ESTIMATING MAXIMA OF GENERALIZED CROSS AMBIGUITY FUNCTIONS, AND UNCERTAINTY PRINCIPLES

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Abstract. In certain signal processing problems, it is customary to estimate parameters in distorted signals by approximating what is termed a cross ambiguity function and estimating where it attains its maximum modulus. To unify and generalize these procedures, we consider a generalized form of the cross ambiguity function and give error bounds for estimating the parameters, showing that these bounds are lower if we maximize the real part rather than the modulus. We also reveal a connection between these bounds and certain uncertainty principles, which leads to a new type of uncertainty principle.

1. Introduction

For various applications in signal processing, it is of interest to jointly estimate the time delay and Doppler shift of a signal. Suppose that a given signal $F : \mathbb{R} \longrightarrow \mathbb{C}$ is subjected to a time delay t_d and Doppler shift ω_d, which results in the distorted signal

$$G(t) = e^{-2\pi i \omega_d t} F(t + t_d), \quad t \in \mathbb{R}. \tag{1.1}$$

The problem is to estimate the parameters t_d and ω_d from observed values of F and G. A standard approach is to use the fact that (t_d, ω_d) is the unique point in $\mathbb{R}^2$ at which the *cross ambiguity function*

$$A(F,G)(t,\omega) := \int_{-\infty}^{\infty} F(\tau) e^{-2\pi i \omega \tau} \overline{G(\tau - t)}\, d\tau, \quad t, \omega \in \mathbb{R}, \tag{1.2}$$

attains its maximum modulus, e.g. see [3, 8, 13, 19, 21, 22]. The cross ambiguity function is well studied in the literature on various fronts, such as radar and sonar signal processing, Fourier optics, and time-frequency analysis.

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A similar problem in wideband signal processing is to estimate the time delay t_d and time scale $\alpha_d > 0$ from the distortion

$$G(t) = \frac{1}{\sqrt{\alpha_d}} F\left(\frac{t+t_d}{\alpha_d}\right), \quad t \in \mathbb{R}, \tag{1.3}$$

of a given signal F. In this case (t_d, α_d) is the unique point in $\mathbb{R} \times (0,\infty)$ at which the modulus of the *wideband cross ambiguity function*

$$W(F,G)(t,\alpha) := \sqrt{\alpha} \int_{-\infty}^{\infty} F(\tau)\overline{G(\alpha\tau - t)}\, d\tau, \quad t \in \mathbb{R},\ \alpha > 0, \tag{1.4}$$

attains its maximum modulus, e.g. see [8, 18, 20, 23]. While the cross ambiguity function (1.2) is of the same form as the short-time Fourier transform, the wideband cross ambiguity function (1.4) is related to the continuous wavelet transform.

In this paper we analyze a more general problem which covers both the above cases of parameter estimation. Corresponding to (1.1) and (1.3), the distorted signal G is of the form

$$G = U(t_d)F, \tag{1.5}$$

where F is a given signal in a Hilbert space X, t_d lies in some set $S \subset \mathbb{R}^s$, and for every $t \in S$, $U(t)$ is a unitary operator on X. Corresponding to (1.2) and (1.4), we define a *generalized cross ambiguity function*

$$A(F,G)(t) := \langle F, U(t)^* G\rangle, \quad t \in S, \tag{1.6}$$

which gains its maximum modulus at $t = t_d$.

This general setting not only allows unification of the above two problems, but also the construction of other choices of the operator $U(t)$. For example, we could consider simultaneously time delay, Doppler shift and time scale, which is studied in Example 2.1. One could also make a change of parameter in the operator $U(t)$, or equivalently in the function $A(F,G)$. Indeed, by making a change of parameter in Example 2.1, we can obtain the two practical cases (1.2) and (1.4) discussed above. Similarly one could obtain the simple case $U(t)f = f(\cdot + t)$, $t \in \mathbb{R}$, for $f : \mathbb{R} \longrightarrow \mathbb{C}$, corresponding to estimating time delay alone, which is a frequently encountered problem in signal processing, e.g. see [5]. Alternatively we could consider estimating Doppler shift alone. In all these cases, t_d is the *unique* point at which $A(F,G)$ attains its maximum modulus, as is shown in the Appendix.

In [1], the problem is considered of estimating the polar coordinates (r,θ) of an emitter from observed values of the Fourier transform of the received signal at an array of receivers. This can also be put in the above general setting except that in this case the *range-angle ambiguity function* $\mathcal{A}(r,\theta)$ to be maximized is a weighted *sum* of functions of the form (1.6), where the terms in the sum, corresponding to different receivers, are obtained by different changes of parameter from the above example of Doppler shift alone. Here the terms in the sum are simultaneously maximized at the required coordinates (r_d, θ_d) and this is the *unique* point at which $\mathcal{A}(r, \theta)$ has maximum modulus if the receivers are not collinear.

All of the above is discussed in detail in Section 2. Now recall that the problem we are considering is to estimate the parameter t_d from certain observed values of the signals F and G. For the case (1.1), this is achieved very effectively in [10] for band-limited signals by approximating F and G by their truncated Shannon series F_N and G_N. Then (t_d, ω_d) is approximated by the point (t_d^N, ω_d^N) at which the approximate cross ambiguity function $A(F_N, G_N)$ attains its maximum modulus. An analogous procedure for the case (1.3) is developed in [11] and also proves highly successful. In this paper we consider the general situation of (1.5) and take general approximations F_N, G_N to F, G. We then derive, in Section 3, error estimates in approximating t_d by a point t_d^N at which $A(F_N, G_N)$, defined as in (1.6), attains its maximum modulus. These results require the unitary operator $U(t)$ to satisfy certain properties; these are described in Section 2 and shown to hold in the examples considered.

The error estimates given in Section 3 extend work in [10] in three ways. Firstly [10] considers only the case of time delay and Doppler shift, maximizing (1.2), whereas here we allow the general situation of maximizing (1.6), thus including all the examples mentioned earlier, as well as other possible choices of $U(t)$ such as rotation in $\mathbb{R}^2$. Secondly, in [10] the functions F, G are assumed to be band-limited and F_N, G_N are truncated Shannon series. Here we allow general approximations F_N, G_N to F, G. In Section 5, we examine in more detail two special cases: the above truncated Shannon series, and the case where F, G are compactly supported functions and F_N, G_N are defined as Riemann sums of convolutions of F, G with a suitable kernel function.

Thirdly we note that [10] and, as far as we are aware, all previous work in the literature, consider maximizing the modulus of the cross ambiguity function. However at the required point t_d, $A(F, G)$ attains not only its maximum modulus but also its maximum real part. In Theorem 3.2 we give an error bound for estimating t_d by maximizing the modulus of $A(F, G)$, and in Theorem 3.1 we give an error bound gained by maximizing the real part. In Section 4, we use a general result on symmetric operators to show, in Corollary 4.1, that the error estimate in Theorem 3.1 is better than that in Theorem 3.2. Taken together with forthcoming work on noisy data, this suggests that maximizing the real part could be better in practice than maximizing the modulus.

We also see in Section 4, extending work in [10], that for the case $s = 2$, i.e. $t_d \in \mathbb{R}^2$, upper bounds on the areas of the estimation regions in Theorems 3.1 and 3.2 can be gained from an uncertainty principle for two symmetric operators given in Theorem 4.2. Although the uncertainty principle can be quite easily derived from a known generalization of Heisenberg's uncertainty principle, the form in which it is expressed suggests extension to an uncertainty principle involving s symmetric operators, $s \geq 3$. This is achieved in Theorem 4.3 for $s = 3$. This gives upper bounds on the volumes of the estimation regions in Theorems 3.1 and 3.2 for $s = 3$, e.g. Example 2.1, but we also feel, since it is to our knowledge a completely new type of uncertainty principle, that Theorem 4.3 is also of general theoretical interest.

2. Generalized cross ambiguity functions

For f, g in a Hilbert space X and $t \in S$, where S is an open set in $\mathbb{R}^s$, $s \geq 1$, we define the *generalized cross ambiguity function*

$$A(f, g)(t) := \langle f, U(t)^* g \rangle, \tag{2.1}$$

where for every $t \in S$, $U(t)$ is a unitary operator on X and for some $t_0 \in S$, $U(t_0) = I$. Note that for $t \in S$,

$$|A(f, g)(t)| \leq \|f\| \|U(t)^* g\| = \|f\| \|g\|. \tag{2.2}$$

For simplicity in subsequent derivations, we shall assume that for all $f, g \in X$ and $t \in S$, $A(f, g)(t)$ is continuous in t.

Now take $F \in X$ and for some $t_d \in S$, let G be defined by (1.5). Then for any $f \in X$,

$$A(f, G)(t_d) = \langle f, U(t_d)^* U(t_d) F \rangle = \langle f, F \rangle. \tag{2.3}$$

For all $t \in S$, by (2.2),

$$\operatorname{Re} A(F, G)(t) \leq |A(F, G)(t)| \leq \|F\| \|G\| = \|F\|^2,$$

and by (2.3),

$$\operatorname{Re} A(F, G)(t_d) = |A(F, G)(t_d)| = \|F\|^2.$$

Thus both $\operatorname{Re} A(F, G)$ and $|A(F, G)|$ attain their maximum at t_d.

