CAUSALITY PROPERTIES OF REFINABLE FUNCTIONS AND SEQUENCES

SAY SONG GOH*, TIM N. T. GOODMAN† AND S. L. LEE*

In commemoration of the sixtieth birthday of Mariano Gasca

Abstract.
We show that the scale-space operators defined by a class of refinable kernels satisfy a version of the causality property, and a sequence of such operators converges to the corresponding operator with the Gaussian kernel, if the sequence of refinable kernels converges to the Gaussian function. In addition, we consider discrete analogs of these operators and show that a class of refinable sequences satisfies a discrete version of the causality property. The solutions of the corresponding discrete refinement equations are also investigated in detail.

1. Introduction

The Gaussian kernel has been used as a filter for a long time for scale-space analysis because of its optimality in time-frequency localization as well as its causality property. It is known that the class of refinable functions, which includes the uniform B-splines, whose symbols have all roots in the left half-plane, approximates the Gaussian [3]. It is also known that these refinable functions enjoy, at least approximately, many properties of the Gaussian kernel, including the asymptotic optimality in time-frequency localization [6]. However the causality property of the Gaussian kernel as defined in (1.4) below is not shared by other functions, since the Gaussian is the unique kernel that satisfies the causal condition (1.4). Our main object is to show that the scale-space operators defined by this class of refinable kernels satisfy a version of the causality property, and a sequence of such operators converges to the corresponding operator with Gaussian kernel, if the sequence of refinable kernels converges to the Gaussian function.

Mathematically, causality means the diminishing of variation under scaling, as we proceed to describe. For any sequence s (finite or infinite) of real numbers, we denote by $V(s)$ the number of strict sign changes in s. Similarly for any continuous real-valued function f on an interval I, we denote by $V(f)$ the number of strict sign changes in f, i.e.

$$V(f) = \sup V\left(f(x_1), \ldots, f(x_N)\right),$$

*Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Republic of Singapore. †Department of Mathematics, The University of Dundee, Dundee DD1 4HN, Scotland, United Kingdom.

2000 *Mathematics Subject Classification.* 94A12, 47B38, 41A25, 47B37.

Key words and phrases. Variation-diminishing, causality property, refinable functions, refinable sequences.

where the supremum is taken over all N and all $x_1 < \ldots < x_N$ in I.

For a function K in $L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} K = 1$, we define the operator $T : L^\infty(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ by

$$Tf(x) := \int_{-\infty}^{\infty} f(t)K(x-t)\,dt, \quad x \in \mathbb{R}. \tag{1.1}$$

This operator is called *variation-diminishing* if

$$V(Tf) \le V(f), \tag{1.2}$$

when f is continuous. This implies that, in some senses, the graph of Tf does not oscillate more than that of f. In [10] Schoenberg characterizes all kernels K for which T is variation-diminishing. In particular this holds when K is the Gaussian function

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Now for $j \in \mathbb{Z}$, define the operator

$$T_j f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, K\left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R}. \tag{1.3}$$

Then we say the kernel K has the *causality property* if

$$V(T_{j+1}f) \le V(T_j f), \quad j \in \mathbb{Z}, \tag{1.4}$$

when f is continuous. Now for any f in $C(\mathbb{R})$,

$$\lim_{j \to -\infty} T_j f(x) = f(x), \quad x \in \mathbb{R},$$

and hence the causality property implies that the operator $T = T_0$ is variation-diminishing. Furthermore, it is shown in [2], [13] that K has the causality property if and only if it is a shift and dilation of the Gaussian function, i.e. $K(x) = \lambda\, G(\lambda x - a)$, $x \in \mathbb{R}$, for some $\lambda > 0$, $a \in \mathbb{R}$. In [12], it is noted that a discrete sense of the causality property holds for discrete B-splines. Physically, a causal kernel does not introduce new features into the transformed images as the scale increases, which is an important property in scale-space analysis of images and signals. Mathematically, a causal kernel is variation-diminishing in the sense of (1.4). Since the Gaussian kernel is the only kernel that is causal in this sense, it has been the traditional favorite in scale-space analysis for a long time. It is known that uniform B-splines, standardized with mean 0 and variance 1, approximate the Gaussian ([4], [11]), and recent work in [12] has shown that B-spline kernels can play the same role as the Gaussian in scale-space analysis with an added advantage that the filtering can be implemented with a fast algorithm. Although B-spline kernels are not causal in the sense of (1.4), computationally its scale-space does exhibit the property of causality. The object of this paper is to show that indeed a class of refinable functions, which includes the uniform B-splines, is causal in a weaker sense.

Towards this end, let K be a continuous function with compact support such that for any $N \ge 1$, $t_1 < \ldots < t_N$, and integers $k_1 < \ldots < k_N$,

$$\det\left(K(k_i - t_\ell)\right)_{i,\ell=1}^{N} \ge 0.$$

Such a kernel is called a *ripplet* in [7]. It follows that for any $t_1 < \ldots < t_N$ and continuous function f,

$$V\left(\sum_{i=1}^{N} f(t_i)K(k-t_i)\right)_{k\in\mathbb{Z}} \leq V(f(k))_{k\in\mathbb{Z}}.$$

By considering Riemann sums, we see that in (1.1), for any $a \in \mathbb{R}$,

$$V(Tf(k+a))_{k\in\mathbb{Z}} \leq V(f),$$

which is a weaker form of the variation-diminishing property (1.2). It is shown in [7] that a continuous function ϕ with compact support is a ripplet if it satisfies a refinement equation whose symbol has all zeros in the open left half-plane, in particular when ϕ is a uniform B-spline. This is explained further in Section 2, where we define, for $j \in \mathbb{Z}$, the operators

$$S_j f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, \phi\left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R},$$

and show that they satisfy the following version of the causality property (1.4): for any $a \in \mathbb{R}$, $j \in \mathbb{Z}$,

$$V\left(S_{j+1} f(2^{j+1}k + a)\right)_{k\in\mathbb{Z}} \leq V\left(S_j f(2^j k + a)\right)_{k\in\mathbb{Z}}.$$

If we take a sequence $\widetilde{\phi}_n$ of such refinable functions, standardized with mean 0 and variance 1, then it is shown in [3] that $\widetilde{\phi}_n$ converges to G in certain senses as $n \to \infty$. In the rest of Section 2 we study the convergence as $n \to \infty$ of the operators

$$\widetilde{S}_{j,n} f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, \widetilde{\phi}_n\left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R},$$

for $j \in \mathbb{Z}$, to the operators

$$T_j f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, G\left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R}.$$

These results explain and support computational observations that B-splines can play a more effective role than the Gaussian kernel in scale-space analysis. In [3] it is also shown that a class of refinable functions converges faster than the B-splines to the Gaussian, and we expect this class to do better in scale-space analysis.

In practice, one is dealing with digital sets of data. Therefore, it is sometimes useful to understand the discrete version of the scale-space filtering. In Section 3 we consider the discrete version of the operators (1.3):

$$(\tau_j s)_i := \sum_{\ell=-\infty}^{\infty} s_\ell\, \kappa(i - 2^{-j}\ell), \quad i \in \mathbb{Z},$$

where $j \in \mathbb{Z}$, $j \leq 0$, and κ is a non-negative sequence with finite support. We show that they satisfy a discrete version of the causality property (1.4), i.e.

$$V(\tau_{j+1} s) \leq V(\tau_j s), \quad j \in \mathbb{Z},\ j \leq -1,$$

for any sequence s, provided that κ satisfies a discrete refinement equation whose symbol has all its zeros in the left half-plane and $\sum_{i=-\infty}^{\infty} \kappa(i) = 1$. The remainder of Section 3 is devoted to studying solutions of such refinement equations. These solutions provide filters with the causality property, and it is possible to construct filters with additional properties that are useful in scale-space analysis.

