructing multiwavelets from multiresolutions generated by scaling functions. In specific cases, using special properties of the scaling functions involved, it is possible to obtain multiwavelets through more direct approaches. For instance, see [1].

In this paper, we give a constructive solution of the matrix extension problem for the construction of biorthogonal multiwavelets, and our solution is easily implementable in the computer. Let $\mathcal{R}[z]$ denote the ring of univariate Laurent polynomials over the complex field, and let $\mathbb{T}$ denote the unit circle in the complex plane. The matrix extension problem that we are concerned with is as follows. Consider two $r \times n$ matrices $P(z)$ and $\tilde{P}(z)$ over $\mathcal{R}[z]$, where $r < n$, satisfying the identity $P(z)\tilde{P}(z)^* = I_r$ on $\mathbb{T}$. Can we find two $n \times n$ matrices $Q(z)$ and $\tilde{Q}(z)$ over $\mathcal{R}[z]$, which satisfy $Q(z)\tilde{Q}(z)^* = I_n$ on $\mathbb{T}$, such that the submatrices formed by the first r rows of $Q(z)$ and $\tilde{Q}(z)$ are $P(z)$ and $\tilde{P}(z)$ respectively? We shall show that the answer to this question is always affirmative and the matrices $Q(z)$ and $\tilde{Q}(z)$ are extensions of $P(z)$ and $\tilde{P}(z)$ respectively.

First, let us see how the construction of biorthogonal multiwavelets from the multiresolutions generated by two sets of scaling functions that are biorthogonal can be reduced to the above matrix extension problem. Suppose that ϕ_i and $\tilde{\phi}_i$, $i = 1, \dots, r$, are compactly supported functions in $L^2(\mathbb{R})$, the space of square-integrable measurable functions over the real line. Let V_0 and $\tilde{V}_0$ be the closed subspaces generated by the integer shifts of ϕ_i and $\tilde{\phi}_i$, $i = 1, \dots, r$, respectively:

$$V_0 = \overline{\langle \{\phi_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r\} \rangle},$$

$$\tilde{V}_0 = \overline{\langle \{\tilde{\phi}_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r\} \rangle}.$$

For each $\nu \in \mathbb{Z}$, we define the subspaces V_ν and $\tilde{V}_\nu$ by

$$V_\nu = \{f(m^\nu \cdot) : f \in V_0\}, \qquad \tilde{V}_\nu = \{f(m^\nu \cdot) : f \in \tilde{V}_0\}.$$

The two sets of functions $\{\phi_1, \dots, \phi_r\}$ and $\{\tilde{\phi}_1, \dots, \tilde{\phi}_r\}$ are called *biorthogonal scaling functions* if there are finitely supported sequences h_{ij} and $\tilde{h}_{ij}$ such that

$$\phi_i(x) = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}} h_{ij}(k) \phi_j(mx - k), \quad i = 1, \dots, r, \tag{1.1}$$

and

$$\tilde{\phi}_i(x) = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}} \tilde{h}_{ij}(k) \tilde{\phi}_j(mx - k), \quad i = 1, \dots, r, \tag{1.2}$$

with

$$\langle \phi_i(\cdot - k), \tilde{\phi}_j(\cdot - \ell) \rangle = \delta_{i,j} \delta_{k,\ell}, \quad i, j = 1, \dots, r, \quad k, \ell \in \mathbb{Z}. \tag{1.3}$$

It is well known that if ϕ_i and $\tilde{\phi}_i$, $i = 1, \dots, r$, satisfy (1.1)–(1.2) such that $\{\phi_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r\}$ and $\{\tilde{\phi}_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r\}$ form Riesz bases of V_0 and $\tilde{V}_0$ respectively, then both $\{V_\nu\}_{\nu \in \mathbb{Z}}$ and $\{\tilde{V}_\nu\}_{\nu \in \mathbb{Z}}$ are multiresolutions of $L^2(\mathbb{R})$.

Now, suppose that W_0 is an algebraic complement of V_0 in V_1 which is orthogonal to $\tilde{V}_0$, and $\tilde{W}_0$ is an algebraic complement of $\tilde{V}_0$ in $\tilde{V}_1$ which is orthogonal to V_0. We say that the compactly supported functions $\psi_1, \dots, \psi_{(m-1)r}$ and $\tilde{\psi}_1, \dots, \tilde{\psi}_{(m-1)r}$ are *biorthogonal multiwavelets* if $\{\psi_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, (m-1)r\}$ and $\{\tilde{\psi}_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, (m-1)r\}$ form Riesz bases of W_0 and $\tilde{W}_0$ respectively, and

$$\langle \psi_i(\cdot - k), \tilde{\psi}_j(\cdot - \ell)\rangle = \delta_{i,j}\delta_{k,\ell}, \quad i, j = 1, \dots, (m-1)r, \quad k, \ell \in \mathbb{Z}. \tag{1.4}$$

Let $\Phi = (\phi_1, \dots, \phi_r)^T$ and $H(k) = (h_{ij}(k))$ for $k \in \mathbb{Z}$. Then (1.1) can be rewritten as a *matrix dilation equation*:

$$\Phi(x) = \sum_{k\in\mathbb{Z}} H(k)\Phi(mx - k).$$

For $\ell = 0, 1, \dots, m-1$, define an $r \times r$ matrix $H^\ell(z)$ over $\mathcal{R}[z]$ by

$$H^\ell(z) = \frac{1}{\sqrt{m}} \sum_{k\in\mathbb{Z}} H(mk + \ell)z^k.$$

Then form the $r \times mr$ matrix

$$P(z) = (H^0(z), \dots, H^{m-1}(z)),$$

which is called a *polyphase matrix*. Similarly, for $\tilde{\Phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_r)^T$ and $\tilde{H}(k) = (\tilde{h}_{ij}(k))$ where $k \in \mathbb{Z}$, we obtain another polyphase matrix

$$\tilde{P}(z) = (\tilde{H}^0(z), \dots, \tilde{H}^{m-1}(z)).$$

The biorthogonality condition (1.3) implies that

$$P(z)\tilde{P}(z)^* = I_r, \quad z \in \mathbb{T}.$$

In Section 2, we develop an algorithm to find two $mr \times mr$ matrices $Q(z)$ and $\tilde{Q}(z)$, which are extensions of $P(z)$ and $\tilde{P}(z)$ respectively, satisfying

$$Q(z)\tilde{Q}(z)^* = I_{mr}, \quad z \in \mathbb{T}. \tag{1.5}$$

The $mr \times mr$ matrices $Q(z)$ and $\tilde{Q}(z)$ can be written as

$$Q(z) = \begin{pmatrix} P(z) \\ P'(z) \end{pmatrix}, \qquad \tilde{Q}(z) = \begin{pmatrix} \tilde{P}(z) \\ \tilde{P}'(z) \end{pmatrix},$$