For the approximation properties considered later, we shall need the following properties. For $1 \leq j \leq s$, $t = (t_1, \ldots, t_s) \in S$, there is a (possibly unbounded) linear operator $T_j(t) : \mathcal{D}(T_j(t)) \subset X \longrightarrow X$ with

$$\langle T_j(t) f, g \rangle = -\langle f, T_j(t) g \rangle, \quad f, g \in \mathcal{D}(T_j(t)), \tag{2.4}$$

and for $f \in \mathcal{D}(T_j(t))$, $g \in X$,

$$\frac{\partial}{\partial t_j} A(f, g)(t) = A(T_j(t) f, g)(t). \tag{2.5}$$

In addition, for F and G as in (1.5) and $F \in \mathcal{D}(T_j(t_d)T_k(t_d))$, $j,k = 1,\dots,s$, we assume that

$$\operatorname{Re} \frac{\partial^2}{\partial t_j \partial t_k} A(F,G)(t_d) = \operatorname{Re} A(T_j(t_d)T_k(t_d)F,G)(t_d). \tag{2.6}$$

Note that since $\operatorname{Re} A(F,G)$ attains its maximum at t_d, (2.5) and (2.3) give

$$\operatorname{Re} A(T_j(t_d)F,G)(t_d) = \operatorname{Re} \langle T_j(t_d)F, F\rangle = 0. \tag{2.7}$$

Let us identify some sufficient conditions for the properties (2.5) and (2.6). If we have for $1 \le j \le s$, $t \in S$, $f \in \mathcal{D}(T_j(t))$,

$$\frac{\partial}{\partial t_j} U(t)f = U(t)T_j(t)f, \tag{2.8}$$

where the convergence is in X, then (2.5) follows from (2.8) and (2.1).

As for (2.6), for $1 \le j,k \le s$, assume that $\frac{\partial}{\partial t_j}(T_k(t_d)F)$ exists in X, where $\frac{\partial}{\partial t_j}(T_k(t_d)F)$ denotes $\frac{\partial}{\partial t_j}(T_k(t)F)$ evaluated at $t = t_d$. Then

$$\begin{aligned} \frac{\partial^2}{\partial t_j \partial t_k} A(F,G)(t_d) &= \frac{\partial}{\partial t_j} A(T_k(t_d)F,G)(t_d) \\ &= A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d) + A(T_j(t_d)T_k(t_d)F,G)(t_d). \end{aligned} \tag{2.9}$$

The first equality is due to (2.5). For the second equality, we argue as follows. Let $v_{r,j}$ be a vector in $\mathbb{R}^s$ for which only the jth component is nonzero and given by r. Then

$$\begin{aligned} \frac{\partial}{\partial t_j} A(T_k(t_d)F,G)(t_d) =& \lim_{r\to 0} \frac{A\big(T_k(t_d+v_{r,j})F - T_k(t_d)F, G\big)(t_d+v_{r,j})}{r} \\ &+ \lim_{r\to 0} \frac{A(T_k(t_d)F,G)(t_d+v_{r,j}) - A(T_k(t_d)F,G)(t_d)}{r} =: L_1 + L_2. \end{aligned}$$

Observe that

$$\begin{aligned} &\left| L_1 - A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d)\right| \\ &\le \lim_{r\to 0} \left| \frac{A\big(T_k(t_d+v_{r,j})F - T_k(t_d)F, G\big)(t_d+v_{r,j})}{r} - A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d+v_{r,j})\right| \\ &\quad + \lim_{r\to 0} \left| A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d+v_{r,j}) - A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d)\right| = 0, \end{aligned}$$

by (2.2) and the continuity of $A\big(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\big)(t)$ at $t = t_d$. On the other hand, (2.5) implies that $L_2 = A(T_j(t_d)T_k(t_d)F,G)(t_d)$. Therefore (2.9) holds. Consequently, to obtain (2.6), it is sufficient to show

$$\operatorname{Re} A\left(\frac{\partial}{\partial t_j}(T_k(t_d)F), G\right)(t_d) = 0,$$

which, by (2.3), is equivalent to

$$\mathrm{Re}\left\langle \frac{\partial}{\partial t_j}(T_k(t_d)F), F \right\rangle = 0. \tag{2.10}$$

For the rest of this section we consider some examples with $X = L^2(\mathbb{R})$. For each example, in order to carry out the error analysis in the paper, we need to identify the operators $T_j(t)$ satisfying (2.8), (2.4) and (2.10).

Example 2.1. (Translation, modulation and scaling)
Take $S = \mathbb{R}^2 \times (0, \infty)$ and define for $f \in L^2(\mathbb{R})$,

$$U(t,\omega,\alpha)f(\tau) := \frac{1}{\sqrt{\alpha}} e^{-2\pi i\omega\tau} f\left(\frac{\tau+t}{\alpha}\right), \quad t,\omega,\tau \in \mathbb{R},\ \alpha > 0.$$

Then for $t, \omega, \tau \in \mathbb{R}$, $\alpha > 0$,

$$U(t,\omega,\alpha)^* f(\tau) = \sqrt{\alpha} e^{2\pi i\omega(\alpha\tau - t)} f(\alpha\tau - t),$$

and so

$$A(f,g)(t,\omega,\alpha) = \sqrt{\alpha} e^{2\pi i\omega t} \int_{-\infty}^{\infty} f(\tau) e^{-2\pi i\omega\alpha\tau} \overline{g(\alpha\tau - t)}\, d\tau.$$

Suppose that $f, f', \tau f(\tau), \tau f'(\tau) \in L^2(\mathbb{R})$, $\tau \in \mathbb{R}$. Then

$$\frac{\partial}{\partial t} U(t,\omega,\alpha)f(\tau) = \frac{1}{\alpha^{3/2}} e^{-2\pi i\omega\tau} f'\left(\frac{\tau+t}{\alpha}\right),$$

$$\frac{\partial}{\partial \omega} U(t,\omega,\alpha)f(\tau) = -2\pi i\tau U(t,\omega,\alpha)f(\tau),$$

$$\frac{\partial}{\partial \alpha} U(t,\omega,\alpha)f(\tau) = -\frac{1}{2\alpha} U(t,\omega,\alpha)f(\tau) - \frac{\tau+t}{\alpha^{5/2}} e^{-2\pi i\omega\tau} f'\left(\frac{\tau+t}{\alpha}\right),$$

and it follows that (2.8) holds with

$$T_1(t,\omega,\alpha)f := T_1(\alpha)f := \frac{1}{\alpha} f',$$

$$T_2(t,\omega,\alpha)f := T_2(t,\alpha)f := -2\pi i(\alpha \cdot - t)f,$$

$$T_3(t,\omega,\alpha)f(\tau) := T_3(\alpha)f(\tau) := -\frac{1}{\alpha}\left(\tau f'(\tau) + \frac{1}{2} f(\tau)\right).$$

For $j = 1, 3$, (2.4) holds by integration by parts, while for $j = 2$, (2.4) clearly holds. Also

$$\frac{\partial}{\partial t}(T_2(t,\alpha)f) = 2\pi i f, \qquad \frac{\partial}{\partial \alpha}(T_2(t,\alpha)f)(\tau) = -2\pi i\tau f(\tau), \quad \tau \in \mathbb{R},$$

$$\frac{\partial}{\partial \alpha}(T_j(\alpha)f) = -\frac{1}{\alpha} T_j(\alpha)f, \quad j = 1, 3, \tag{2.11}$$

and all other such derivatives are zero. It follows that in all cases (2.10) holds, where for the case (2.11) we recall (2.7).

We note that in the above example the operators $U(t)$, $t \in \mathbb{R}^2 \times (0,\infty)$, form a Lie group. Moreover it can be seen that each operator $T_j(t)$ can be written in the form $T_j(t) = U(t)^* R_j U(t)$, where R_j is a generator of the Lie group. However the assumptions and results of Lie group theory are not necessary here and to assume the operators form a Lie group would be unnecessarily restrictive, e.g. not allowing a general change of parameter as given in the next result.

Proposition 2.1. *Suppose that the unitary operators $U(t)$, $t \in S$, as in* (2.1)*, satisfy* (2.8)*,* (2.4) *and* (2.10) *for some operators $T_j(t)$, $j = 1,\dots,s$, $t \in S$, and take $t_d \in S$, F, G as before. Let $\tilde{S}$ be an open set in $\mathbb{R}^m$, $m \geq 1$, and $\varphi : \tilde{S} \to S$ a differentiable function which is twice differentiable at $\tilde{t}_d \in \tilde{S}$ with $\varphi(\tilde{t}_d) = t_d$. Then the operators $\tilde{U}(t) := U(\varphi(t))$, $t \in \tilde{S}$, satisfy* (2.8), (2.4) *and* (2.10) *with U, t_d replaced by $\tilde{U}$, $\tilde{t}_d$ there and in* (1.5)*, and T_j replaced by $\tilde{T}_j$, $j = 1,\dots,m$, where*

$$\tilde{T}_j(t) := \sum_{\ell=1}^{s} \frac{\partial \varphi_\ell(t)}{\partial t_j} T_\ell(\varphi(t)), \quad t \in \tilde{S}, \tag{2.12}$$

and $\varphi = (\varphi_1,\dots,\varphi_s)$.