2. Refinable functions

For $n \geq 2$, let

$$P_n(z) = \sum_{\ell=0}^{n} a_{n,\ell} z^\ell$$

be a polynomial with real coefficients, $a_{n,0}, a_{n,n} > 0$, with all its roots in the left half-plane $\{z : \operatorname{Re} z \leq 0\}$ and at least two of them in $\{z : \operatorname{Re} z < 0\}$, and satisfying

$$P_n(1) = 1, \qquad P_n(-1) = 0.$$

It is shown in [7] that the *refinement equation*

$$\phi_n(x) = \sum_{\ell=0}^{n} 2a_{n,\ell}\phi_n(2x-\ell), \quad x \in \mathbb{R}, \tag{2.1}$$

has a unique solution satisfying $\int_{-\infty}^{\infty} \phi_n = 1$, and that ϕ_n is continuous, non-negative and has support in $[0, n]$. We shall refer to ϕ_n as a *refinable function* with *symbol* P_n. In the special case $P_n(z) = 2^{-n}(z+1)^n$, ϕ_n is the uniform B-spline B_n of degree $n-1$ with knots $0, \ldots, n$.

We now show that the refinable function ϕ_n satisfies a version of the causality property. For $j \in \mathbb{Z}$, define the operator on $L^\infty(\mathbb{R})$:

$$S_j f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, \phi_n\left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R}. \tag{2.2}$$

Theorem 2.1. *For any $a \in \mathbb{R}$, $j \in \mathbb{Z}$, $f \in C(\mathbb{R})$,*

$$V\left(S_{j+1}f(2^{j+1}k+a)\right)_{k\in\mathbb{Z}} \leq V\left(S_j f(2^j k+a)\right)_{k\in\mathbb{Z}}, \tag{2.3}$$

while if P_n has real negative roots,

$$V\left(S_{j+1}f(2^{j}k+a)\right)_{k\in\mathbb{Z}} \leq V\left(S_j f(2^j k+a)\right)_{k\in\mathbb{Z}}. \tag{2.4}$$

Proof: For $x \in \mathbb{R}$, by (2.1) and (2.2),

$$\begin{aligned} S_{j+1}f(x) &= 2^{-j-1} \int_{-\infty}^{\infty} f(t) \sum_{\ell=0}^{n} 2a_{n,\ell}\, \phi_n\left(\frac{x-t}{2^j} - \ell\right) dt \\ &= \sum_{\ell=0}^{n} a_{n,\ell} 2^{-j} \int_{-\infty}^{\infty} f(t)\, \phi_n\left(\frac{x-t-2^j\ell}{2^j}\right) dt \\ &= \sum_{\ell=0}^{n} a_{n,\ell} S_j f(x - 2^j \ell). \end{aligned} \tag{2.5}$$

So for $k \in \mathbb{Z}$,

$$S_{j+1}f(2^{j+1}k+a) = \sum_{\ell=0}^{n} a_{n,\ell} S_j f(2^j(2k-\ell)+a) = \sum_{i=-\infty}^{\infty} a_{n,2k-i} S_j f(2^j i + a), \tag{2.6}$$

where we define $a_{n,\ell} := 0$ for $\ell < 0$ and for $\ell > n$. Now it is shown in [9] that the matrix $(a_{n,2k-i})_{k,i\in\mathbb{Z}}$ is totally positive and thus from the variation-diminishing property of totally positive matrices [8], (2.6) implies (2.3).

Also from (2.5), for $k \in \mathbb{Z}$,

$$S_{j+1}f(2^j k + a) = \sum_{\ell=0}^{n} a_{n,\ell} S_j f(2^j(k-\ell) + a) = \sum_{i=-\infty}^{\infty} a_{n,k-i} S_j f(2^j i + a). \tag{2.7}$$

It is shown in [1] that if P_n has real negative roots, then the matrix $(a_{n,k-i})_{k,i\in\mathbb{Z}}$ is totally positive and so from the variation-diminishing property, (2.7) implies (2.4). ∎

Remark 2.1. For any f in $C(\mathbb{R})$, $\lim_{j\to-\infty} S_j f(x) = f(x)$, $x \in \mathbb{R}$, and so

$$\lim_{j\to-\infty} V\left(S_j f(2^j k + a)\right)_{k\in\mathbb{Z}} = V(f).$$

Then it follows from (2.3) that for $j \in \mathbb{Z}$,

$$V\left(S_j f(2^j k + a)\right)_{k\in\mathbb{Z}} \le V(f).$$

Remark 2.2. If P_n has real negative roots, then since $(2^{j+1}k+a)_{k\in\mathbb{Z}}$ is a subsequence of $(2^j k+a)_{k\in\mathbb{Z}}$,

$$V\left(S_{j+1}f(2^{j+1}k + a)\right)_{k\in\mathbb{Z}} \le V\left(S_{j+1}f(2^j k + a)\right)_{k\in\mathbb{Z}} \le V\left(S_j f(2^j k + a)\right)_{k\in\mathbb{Z}}.$$

Thus in this case, (2.3) also follows from (2.4).

Now suppose that the roots of P_n are $-r_{n,\ell}$, $\ell = 1,\dots,n$. It is shown in [3] that if we assume the stronger condition that for some β in $[0, \frac{\pi}{2})$,

$$|\arg r_{n,\ell}| \le \beta, \quad \ell = 1,\dots,n,\ n = 2,3,\dots, \tag{2.8}$$

and furthermore

$$\sigma_n^2 := \frac{1}{3}\sum_{\ell=1}^{n} \frac{r_{n,\ell}}{(1+r_{n,\ell})^2} \to \infty \qquad \text{as } n \to \infty, \tag{2.9}$$

then a suitable shift and scaling of ϕ_n converges to the Gaussian function G. To be precise, let

$$\mu_n := \sum_{\ell=1}^{n} \frac{1}{1+r_{n,\ell}} \tag{2.10}$$

and

$$\widetilde{\phi}_n(x) := \sigma_n \phi_n(\sigma_n x + \mu_n), \quad x \in \mathbb{R}. \tag{2.11}$$

Then $\widetilde{\phi}_n$ has mean 0 and standard deviation 1. Our next result from [3] describes convergence of $\widetilde{\phi}_n$ to G as $n \to \infty$. We denote by $\widehat{f}$ the Fourier transform

$$\widehat{f}(u) := \int_{-\infty}^{\infty} e^{-\mathrm{i}ux} f(x)\,dx,$$

and note that $\widehat{G}(u) = e^{-u^2/2}$, $u \in \mathbb{R}$.

Theorem 2.2 ([3])**.** *If* (2.8), (2.9) *are satisfied, then the following hold as* $n \to \infty$.