where $P'(z)$ and $\tilde{P}'(z)$ are $(m-1)r \times mr$ matrices of the form

$$P'(z) = (G^0(z), \dots, G^{m-1}(z)), \qquad \tilde{P}'(z) = (\tilde{G}^0(z), \dots, \tilde{G}^{m-1}(z)),$$

and each $G^\ell(z), \tilde{G}^\ell(z)$, $\ell = 0, \dots, m-1$, is an $(m-1)r \times r$ matrix over $\mathcal{R}[z]$ with entries given by

$$(G^\ell(z))_{ij} = \frac{1}{\sqrt{m}} \sum_{k\in\mathbb{Z}} g_{ij}(mk+\ell)z^k, \quad i = 1, \dots, (m-1)r, \quad j = 1, \dots, r,$$

$$(\tilde{G}^\ell(z))_{ij} = \frac{1}{\sqrt{m}} \sum_{k\in\mathbb{Z}} \tilde{g}_{ij}(mk+\ell)z^k, \quad i = 1, \dots, (m-1)r, \quad j = 1, \dots, r.$$

Then the functions

$$\psi_i(x) = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}} g_{ij}(k)\phi_j(mx-k), \quad i = 1, \dots, (m-1)r, \tag{1.6}$$

$$\tilde{\psi}_i(x) = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}} \tilde{g}_{ij}(k)\tilde{\phi}_j(mx-k), \quad i = 1, \dots, (m-1)r, \tag{1.7}$$

form biorthogonal multiwavelets. Indeed, let

$$W_0 = \overline{\langle\{\psi_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, (m-1)r\}\rangle},$$

$$\tilde{W}_0 = \overline{\langle\{\tilde{\psi}_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, (m-1)r\}\rangle}.$$

Since $Q(z)\tilde{Q}(z)^* = I_{mr}$ for all z on $\mathbb{T}$, both $Q(z)$ and $\tilde{Q}(z)$ have rank mr for all z on $\mathbb{T}$. Now it follows from a result in [4] that the integer shifts of $\phi_1, \dots, \phi_r, \psi_1, \dots, \psi_{(m-1)r}$ form a Riesz basis of V_1. Hence V_1 is the algebraic direct sum of V_0 and W_0. Consequently, the integer shifts of $\psi_1, \dots, \psi_{(m-1)r}$ form a Riesz basis of W_0. Similarly, we conclude that $\tilde{V}_1$ is the algebraic direct sum of $\tilde{V}_0$ and $\tilde{W}_0$ and the integer shifts of $\tilde{\psi}_1, \dots, \tilde{\psi}_{(m-1)r}$ form a Riesz basis of $\tilde{W}_0$. By (1.1)–(1.2) and (1.5)–(1.7), $W_0 \perp \tilde{V}_0$ and $\tilde{W}_0 \perp V_0$. Also, the biorthogonality relation (1.4) follows from (1.5)–(1.7).

It should be mentioned that in the special case where $\tilde{\phi}_i = \phi_i$, $i = 1, \dots, r$, and $\{\phi_i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r\}$ is an orthonormal basis of V_0, our algorithm reduces to the paraunitary matrix extension in [2].

In Section 3 of the paper, we apply our algorithm to obtain biorthogonal multiwavelets based on the biorthogonal scaling functions constructed in [3]. Explicit values for the entries of the matrices $G(k) = (g_{ij}(k))$ and $\tilde{G}(k) = (\tilde{g}_{ij}(k))$, $k \in \mathbb{Z}$, are provided, together with the graphs of the scaling functions and multiwavelets.

2. Main results

For any two column vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{C}^n$, the inner product of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

We denote the group of invertible $n \times n$ complex matrices by $GL_n(\mathbb{C})$ and the ring of $n \times n$ matrices over $\mathcal{R}[z]$ by $M_n(\mathcal{R}[z])$.

Lemma 2.1. *Let $\mathbf{v}, \tilde{\mathbf{v}} \in \mathbb{C}^n$ be nonzero column vectors, where $n \geq 2$.*

(a) *If $\langle \mathbf{v}, \tilde{\mathbf{v}} \rangle \neq 0$, then there exist matrices $S, \tilde{S} \in GL_n(\mathbb{C})$ such that $\tilde{S}^* S = I_n$ and*

$$S\mathbf{v} = \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle \mathbf{e}_1, \qquad \tilde{S}\tilde{\mathbf{v}} = \mathbf{e}_1. \tag{2.1}$$

(b) *If $\langle \mathbf{v}, \tilde{\mathbf{v}} \rangle = 0$, then there exist matrices $S, \tilde{S} \in GL_n(\mathbb{C})$ such that $\tilde{S}^* S = I_n$ and*

$$S\mathbf{v} = \langle \mathbf{v}, \mathbf{v} \rangle \mathbf{e}_2, \qquad \tilde{S}\tilde{\mathbf{v}} = \mathbf{e}_1. \tag{2.2}$$

Here, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the column vectors $(1, 0, 0, \ldots, 0)^T$ and $(0, 1, 0, \ldots, 0)^T$ in $\mathbb{C}^n$ respectively.

Proof. (a) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ be a basis of the orthogonal complement of the linear span of $\{\mathbf{v}\}$ in $\mathbb{C}^n$. Define $S = (\tilde{\mathbf{v}}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1})^*$, the conjugate transpose of the matrix formed by the column vectors $\tilde{\mathbf{v}}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$. To show that S is invertible, it suffices to show that $\{\tilde{\mathbf{v}}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ is linearly independent. Suppose that

$$\tilde{c}\tilde{\mathbf{v}} + c_1\mathbf{v}_1 + \ldots + c_{n-1}\mathbf{v}_{n-1} = 0,$$

for some $\tilde{c}, c_1, \ldots, c_{n-1} \in \mathbb{C}$. Taking inner product with $\mathbf{v}$, we obtain $\tilde{c}\langle \tilde{\mathbf{v}}, \mathbf{v} \rangle = 0$, and hence $\tilde{c} = 0$. Then it follows from the linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ that $c_1 = \ldots = c_{n-1} = 0$. Now, define $\tilde{S} = (S^*)^{-1}$. Clearly, $S\mathbf{v} = \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle \mathbf{e}_1$ and $\tilde{S}^* S = I_n$. Furthermore, $\tilde{S}S^* = I_n$ yields $\tilde{S}\tilde{\mathbf{v}} = \mathbf{e}_1$.