Proof. For $1 \leq j \leq m$, $t \in \tilde{S}$, $f \in \mathcal{D}(\tilde{T}_j(t))$, recalling (2.8),

$$\frac{\partial}{\partial t_j}\tilde{U}(t)f = \frac{\partial}{\partial t_j}U(\varphi(t))f = \sum_{\ell=1}^{s} \frac{\partial \varphi_\ell(t)}{\partial t_j} U(\varphi(t))T_\ell(\varphi(t))f = \tilde{U}(t)\tilde{T}_j(t)f,$$

by (2.12), which is the required analogue of (2.8). From (2.4), it is easily seen that for $1 \leq j \leq m$, $\tilde{T}_j$ satisfies the analogue of (2.4). Finally for $1 \leq j,k \leq m$, (2.12) gives

$$\mathrm{Re}\left\langle \frac{\partial}{\partial t_j}(\tilde{T}_k(\tilde{t}_d)F), F \right\rangle = \sum_{\ell=1}^{s} \frac{\partial^2 \varphi_\ell(\tilde{t}_d)}{\partial t_j \partial t_k} \mathrm{Re}\, \langle T_\ell(t_d)F, F\rangle + \sum_{\ell=1}^{s} \frac{\partial \varphi_\ell(\tilde{t}_d)}{\partial t_k} \mathrm{Re}\left\langle \frac{\partial}{\partial t_j}(T_\ell(t_d)F), F \right\rangle.$$

The two terms on the right vanish by (2.7) and (2.10) respectively, which gives the required analogue of (2.10). ■

Remark 2.1. Note that for $t \in \tilde{S}$,

$$\langle f, \tilde{U}(t)^* g\rangle = \langle f, U(\varphi(t))^* g\rangle = A(f,g)(\varphi(t)),$$

by (2.1), and so Proposition 2.1 is essentially considering a change of variable in $A(f,g)$.

The next three examples correspond to the practical scenarios highlighted in Section 1 and all are obtained by a change of variable in Example 2.1.

Example 2.2. (Translation)
Define $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2 \times (0,\infty)$ by $\varphi(t) = (t,0,1)$. Then defining $\tilde{U}(t) = U(\varphi(t))$ as in Proposition 2.1 and Example 2.1 gives

$$A(f,g)(t) := \langle f, \tilde{U}(t)^* g\rangle = \int_{-\infty}^{\infty} f(\tau)\overline{g(\tau - t)}\, d\tau, \quad t \in \mathbb{R}.$$

From (2.12), the operator $\tilde{T}_1(t)$ equals $\tilde{T}_1$, $t \in \mathbb{R}$, where $\tilde{T}_1 f := f'$.

Example 2.3. (Translation and modulation)
Define $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times (0,\infty)$ by $\varphi(t) = (t,\omega,1)$. Then as in Example 2.2, we get

$$A(f,g)(t,\omega) = e^{2\pi i\omega t}\int_{-\infty}^{\infty} f(\tau)e^{-2\pi i\omega\tau}\overline{g(\tau - t)}\, d\tau, \quad t,\omega \in \mathbb{R}. \tag{2.13}$$

Here for $t,\omega \in \mathbb{R}$,

$$\tilde{T}_1(t,\omega)f := \tilde{T}_1 f := f', \qquad \tilde{T}_2(t,\omega)f := \tilde{T}_2(t)f := -2\pi i(\cdot - t)f. \tag{2.14}$$

We note that in the literature, the term $e^{2\pi i\omega t}$ in (2.13) is usually omitted, but since they only consider $|A(f,g)|$, this is irrelevant. On the other hand, by including the term $e^{2\pi i\omega t}$ in (2.13), we gain the alternative of maximizing $\operatorname{Re} A(f,g)$ in the joint estimation of time delay and Doppler shift.

Example 2.4. (Translation and scaling)
Define $\varphi : \mathbb{R} \times (0,\infty) \longrightarrow \mathbb{R}^2 \times (0,\infty)$ by $\varphi(t,\alpha) = (t,0,\alpha)$. Then as in Example 2.2, we get

$$A(f,g)(t,\alpha) = \sqrt{\alpha}\int_{-\infty}^{\infty} f(\tau)\overline{g(\alpha\tau - t)}\, d\tau, \quad t \in \mathbb{R},\ \alpha > 0.$$

Here for $t \in \mathbb{R}$, $\alpha > 0$,

$$\tilde{T}_1(t,\alpha)f := \tilde{T}_1(\alpha)f := \frac{1}{\alpha}f',$$

$$\tilde{T}_2(t,\alpha)f(\tau) := \tilde{T}_2(\alpha)f(\tau) := -\frac{1}{\alpha}\left(\tau f'(\tau) + \frac{1}{2}f(\tau)\right), \quad \tau \in \mathbb{R}.$$

Our next example is also gained by a change of variable in Example 2.1 and we shall see shortly that it is central to target estimation in sonar and radar as in [1].

Example 2.5. (Modulation)
Define $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2 \times (0,\infty)$ by $\varphi(\omega) = (0,\omega,1)$. Then as in Example 2.2, we get

$$A(f,g)(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-2\pi i\omega\tau}\overline{g(\tau)}\, d\tau, \quad \omega \in \mathbb{R}.$$

Here for $\omega \in \mathbb{R}$,

$$\tilde{T}_1(\omega)f(\tau) := \tilde{T}_1 f(\tau) := -2\pi i\tau f(\tau), \quad \tau \in \mathbb{R}.$$

We note that this cross ambiguity function could also be obtained by applying the cross ambiguity function in Example 2.2 to the Fourier transforms of f and g.

In [1], they consider the *range-angle ambiguity function* $\mathcal{A}(r,\theta)$ which can be expressed as follows. For $k = 1,\ldots,K$, take $c_k > 0$, $-\pi < \psi_k \le \pi$, and define

$$\varphi_k(r,\theta) := \left(r^2 + c_k^2 - 2rc_k\cos(\psi_k - \theta)\right)^{1/2}, \quad r > 0,\ \theta \in \mathbb{R}.$$

Take $r_d > 0$, $-\pi < \theta_d \le \pi$, and for $A(f,g)$ as in Example 2.5, define

$$\mathcal{A}(r,\theta) := \sum_{k=1}^{K} |\zeta_k|^2 A(F,G_k)(\varphi_k(r,\theta)), \tag{2.15}$$

where for $k = 1,\ldots,K$, $G_k(\tau) := \exp(-2\pi i\tau\varphi_k(r_d,\theta_d))\,F(\tau)$, $\tau \in \mathbb{R}$, and $\zeta_k \in \mathbb{C}\backslash\{0\}$. Here $G_1,\ldots,G_K$ are the received waveforms by an array of K receivers of the waveform F, all of which are in the frequency domain.

By Proposition 2.1 and Remark 2.1, each term in the summation in (2.15) is a generalized cross ambiguity function satisfying our earlier conditions. Moreover all these terms attain their maximum real part and modulus at (r_d,θ_d). In [1], these terms arise from different receivers placed in a planar array. We note that, unlike Examples 2.2–2.5, each function φ_k is not injective. However it can be shown that if the array is not collinear, then (r_d,θ_d) is the only point in $\mathbb{R}\times(-\pi,\pi]$ at which $\mathcal{A}(r,\theta)$ attains its maximum real part or modulus.

As seen above, the formulation (2.1) covers a variety of situations. Clearly we could also consider scaling on its own, or modulation and scaling together. In addition, we may study similar operators on $L^2(\mathbb{R}^s)$, $s \ge 2$. Other examples could be gained by, for instance, rotation on $\mathbb{R}^2$.

In [7], a notion of generalized ambiguity functions for $f, g \in L^2(\mathbb{R})$ is introduced, which is defined as

$$A(f,g)(t,\omega) := \int_{-\infty}^{\infty} f\left(\frac{b_{12}\tau + b_{22}t}{b_{11}b_{22} - b_{12}b_{21}}\right) e^{-2\pi i\omega\tau}\,\overline{g\left(\frac{-b_{11}\tau - b_{21}t}{b_{11}b_{22} - b_{12}b_{21}}\right)}\,d\tau, \quad t,\omega \in \mathbb{R}, \tag{2.16}$$

where $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is a 2×2 real matrix with determinant ± 1. This extends the symmetric version of (2.13) given by $b_{11} = 1$, $b_{12} = -1$, $b_{21} = b_{22} = -1/2$. While the formulation in [7] is different from here, it is interesting to note that the nondegenerate cases of (2.16), i.e. $b_{11}b_{12} \ne 0$, are also obtainable from (2.1). More precisely, by multiplying the normalizing constant $\sqrt{|b_{11}b_{12}|}$ to (2.16), the resulting cross ambiguity function may be written in the form of (2.1) for appropriate unitary operators $U(t,\omega)$, $t,\omega \in \mathbb{R}$. Further, the corresponding operators $T_1(t,\omega)$ and $T_2(t,\omega)$ can be identified and shown to satisfy (2.4), (2.8) and (2.10).

It is also worthwhile mentioning that in [2, 14], ideas on unitary equivalence in signal analysis and processing tools, especially time-frequency representations, are introduced and investigated. They generate new classes of such tools from existing ones by applying a unitary transformation to the signal before performing standard processing to the transformed signal. In a similar vein, we may apply a unitary operator V to $f, g \in X$ in (2.1) to obtain

$$A(Vf, Vg)(t) = \langle Vf, U(t)^* Vg\rangle = \langle f, V^* U(t)^* Vg\rangle,$$

which is again a generalized cross ambiguity function of the form (2.1).