(a) *We have*

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_\infty = O(\sigma_n^{-1}), \qquad \|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-1/2}).$$

(b) *If P_n is reciprocal, i.e. $a_{n,0} \neq 0$ and $a_{n,\ell} = a_{n,n-\ell}$, $\ell = 0, \ldots, n$, then*

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_\infty = O(\sigma_n^{-2}), \qquad \|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-1}).$$

(c) *If P_n is reciprocal and*

$$\sigma_n^{-1} \sum_{\ell=1}^{n} r_{n,\ell}(r_{n,\ell}^2 - 4r_{n,\ell} + 1)/(1 + r_{n,\ell})^4 \text{ is bounded,} \tag{2.12}$$

then

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_\infty = O(\sigma_n^{-3}), \qquad \|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-3/2}).$$

Now for $j \in \mathbb{Z}$, $n = 2, 3, \ldots$, define the operators on $L^\infty(\mathbb{R})$:

$$\widetilde{S}_{j,n} f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, \widetilde{\phi}_n \left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R}, \tag{2.13}$$

$$T_j f(x) := 2^{-j} \int_{-\infty}^{\infty} f(t)\, G \left(\frac{x-t}{2^j}\right) dt, \quad x \in \mathbb{R}. \tag{2.14}$$

Note that from (2.11), for $j \in \mathbb{Z}$, $f \in L^\infty(\mathbb{R})$, $x \in \mathbb{R}$,

$$\widetilde{S}_{j,n} f(x) = 2^{-j} \int_{-\infty}^{\infty} f\left(\frac{w}{\sigma_n}\right) \phi_n \left(\frac{\sigma_n x - w}{2^j} + \mu_n\right) dw = S_j h(\sigma_n x + 2^j \mu_n),$$

where S_j is given by (2.2) and $h(x) := f(\frac{x}{\sigma_n})$, $x \in \mathbb{R}$. So from Theorem 2.1 we see that for any $a \in \mathbb{R}$, $j \in \mathbb{Z}$, $f \in C(\mathbb{R})$,

$$V\left(\widetilde{S}_{j+1,n} f\left(\frac{2^{j+1}(k-\mu_n)}{\sigma_n} + a\right)\right)_{k\in\mathbb{Z}} \leq V\left(\widetilde{S}_{j,n} f\left(\frac{2^{j}(k-\mu_n)}{\sigma_n} + a\right)\right)_{k\in\mathbb{Z}},$$

while if P_n has real negative roots,

$$\begin{aligned} V\left(\widetilde{S}_{j+1,n} f\left(\frac{2^{j+1}(k-\mu_n)}{\sigma_n} + a\right)\right)_{k\in\mathbb{Z}} &\leq V\left(\widetilde{S}_{j+1,n} f\left(\frac{2^{j}k - 2^{j+1}\mu_n}{\sigma_n} + a\right)\right)_{k\in\mathbb{Z}} \\ &\leq V\left(\widetilde{S}_{j,n} f\left(\frac{2^{j}(k-\mu_n)}{\sigma_n} + a\right)\right)_{k\in\mathbb{Z}}. \end{aligned}$$

If $f \in C(\mathbb{R})$ and $V(f) < \infty$ then, since $\lim_{n\to\infty} \sigma_n = \infty$, for each integer j,

$$V(\widetilde{S}_{j+1,n} f) \leq V(\widetilde{S}_{j,n} f)$$

for all large enough n (depending on f). So in this sense asymptotically the operator $\widetilde{S}_{j,n}$ shares the causality property of the operator T_j. For the rest of this section we shall examine convergence of $\widetilde{S}_{j,n}$ to T_j as $n \to \infty$.

Theorem 2.3. *There is a constant $C > 0$ such that for any j in $\mathbb{Z}$ and any function f which is the Fourier transform of an integrable distribution g,*

$$\|\widetilde{S}_{j,n}f - T_jf\|_\infty \leq C\|g\|_1\sigma_n^{-1},$$

while if P_n is reciprocal, then

$$\|\widetilde{S}_{j,n}f - T_jf\|_\infty \leq C\|g\|_1\sigma_n^{-2},$$

and if, in addition, (2.12) *holds then*

$$\|\widetilde{S}_{j,n}f - T_jf\|_\infty \leq C\|g\|_1\sigma_n^{-3}.$$

Proof: By (2.13) and (2.14), for any $x \in \mathbb{R}$,

$$\begin{aligned}
\widetilde{S}_{j,n}f(x) - T_jf(x) &= 2^{-j}\int_{-\infty}^{\infty} \widehat{g}(t)\,(\widetilde{\phi}_n - G)\left(\frac{x-t}{2^j}\right)\,dt \\
&= \int_{-\infty}^{\infty} \widehat{g}(x - 2^jw)(\widetilde{\phi}_n - G)(w)\,dw \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\mathrm{i}yx}g(y)e^{\mathrm{i}y2^jw}(\widetilde{\phi}_n - G)(w)\,dy\,dw.
\end{aligned}$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|e^{-\mathrm{i}yx}g(y)e^{\mathrm{i}y2^jw}(\widetilde{\phi}_n - G)(w)\right|\,dy\,dw = \int_{-\infty}^{\infty}|g(y)|\,dy\ \int_{-\infty}^{\infty}\left|(\widetilde{\phi}_n - G)(w)\right|\,dw < \infty.$$

Thus for $x \in \mathbb{R}$,

$$\begin{aligned}
\left|\widetilde{S}_{j,n}f(x) - T_jf(x)\right| &= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\mathrm{i}y2^jw}(\widetilde{\phi}_n - G)(w)e^{-\mathrm{i}yx}g(y)\,dw\,dy\right| \\
&= \left|\int_{-\infty}^{\infty}(\widehat{\widetilde{\phi}}_n - \widehat{G})\left(-2^jy\right)e^{-\mathrm{i}yx}g(y)\,dy\right| \\
&\leq \|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_\infty\|g\|_1.
\end{aligned}$$

The result then follows from Theorem 2.2. ∎

The conditions of Theorem 2.3 hold, for example, if $f \in L^2(\mathbb{R})$ with $\widehat{f} \in L^1(\mathbb{R})$, or if $f(x) = \sum_{\ell=-\infty}^{\infty} a_\ell e^{\mathrm{i}\ell x}$ with $\sum_{\ell=-\infty}^{\infty}|a_\ell| < \infty$. In order to consider convergence of $\widetilde{S}_{j,n}f$ to T_jf for all f in $L^\infty(\mathbb{R})$, we note that for $x \in \mathbb{R}$,

$$\begin{aligned}
\left|\widetilde{S}_{j,n}f(x) - T_jf(x)\right| &= 2^{-j}\left|\int_{-\infty}^{\infty} f(t)\,(\widetilde{\phi}_n - G)\left(\frac{x-t}{2^j}\right)\,dt\right| \\
&\leq \|f\|_\infty\|\widetilde{\phi}_n - G\|_1. \qquad (2.15)
\end{aligned}$$

In order to apply (2.15) we shall use the following.

Lemma 2.1. *Suppose that for a sequence (μ_n) with $\lim_{n\to\infty}\mu_n = \infty$, $\|\widetilde{\phi}_n - G\|_\infty = O(\mu_n^{-1})$ as $n \to \infty$. Then as $n \to \infty$,*

$$\|\widetilde{\phi}_n - G\|_1 = O(\mu_n^{-1}\log\mu_n).$$

Proof: Take $C \geq 2$. Since $\widetilde{\phi}_n \geq 0$ and $\int_{-\infty}^{\infty} \widetilde{\phi}_n = \int_{-\infty}^{\infty} G = 1$, we have

$$\begin{aligned}
\|\widetilde{\phi}_n - G\|_1 &\leq \int_{-C}^{C} |\widetilde{\phi}_n - G| + \int_{|x|\geq C} G + \int_{|x|\geq C} \widetilde{\phi}_n \\
&\leq 2C\|\widetilde{\phi}_n - G\|_\infty + 2\int_{|x|\geq C} G + \left|\int_{|x|\geq C} (\widetilde{\phi}_n - G)\right| \\
&\leq 2C\|\widetilde{\phi}_n - G\|_\infty + 2\sqrt{\frac{2}{\pi}}\int_{C}^{\infty} e^{-x}\,dx + \left|\int_{-C}^{C} (\widetilde{\phi}_n - G)\right| \\
&\leq 4C\|\widetilde{\phi}_n - G\|_\infty + 2\sqrt{\frac{2}{\pi}}e^{-C}.
\end{aligned}$$

Putting $C = \log \mu_n$ yields, for large enough n,

$$\|\widetilde{\phi}_n - G\|_1 \leq 4\log\mu_n\|\widetilde{\phi}_n - G\|_\infty + \sqrt{\frac{2}{\pi}}\frac{2}{\mu_n},$$

which gives the result. ∎

Then Theorem 2.2, Lemma 2.1 and (2.15) give the following.