(b) Let $S = (\tilde{\mathbf{v}}, \mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-2})^*$, where $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-2}\}$ is a basis of the orthogonal complement of the linear span of $\{\mathbf{v}, \tilde{\mathbf{v}}\}$ in $\mathbb{C}^n$. Suppose that

$$\tilde{c}\tilde{\mathbf{v}} + c\mathbf{v} + c_1\mathbf{v}_1 + \dots + c_{n-2}\mathbf{v}_{n-2} = 0,$$

for some $\tilde{c}, c, c_1, \dots, c_{n-2} \in \mathbb{C}$. As in (a), by taking inner products with $\tilde{\mathbf{v}}$ and $\mathbf{v}$, we conclude that $\tilde{c} = c = 0$, and subsequently, $c_1 = \dots = c_{n-2} = 0$. Then setting $\tilde{S} = (S^*)^{-1}$ gives the desired result. $\square$

Suppose that $A(z)$ is an $n \times 1$ matrix over $\mathcal{R}[z]$ given by

$$A(z) = \sum_{i=N_1}^{N_2} \mathbf{a}_i z^i,$$

where $\mathbf{a}_i \in \mathbb{C}^n$, $\mathbf{a}_{N_1}, \mathbf{a}_{N_2} \neq \mathbf{0}$. Define the degree of $A(z)$ by

$$d(A) = N_2 - N_1.$$

Lemma 2.2. *For $n \geq 2$, suppose that $A(z) = \sum_{i=N_1}^{N_2} \mathbf{a}_i z^i$ and $\tilde{A}(z) = \sum_{i=\tilde{N}_1}^{\tilde{N}_2} \tilde{\mathbf{a}}_i z^i$ are $n \times 1$ matrices over $\mathcal{R}[z]$, with $\mathbf{a}_i, \tilde{\mathbf{a}}_i \in \mathbb{C}^n$, $\mathbf{a}_{N_1}, \mathbf{a}_{N_2}, \tilde{\mathbf{a}}_{\tilde{N}_1}, \tilde{\mathbf{a}}_{\tilde{N}_2} \neq \mathbf{0}$, $d(A) > 0$, and $\langle A(z), \tilde{A}(z)\rangle = 1$ for all $z \in \mathbb{T}$. If $\langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1}\rangle \neq 0$, then there exist matrices $F(z), \tilde{F}(z) \in M_n(\mathcal{R}[z])$ such that*

$$d(FA) < d(A), \tag{2.3}$$

and

$$\tilde{F}(z)^* F(z) = I_n, \quad z \in \mathbb{T}. \tag{2.4}$$

In particular,

$$\langle F(z)A(z), \tilde{F}(z)\tilde{A}(z)\rangle = 1, \quad z \in \mathbb{T}. \tag{2.5}$$

Proof. Using Lemma 2.1, we obtain $S, \tilde{S} \in GL_n(\mathbb{C})$ such that $\tilde{S}^* S = I_n$, and

$$S\mathbf{a}_{N_1} = \langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1}\rangle \mathbf{e}_1, \qquad \tilde{S}\tilde{\mathbf{a}}_{\tilde{N}_1} = \mathbf{e}_1. \tag{2.6}$$

Since $\langle \mathbf{a}_{N_2}, \tilde{\mathbf{a}}_{\tilde{N}_1}\rangle$ is the coefficient of $z^{N_2 - \tilde{N}_1}$ in $\langle A(z), \tilde{A}(z)\rangle$ and $\langle A(z), \tilde{A}(z)\rangle = 1$ for all $z \in \mathbb{T}$, it follows that

$$\langle S\mathbf{a}_{N_2}, \tilde{S}\tilde{\mathbf{a}}_{\tilde{N}_1}\rangle = \langle \mathbf{a}_{N_2}, \tilde{\mathbf{a}}_{\tilde{N}_1}\rangle = 0.$$

Consequently, (2.6) implies that $(S\mathbf{a}_{N_2})_1$, the first component of $S\mathbf{a}_{N_2}$, is zero. Now, let $D(z) \in M_n(\mathcal{R}[z])$ be the diagonal matrix given by $D(z) = \operatorname{diag}(1, z^{-1}, \dots, z^{-1})$. Then defining $F(z) = D(z)S$ and $\tilde{F}(z) = D(z)\tilde{S}$ gives (2.3)–(2.5). □

We are now ready for

Theorem 2.1. *For $n \geq 2$, suppose that*

$$A(z) = \sum_{i=N_1}^{N_2} \mathbf{a}_i z^i, \qquad \tilde{A}(z) = \sum_{i=\tilde{N}_1}^{\tilde{N}_2} \tilde{\mathbf{a}}_i z^i, \tag{2.7}$$

are $n \times 1$ matrices over $\mathcal{R}[z]$, with $\mathbf{a}_i, \tilde{\mathbf{a}}_i \in \mathbb{C}^n$, $\mathbf{a}_{N_1}, \mathbf{a}_{N_2}, \tilde{\mathbf{a}}_{\tilde{N}_1}, \tilde{\mathbf{a}}_{\tilde{N}_2} \neq \mathbf{0}$, and

$$\langle A(z), \tilde{A}(z)\rangle = 1, \quad z \in \mathbb{T}. \tag{2.8}$$

Then there exist $n \times n$ matrices $C(z), \tilde{C}(z) \in M_n(\mathcal{R}[z])$ such that

$$C(z)A(z) = \tilde{C}(z)\tilde{A}(z) = \mathbf{e}_1, \quad z \in \mathbb{T}, \tag{2.9}$$

and

$$\tilde{C}(z)^* C(z) = I_n, \quad z \in \mathbb{T}. \tag{2.10}$$

Proof. We shall proceed by induction on n.

For $n = 2$, $A(z)$ and $\tilde{A}(z)$ are 2×1 matrices which can be expressed as

$$A(z) = (A_{11}(z), A_{21}(z))^T, \qquad \tilde{A}(z) = (\tilde{A}_{11}(z), \tilde{A}_{21}(z))^T, \tag{2.11}$$

for some scalars $A_{11}(z), A_{21}(z), \tilde{A}_{11}(z), \tilde{A}_{21}(z)$. It follows from (2.8) that

$$A_{11}(z)\overline{\tilde{A}_{11}(z)} + A_{21}(z)\overline{\tilde{A}_{21}(z)} = 1, \quad z \in \mathbb{T}. \tag{2.12}$$

Now, let

$$C(z) = \begin{pmatrix} \overline{\tilde{A}_{11}(z)} & \overline{\tilde{A}_{21}(z)} \\ A_{21}(z) & -A_{11}(z) \end{pmatrix}, \qquad \tilde{C}(z) = \begin{pmatrix} \overline{A_{11}(z)} & \overline{A_{21}(z)} \\ \tilde{A}_{21}(z) & -\tilde{A}_{11}(z) \end{pmatrix}. \tag{2.13}$$

Then a direct calculation shows that (2.11)–(2.13) yield (2.9)–(2.10).