3. Approximation

We return to the general situation of Section 2, where we take $F \in X$ and define G by (1.5) for some $t_d \in S$. Suppose that there are F_N, G_N in X, $N = 1, 2, \ldots$, with

$$\lim_{N\to\infty} F_N = F, \qquad \lim_{N\to\infty} G_N = G, \tag{3.1}$$

in X. The approaches developed in [10, 11] are special cases of this setting. For $h = \operatorname{Re} A(F, G)$ or $|A(F, G)|$, we recall that h attains its maximum at t_d. We shall suppose that either for $h_N = \operatorname{Re} A(F_N, G_N)$, $N \geq 1$, or for $h_N = |A(F_N, G_N)|$, $N \geq 1$, h_N attains its maximum at t_d^N, and we shall investigate the convergence of t_d^N to t_d. First we give some more general results.

Lemma 3.1. *For $t \in S$,*

$$|A(F, G)(t) - A(F_N, G_N)(t)| \leq \|F - F_N\|\|G\| + \|G - G_N\|\|F\| + \|F - F_N\|\|G - G_N\|.$$

Proof. By (2.1) and (2.2), for $t \in S$,

$$\begin{aligned} |A(F, G)(t) - A(F_N, G_N)(t)| &\leq |A(F - F_N, G)(t)| + |A(F_N, G - G_N)(t)| \\ &\leq \|F - F_N\|\|G\| + \|F_N\|\|G - G_N\| \\ &\leq \|F - F_N\|\|G\| + (\|F_N - F\| + \|F\|)\|G - G_N\|, \end{aligned}$$

which gives the result. ■

Lemma 3.1 was proved in [10] for the special case of Example 2.3. The following result was also proved in [10].

Lemma 3.2. *Let h and h_N, $N = 1, 2, \ldots$, be real-valued functions on a metric space (Y, d). Suppose that for a point a in Y,*

$$\sup\{h(y) : y \in Y, d(y, a) \geq c\} < h(a) \tag{3.2}$$

for any $c > 0$. If $h_N \to h$ uniformly on Y as $N \to \infty$ and for $N = 1, 2, \ldots$, h_N has a maximum at a_N in Y, then $\lim_{N\to\infty} a_N = a$.

Proof. We give a proof here for completeness. The condition (3.2) is clearly equivalent to h attaining its maximum at the unique point a, and for any sequence (y_N) in Y, $\lim_{N\to\infty} h(y_N) = h(a)$ implies $\lim_{N\to\infty} y_N = a$. Then for any $\epsilon > 0$, for all large enough N,

$$h(a) \geq h(a_N) > h_N(a_N) - \epsilon \geq h_N(a) - \epsilon > h(a) - 2\epsilon.$$

Thus $\lim_{N\to\infty} h(a_N) = h(a)$ and so $\lim_{N\to\infty} a_N = a$. ∎

Although Lemma 3.2 can be proved so easily, it is very useful for us as it requires very few conditions, e.g. no continuity of the functions h, h_N and no uniqueness of the point a_N where h_N attains its maximum.

Now suppose $a \in Y \subset \mathbb{R}^s$, $s \geq 1$, $h \in C^2(Y)$ and denote by M the $s \times s$ matrix given by

$$M_{jk} := -\frac{\partial^2 h}{\partial t_j \partial t_k}(a), \quad j, k = 1, \ldots, s. \tag{3.3}$$

Since h has a maximum value at a, $M \geq 0$, i.e. M is positive semi-definite. We shall assume further that $M > 0$, i.e. M is positive definite. For any $c > 0$ we denote by $E_M(c)$ the ellipsoidal region

$$E_M(c) := \{x \in \mathbb{R}^s : x^T M x < c\}. \tag{3.4}$$

Suppose that M has orthonormal eigenvectors $v_1, \ldots, v_s$ with eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_s$. Then $E_M(c)$ comprises $x \in \mathbb{R}^s$ satisfying

$$\sum_{j=1}^{s} \lambda_j (v_j^T x)^2 < c. \tag{3.5}$$

For $x \in E_M(c)$,

$$|x|^2 = \sum_{j=1}^{s} (v_j^T x)^2 < \frac{c}{\lambda_1}.$$

The following is a simple extension of a result in [10].

Lemma 3.3. *As in Lemma 3.2, let Y be an open neighborhood of a in $\mathbb{R}^s$, $s \geq 1$, and $h \in C^2(Y)$. Suppose $M > 0$, with M as in (3.3). Take any $\mu > 1$. Then for all large enough N,*

$$a_N - a \in E_M(4\mu \|h_N - h\|_\infty),$$

where $E_M(4\mu\|h_N - h\|_\infty)$ is defined as in (3.4) and $\|h_N - h\|_\infty = \sup\{|h_N(y) - h(y)| : y \in Y\}$.

Proof. Applying Taylor's theorem to h about a, we see that for all x in some neighborhood of a,

$$h(a) - h(x) > \frac{1}{2\mu}(x-a)^T M(x-a).$$

Since $\lim_{N\to\infty} a_N = a$, for all large enough N, a_N lies in this neighborhood and so

$$\frac{1}{2\mu}(a_N - a)^T M(a_N - a) < h(a) - h(a_N). \tag{3.6}$$

Now

$$h(a) - \|h_N - h\|_\infty \le h_N(a) \le h_N(a_N) \le h(a_N) + \|h_N - h\|_\infty$$

and so

$$h(a) - h(a_N) \le 2\|h_N - h\|_\infty. \tag{3.7}$$

The result then follows from (3.6), (3.7) and (3.4). ■

We now consider the general situation of Section 2 and study two cases: $Y = S \subset \mathbb{R}^s$ and

Case I: $h = \operatorname{Re} A(F,G)$, $h_N = \operatorname{Re} A(F_N, G_N)$,

Case II: $h = |A(F,G)|$, $h_N = |A(F_N, G_N)|$.

In both cases we know that h attains its maximum at t_d and we assume that h_N attains its maximum at t_d^N, $N = 1, 2, \ldots$. From (3.1) and Lemma 3.1, $h_N \to h$ uniformly on S. We assume that (3.2) holds at the point t_d. (For the examples in Section 2 we shall show this in the Appendix.) Then from Lemma 3.2,

$$\lim_{N\to\infty} t_d^N = t_d.$$

We now apply Lemma 3.3 to consider the error $t_d^N - t_d$. We assume that for $j, k = 1, \ldots, s$, $F \in \mathcal{D}(T_j(t_d)T_k(t_d))$ and then from (2.6), (2.3) and (2.4),

$$\operatorname{Re} \frac{\partial^2}{\partial t_j \partial t_k} A(F,G)(t_d) = -\operatorname{Re} \langle T_j(t_d)F, T_k(t_d)F\rangle. \tag{3.8}$$

Case I: Denote by B the $s \times s$ matrix given by

$$B_{jk} := \operatorname{Re} \langle T_j(t_d)F, T_k(t_d)F\rangle, \quad j, k = 1, \ldots, s. \tag{3.9}$$

Note that for $x \in \mathbb{C}^s$,

$$\begin{aligned}\overline{x}^T B x &= \frac{1}{2}\sum_{j,k=1}^{s} \overline{x_j}\, \langle T_j(t_d)F, T_k(t_d)F\rangle\, x_k + \frac{1}{2}\sum_{j,k=1}^{s} \overline{x_j}\, \langle T_k(t_d)F, T_j(t_d)F\rangle\, x_k \\ &= \frac{1}{2}\left\| \sum_{j=1}^{s} \overline{x_j} T_j(t_d)F \right\|^2 + \frac{1}{2}\left\| \sum_{j=1}^{s} x_j T_j(t_d)F \right\|^2. \end{aligned} \tag{3.10}$$

Thus $B \ge 0$. We assume $B > 0$, which will be the case if $T_j(t_d)F$, $j = 1, \ldots, s$, are linearly independent. Then the result below follows from Lemmas 3.1 and 3.3.