Theorem 2.4. *There is a constant $C > 0$ such that for any j in $\mathbb{Z}$ and f in $L^\infty(\mathbb{R})$,*

$$\|\widetilde{S}_{j,n}f - T_j f\|_\infty \leq C\|f\|_\infty \sigma_n^{-1/2}\log\sigma_n,$$

while if P_n is reciprocal, then

$$\|\widetilde{S}_{j,n}f - T_j f\|_\infty \leq C\|f\|_\infty \sigma_n^{-1}\log\sigma_n,$$

and if, in addition, (2.12) *holds then*

$$\|\widetilde{S}_{j,n}f - T_j f\|_\infty \leq C\|f\|_\infty \sigma_n^{-3/2}\log\sigma_n.$$

We can improve the order of convergence in Theorem 2.4 if we impose stronger restrictions on the polynomial P_n. We shall further assume P_n is reciprocal, that its roots $-r_{n,\ell}$, $\ell = 1, \ldots, n$, are real and negative, and $P_n(-1) = P_n'(-1) = 0$. We write

$$\lambda_{n,\ell} := \frac{1}{2}(r_{n,\ell} + r_{n,\ell}^{-1})$$

and denote it by λ_ℓ when this is unambiguous. Then proceeding as in [6],

$$P_n(e^{-\mathrm{i}u}) = e^{-\mathrm{i}nu/2}\prod_{\ell=1}^{n}\left(\frac{\lambda_\ell + \cos u}{\lambda_\ell + 1}\right)^{1/2}, \quad u \in \mathbb{R}, \tag{2.16}$$

where $\lambda_\ell \geq 1$, $\ell = 1, \ldots, n$. Taking Fourier transforms of the refinement equation (2.1) gives

$$\widehat{\phi}_n(u) = P_n(e^{-\mathrm{i}u/2})\widehat{\phi}_n\left(\frac{u}{2}\right), \quad u \in \mathbb{R},$$

and it follows that

$$\widehat{\phi}_n(u) = \prod_{j=1}^{\infty} P_n(e^{-\mathrm{i}u/2^j}), \quad u \in \mathbb{R},$$

where the infinite product converges locally uniformly. So from (2.16),

$$\widehat{\phi}_n(u) = e^{-\mathrm{i}nu/2} \prod_{j=1}^{\infty} \prod_{\ell=1}^{n} \left(\frac{\lambda_\ell + \cos(u/2^j)}{\lambda_\ell + 1} \right)^{1/2}, \quad u \in \mathbb{R}. \tag{2.17}$$

From (2.9) and (2.10),

$$\sigma_n^2 = \frac{1}{3} \sum_{\ell=1}^{n} \frac{r_{n,\ell}}{(1+r_{n,\ell})^2} = \frac{1}{3} \sum_{\ell=1}^{n} \frac{1}{r_{n,\ell}^{-1} + 2 + r_{n,\ell}} = \frac{1}{6} \sum_{\ell=1}^{n} \frac{1}{1+\lambda_\ell}, \tag{2.18}$$

$$\mu_n = \sum_{\ell=1}^{n} \frac{1}{1+r_{n,\ell}} = \frac{n}{2}. \tag{2.19}$$

Then from (2.11), (2.17) and (2.19),

$$\widehat{\widetilde{\phi}}_n(u) = \prod_{j=1}^{\infty} \prod_{\ell=1}^{n} \left(\frac{\lambda_\ell + \cos(\frac{u}{2^j \sigma_n})}{\lambda_\ell + 1} \right)^{1/2} = e^{\mathrm{i}nu/(2\sigma_n)} \widehat{\phi}_n \left(\frac{u}{\sigma_n} \right), \quad u \in \mathbb{R}. \tag{2.20}$$

Following [6] we shall also make the following mild assumption. There are constants $A > 0$ and $0 < \alpha \le 1$ such that

$$|\{\ell : \lambda_{n,\ell} \le A\}| \ge \alpha n, \quad n = 2, 3, \ldots. \tag{2.21}$$

Then from (2.18),

$$\frac{\alpha n}{6(1+A)} \le \sigma_n^2 \le \frac{n}{12}, \quad n = 2, 3, \ldots. \tag{2.22}$$

By Lemma 4 of [6] we know there exist $B > 0$, $0 < \rho < 1$, such that for $n = 2, 3, \ldots$,

$$|\widehat{\phi}_n(u)| \le \frac{B\rho^n}{u^2}, \quad u \ge \frac{3\pi}{2}. \tag{2.23}$$

Theorem 2.5. *Under the above conditions, as $n \to \infty$,*

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_1 = O\left(\frac{1}{n}\right), \tag{2.24}$$

and if, in addition, (2.12) *holds then*

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_1 = O\left(\frac{1}{n^{3/2}}\right). \tag{2.25}$$

Proof: In the proof of Proposition 4.1 of [3], it is shown that there are constants $a, C > 0$ such that for $|u| \le a\sigma_n$, $n = 2, 3, \ldots$,

$$|\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}| \le C\sigma_n^{-L} |u|^{L+2} e^{-u^2/2},$$

where $L = 2$ if P_n is reciprocal, and $L = 3$ if (2.12) also holds. Thus as $n \to \infty$,

$$\int_{-a\sigma_n}^{a\sigma_n} |\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}| \, du = O(\sigma_n^{-L}),$$

and so

$$\begin{aligned} \|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_1 &\leq \int_{|u|\geq a\sigma_n} |\widehat{\widetilde{\phi}}_n| + \int_{|u|\geq a\sigma_n} e^{-u^2/2}\,du + O(\sigma_n^{-L}) \\ &= \int_{|u|\geq a\sigma_n} |\widehat{\widetilde{\phi}}_n| + O(\sigma_n^{-L}) \\ &= \int_{|u|\geq a\sigma_n} |\widehat{\widetilde{\phi}}_n| + O(n^{-L/2}), \end{aligned} \tag{2.26}$$

by (2.22). Now by (2.20),

$$\int_{|u|\geq a\sigma_n} |\widehat{\widetilde{\phi}}_n| = 2\int_{a\sigma_n}^{\infty} \left|\widehat{\phi}_n\left(\frac{u}{\sigma_n}\right)\right| du = 2\sigma_n \int_a^{\infty} |\widehat{\phi}_n|. \tag{2.27}$$

If $a < \frac{3\pi}{2}$, then for $a \leq u \leq \frac{3\pi}{2}$, (2.17) gives

$$|\widehat{\phi}_n(u)| \leq \prod_{\ell=1}^{n} \left(\frac{\lambda_\ell + \cos(u/2)}{\lambda_\ell + 1}\right)^{1/2} \leq \left(\frac{A + \cos(a/2)}{A+1}\right)^{\alpha n/2}$$

by (2.21), and so there is a number β, $0 < \beta < 1$, such that as $n \to \infty$,

$$\int_a^{3\pi/2} |\widehat{\phi}_n| = O(\beta^n). \tag{2.28}$$

Also by (2.23),

$$\int_{3\pi/2}^{\infty} |\widehat{\phi}_n| \leq B\rho^n \int_{3\pi/2}^{\infty} \frac{du}{u^2} = O(\rho^n) \tag{2.29}$$

as $n \to \infty$. So by (2.22), (2.26)–(2.29), we have

$$\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_1 = O(n^{-L/2})$$

as $n \to \infty$, which gives (2.24) and (2.25). ∎

It follows from Theorem 2.5 that (2.24) holds when $\phi_n = B_n$. Also (2.25) holds, in particular if $P_{2n}(z) = \frac{1}{4}(z+1)^2\{\frac{1}{6}(z^2+4z+1)\}^{n-1}$.