For $n \geq 3$, we shall obtain two finite sequences of matrices $F_i(z)$ and $\tilde{F}_i(z)$, $i = 1, \dots, \ell$, in $M_n(\mathcal{R}[z])$ such that $d(F_\ell \cdots F_1 A) = 0$, and $\tilde{F}_i(z)^* F_i(z) = I_n$ on $\mathbb{T}$. Since this is trivial for $d(A) = 0$, we may assume that $d(A) > 0$.

With $A(z)$ and $\tilde{A}(z)$ as in (2.7), if $\langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1} \rangle \neq 0$, then it follows from Lemma 2.2 that there exist matrices $F_1(z), \tilde{F}_1(z) \in M_n(\mathcal{R}[z])$ such that $d(F_1A) < d(A)$ and $\tilde{F}_1(z)^* F_1(z) = I_n$ for all $z \in \mathbb{T}$.

On the other hand, if $\langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1} \rangle = 0$, by Lemma 2.1, there exist matrices $S_1, \tilde{S}_1$ in $GL_n(\mathbb{C})$ such that $S_1 \mathbf{a}_{N_1} = \langle \mathbf{a}_{N_1}, \mathbf{a}_{N_1} \rangle \mathbf{e}_2$, $\tilde{S}_1 \tilde{\mathbf{a}}_{\tilde{N}_1} = \mathbf{e}_1$, and $\tilde{S}_1^* S_1 = I_n$. Writing

$$S_1 A(z) = \sum_{i=N_1}^{N_2} \mathbf{a}_i^{(1)} z^i, \qquad \tilde{S}_1 \tilde{A}(z) = \sum_{i=\tilde{N}_1}^{\tilde{N}_2} \tilde{\mathbf{a}}_i^{(1)} z^i, \tag{2.14}$$

we have $\mathbf{a}_{N_1}^{(1)} = (0, \alpha, 0, \ldots, 0)^T$ and $\tilde{\mathbf{a}}_{\tilde{N}_1}^{(1)} = (1, 0, 0, \ldots, 0)^T$ where $\alpha = \langle \mathbf{a}_{N_1}, \mathbf{a}_{N_1} \rangle \neq 0$. For the first component of $S_1 A(z)$, either there exists an integer K, where $N_1 < K \leq N_2$, such that $(\mathbf{a}_K^{(1)})_1$, the first component of $\mathbf{a}_K^{(1)}$, is nonzero, or $(\mathbf{a}_i^{(1)})_1$ is zero for every $i = N_1, \ldots, N_2$.

First, suppose that K is an integer, where $N_1 < K \leq N_2$, such that $(\mathbf{a}_K^{(1)})_1 = \beta \neq 0$ and $(\mathbf{a}_i^{(1)})_1 = 0$ for $i = N_1, \ldots, K-1$. Let $D_1(z) = \operatorname{diag}(z^{N_1 - K}, 1, \ldots, 1) \in M_n(\mathcal{R}[z])$. Then (2.14) yields

$$D_1(z) S_1 A(z) = \sum_{i=N_1}^{N_2} \mathbf{a}_i^{(2)} z^i, \qquad D_1(z) \tilde{S}_1 \tilde{A}(z) = \sum_{i=\tilde{N}_1 + N_1 - K}^{\tilde{N}_2} \tilde{\mathbf{a}}_i^{(2)} z^i,$$

where $\mathbf{a}_{N_1}^{(2)} = (\beta, \alpha, 0, \ldots, 0)^T$ and $\tilde{\mathbf{a}}_{\tilde{N}_1 + N_1 - K}^{(2)} = (1, 0, 0, \ldots, 0)^T$. Now, observe that $\langle \mathbf{a}_{N_1}^{(2)}, \tilde{\mathbf{a}}_{\tilde{N}_1 + N_1 - K}^{(2)} \rangle = \beta \neq 0$ and $\langle D_1(z) S_1 A(z), D_1(z) \tilde{S}_1 \tilde{A}(z) \rangle = 1$ on $\mathbb{T}$. Thus we can apply Lemma 2.2 to obtain $T_1(z), \tilde{T}_1(z) \in M_n(\mathcal{R}[z])$ such that

$$d(T_1 D_1 S_1 A) < d(D_1 S_1 A) \leq N_2 - N_1 = d(A),$$

and $(\tilde{T}_1(z) D_1(z) \tilde{S}_1)^* (T_1(z) D_1(z) S_1) = I_n$ for all $z \in \mathbb{T}$.

Next, suppose that $(\mathbf{a}_i^{(1)})_1 = 0$ for every $i = N_1, \ldots, N_2$, where $\mathbf{a}_i^{(1)}$ is as in (2.14). Then consider the $(n-1) \times 1$ matrices $B(z)$ and $\tilde{B}(z)$ formed by the last $(n-1)$ rows of $S_1 A(z)$ and $\tilde{S}_1 \tilde{A}(z)$ respectively. More precisely, let

$$B(z) = \sum_{i=N_1}^{N_2} \mathbf{b}_i z^i, \qquad \tilde{B}(z) = \sum_{i=\tilde{N}_1}^{\tilde{N}_2} \tilde{\mathbf{b}}_i z^i,$$

where $\mathbf{b}_i = ((\mathbf{a}_i^{(1)})_2, \ldots, (\mathbf{a}_i^{(1)})_n)^T$ and $\tilde{\mathbf{b}}_i = ((\tilde{\mathbf{a}}_i^{(1)})_2, \ldots, (\tilde{\mathbf{a}}_i^{(1)})_n)^T$. Since $(\mathbf{a}_i^{(1)})_1 = 0$ for every $i = N_1, \ldots, N_2$, it follows from (2.8) that $\langle B(z), \tilde{B}(z) \rangle = 1$ for all $z \in \mathbb{T}$.

By the induction hypothesis, there exist $(n-1)\times(n-1)$ matrices $T_0(z), \tilde{T}_0(z) \in M_{n-1}(\mathcal{R}[z])$ such that $T_0(z)B(z) = \tilde{T}_0(z)\tilde{B}(z) = (1,0,\dots,0)^T$ and $\tilde{T}_0(z)^* T_0(z) = I_{n-1}$ for all $z \in \mathbb{T}$. Now, define

$$T_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & T_0(z) \end{pmatrix}, \qquad \tilde{T}_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{T}_0(z) \end{pmatrix}.$$

Then $T_1(z)S_1A(z) = \mathbf{e}_2$ on $\mathbb{T}$, and hence, $d(T_1S_1A) = 0$. In addition, a simple calculation shows that $(\tilde{T}_1(z)\tilde{S}_1)^*(T_1(z)S_1) = I_n$ for all $z \in \mathbb{T}$.