Theorem 3.1. *Consider* $h = \operatorname{Re} A(F,G)$ *for which* (3.2) *holds at* t_d. *Suppose that* $h_N = \operatorname{Re} A(F_N, G_N)$ *attains its maximum value at* t_d^N, $N = 1, 2, \ldots$. *Then for* $\mu > 1$ *and all large enough* N,

$$t_d^N - t_d \in E_B(c), \tag{3.11}$$

where $E_B(c)$ *is defined as in* (3.4), B *given by* (3.9) *and*

$$c = 4\mu \left(\|F - F_N\|\|F\| + \|G - G_N\|\|F\| + \|F - F_N\|\|G - G_N\|\right). \tag{3.12}$$

In (3.12), we have used $\|G\| = \|F\|$. We note that by (3.4) and (3.10),

$$E_B(c) = \left\{ x \in \mathbb{R}^s : \left\| \sum_{j=1}^{s} x_j T_j(t_d) F \right\|^2 < c \right\}.$$

Case II: Here for $j = 1, \ldots, s$, $A(F,G)(t) \neq 0$,

$$\frac{\partial h}{\partial t_j}(t) = \frac{1}{h(t)} \operatorname{Re} \left\{ A(F,G)(t) \overline{\frac{\partial}{\partial t_j} A(F,G)(t)} \right\}$$

and $\frac{\partial h}{\partial t_j}(t_d) = 0$. Thus for $j, k = 1, \ldots, s$,

$$\begin{aligned}
\frac{\partial^2 h}{\partial t_j \partial t_k}(t_d) &= \frac{1}{h(t_d)} \operatorname{Re} \left\{ \frac{\partial}{\partial t_j} A(F,G)(t_d) \overline{\frac{\partial}{\partial t_k} A(F,G)(t_d)} \right\} \\
&\quad + \frac{1}{h(t_d)} \operatorname{Re} \left\{ A(F,G)(t_d) \overline{\frac{\partial^2}{\partial t_j \partial t_k} A(F,G)(t_d)} \right\} \\
&= \frac{1}{\|F\|^2} \operatorname{Re} \left\{ \langle T_j(t_d)F, F \rangle \overline{\langle T_k(t_d)F, F \rangle} - \|F\|^2 \langle T_j(t_d)F, T_k(t_d)F \rangle \right\},
\end{aligned}$$

by (2.3), (2.5) and (3.8). Writing

$$\lambda_j := \langle T_j(t_d)F, F \rangle / \|F\|^2, \quad j = 1, \ldots, s, \tag{3.13}$$

we have

$$\frac{\partial^2 h}{\partial t_j \partial t_k}(t_d) = -\operatorname{Re} \langle T_j(t_d)F - \lambda_j F,\, T_k(t_d)F - \lambda_k F \rangle$$

for $j, k = 1, \ldots, s$. Denote by C the $s \times s$ matrix given by

$$C_{jk} := \operatorname{Re} \langle T_j(t_d)F - \lambda_j F,\, T_k(t_d)F - \lambda_k F \rangle, \quad j, k = 1, \ldots, s. \tag{3.14}$$

As for Case I, we have $C \geq 0$ and assume $C > 0$, which is the case if $T_j(t_d)F - \lambda_j F$, $j = 1, \ldots, s$, are linearly independent. Then by Lemmas 3.1 and 3.3, we obtain the following result.

Theorem 3.2. *Consider* $h = |A(F,G)|$ *for which* (3.2) *holds at* t_d. *Suppose that* $h_N = |A(F_N, G_N)|$ *attains its maximum value at* t_d^N, $N = 1, 2, \ldots$. *Then for* $\mu > 1$ *and all large enough* N,

$$t_d^N - t_d \in E_C(c), \tag{3.15}$$

where $E_C(c)$ *is defined as in* (3.4)*,* C *given by* (3.14) *and* (3.13)*, and* c *by* (3.12)*.*

4. Inequalities

We now see that the error bounds are smaller for Case I than for Case II, i.e., recalling (3.11) and (3.15), this amounts to the ellipsoidal region $E_B(c)$ being a subset of the ellipsoidal region $E_C(c)$. From the definition (3.4), this will hold if $B \geq C$. To see this, we establish a general result involving symmetric operators S_j, $j = 1, \ldots, s$, on X.

Theorem 4.1. *For* $j = 1, \ldots, s$*, let* $S_j : \mathcal{D}(S_j) \subset X \longrightarrow X$ *be symmetric operators on the Hilbert space* X*. For* $F \in \mathcal{D}(S_j)$*,* $j = 1, \ldots, s$*, suppose that* B *and* C *are the* $s \times s$ *real symmetric matrices defined by*

$$B_{jk} := \operatorname{Re} \langle S_j F, S_k F \rangle, \tag{4.1}$$

$$C_{jk} := \operatorname{Re} \langle S_j F - \widetilde{\mu}_j F,\ S_k F - \widetilde{\mu}_k F \rangle, \tag{4.2}$$

where

$$\widetilde{\mu}_j := \langle S_j F, F \rangle / \|F\|^2, \tag{4.3}$$

for $j, k = 1, \ldots, s$*. Then* $B \geq C$*.*

Proof. For $\mu \in \mathbb{R}^s$, define the real symmetric $s \times s$ matrix $P(\mu)$ by

$$P_{jk}(\mu) := \operatorname{Re} \langle S_j F - \mu_j F, S_k F - \mu_k F \rangle, \quad j, k = 1, \ldots, s.$$

Fix $x \in \mathbb{C}^s$ and define

$$\psi(\mu; x) := \overline{x}^T P(\mu) x, \quad \mu \in \mathbb{R}^s.$$

As in the calculation leading to (3.10), for $\mu \in \mathbb{R}^s$,

$$\psi(\mu; x) = \frac{1}{2} \left\| \sum_{j=1}^{s} \overline{x_j} (S_j F - \mu_j F) \right\|^2 + \frac{1}{2} \left\| \sum_{j=1}^{s} x_j (S_j F - \mu_j F) \right\|^2.$$

Also for $1 \leq j \leq s$, $\mu \in \mathbb{R}^s$,

$$\begin{aligned} D_j \psi(\mu; x) &:= \frac{\partial}{\partial \mu_j} \psi(\mu; x) \\ &= x_j \sum_{k=1}^{s} \overline{x_k} \left(\mu_k \|F\|^2 - \langle S_k F, F \rangle \right) + \overline{x_j} \sum_{k=1}^{s} x_k \left(\mu_k \|F\|^2 - \langle S_k F, F \rangle \right) \\ &= 2 \sum_{k=1}^{s} \operatorname{Re} (x_j \overline{x_k}) \left(\mu_k \|F\|^2 - \langle S_k F, F \rangle \right). \end{aligned}$$

Note that for $w \in \mathbb{R}^s$,

$$\begin{aligned}\sum_{j,k=1}^{s} w_j \operatorname{Re}(x_j \overline{x_k}) w_k &= \frac{1}{2}\sum_{j,k=1}^{s} w_j x_j \overline{x_k} w_k + \frac{1}{2}\sum_{j,k=1}^{s} w_j \overline{x_j} x_k w_k \\ &= \sum_{j=1}^{s} w_j x_j \sum_{k=1}^{s} w_k \overline{x_k} = \left|\sum_{j=1}^{s} w_j x_j\right|^2.\end{aligned}$$

Then for $\mu \in \mathbb{R}^s$, and $\widetilde{\mu} \in \mathbb{R}^s$ defined by (4.3),

$$\begin{aligned}D_{\mu-\widetilde{\mu}}\psi(\mu;x) &:= \sum_{j=1}^{s}(\mu_j - \widetilde{\mu}_j) D_j \psi(\mu;x) \\ &= 2\sum_{j,k=1}^{s}(\mu_j - \widetilde{\mu}_j)\operatorname{Re}(x_j\overline{x_k})\big(\mu_k\|F\|^2 - \langle S_k F, F\rangle\big) \\ &= \frac{2}{\|F\|^2}\sum_{j,k=1}^{s} w_j \operatorname{Re}(x_j\overline{x_k})\, w_k \geq 0,\end{aligned}$$

where $w_j := \mu_j\|F\|^2 - \langle S_j F, F\rangle$, $j = 1,\ldots,s$. Thus for $\mu \in \mathbb{R}^s$, $x \in \mathbb{C}^s$, $\psi(\mu;x) \geq \psi(\widetilde{\mu};x)$, i.e. $\overline{x}^T P(\mu) x \geq \overline{x}^T P(\widetilde{\mu}) x$, and so $P(\mu) \geq P(\widetilde{\mu})$. Putting $\mu = 0$ gives $B \geq C$. ∎

The general result Theorem 4.1 can be readily applied to analyze the error bounds of the approximations in Cases I and II of Section 3.

Corollary 4.1. *For $F \in \mathcal{D}(T_j(t_d))$, $j = 1,\ldots,s$, let B and C be given by* (3.9), (3.14) *and* (3.13)*. Then for any $c > 0$, with $E_B(c)$ and $E_C(c)$ defined as in* (3.4)*,*

$$E_B(c) \subset E_C(c). \tag{4.4}$$

Proof. We write $S_j = iT_j(t_d)$, $j = 1,\ldots,s$, which are symmetric operators by (2.4). Then by (3.9), (3.14) and (3.13), for $j,k = 1,\ldots,s$, B_{jk} and C_{jk} satisfy (4.1)–(4.3) with $\widetilde{\mu}_j = i\lambda_j$. Thus (4.4) is a consequence of Theorem 4.1. ∎

We now take $s = 2$ and consider a bound on the areas of the regions $E_B(c)$ and $E_C(c)$ in (3.11) and (3.15). Recalling (3.5), we see that

$$\operatorname{area} E_M(c) = \frac{c\pi}{\sqrt{\lambda_1\lambda_2}} = \frac{c\pi}{\sqrt{\det M}}. \tag{4.5}$$

To gain a lower bound on $\det C$, we recall the following generalization of Heisenberg's uncertainty principle which appears in [17], extending work in [6]. Take symmetric operators S_1, S_2 on a Hilbert space X and define $[S_1,S_2] := S_1S_2 - S_2S_1$, $[S_1,S_2]_+ := S_1S_2 + S_2S_1$. Then for $F \in \mathcal{D}(S_1S_2) \cap \mathcal{D}(S_2S_1) \subset X$,

$$|\langle [S_1,S_2]F, F\rangle|^2 + |\langle [S_1,S_2]_+F, F\rangle|^2 \leq 4\|S_1F\|^2\|S_2F\|^2. \tag{4.6}$$

Theorem 4.2. *Let S_1 and S_2 be symmetric operators on a Hilbert space X. Then for $F \in \mathcal{D}(S_1S_2) \cap \mathcal{D}(S_2S_1) \subset X$,*

$$\det B \geq \frac{1}{4} |\langle [S_1, S_2]F, F\rangle|^2, \tag{4.7}$$

$$\det C \geq \frac{1}{4} |\langle [S_1, S_2]F, F\rangle|^2, \tag{4.8}$$

where B and C are 2×2 matrices defined by (4.1)–(4.3).