Since $\|\widetilde{\phi}_n - G\|_\infty \leq \frac{1}{2\pi}\|\widehat{\widetilde{\phi}}_n - \widehat{G}\|_1$, Theorem 2.5, Lemma 2.1 and (2.15) give the following.

Theorem 2.6. *Under the conditions of Theorem* 2.5*, there is a constant* $C > 0$ *such that for any* j *in* $\mathbb{Z}$ *and* f *in* $L^\infty(\mathbb{R})$*,*

$$\|\widetilde{S}_{j,n} f - T_j f\|_\infty \leq C\|f\|_\infty \frac{\log n}{n},$$

and if, in addition, (2.12) *holds then*

$$\|\widetilde{S}_{j,n} f - T_j f\|_\infty \leq C\|f\|_\infty \frac{\log n}{n^{3/2}}.$$

3. Refinable sequences

We now consider a discrete version of scale-space filtering. We are interested in a class of filters with the causality property. Let $\kappa = (\kappa(i))_{i=-\infty}^{\infty}$ be a non-negative sequence with finite support and $\sum_{i=-\infty}^{\infty} \kappa(i) = 1$. For $j \in \mathbb{Z}$, $j \leq 0$, define the scale-space operator τ_j on $\ell(\mathbb{Z})$, the space of all real sequences, by

$$(\tau_j s)_i := \sum_{\ell=-\infty}^{\infty} s_\ell \, \kappa(i - 2^{-j}\ell), \quad i \in \mathbb{Z}. \tag{3.1}$$

This is a discrete analog of T_j in (1.3), and analogously we say that the filter or discrete kernel κ has the *causality property* if

$$V(\tau_{j+1}s) \leq V(\tau_j s), \quad j \in \mathbb{Z},\ j \leq -1, \tag{3.2}$$

for any s in $\ell(\mathbb{Z})$. However, unlike the continuous case where only the Gaussian kernel satisfies the causality property, the discrete case exhibits a large class of filters with the causality property. These filters are a discrete analog of ripplets, i.e. they are solutions of a discrete analog of refinement equations whose symbols have all their roots in the left half-plane.

Observe that if κ has support of length ν, i.e. $\kappa(i)\kappa(\ell) = 0$ for $|i - \ell| \geq \nu$, then for $2^{-j} \geq \nu$, the sequence $(\kappa(i - 2^{-j}\ell))_{\ell=-\infty}^{\infty}$ has at most one non-zero element for any integer i, and since κ is non-negative,

$$V(\tau_j s) = V(s)$$

for any s in $\ell(\mathbb{Z})$. Hence (3.2) implies that for any s in $\ell(\mathbb{Z})$,

$$V(\tau_j s) \leq V(s), \quad j \in \mathbb{Z},\ j \leq 0.$$

Considering the case $j = 0$, we see that the matrix $(\kappa(i - \ell))_{i,\ell=-\infty}^{\infty}$ is totally positive and so, by [1], the Laurent polynomial

$$Q(z) := \sum_{i=-\infty}^{\infty} \kappa(i) z^i \tag{3.3}$$

has all real negative zeros.

Now for $n \geq 2$, let

$$P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$$

be a polynomial with real coefficients and $a_0 a_n \neq 0$. We shall consider the discrete refinement equation

$$\kappa(i) = \sum_{\ell=0}^{n} a_\ell \kappa(2i - \ell), \quad i \in \mathbb{Z}. \tag{3.4}$$

Theorem 3.1. *Suppose P has all its roots in the left half-plane $\{z : \operatorname{Re} z \leq 0\}$. If κ satisfies* (3.4) *and is non-negative with finite support and $\sum_{i=-\infty}^{\infty} \kappa(i) = 1$, then it satisfies the causality property* (3.2)*.*

Proof: By (3.1) and (3.4), for any $j \in \mathbb{Z}$, $j \leq -1$, $s \in \ell(\mathbb{Z})$,

$$\begin{aligned}(\tau_{j+1}s)_i &= \sum_{\ell=-\infty}^{\infty} s_\ell\, \kappa(i - 2^{-j-1}\ell) \\ &= \sum_{\ell=-\infty}^{\infty} s_\ell \sum_{k=0}^{n} a_k \kappa(2i - 2^{-j}\ell - k) \\ &= \sum_{k=0}^{n} a_k (\tau_j s)_{2i-k} \\ &= \sum_{k=-\infty}^{\infty} a_{2i-k} (\tau_j s)_k,\end{aligned}$$

where we define $a_\ell := 0$ for $\ell < 0$ and for $\ell > n$. As in the proof of Theorem 2.1, from [9] and [8], (3.2) follows. ■

For the rest of this section we shall consider solutions κ of the discrete refinement equation (3.4). By a solution of (3.4) we shall mean a non-trivial solution with finite support. Because of Theorem 3.1, we are particularly interested in non-negative solutions when P has all roots in the left half-plane. With Q as in (3.3), we note that (3.4) gives

$$Q(z^2) = \sum_{i=-\infty}^{\infty} \kappa(i) z^{2i} = \sum_{i,\ell=-\infty}^{\infty} a_\ell z^\ell \kappa(2i - \ell) z^{2i-\ell},$$

and so (3.4) is equivalent to the polynomial identity

$$2Q(z^2) = P(z)Q(z) + P(-z)Q(-z). \tag{3.5}$$

Lemma 3.1. *If κ is a solution of* (3.4)*, then it has support in $\{0, \ldots, n\}$. If there is a solution κ with $\kappa(0) \neq 0$* (*respectively $\kappa(n) \neq 0$*)*, then $a_0 = 1$* (*respectively $a_n = 1$*)*, while if $a_0 = 1$* (*respectively $a_n = 1$*)*, then there is a solution κ with support in $\{0, \ldots, n-1\}$* (*respectively $\{1, \ldots, n\}$*)*.*

Proof: Let $\mu := \min\{i : \kappa(i) \neq 0\}$. Then by (3.4), $\kappa(\mu)$ is a linear combination of $\kappa(2\mu - \ell)$, $\ell = 0, \ldots, n$, and so $2\mu \geq \mu$, i.e. $\mu \geq 0$. Similarly $\max\{i : \kappa(i) \neq 0\} \leq n$.