In all the above cases, we can find matrices $F_1(z), \tilde{F}_1(z) \in M_n(\mathcal{R}[z])$ such that $\tilde{F}_1(z)^* F_1(z) = I_n$ which leads to $\langle F_1(z)A(z), \tilde{F}_1(z)\tilde{A}(z)\rangle = 1$ for all $z \in \mathbb{T}$. Furthermore, either $d(F_1A) < d(A)$ or $d(F_1A) = 0$. Thus repeating the above process, if necessary, gives two finite sequences of matrices $F_i(z)$ and $\tilde{F}_i(z)$, $i = 1,\dots,\ell$, in $M_n(\mathcal{R}[z])$ such that $d(F_\ell \cdots F_1 A) = 0$ and $\tilde{F}_i(z)^* F_i(z) = I_n$ on $\mathbb{T}$. In particular,

$$\langle F_\ell(z)\cdots F_1(z)A(z), \tilde{F}_\ell(z)\cdots\tilde{F}_1(z)\tilde{A}(z)\rangle = 1, \quad z \in \mathbb{T}. \tag{2.15}$$

Without loss of generality, we may assume that

$$F_\ell(z)\cdots F_1(z)A(z) = \mathbf{a}_0^{(\ell)}, \qquad \tilde{F}_\ell(z)\cdots\tilde{F}_1(z)\tilde{A}(z) = \sum_{i=\tilde{M}_1}^{\tilde{M}_2} \tilde{\mathbf{a}}_i^{(\ell)} z^i,$$

where $\mathbf{a}_0^{(\ell)}, \tilde{\mathbf{a}}_i^{(\ell)} \in \mathbb{C}^n$, $\mathbf{a}_0^{(\ell)}, \tilde{\mathbf{a}}_{\tilde{M}_1}^{(\ell)}, \tilde{\mathbf{a}}_{\tilde{M}_2}^{(\ell)} \neq \mathbf{0}$. Then (2.15) implies that $\tilde{M}_1 \le 0 \le \tilde{M}_2$, and

$$\langle \mathbf{a}_0^{(\ell)}, \tilde{\mathbf{a}}_i^{(\ell)}\rangle = \delta_{0,i}, \tag{2.16}$$

for $i = \tilde{M}_1,\dots,\tilde{M}_2$. By Lemma 2.1, there exist matrices $S_{\ell+1}, \tilde{S}_{\ell+1} \in GL_n(\mathbb{C})$ such that $\tilde{S}_{\ell+1}^* S_{\ell+1} = I_n$ and $S_{\ell+1}\mathbf{a}_0^{(\ell)} = \tilde{S}_{\ell+1}\tilde{\mathbf{a}}_0^{(\ell)} = \mathbf{e}_1$. Furthermore, by (2.16), for every $i = \tilde{M}_1,\dots,\tilde{M}_2$, where $i \neq 0$, the first component of $\tilde{S}_{\ell+1}\tilde{\mathbf{a}}_i^{(\ell)}$ is zero. Let $D_{\ell+1}(z) = \mathrm{diag}(1, z^{-\tilde{M}_1}, \dots, z^{-\tilde{M}_1}) \in M_n(\mathcal{R}[z])$. Then setting $F_{\ell+1}(z) = D_{\ell+1}(z)S_{\ell+1}$ and $\tilde{F}_{\ell+1}(z) = D_{\ell+1}(z)\tilde{S}_{\ell+1}$ gives

$$F_{\ell+1}(z)\cdots F_1(z)A(z) = \mathbf{e}_1, \qquad \tilde{F}_{\ell+1}(z)\cdots\tilde{F}_1(z)\tilde{A}(z) = \sum_{i=0}^{\tilde{M}_2-\tilde{M}_1} \tilde{\mathbf{a}}_i^{(\ell+1)} z^i,$$

where the first component of $\tilde{\mathbf{a}}_0^{(\ell+1)}$ is 1. As in the proof of Lemma 2.2, we can find matrices $F_{\ell+2}(z), \tilde{F}_{\ell+2}(z) \in M_n(\mathcal{R}[z])$ such that $d(\tilde{F}_{\ell+2}\cdots\tilde{F}_1\tilde{A}) < d(\tilde{F}_{\ell+1}\cdots\tilde{F}_1\tilde{A})$, $F_{\ell+2}(z)\cdots F_1(z)A(z) = \mathbf{e}_1$, and $\tilde{F}_{\ell+2}(z)^* F_{\ell+2}(z) = I_n$ for all $z \in \mathbb{T}$. Continuing this procedure, we obtain two finite sequences of matrices $F_i(z)$ and $\tilde{F}_i(z)$, $i = 1, \ldots, m$, in $M_n(\mathcal{R}[z])$ such that on $\mathbb{T}$,

$$F_m(z) \cdots F_1(z) A(z) = \mathbf{e}_1, \qquad \tilde{F}_m(z) \cdots \tilde{F}_1(z) \tilde{A}(z) = \mathbf{e}_1,$$

and $\tilde{F}_i(z)^* F_i(z) = I_n$. Thus the matrices $C(z)$ and $\tilde{C}(z)$ defined by

$$C(z) = F_m(z) \cdots F_1(z), \qquad \tilde{C}(z) = \tilde{F}_m(z) \cdots \tilde{F}_1(z), \tag{2.17}$$

give the desired result. □

The main steps of the algorithm described in the above proof can be summarized as follows. For $n \geq 2$, given two $n \times 1$ matrices $A(z)$ and $\tilde{A}(z)$ of the form (2.7) satisfying (2.8), we seek two $n \times n$ matrices $C(z)$ and $\tilde{C}(z)$ in $M_n(\mathcal{R}[z])$ such that (2.9) and (2.10) hold.

Step 1: For $n = 2$, $C(z)$ and $\tilde{C}(z)$ are given by (2.13).

Step 2: For $n \geq 3$, construct two matrices $F_1(z)$ and $\tilde{F}_1(z)$ in $M_n(\mathcal{R}[z])$ such that $d(F_1 A) < d(A)$ and $\tilde{F}_1(z)^* F_1(z) = I_n$ on $\mathbb{T}$. In this connection, there are two possibilities.

(a) If $\langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1} \rangle \neq 0$, apply Lemma 2.2 to obtain $F_1(z)$ and $\tilde{F}_1(z)$.

(b) If $\langle \mathbf{a}_{N_1}, \tilde{\mathbf{a}}_{\tilde{N}_1} \rangle = 0$, first apply Lemma 2.1(b). Then after some manipulation, either use Lemma 2.2 or reduce the problem to the $(n-1)$th case to obtain $F_1(z)$ and $\tilde{F}_1(z)$.

Step 3: If $d(F_1 A) > 0$, repeat **Step 2** with $A(z)$ and $\tilde{A}(z)$ replaced by $F_1(z) A(z)$ and $\tilde{F}_1(z) \tilde{A}(z)$ respectively. If $d(F_1 A) = 0$, proceed to **Step 4**.