Proof. Now

$$\langle [S_1, S_2]_+ F, F\rangle = \langle S_2F, S_1F\rangle + \langle S_1F, S_2F\rangle = 2\,\mathrm{Re}\,\langle S_1F, S_2F\rangle$$

and so (4.6) can be written as

$$\|S_1F\|^2\|S_2F\|^2 - (\mathrm{Re}\,\langle S_1F, S_2F\rangle)^2 \geq \frac{1}{4} |\langle [S_1, S_2]F, F\rangle|^2.$$

Recalling (4.1), for $s = 2$, we get (4.7). In addition, replacing S_j by $S_j - \widetilde{\mu}_j I$, $j = 1, 2$, and using (4.2) and (4.3) gives (4.8). ■

While (4.7) and (4.8) follow easily from (4.6), they can be viewed as a new type of uncertainty principle which involves the determinants of the matrices B and C. For $s = 2$, the diagonal entries of B are $\|S_jF\|^2$, $j = 1, 2$, and those of C are $\|S_jF\|^2 - \frac{|\langle S_jF,F\rangle|^2}{\|F\|^2}$, $j = 1, 2$. These are exactly the quantities appearing in uncertainty principles of the Heisenberg type, e.g. see [12].

Now applying Corollary 4.1 with $s = 2$, we see from (4.4) and (4.5) that for any $c > 0$,

$$\mathrm{area}\, E_B(c) \leq \mathrm{area}\, E_C(c) \leq 2c\pi |\langle [T_1(t_d), T_2(t_d)]F, F\rangle|^{-1}, \tag{4.9}$$

where the second inequality is given by Theorem 4.2 with $S_j = iT_j(t_d)$, $j = 1, 2$. We illustrate this with Example 2.3, where c is given by (3.12). In this case, it follows from (2.14) that $[T_1(t_d), T_2(t_d)]F = -2\pi i F$, and (4.9) becomes

$$\begin{aligned}\mathrm{area}\, E_B(c) \leq \mathrm{area}\, E_C(c) &\leq c\|F\|^{-2}\\ &= 4\mu \left(\frac{\|F - F_N\|}{\|F\|} + \frac{\|G - G_N\|}{\|F\|} + \frac{\|F - F_N\|\|G - G_N\|}{\|F\|^2} \right),\end{aligned}$$

by (3.12). Unlike the uncertainty principles in [16] that address how sharply peaked the cross ambiguity function (1.2) could be, the inequalities obtained from Theorem 4.2 focus on the interplay of the functions $T_j(t_d)F = U(t_d)^* \frac{\partial}{\partial t_j} U(t_d)F$, $j = 1, 2$, from (2.8), which lead to bounds for the areas of the regions $E_B(c)$ and $E_C(c)$.

The above work suggests extending the uncertainty principles (4.7) and (4.8) to $s \geq 3$. We shall do this for $s = 3$. These uncertainty principles are of a different type from the uncertainty principles in [9] on multiple pairs of symmetric operators.

Theorem 4.3. *Let S_1, S_2 and S_3 be symmetric operators on a Hilbert space X. Then for $F \in \mathcal{D}(S_jS_k) \cap \mathcal{D}(S_kS_j) \subset X$, $1 \leq j,k \leq 3$,*

$$\det B \geq \frac{1}{4}\,\overline{\xi}^T B\xi, \tag{4.10}$$

$$\det C \geq \frac{1}{4}\,\overline{\xi}^T C\xi, \tag{4.11}$$

where B and C are 3×3 matrices defined by (4.1)–(4.3)*, and*

$$\xi := \big(\langle [S_2,S_3]F,F\rangle, \langle [S_3,S_1]F,F\rangle, \langle [S_1,S_2]F,F\rangle\big)^T. \tag{4.12}$$

Proof. Consider the 3×3 matrix B as defined in (4.1) and also the 3×3 matrix Q whose entries are given by

$$Q_{jk} := \langle S_jF, S_kF\rangle, \qquad 1 \leq j,k \leq 3.$$

Recall that B is real symmetric and note that Q is Hermitian. For $x \in \mathbb{C}^3$,

$$\overline{x}^T Bx = \frac{1}{2}\left\| \sum_{j=1}^{3} x_jS_jF\right\|^2 + \frac{1}{2}\left\| \sum_{j=1}^{3} \overline{x_j}S_jF\right\|^2, \qquad \overline{x}^T Qx = \left\| \sum_{j=1}^{3} \overline{x_j}S_jF\right\|^2,$$

and so $B \geq 0$, $Q \geq 0$. Now

$$\det B = B_{11}B_{22}B_{33} - B_{11}B_{23}^2 - B_{22}B_{31}^2 - B_{33}B_{12}^2 + 2B_{12}B_{23}B_{31},$$

$$\det Q = Q_{11}Q_{22}Q_{33} - Q_{11}|Q_{23}|^2 - Q_{22}|Q_{31}|^2 - Q_{33}|Q_{12}|^2 + 2\,\mathrm{Re}\,(Q_{12}Q_{23}Q_{31}),$$

and so, putting $w_1 = \mathrm{Im}\,Q_{23}$, $w_2 = \mathrm{Im}\,Q_{31}$, $w_3 = \mathrm{Im}\,Q_{12}$,

$$\begin{aligned}\det B &= \det Q + B_{11}w_1^2 + B_{22}w_2^2 + B_{33}w_3^2 + 2\,(B_{12}w_1w_2 + B_{23}w_2w_3 + B_{31}w_3w_1)\\ &= \det Q + w^TBw = \det Q + \left\| \sum_{j=1}^{3} w_jS_jF\right\|^2.\end{aligned}$$

Since $Q \geq 0$, we have $\det Q \geq 0$. Then recalling (4.1) gives the uncertainty principle

$$\begin{aligned}\det B &\geq \|\mathrm{Im}\,\langle S_2F,S_3F\rangle\, S_1F + \mathrm{Im}\,\langle S_3F,S_1F\rangle\, S_2F + \mathrm{Im}\,\langle S_1F,S_2F\rangle\, S_3F\|^2\\ &= \frac{1}{4}\,\|\langle [S_2,S_3]F,F\rangle\, S_1F + \langle [S_3,S_1]F,F\rangle\, S_2F + \langle [S_1,S_2]F,F\rangle\, S_3F\|^2,\end{aligned}$$

which we can also write as (4.10) with ξ as in (4.12). Furthermore, replacing S_j by $S_j - \widetilde{\mu}_j I$, $j = 1, 2, 3$, with $\widetilde{\mu}_j$ as in (4.3), and recalling (4.2) gives (4.11). ■

Unlike the case $s = 2$, the right-hand sides of (4.10) and (4.11) may be different, though since $B \geq C$,

$$\overline{\xi}^T B\xi \geq \overline{\xi}^T C\xi.$$

We may employ Theorem 4.3 to obtain an inequality analogous to (4.9) for the case $s = 3$ in terms of the volumes of the regions $E_B(c)$ and $E_C(c)$ in (3.11) and (3.15). This is possible because by (3.5),

$$\text{volume}\, E_M(c) = \frac{4c^{3/2}\pi}{3\sqrt{\lambda_1\lambda_2\lambda_3}} = \frac{4c^{3/2}\pi}{3\sqrt{\det M}}.$$

Then taking $S_j = iT_j(t_d)$, $j = 1, 2, 3$, as before and using (4.4) yields such an inequality.

It would be interesting to extend Theorem 4.3 to $s \geq 4$, but we have as yet been unable to do it.

5. Interpolation

Recall from (3.1) that we are approximating functions F, G in a Hilbert space X by sequences (F_N), (G_N) in X. In this section we assume, as in the examples at the end of Section 2, that $X \subset L^2(\mathbb{R})$. We assume that the functions F_N, G_N depend on F, G respectively only on the values of F, G at a finite number of uniformly spaced points. We shall consider two situations commonly encountered in practice: band-limited functions and compactly supported functions.

5.1. **Band-limited functions.** Take $\Omega > 0$ and let X denote all functions $f \in L^2(\mathbb{R})$ for which the Fourier transform $\widehat{f}$ has support in $[-\Omega, \Omega]$, where formally

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\, dt, \quad \omega \in \mathbb{R}.$$

We choose $\Delta > 0$ with $2\Delta\Omega \leq 1$. Then Shannon's sampling theorem states that for $f \in X$,

$$f = 2\Delta\Omega \sum_{n=-\infty}^{\infty} f(n\Delta)u(\cdot - n\Delta), \tag{5.1}$$

where u denotes the sinc function

$$u(\tau) := \frac{\sin(2\pi\Omega\tau)}{2\pi\Omega\tau}, \quad \tau \in \mathbb{R}.$$

The convergence in (5.1) is in $L^2(\mathbb{R})$, indeed

$$\|f\|^2 = \Delta \sum_{n=-\infty}^{\infty} |f(n\Delta)|^2$$

and

$$\Big\|2\Delta\Omega\sum_{|n|>N} f(n\Delta)u(\cdot - n\Delta)\Big\|^2 \le \Delta \sum_{|n|>N} |f(n\Delta)|^2.$$

The truncation error in the $L^2(\mathbb{R})$-norm, which we require here, is estimated simply by applying Parseval's identity followed by a direct calculation, while more involved pointwise estimates are extensively studied in the literature (see for instance [15]).