Now (3.4) is equivalent to

$$\kappa(0) = a_0 \kappa(0), \tag{3.6}$$

$$\kappa(i) = \sum_{\ell=0}^{n} a_\ell \kappa(2i - \ell) = \sum_{\ell=0}^{n} a_{2i-\ell} \kappa(\ell), \quad i = 1, \ldots, n-1, \tag{3.7}$$

$$\kappa(n) = a_n \kappa(n), \tag{3.8}$$

where we define $a_\ell := 0$ for $\ell < 0$ and for $\ell > n$. Thus $\kappa(0) \neq 0$ implies $a_0 = 1$, by (3.6). Conversely, if $a_0 = 1$ and $\kappa(n) = 0$, then (3.6) and (3.8) are trivially satisfied, and (3.7) gives a homogeneous system of $n - 1$ equations in n unknowns. So there is a solution κ with support in $\{0, \ldots, n-1\}$. The other results follow similarly. ■

Lemma 3.2. *The equation* (3.4) *has a solution* κ *with support in* $\{1,\dots,n-1\}$ *if and only if the matrix*

$$A := (a_{2i-\ell})_{i,\ell=1}^{n-1}$$

has 1 *as an eigenvalue and* $(\kappa(1),\dots\kappa(n-1))^T$ *as a corresponding eigenvector.*

Proof: Suppose κ has support in $\{1,\dots,n-1\}$. Then (3.4) is trivially satisfied for $i \le 0$ and for $i \ge n$. Thus (3.4) is equivalent to (3.7), i.e. $(\kappa(1),\dots\kappa(n-1))^T$ is an eigenvector of A with eigenvalue 1. ■

Lemma 3.3. *If* $P(1) = 2$, $P(-1) = 0$, *then* (3.4) *has a solution* κ *with support in* $\{1,\dots,n-1\}$.

Proof: The conditions on P are equivalent to

$$\sum_{\ell\in\mathbb{Z}} a_{2\ell} = \sum_{\ell\in\mathbb{Z}} a_{2\ell+1} = 1.$$

Then with $e := (1,1,\dots,1)$,

$$(eA)_i = \sum_{\ell=1}^{n-1} a_{2\ell-i} = 1, \quad i = 1,\dots,n-1.$$

Thus 1 is an eigenvalue of A, and the result follows from Lemma 3.2. ■

We note that if P satisfies the conditions of Lemma 3.3 and has all its roots in the left half-plane $\{z : \operatorname{Re} z \le 0\}$ with at least two in $\{z : \operatorname{Re} z < 0\}$ then, as mentioned at the beginning of Section 2, there is a continuous non-negative function ϕ with support on $[0,n]$ satisfying

$$\phi(x) = \sum_{\ell=0}^{n} a_\ell \phi(2x-\ell), \quad x \in \mathbb{R}.$$

Thus the sequence κ given by $\kappa(i) := \phi(i)$, $i \in \mathbb{Z}$, is a solution of (3.4) with support in $\{1,\dots,n-1\}$. However the condition $P(-1) = 0$ is not necessary for solutions of (3.4); indeed we shall see later that if P is *any* polynomial with all roots in the left half-plane, then for a suitable normalization of P, (3.4) has a non-negative solution.

Lemma 3.4. *Suppose that* $P(z) = (z+1)\sum_{\ell=0}^{n-1} \widetilde{a}_\ell z^\ell$ *and*

$$\widetilde{\kappa}(i) = \sum_{\ell=0}^{n-1} \widetilde{a}_\ell \widetilde{\kappa}(2i-\ell), \quad i \in \mathbb{Z}.$$

Then κ *satisfies* (3.4), *where* $\kappa(i) := \widetilde{\kappa}(i) - \widetilde{\kappa}(i-1)$, $i \in \mathbb{Z}$.

Proof: Putting $\widetilde{P}(z) = \sum_{\ell=0}^{n-1} \widetilde{a}_\ell z^\ell$, $\widetilde{Q}(z) = \sum_{\ell=-\infty}^{\infty} \widetilde{\kappa}(\ell) z^\ell$, we have as in (3.5),

$$2\widetilde{Q}(z^2) = \widetilde{P}(z)\widetilde{Q}(z) + \widetilde{P}(-z)\widetilde{Q}(-z).$$

Then multiplying by z^2-1 gives

$$2Q(z^2) = P(z)Q(z) + P(-z)Q(-z),$$

where $Q(z) := (z-1)\widetilde{Q}(z) = \sum_{\ell=-\infty}^{\infty}(\widetilde{\kappa}(\ell-1)-\widetilde{\kappa}(\ell))z^\ell$, and the result follows. ■

Theorem 3.2. *Suppose that $p(z) = \sum_{\ell=0}^{n} b_\ell z^\ell$, with $b_\ell \in \mathbb{R}$, $\ell = 0,\ldots,n$, $b_0 b_n \neq 0$, $p(1) \neq 0$, has a factor of $(z+1)^k$ for some k, $1 \leq k \leq n-1$. Then for $i = 1,\ldots,k$, the equation* (3.4) *with $a_\ell = 2^i p(1)^{-1} b_\ell$, $\ell = 0,\ldots,n$, has a solution κ with support in $\{1,\ldots,n-1\}$.*

Proof: Take i, $1 \leq i \leq k$. Then $p(z) = (z+1)^{i-1} q_i(z)$, where q_i is a polynomial of degree $n-i+1 \geq 2$ with $q_i(-1) = 0$, $q_i(1) = 2^{1-i}p(1)$. Then by Lemma 3.3, there is a solution with support in $\{1,\ldots,n-i\}$ of (3.4) with P replaced by $2^i p(1)^{-1} q_i$. So by repeated application of Lemma 3.4, there is a solution with support κ in $\{1,\ldots,n-1\}$ of (3.4) with P replaced by $2^i p(1)^{-1} p$. ■

Theorem 3.3. *Suppose that $p(z) = \sum_{\ell=0}^{n} b_\ell z^\ell$, with $b_\ell \in \mathbb{R}$, $\ell = 0,\ldots,n$, $b_0, b_n > 0$, is a Hurwitz polynomial, i.e. all its zeros lie in $\{z : \operatorname{Re} z < 0\}$. Then there are numbers $\lambda_1 > \ldots > \lambda_{n-1} > 0$ such that for $i = 1,\ldots,n-1$, the equation* (3.4) *with $a_\ell = \lambda_i^{-1} b_\ell$, $\ell = 0,\ldots,n$, has a solution κ with support in $\{1,\ldots,n-1\}$ and $V(\kappa) = i-1$. Furthermore if p has a factor of $(z+1)^k$ for some k, $1 \leq k \leq n-1$, then $\lambda_i = 2^{-i}p(1)$, $i = 1,\ldots,k$.*

Let $\lambda_0 = \infty$, $\lambda_n = 0$. If $\lambda_i > b_0 > \lambda_{i+1}$ for i, $0 \leq i \leq n-1$, then (3.4) *with $a_\ell = b_0^{-1} b_\ell$, $\ell = 0,\ldots n$, has a solution κ with support in $\{0,\ldots,n-1\}$, $\kappa(0) > 0$, and $V(\kappa) \leq i$. If $\lambda_i > b_n > \lambda_{i+1}$ for some i, $0 \leq i \leq n-1$, then* (3.4) *with $a_\ell = b_n^{-1} b_\ell$, $\ell = 0,\ldots n$, has a solution κ with support in $\{1,\ldots,n\}$, $\kappa(n) > 0$, and $V(\kappa) \leq i$.*

Proof: Let B denote the matrix $(b_{2i-\ell})_{i,\ell=1}^{n-1}$, where we put $b_\ell := 0$ if $\ell < 0$ or $\ell > n$. By [9], B is totally positive and moreover $b_\ell > 0$, $\ell = 0,\ldots,n$. Since the elements on the main diagonal of B, and also on those immediately above and below, are strictly positive, it follows from [5] that B is an oscillation matrix, i.e. some power of B has all minors strictly positive. Then by [5], B has eigenvalues $\lambda_1 > \ldots > \lambda_{n-1} > 0$ and for $i = 1,\ldots,n-1$, the eigenvector of B corresponding to λ_i has exactly $i-1$ changes of sign. Thus for $i = 1,\ldots,n-1$, and $a_\ell = \lambda_i^{-1} b_\ell$, $\ell \in \mathbb{Z}$, the matrix A in Lemma 3.2 equals $\lambda_i^{-1} B$ and so has 1 as an eigenvalue and hence, by Lemma 3.2, (3.4) has a solution κ with $V(\kappa) = i-1$.