Step 4: Let $F_i(z)$ and $\tilde{F}_i(z)$, $i = 1, \ldots, \ell$, be matrices in $M_n(\mathcal{R}[z])$ such that $d(F_\ell \cdots F_1 A) = 0$ and $\tilde{F}_i(z)^* F_i(z) = I_n$ on $\mathbb{T}$. Apply Lemma 2.2 repeatedly with some manipulation to obtain matrices $F_i(z)$ and $\tilde{F}_i(z)$, $i = \ell + 1, \ldots, m$, such that $F_m(z) \cdots F_1(z) A(z) = \tilde{F}_m(z) \cdots \tilde{F}_1(z) \tilde{A}(z) = \mathbf{e}_1$ and $\tilde{F}_i(z)^* F_i(z) = I_n$ on $\mathbb{T}$. The required matrices $C(z)$ and $\tilde{C}(z)$ are given by (2.17).

Theorem 2.2. *For $n \geq 2$, suppose that $A_i(z)$ and $\tilde{A}_i(z)$, $i = 1, \ldots, r$, are $n \times 1$ matrices over $\mathcal{R}[z]$, $r < n$, such that the $n \times r$ matrices $A(z)$ and $\tilde{A}(z)$ given by*

$$A(z) = (A_1(z), \ldots, A_r(z)), \qquad \tilde{A}(z) = (\tilde{A}_1(z), \ldots, \tilde{A}_r(z)), \tag{2.18}$$

satisfy

$$\tilde{A}(z)^* A(z) = I_r, \quad z \in \mathbb{T}. \tag{2.19}$$

Then there exist $n \times n$ matrices $Q(z), \tilde{Q}(z) \in M_n(\mathcal{R}[z])$ such that

$$Q(z)A(z) = \tilde{Q}(z)\tilde{A}(z) = \begin{pmatrix} I_r \\ 0 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{2.20}$$

and

$$\tilde{Q}(z)^* Q(z) = I_n, \quad z \in \mathbb{T}. \tag{2.21}$$

Furthermore, $Q(z)$ and $\tilde{Q}(z)$ are given by

$$Q(z) = Q_r(z) \cdots Q_1(z), \qquad \tilde{Q}(z) = \tilde{Q}_r(z) \cdots \tilde{Q}_1(z), \tag{2.22}$$

where for every $i = 1, \ldots, r$,

$$Q_i(z) = \begin{pmatrix} I_{i-1} & 0 \\ 0 & C_i(z) \end{pmatrix}, \qquad \tilde{Q}_i(z) = \begin{pmatrix} I_{i-1} & 0 \\ 0 & \tilde{C}_i(z) \end{pmatrix}, \tag{2.23}$$

and $C_i(z)$ and $\tilde{C}_i(z)$ are $(n-i+1) \times (n-i+1)$ matrices of the form (2.17) satisfying $\tilde{C}_i(z)^* C_i(z) = I_{n-i+1}$ on $\mathbb{T}$.

Proof. The condition (2.19) is equivalent to

$$\langle A_i(z), \tilde{A}_j(z) \rangle = \delta_{i,j}, \quad z \in \mathbb{T}, \tag{2.24}$$

for $i, j = 1, \ldots, r$. In particular, $\langle A_1(z), \tilde{A}_1(z) \rangle = 1$ and by Theorem 2.1, there are matrices $Q_1(z), \tilde{Q}_1(z) \in M_n(\mathcal{R}[z])$ such that

$$Q_1(z)A(z) = \tilde{Q}_1(z)\tilde{A}(z) = \mathbf{e}_1, \tag{2.25}$$

and $\tilde{Q}_1(z)^* Q(z) = I_n$ on $\mathbb{T}$. Thus it follows that for $i, j = 1, \ldots, r$,

$$\langle Q_1(z)A_i(z), \tilde{Q}_1(z)\tilde{A}_j(z) \rangle = \delta_{i,j}, \quad z \in \mathbb{T}. \tag{2.26}$$

The equations (2.25)–(2.26) then imply that the first components of $Q_1(z)A_i(z)$ and $\tilde{Q}_1(z)\tilde{A}_i(z)$ are zero for $i = 2, \dots, r$. Consequently,

$$Q_1(z)A(z) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & A_2^{(1)}(z) & \dots & A_r^{(1)}(z) \end{pmatrix},$$

$$\tilde{Q}_1(z)\tilde{A}(z) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \tilde{A}_2^{(1)}(z) & \dots & \tilde{A}_r^{(1)}(z) \end{pmatrix},$$

where $A_i^{(1)}(z)$ and $\tilde{A}_i^{(1)}(z)$, $i = 2, \dots, r$, are $(n-1) \times 1$ matrices over $\mathcal{R}[z]$ satisfying $\langle A_i^{(1)}(z), \tilde{A}_j^{(1)}(z)\rangle = \delta_{i,j}$ on $\mathbb{T}$ for $i, j = 2, \dots, r$.

Next, suppose that for $k < r$, there exist matrices $Q_i(z)$ and $\tilde{Q}_i(z)$, $i = 1, \dots, k-1$, in $M_n(\mathcal{R}[z])$ such that

$$Q_{k-1}(z) \cdots Q_1(z)A(z) = \begin{pmatrix} I_{k-1} & 0 & \dots & 0 \\ 0 & A_k^{(k-1)}(z) & \dots & A_r^{(k-1)}(z) \end{pmatrix},$$

$$\tilde{Q}_{k-1}(z) \cdots \tilde{Q}_1(z)\tilde{A}(z) = \begin{pmatrix} I_{k-1} & 0 & \dots & 0 \\ 0 & \tilde{A}_k^{(k-1)}(z) & \dots & \tilde{A}_r^{(k-1)}(z) \end{pmatrix},$$

with $\tilde{Q}_i(z)^* Q_i(z) = I_n$ on $\mathbb{T}$ for $i = 1, \dots, k-1$, and

$$\langle A_i^{(k-1)}(z), \tilde{A}_j^{(k-1)}(z)\rangle = \delta_{i,j}, \quad z \in \mathbb{T}, \tag{2.27}$$

for $i, j = k, \dots, r$.