Thus for $F \in X$, $N = 1, 2, \ldots$, we define

$$F_N := 2\Delta\Omega \sum_{n=-N}^{N} F(n\Delta)u(\cdot - n\Delta), \tag{5.2}$$

and we have the error estimate

$$\|F - F_N\|^2 \le \Delta \sum_{|n|>N} |F(n\Delta)|^2.$$

Making the corresponding definition for G_N, (3.12) gives

$$\begin{aligned} c = 4\mu\Delta\Bigg\{ &\Big(\sum_{|n|>N} |F(n\Delta)|^2\Big)^{1/2} \Big(\sum_{n=-\infty}^{\infty} |F(n\Delta)|^2\Big)^{1/2} \\ &+ \Big(\sum_{|n|>N} |G(n\Delta)|^2\Big)^{1/2} \Big(\sum_{n=-\infty}^{\infty} |F(n\Delta)|^2\Big)^{1/2} \\ &+ \Big(\sum_{|n|>N} |F(n\Delta)|^2\Big)^{1/2} \Big(\sum_{|n|>N} |G(n\Delta)|^2\Big)^{1/2} \Bigg\}. \end{aligned}$$

For Examples 2.3 and 2.4, this approach is used respectively in [10, 11] to give practical algorithms for estimating t_d.

5.2. **Compactly supported functions.** Take $L > 0$ and let X denote all functions with support on $[-L, L]$ whose restrictions to $[-L, L]$ are continuous. We take a continuous function $\phi \in L^2(\mathbb{R})$ for which $\widehat{\phi}$ is essentially bounded and continuous at 0 with $\widehat{\phi}(0) = 1$. We shall also assume that ϕ is bounded and

$$\int_{-\infty}^{0} \left(\sup\{|\phi(y)| : y \le x\}\right)^2 \, dx < \infty, \qquad \int_{0}^{\infty} \left(\sup\{|\phi(y)| : y \ge x\}\right)^2 \, dx < \infty. \tag{5.3}$$

Note that conditions (5.3) are satisfied provided that

$$|\phi(x)| \le c|x|^{-\gamma}, \quad x \in \mathbb{R}\backslash\{0\},$$

for some $c > 0$, $\gamma > 1/2$.

Now for $\sigma > 0$ and an integer $K \ge 1$, define for $f \in X$, $x \in \mathbb{R}$,

$$V(\sigma)f(x) := \int_{-L}^{L} \sigma f(t)\phi(\sigma(x - t))\, dt, \tag{5.4}$$

$$W(\sigma,K)f(x) := \frac{L}{K}\sum_{n=-K}^{K}\sigma f\left(\frac{nL}{K}\right)\phi\left(\sigma\left(x-\frac{nL}{K}\right)\right). \tag{5.5}$$

The convolution integral (5.4) and its discretized version (5.5) are rather standard formulations (see for instance [4]), and their convergence in $L^2(\mathbb{R})$ can be proved easily.

Lemma 5.1. *For $f \in X$, $V(\sigma)f \to f$ as $\sigma \to \infty$ in $L^2(\mathbb{R})$.*

Proof. For $\omega \in \mathbb{R}$,

$$(\widehat{V(\sigma)f})(\omega) = \widehat{f}(\omega)\widehat{\phi}\left(\frac{\omega}{\sigma}\right)$$

and

$$\lim_{\sigma\to\infty}\widehat{\phi}\left(\frac{\omega}{\sigma}\right) = \widehat{\phi}(0) = 1,$$

and thus

$$\lim_{\sigma\to\infty}(\widehat{V(\sigma)f})(\omega) = \widehat{f}(\omega).$$

Since $|(\widehat{V(\sigma)f})(\omega)| \le \|\widehat{\phi}\|_\infty|\widehat{f}(\omega)|$, $\omega \in \mathbb{R}$, we can deduce from the Dominated Convergence Theorem that $(\widehat{V(\sigma)f}) \to \widehat{f}$ in $L^2(\mathbb{R})$ as $\sigma \to \infty$, which gives the result. ■

Lemma 5.2. *For $\sigma > 0$, $f \in X$, $W(\sigma,K)f \to V(\sigma)f$ as $K \to \infty$ in $L^2(\mathbb{R})$.*

Proof. Since ϕ is continuous and f is continuous on $[-L, L]$,

$$\lim_{K\to\infty} W(\sigma,K)f(x) = V(\sigma)f(x), \quad x \in \mathbb{R}.$$

Now for $x \in \mathbb{R}$,

$$\max\{|W(\sigma,K)f(x)|, |V(\sigma)f(x)|\} \le 3L\sigma\|f\|_\infty\Phi(x),$$

where

$$\Phi(x) := \sup\{|\phi(y)| : \sigma(x-L) \le y \le \sigma(x+L)\}.$$

We shall show that $\Phi \in L^2(\mathbb{R})$ and then the result follows from the Dominated Convergence Theorem. Now

$$\begin{aligned}\int_{-\infty}^{\infty}\Phi^2 &= \int_{-\infty}^{-L}\Phi^2 + \int_{-L}^{L}\Phi^2 + \int_{L}^{\infty}\Phi^2 \\ &\le \int_{-\infty}^{0}\Phi(x-L)^2\,dx + 2L\|\phi\|_\infty^2 + \int_0^\infty \Phi(x+L)^2\,dx.\end{aligned}$$

By (5.3),

$$\begin{aligned}\infty > \int_0^\infty (\sup\{|\phi(y)| : y \ge x\})^2\,dx &= \sigma\int_0^\infty (\sup\{|\phi(y)| : y \ge \sigma x\})^2\,dx \\ \ge \sigma\int_0^\infty (\sup\{|\phi(y)| : \sigma x \le y \le \sigma(x+2L)\})^2\,dx &= \sigma\int_0^\infty \Phi(x+L)^2\,dx.\end{aligned}$$

So $\int_0^\infty \Phi(x+L)^2\,dx < \infty$ and similarly $\int_{-\infty}^0 \Phi(x-L)^2\,dx < \infty$. Therefore $\Phi \in L^2(\mathbb{R})$ and the proof is complete. ■

Take $F \in X$ and $\epsilon > 0$. By Lemma 5.1, we may choose $\sigma > 0$ with $\|V(\sigma)F - F\| < \epsilon/2$, and by Lemma 5.2 we can select $K \geq 1$ with $\|W(\sigma,K)F - V(\sigma)F\| < \epsilon/2$, and so $\|W(\sigma,K)F - F\| < \epsilon$. Thus we may choose $\sigma_N > 0$, $K_N \geq 1$, $N = 1, 2, \ldots$, such that with

$$F_N := W(\sigma_N, K_N)F, \quad N = 1, 2, \ldots, \tag{5.6}$$

$$\lim_{N\to\infty} \|F_N - F\| = 0, \tag{5.7}$$

and similarly for $G \in X$.

Remark 5.1. As an example we could take

$$\phi(\tau) = \frac{\sin(2\pi\tau)}{\pi\tau}, \quad \tau \in \mathbb{R}.$$

In this case, $W(\sigma,K)F = F_K$, where F_K is given by (5.2) with $N = K$, $\Omega = \sigma$ and $\Delta = L/K$. Another simple choice is to take ϕ as the Gaussian,

$$\phi(\tau) = e^{-\pi\tau^2}, \quad \tau \in \mathbb{R}.$$

Remark 5.2. For smooth f and ϕ, a better approximation in Lemma 5.2 could be gained by replacing the terms $f(-L)$ and $f(L)$ in (5.5), for $n = -K$ and K, by $\frac{1}{2}f(-L)$ and $\frac{1}{2}f(L)$.

Remark 5.3. In practice we may wish to consider a function F which is not compactly supported. Assuming F and G are continuous on $[-L, L]$, we may apply the above approximation procedure to their truncations $\Lambda_L F$ and $\Lambda_L G$, where for a function f on $\mathbb{R}$ we define $\Lambda_L f := f\chi_{[-L,L]}$.

We now make Remark 5.3 precise. Suppose $G = U(t_d)F$, as before. Since $\Lambda_L G = U(t_d)U(t_d)^*\Lambda_L G$, the function $A(U(t_d)^*\Lambda_L G, \Lambda_L G)$ attains its maximum modulus (or real part) at t_d. We now calculate the point t_d^N at which $A((\Lambda_L F)_N, (\Lambda_L G)_N)$ attains its maximum modulus (or real part), where $(\Lambda_L F)_N$, $(\Lambda_L G)_N$ are defined as in (5.6). Then the error $t_d^N - t_d$ can be estimated, as in Section 3, in terms of $\|\Lambda_L G - (\Lambda_L G)_N\|$ and

$$\|U(t_d)^*\Lambda_L G - (\Lambda_L F)_N\| \leq \|U(t_d)^*\Lambda_L G - \Lambda_L F\| + \|\Lambda_L F - (\Lambda_L F)_N\|.$$

On recalling (5.7), it remains to consider $\|U(t_d)^*\Lambda_L G - \Lambda_L F\|$.