Now suppose that p has a factor of $(z+1)^k$ for some k, $1 \leq k \leq n-1$. Then as in the proof of Lemma 3.3, B has an eigenvalue $2^{-1}p(1)$ with left eigenvector $(1,1,\ldots,1)$. It follows from [5], as above, that $2^{-1}p(1)$ is the maximal eigenvalue of B, i.e. $\lambda_1 = 2^{-1}p(1)$.

Take i, $1 \leq i \leq k$. As in the proof of Theorem 3.2, we write $p(z) = (z+1)^{i-1} q_i(z)$. Then from the first part of Theorem 3.3 already established with p replaced by q_i, there are numbers $\lambda_i^i > \ldots > \lambda_{n-1}^i > 0$ such that for $\ell = 1,\ldots,n-1$, the equation (3.4) with P replaced by $(\lambda_\ell^i)^{-1} q_i$ has a solution with support in $\{1,\ldots,n-i\}$. By our argument above, we have $\lambda_i^i = 2^{-1}q_i(1) = 2^{-i}p(1)$.

Also, by Lemma 3.4, we see that for $1 \le i \le k-1$, $\{\lambda_i^i, \dots, \lambda_{n-1}^i\} \supset \{\lambda_{i+1}^{i+1}, \dots, \lambda_{n-1}^{i+1}\}$ and hence $\lambda_\ell^i = \lambda_\ell^{i+1}$, $\ell = i+1, \dots, n-1$. Thus for $i = 1, \dots, k$, $\lambda_i = \lambda_i^1 = \lambda_i^i = 2^{-i}p(1)$.

Now suppose $\lambda_i > b_0 > \lambda_{i+1}$ for some i, $0 \le i \le n-1$, and put $a_\ell = b_0^{-1} b_\ell$, $\ell \in \mathbb{Z}$. If κ has support in $\{0, \dots, n-1\}$, then (3.4) is trivially satisfied for $i = 0$, since $a_0 = 1$. Thus (3.4) is equivalent to (3.7), which is a homogeneous system of $n-1$ equations in n unknowns and so must have a non-trivial solution $v = (\kappa(0), \dots, \kappa(n-1))^T$. If $\kappa(0) = 0$, then $(\kappa(1), \dots, \kappa(n-1))^T$ would be an eigenvector of B with eigenvalue b_0, which is not the case. So $\kappa(0) \neq 0$.

We now show that $V(v) \le i$. Let C denote the matrix $(b_{2\ell-j})_{\ell,j=0}^{n-1}$. We have seen that C has an eigenvector v with eigenvalue b_0. Also we know that C has eigenvalues $\lambda_1 > \dots > \lambda_{n-1}$. The matrix C is totally positive but is not an oscillation matrix since its $(0,1)$-entry equals b_{-1} which is 0. However we can construct a sequence of oscillation matrices C_m such that $\lim_{m\to\infty} C_m = C$. Then C_m has eigenvalues $\lambda_0^m > \dots > \lambda_{n-1}^m$ with $\lim_{m\to\infty} \lambda_\ell^m = \lambda_{\ell+1}$, $0 \le \ell \le i-1$, $\lim_{m\to\infty} \lambda_i^m = b_0$, $\lim_{m\to\infty} \lambda_\ell^m = \lambda_\ell$, $i+1 \le \ell \le n-1$. Corresponding to the eigenvalue λ_i^m is an eigenvector v_m such that $\lim_{m\to\infty} v_m = v$. Since $V(v_m) \le i$, it follows that $V(v) \le i$.

The case when $\lambda_i > b_n > \lambda_{i+1}$ follows similarly. ■

To illustrate, consider the case $p(z) = b_0 + b_1 z + b_2 z^2$, where $b_0, b_1, b_2 > 0$. For any solution κ of (3.4), we write $v = (\kappa(0), \kappa(1), \kappa(2))$ and refer to (3.6)–(3.8). Here $\lambda_1 = b_1$ and $P = b_1^{-1}p$ gives solution $v = (0, 1, 0)$. Putting $P = b_0^{-1}p$ gives $v = (b_0 - b_1, b_2, 0)$, while putting $P = b_2^{-1}p$ gives $v = (0, b_0, b_2 - b_1)$. Note that when $b_0 = b_2 \neq b_1$, $P = b_0^{-1}p$ gives two linearly independent solutions $(b_0 - b_1, b_0, 0)$ and $(0, b_0, b_0 - b_1)$ which are both non-negative when $b_0 > b_1$.

Theorem 3.4. *Suppose that $p(z) = \sum_{\ell=0}^n b_\ell z^\ell$, with $b_\ell \in \mathbb{R}$, $\ell = 0, \dots, n$, $b_0, b_n > 0$, has all its zeros in $\{z : \operatorname{Re} z \le 0\}$. Then there is some $\lambda > 0$ such that the equation* (3.4) *with $a_\ell = \lambda^{-1} b_\ell$, $\ell = 0, \dots, n$, has a non-negative solution κ with support equal to $\{1, \dots, n-1\}$ and $\sum_{i=1}^{n-1} \kappa(i) z^{i-1}$ has $n-2$ strictly negative zeros. Moreover if $b_0 > \lambda$, then* (3.4) *with $a_\ell = b_0^{-1} b_\ell$, $\ell = 0, \dots, n$, has a non-negative solution κ with support $\{0, \dots, n-1\}$ and $\sum_{i=0}^{n-1} \kappa(i) z^i$ has $n-1$ strictly negative zeros, while if $b_n > \lambda$, then* (3.4) *with $a_\ell = b_n^{-1} b_\ell$, $\ell = 0, \dots, n$, has a non-negative solution κ with support $\{1, \dots, n\}$ and $\sum_{i=1}^n \kappa(i) z^{i-1}$ has $n-1$ strictly negative zeros.*

Proof: We may take a sequence of Hurwitz polynomials $p_m(z) = \sum_{\ell=0}^n b_\ell^{(m)} z^\ell$ with $b_0^{(m)}, b_n^{(m)} > 0$ and $\lim_{m\to\infty} p_m = p$. For each m, let B_m denote the matrix $(b_{2i-\ell}^{(m)})_{i,\ell=1}^{n-1}$. Then as in the proof of Theorem 3.3, B_m has eigenvalues $\lambda_1^{(m)} > \dots > \lambda_{n-1}^{(m)} > 0$. So the matrix $B := (b_{2i-\ell})_{i,\ell=1}^{n-1}$ has eigenvalues $\lambda_i = \lim_{m\to\infty} \lambda_i^{(m)}$, $i = 1, \dots, n-1$, $\lambda_1 \ge \dots \ge \lambda_{n-1} \ge 0$. Since B is non-trivial, not all its eigenvalues are zero and so $\lambda_1 > 0$. Put $\lambda = \lambda_1$. Now corresponding to $\lambda_1^{(m)}$, there is an eigenvector $v^{(m)} = (v_1^{(m)}, \dots, v_{n-1}^{(m)})^T$ of B_m with $v_i^{(m)} \ge 0$, $i = 1, \dots, n-1$, and $\sum_{i=1}^{n-1} v_i^{(m)} = 1$. Thus there is a subsequence of $(v^{(m)})$ which converges to an eigenvector $v = (v_1, \dots, v_{n-1})$ of B with eigenvalue λ, where $v_i \ge 0$, $i = 1, \dots, n-1$, $\sum_{i=1}^{n-1} v_i = 1$. So it follows from Lemma 3.2 that the equation (3.4) with $a_\ell = \lambda^{-1}b_\ell$, $\ell = 0,\dots,n$, has a solution κ with support in $\{1,\dots,n-1\}$ and $\kappa(i) = v_i$, $i = 1,\dots,n-1$.