Applying Theorem 2.1, we obtain $(n-k+1) \times (n-k+1)$ matrices $C_k(z), \tilde{C}_k(z) \in M_{n-k+1}(\mathcal{R}[z])$ of the form (2.17) such that

$$C_k(z)A_k^{(k-1)}(z) = \tilde{C}_k(z)\tilde{A}_k^{(k-1)}(z) = \mathbf{e}_1,$$

and $\tilde{C}_k(z)^* C_k(z) = I_{n-k+1}$ for all $z \in \mathbb{T}$. Then (2.27) yields

$$\langle C_k(z)A_i^{(k-1)}(z), \tilde{C}_k(z)\tilde{A}_j^{(k-1)}(z)\rangle = \delta_{i,j}, \quad z \in \mathbb{T},$$

for $i, j = k, \dots, r$. Thus the first components of $C_k(z)A_i^{(k-1)}(z)$ and $\tilde{C}_k(z)\tilde{A}_i^{(k-1)}(z)$ are zero for $i = k+1, \dots, r$. Now, let

$$Q_k(z) = \begin{pmatrix} I_{k-1} & 0 \\ 0 & C_k(z) \end{pmatrix}, \qquad \tilde{Q}_k(z) = \begin{pmatrix} I_{k-1} & 0 \\ 0 & \tilde{C}_k(z) \end{pmatrix}.$$

Then we obtain

$$Q_k(z)\dots Q_1(z)A(z) = \begin{pmatrix} I_k & 0 & \dots & 0 \\ 0 & C_k(z)A_{k+1}^{(k-1)}(z) & \dots & C_k(z)A_r^{(k-1)}(z) \end{pmatrix},$$

$$\tilde{Q}_k(z)\dots \tilde{Q}_1(z)\tilde{A}(z) = \begin{pmatrix} I_k & 0 & \dots & 0 \\ 0 & \tilde{C}_k(z)\tilde{A}_{k+1}^{(k-1)}(z) & \dots & \tilde{C}_k(z)\tilde{A}_r^{(k-1)}(z) \end{pmatrix},$$

and $\tilde{Q}_k(z)^*Q_k(z) = I_n$ for all $z \in \mathbb{T}$. Finally, defining $Q(z)$ and $\tilde{Q}(z)$ as in (2.22) gives (2.20)–(2.21). □

Remark 1. *If $P(z)$ and $\tilde{P}(z)$ are two $r \times n$ matrices over $\mathcal{R}[z]$, where $r < n$, which satisfy the identity $P(z)\tilde{P}(z)^* = I_r$ on $\mathbb{T}$, then applying Theorem* 2.2 *to $A(z) = \tilde{P}(z)^*$ and $\tilde{A}(z) = P(z)^*$ yields two $n \times n$ matrices $Q(z)$ and $\tilde{Q}(z)$ over $\mathcal{R}[z]$ such that the submatrices formed by the first r rows of $Q(z)$ and $\tilde{Q}(z)$ are $P(z)$ and $\tilde{P}(z)$ respectively. Furthermore, it follows from* (2.21) *that $Q(z)\tilde{Q}(z)^* = I_n$ on $\mathbb{T}$.*

3. An example

Consider the biorthogonal sets of scaling functions $\{\phi_1, \phi_2\}$ and $\{\tilde{\phi}_1, \tilde{\phi}_2\}$ with dilation $m = 2$ constructed in [3]. They satisfy the matrix dilation equations

$$\Phi(x) = \sum_{k=-1}^{1} H(k)\Phi(2x-k), \qquad \tilde{\Phi}(x) = \sum_{k=-1}^{1} \tilde{H}(k)\tilde{\Phi}(2x-k),$$

where $\Phi = (\phi_1, \phi_2)^T$, $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$, and

$$H(-1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ -1 & -\frac{2}{5} \end{pmatrix}, \quad H(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad H(1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{2}{5} \end{pmatrix};$$

$$\tilde{H}(-1) = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ -\frac{7}{16} & -\frac{35}{32} \end{pmatrix}, \quad \tilde{H}(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \tilde{H}(1) = \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{7}{16} & -\frac{35}{32} \end{pmatrix}.$$

The graphs of ϕ_1, ϕ_2, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are shown in Figure 1. We note that ϕ_1, $\tilde{\phi}_1$ are symmetric, while ϕ_2, $\tilde{\phi}_2$ are antisymmetric, and all the four scaling functions are real, continuous and supported on $[-1, 1]$.

The corresponding polyphase matrices for $\{\phi_1, \phi_2\}$ and $\{\tilde{\phi}_1, \tilde{\phi}_2\}$ are

$$P(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{10}\frac{1}{z} - \frac{\sqrt{2}}{10} \\ 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2}\frac{1}{z} + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{5}\frac{1}{z} - \frac{\sqrt{2}}{5} \end{pmatrix},$$

$$\tilde{P}(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{5\sqrt{2}}{8}\frac{1}{z} - \frac{5\sqrt{2}}{8} \\ 0 & \frac{\sqrt{2}}{4} & -\frac{7\sqrt{2}}{32}\frac{1}{z} + \frac{7\sqrt{2}}{32} & -\frac{35\sqrt{2}}{64}\frac{1}{z} - \frac{35\sqrt{2}}{64} \end{pmatrix},$$

respectively. Applying Theorem 2.2 to $A(z) = \tilde{P}(z)^*$ and $\tilde{A}(z) = P(z)^*$, we obtain two 4×4 matrices $Q(z), \tilde{Q}(z) \in M_4(\mathcal{R}[z])$ such that

$$Q(z)\tilde{P}(z)^* = \tilde{Q}(z)P(z)^* = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}, \quad z \in \mathbb{T},$$

and

$$\tilde{Q}(z)^*Q(z) = I_4, \quad z \in \mathbb{T}.$$

The matrices $Q_1(z), \tilde{Q}_1(z)$, $Q_2(z), \tilde{Q}_2(z)$ in (2.22) are given by

$$Q_1(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\frac{1}{z} & -\frac{\sqrt{2}}{10} + \frac{\sqrt{2}}{10}\frac{1}{z} \\ 0 & 1 & 0 & 0 \\ -\frac{5}{3} & 0 & -\frac{5}{6} + \frac{5}{2}\frac{1}{z} & \frac{1}{3} + \frac{1}{z} \\ \frac{2}{3} + \frac{1}{z} & 0 & \frac{1}{3} - \frac{2}{z} & -\frac{2}{15} \end{pmatrix},$$

$$\tilde{Q}_1(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}\frac{1}{z} & -\frac{5\sqrt{2}}{8} + \frac{\sqrt{2}}{8}\frac{1}{z} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{10} + \frac{2}{15}\frac{1}{z} & 0 & -\frac{1}{20} + \frac{1}{60}\frac{1}{z} & \frac{1}{8} + \frac{17}{24}\frac{1}{z} \\ \frac{1}{3}\frac{1}{z} & 0 & -\frac{1}{3}\frac{1}{z} & \frac{5}{6}\frac{1}{z} \end{pmatrix},$$

$$Q_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & -\frac{2\sqrt{2}}{15}z - \frac{\sqrt{2}}{5} & -\frac{\sqrt{2}}{3}z \\ 0 & 1 & \frac{8}{35}z & \frac{4}{7}z \\ 0 & 0 & -\frac{4}{3}z + 1 & -\frac{10}{3}z \end{pmatrix},$$