We shall study this for the case of Example 2.3. Here $S = \mathbb{R}^2$ and $t_d^N, t_d \in S$ in the general theory are written explicitly as $(t_d^N, \omega_d^N), (t_d, \omega_d) \in \mathbb{R}^2$. Since

$$G(\tau) = e^{-2\pi i\omega_d\tau}F(\tau + t_d), \quad \tau \in \mathbb{R},$$

it follows that

$$U(t_d,\omega_d)^*\Lambda_L G(\tau) = \begin{cases} F(\tau), & -L+t_d \le \tau \le L+t_d, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\|U(t_d,\omega_d)^*\Lambda_L G - \Lambda_L F\|^2 = \left|\int_{-L}^{-L+t_d} |F|^2\right| + \left|\int_{L}^{L+t_d} |F|^2\right|.$$

In practice, $|t_d|$ is small relative to L and computation using the above method, with ϕ as one of the choices in Remark 5.1, gives a good approximation (t_d^N, ω_d^N) to (t_d, ω_d).

Appendix

We now show that (3.2) is satisfied for $h = \operatorname{Re} A(F,G)$ or $|A(F,G)|$, where $A(F,G)$ is as in the examples in Section 2, F is continuous and, as before, $G = U(t_d)F$. It is sufficient to show this for Example 2.1. It is easily seen that $A(F,G)$ is continuous and so we need only establish the following result.

Lemma A1. *For Example* 2.1, $F \in L^2(\mathbb{R})\backslash\{0\}$ *and* $G = U(t_d,\omega_d,\alpha_d)F$,

(a) $|A(F,G)|$ *attains its maximum value only at* (t_d,ω_d,α_d),

(b) $A(F,G)(t,\omega,\alpha) \to 0$ *as* $|(t,\omega)| \to \infty$ *and* $\alpha \to 0^+$ *or* $\alpha \to \infty$.

Proof. (a) From (2.2), $|A(F,G)|$ attains its maximum value at (t,ω,α) if and only if $G = cU(t,\omega,\alpha)F$ for some nonzero constant c, i.e.

$$\frac{1}{\sqrt{\alpha_d}} e^{-2\pi i\omega_d\tau} F\left(\frac{\tau+t_d}{\alpha_d}\right) = \frac{c}{\sqrt{\alpha}} e^{-2\pi i\omega\tau} F\left(\frac{\tau+t}{\alpha}\right), \quad \tau \in \mathbb{R},$$

i.e.

$$F(\tau) = \lambda e^{-2\pi i\gamma\tau} F(\beta\tau+\delta), \tag{A1}$$

where

$$\lambda := c\sqrt{\frac{\alpha_d}{\alpha}} e^{2\pi i t_d(\omega-\omega_d)}, \qquad \beta := \frac{\alpha_d}{\alpha}, \qquad \gamma := \alpha_d(\omega-\omega_d), \qquad \delta := \frac{t-t_d}{\alpha}. \tag{A2}$$

Suppose $\beta \neq 1$ and define

$$H(\tau) := \left|F\left(\tau + \frac{\delta}{1-\beta}\right)\right|, \quad \tau \in \mathbb{R}.$$

Then from (A1), for $\tau \in \mathbb{R}$,

$$H(\tau) = |\lambda| \left|F\left(\beta\left(\tau+\frac{\delta}{1-\beta}\right)+\delta\right)\right| = |\lambda|\left|F\left(\beta\tau + \frac{\delta}{1-\beta}\right)\right| = |\lambda| H(\beta\tau). \tag{A3}$$

Since $H \in L^2(\mathbb{R})\backslash\{0\}$,

$$0 \neq \int_{-\infty}^{\infty} H(\tau)^2\,d\tau = |\lambda|^2 \int_{-\infty}^{\infty} H(\beta\tau)^2\,d\tau = \frac{|\lambda|^2}{\beta}\int_{-\infty}^{\infty} H(\tau)^2\,d\tau,$$

and so $|\lambda| = \beta^{1/2}$. Then by (A3), for any integer n,

$$\int_{\beta^n}^{\beta^{n+1}} H(\tau)^2 \, d\tau = \int_{\beta^n}^{\beta^{n+1}} \beta H(\beta\tau)^2 \, d\tau = \int_{\beta^{n+1}}^{\beta^{n+2}} H(\tau)^2 \, d\tau$$

and so

$$\int_{\beta^{-n}}^{\beta^n} H(\tau)^2 \, d\tau = 2n \int_1^{\beta} H(\tau)^2 \, d\tau.$$

Similarly,

$$\int_{-\beta^{-n}}^{-\beta^n} H(\tau)^2 \, d\tau = 2n \int_{-1}^{-\beta} H(\tau)^2 \, d\tau.$$

Thus if $\int_1^{\beta} H(\tau)^2 \, d\tau = \int_{-1}^{-\beta} H(\tau)^2 \, d\tau = 0$, then $\int_{-\infty}^{\infty} H(\tau)^2 \, d\tau = 0$. Otherwise $\int_{-\infty}^{\infty} H(\tau)^2 \, d\tau = \infty$. So in either case we have a contradiction and so $\beta = 1$, which from (A2) gives $\alpha = \alpha_d$.

Therefore from (A1),

$$|F(\tau)| = |\lambda||F(\tau + \delta)|, \quad \tau \in \mathbb{R}, \tag{A4}$$

which implies that

$$0 \neq \int_{-\infty}^{\infty} |F(\tau)|^2 \, d\tau = |\lambda|^2 \int_{-\infty}^{\infty} |F(\tau + \delta)|^2 \, d\tau = |\lambda|^2 \int_{-\infty}^{\infty} |F(\tau)|^2 \, d\tau,$$

and so $|\lambda| = 1$. In this case, (A4) would contradict $F \in L^2(\mathbb{R})$ unless $\delta = 0$, which from (A2) gives $t = t_d$. Thus from (A1), $e^{-2\pi i\gamma\tau} = \lambda^{-1}$ for τ on a set of positive measure, and so $\gamma = 0$. Consequently, (A2) yields $\omega = \omega_d$. Hence $(t, \omega, \alpha) = (t_d, \omega_d, \alpha_d)$.

(b) For $K > 0$, $t, \omega \in \mathbb{R}$, $\alpha > 0$,

$$\begin{aligned} &|A(F,G)(t,\omega,\alpha)| \\ &\leq \left| \int_{-K}^{K} F(\tau) e^{-2\pi i\omega\alpha\tau} \sqrt{\alpha}\, \overline{G(\alpha\tau - t)} \, d\tau \right| + \left| \int_{|\tau|>K} F(\tau) e^{-2\pi i\omega\alpha\tau} \sqrt{\alpha}\, \overline{G(\alpha\tau - t)} \, d\tau \right| \\ &\leq \|F\| \left\{ \int_{-K}^{K} \alpha |G(\alpha\tau - t)|^2 \, d\tau \right\}^{1/2} + \|G\| \left\{ \int_{|\tau|>K} |F(\tau)|^2 \, d\tau \right\}^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. So for $\epsilon > 0$, we can choose K so that for $t, \omega \in \mathbb{R}$, $\alpha > 0$,

$$|A(F,G)(t,\omega,\alpha)| < \|F\| \left\{ \int_{-\alpha K - t}^{\alpha K - t} |G(\tau)|^2 \, d\tau \right\}^{1/2} + \epsilon. \tag{A5}$$

Since $\int_{a-\eta}^{a+\eta} |G(\tau)|^2 \, d\tau \to 0$ as $\eta \to 0^+$ uniformly over $a \in \mathbb{R}$, we see that

$$\lim_{\alpha \to 0^+} A(F,G)(t,\omega,\alpha) = 0 \tag{A6}$$

uniformly over $t, \omega \in \mathbb{R}$. Also for any $L > 0$, $0 < \alpha \leq L$, (A5) gives

$$|A(F,G)(t,\omega,\alpha)| < \|F\| \left\{ \int_{-LK-t}^{LK-t} |G(\tau)|^2 \, d\tau \right\}^{1/2} + \epsilon,$$

and so

$$\lim_{|t|\to\infty} A(F,G)(t,\omega,\alpha) = 0 \tag{A7}$$

uniformly over $\omega \in \mathbb{R}$, $0 < \alpha \le L$.

Now it is easily seen that

$$|A(F,G)(t,\omega,\alpha)| = \left| A(G,F)\left(-\frac{t}{\alpha}, -\omega\alpha, \frac{1}{\alpha}\right)\right| = \left| A(\widehat{F},\widehat{G})\left(\omega, -t, \frac{1}{\alpha}\right)\right|.$$

Then from (A6) we have

$$\lim_{\alpha\to\infty} A(F,G)(t,\omega,\alpha) = 0 \tag{A8}$$

uniformly over $t, \omega \in \mathbb{R}$, and from (A7) we have

$$\lim_{|\omega|\to\infty} A(F,G)(t,\omega,\alpha) = 0 \tag{A9}$$

uniformly over $t \in \mathbb{R}$, $\alpha \ge L^{-1}$. The result then follows from (A6)–(A9). ∎

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