$$\tilde{Q}_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & -\frac{35\sqrt{2}}{24} & -\frac{7\sqrt{2}}{16}z + \frac{7\sqrt{2}}{12} \\ 0 & \frac{7}{8} & \frac{35}{48} & \frac{7}{32}z - \frac{7}{24} \\ 0 & \frac{1}{10} & \frac{5}{12} & -\frac{7}{40}z - \frac{1}{6} \end{pmatrix};$$

and the explicit expressions for the matrices $Q(z)$ and $\tilde{Q}(z)$ are

$$Q(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{10}\frac{1}{z} - \frac{\sqrt{2}}{10} \\ 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2}\frac{1}{z} + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{5}\frac{1}{z} - \frac{\sqrt{2}}{5} \\ \frac{4}{7} & 1 & -\frac{4}{7} & \frac{8}{35} \\ -5 & 0 & \frac{5}{2} + \frac{5}{2}\frac{1}{z} & -1 + \frac{1}{z} \end{pmatrix},$$

$$\tilde{Q}(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{5\sqrt{2}}{8}\frac{1}{z} - \frac{5\sqrt{2}}{8} \\ 0 & \frac{\sqrt{2}}{4} & -\frac{7\sqrt{2}}{32}\frac{1}{z} + \frac{7\sqrt{2}}{32} & -\frac{35\sqrt{2}}{64}\frac{1}{z} - \frac{35\sqrt{2}}{64} \\ 0 & \frac{7}{8} & -\frac{7}{64} + \frac{7}{64}\frac{1}{z} & \frac{35}{128} + \frac{35}{128}\frac{1}{z} \\ -\frac{1}{10} & \frac{1}{10} & \frac{3}{80} + \frac{1}{16}\frac{1}{z} & -\frac{3}{32} + \frac{5}{32}\frac{1}{z} \end{pmatrix}.$$

The last two rows of $Q(z)$ and $\tilde{Q}(z)$ lead to biorthogonal sets of multiwavelets $\{\psi_1, \psi_2\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$. To obtain symmetric or antisymmetric multiwavelets, we perform elementary row operations on the last two rows of $Q(z)$ and $\tilde{Q}(z)$. We shall still denote the resulting matrices by $Q(z)$ and $\tilde{Q}(z)$, and they are given by

$$Q(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{10}\frac{1}{z} - \frac{\sqrt{2}}{10} \\ 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2}\frac{1}{z} + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{5}\frac{1}{z} - \frac{\sqrt{2}}{5} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{7} + \frac{\sqrt{2}}{7}\frac{1}{z} & \frac{2\sqrt{2}}{35} + \frac{2\sqrt{2}}{35}\frac{1}{z} \\ \frac{-5\sqrt{2}}{2} & 0 & \frac{5\sqrt{2}}{4} + \frac{5\sqrt{2}}{4}\frac{1}{z} & -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{1}{z} \end{pmatrix},$$

$$\tilde{Q}(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{4}\frac{1}{z} + \frac{\sqrt{2}}{4} & \frac{5\sqrt{2}}{8}\frac{1}{z} - \frac{5\sqrt{2}}{8} \\ 0 & \frac{\sqrt{2}}{4} & -\frac{7\sqrt{2}}{32}\frac{1}{z} + \frac{7\sqrt{2}}{32} & -\frac{35\sqrt{2}}{64}\frac{1}{z} - \frac{35\sqrt{2}}{64} \\ 0 & \frac{7\sqrt{2}}{8} & -\frac{7\sqrt{2}}{64} + \frac{7\sqrt{2}}{64}\frac{1}{z} & \frac{35\sqrt{2}}{128} + \frac{35\sqrt{2}}{128}\frac{1}{z} \\ -\frac{\sqrt{2}}{10} & 0 & \frac{\sqrt{2}}{20} + \frac{\sqrt{2}}{20}\frac{1}{z} & -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8}\frac{1}{z} \end{pmatrix}.$$

In addition, $\tilde{Q}(z)^* Q(z) = I_4$ for all $z \in \mathbb{T}$.

The biorthogonal sets of multiwavelets $\{\psi_1, \psi_2\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ are given by (1.6)–(1.7). Writing (1.6)–(1.7) in matrix form, we have

$$\Psi(x) = \sum_{k=-1}^{1} G(k)\Phi(2x-k), \qquad \tilde{\Psi}(x) = \sum_{k=-1}^{1} \tilde{G}(k)\tilde{\Phi}(2x-k),$$

where $\Psi = (\psi_1, \psi_2)^T$, $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^T$, $G(k) = (g_{ij}(k))$, $\tilde{G}(k) = (\tilde{g}_{ij}(k))$. The matrices $G(k)$ and $\tilde{G}(k)$ are obtained from the last two rows of $Q(z)$ and $\tilde{Q}(z)$ respectively.

They are explicitly given by

$$G(-1) = \begin{pmatrix} \frac{2}{7} & \frac{4}{35} \\ \frac{5}{2} & 1 \end{pmatrix}, \quad G(0) = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}, \quad G(1) = \begin{pmatrix} -\frac{2}{7} & \frac{4}{35} \\ \frac{5}{2} & -1 \end{pmatrix};$$

$$\tilde{G}(-1) = \begin{pmatrix} \frac{7}{32} & \frac{35}{64} \\ \frac{1}{10} & \frac{1}{4} \end{pmatrix}, \quad \tilde{G}(0) = \begin{pmatrix} 0 & \frac{7}{4} \\ -\frac{1}{5} & 0 \end{pmatrix}, \quad \tilde{G}(1) = \begin{pmatrix} -\frac{7}{32} & \frac{35}{64} \\ \frac{1}{10} & -\frac{1}{4} \end{pmatrix}.$$

Figure 2 shows the graphs of ψ_1, ψ_2, $\tilde{\psi}_1$ and $\tilde{\psi}_2$. The functions ψ_1, $\tilde{\psi}_1$ are antisymmetric, while ψ_2, $\tilde{\psi}_2$ are symmetric, and they are real, continuous and supported on $[-1, 1]$.

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References

1. T. N. T. Goodman and S. L. Lee, Wavelets of multiplicity r, *Trans. Amer. Math. Soc.* 342:307–324(1994).
2. W. Lawton, S. L. Lee and Z. Shen, An algorithm for matrix extension and wavelet construction, *Mathematics of Computation* 65:723–737(1996).
3. S. L. Lee and H. H. Tan, Interpolatory multi-scaling functions, manuscript.
4. S. L. Lee, W. S. Tang and H. H. Tan, Wavelet bases for a unitary operator, *Proc. Edinburgh Math. Soc.* 38:233–260(1995).