Now suppose $b_0 > \lambda$. Then for large enough m, $b_0^{(m)} > \lambda_1^{(m)}$, and from Theorem 3.3, (3.4) with $a_\ell = (b_0^{(m)})^{-1}b_\ell^{(m)}$, $\ell = 0,\dots,n$, has a non-negative solution $\kappa^{(m)}$ with support in $\{0,\dots,n-1\}$ and $\sum_{i=0}^{n-1}\kappa^{(m)}(i) = 1$. Taking a subsequence of $(\kappa^{(m)})$ which converges to a limit κ shows that (3.4) with $a_\ell = b_0^{-1}b_\ell$, $\ell = 0,\dots,n$, has a non-negative solution κ with support in $\{0,\dots,n-1\}$. If $\kappa(0) = 0$, then $(\kappa(1),\dots,\kappa(n-1))^T$ is an eigenvector of B with eigenvalue b_0, which contradicts $b_0 > \lambda_i$, $i = 1,\dots,n-1$. So $\kappa(0) > 0$. If $b_n > \lambda$, then similarly we can show that (3.4) with $a_\ell = b_n^{-1}b_\ell$, $\ell = 0,\dots,n$, has a non-negative solution κ with support in $\{1,\dots,n\}$ and $\kappa(n) > 0$.

Next consider again the non-negative solution κ to (3.4) with $a_\ell = \lambda^{-1}b_\ell$, $\ell = 0,\dots,n$. By Theorem 3.1 and the remark before that, we know that the polynomial $Q(z) := \sum_{i=0}^{n}\kappa(i)z^i$ has all its zeros real and non-positive. Let $\mu := \min\{i : \kappa(i) > 0\}$ and write $Q(z) = z^\mu r(z)$ for some polynomial r. Then by (3.5),

$$2z^\mu r(z^2) = \lambda^{-1}p(z)r(z) + (-1)^\mu\lambda^{-1}p(-z)r(-z). \tag{3.9}$$

Since $\mu \geq 1$, putting $z = 0$ gives $0 = p(0)r(0)(1 + (-1)^\mu)$. Since $p(0) = b_0 > 0$ and $r(0) = \kappa(\mu) > 0$, μ must be odd.

Suppose $\mu \geq 3$. Then equating coefficients of z in (3.9) gives

$$0 = b_0\kappa(\mu+1) + b_1\kappa(\mu),$$

and since $b_0, \kappa(\mu) > 0$, $\kappa(\mu+1), b_1 \geq 0$, we must have $\kappa(\mu+1) = 0$. Since $Q(z)$ has all its zeros real and non-positive, it follows that $\kappa(i) = 0$, $i \geq \mu+1$, i.e. $Q(z) = \kappa(\mu)z^\mu$. Substituting in (3.9) gives

$$2z^\mu = \lambda^{-1}(p(z) - p(-z)),$$

and so $b_\mu > 0$, $b_\ell = 0$ for ℓ odd, $\ell \neq \mu$. Now by [9] we know that

$$0 \leq \begin{vmatrix} b_2 & b_\mu \\ b_0 & b_{\mu-2} \end{vmatrix} = -b_0 b_\mu < 0,$$

which is a contradiction.

Thus $\mu = 1$, i.e. $\kappa(1) > 0$. Similarly we can show $\kappa(n-1) > 0$. Therefore $\sum_{i=1}^{n-1}\kappa(i)z^{i-1}$ has $n-2$ strictly negative zeros and so κ has support equal to $\{1,\dots,n-1\}$.

Next suppose $b_0 > \lambda$ and consider again the solution κ to (3.4) with $a_\ell = b_0^{-1}b_\ell$, $\ell = 0,\dots,n$. In a similar manner to above we can show $\kappa(n-1) > 0$. Since we have already shown $\kappa(0) > 0$, the polynomial $\sum_{i=0}^{n-1}\kappa(i)z^i$ has $n-1$ strictly negative zeros and so κ has support $\{0,\dots,n-1\}$.

Finally suppose $b_n > \lambda$ and consider again the solution κ to (3.4) with $a_\ell = b_n^{-1}b_\ell$, $\ell = 0,\dots,n$. In a similar manner we see that $\sum_{i=1}^{n}\kappa(i)z^{i-1}$ has $n-1$ strictly negative zeros and κ has support $\{1,\dots,n\}$. ■

To illustrate Theorem 3.4, consider the case $p(z) = (z+\alpha)(z^2+\beta)$, where $\alpha, \beta > 0$. For any solution κ of (3.4), we write $v = (\kappa(0),\kappa(1),\kappa(2),\kappa(3))$. The matrix B, as in the proof of Theorem

3.4, is

$$\begin{bmatrix} \beta & \alpha\beta \\ 1 & \alpha \end{bmatrix}.$$

Thus $\lambda = \alpha + \beta$ with corresponding eigenvector $(\beta, 1)^T$. So $P = (\alpha + \beta)^{-1} p$ gives solution $v = (0, \beta, 1, 0)$. Now for $P = p$ and $\kappa(0) = 0$, (3.4) is equivalent to

$$\beta\kappa(1) + \alpha\beta\kappa(2) = \kappa(1), \qquad \kappa(1) + \alpha\kappa(2) + \beta\kappa(3) = \kappa(2),$$

which gives solution $v = (0, \alpha\beta^2, \beta(1 - \beta), 1 - \alpha - \beta)$. As predicted by Theorem 3.4, v is non-negative when $1 > \alpha + \beta$. Similarly for $P = (\alpha\beta)^{-1} p$, there is a solution $v = (1 - \alpha^{-1} - \beta^{-1}, \beta^{-1}(1 - \beta^{-1}), \alpha^{-1}\beta^{-2}, 0)$, which is non-negative for $\alpha\beta > \alpha + \beta$.

From Theorem 3.1 and the remark before that, we see that if (3.4) has a non-negative solution κ with finite support, and $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ has all its zeros in the left half-plane, then the Laurent polynomial $Q(z) := \sum_{i=-\infty}^{\infty} \kappa(i) z^i$ has all negative zeros. We have seen in Theorem 3.4 that for *any* such polynomial P, there is at least one, and possibly up to three, normalizations of P for which (3.4) has a non-negative solution κ with finite support. It is natural to ask whether *any* polynomial with negative zeros can be expressed in the form Q for a solution κ of (3.4) for some polynomial P with all zeros in the left half-plane. We shall show that this is false by considering the polynomial $(z + \alpha)^3$, $\alpha > 0$.

First suppose that $n = 5$ and $(z + \alpha)^3 = \sum_{i=1}^{4} \kappa(i) z^{i-1}$. Then (3.4) holds if and only if

$$\sum_{\ell=0}^{5} \kappa(2i - \ell) a_\ell = \kappa(i), \quad i = 1, \ldots, 4. \tag{3.10}$$

Suppose that (3.10) is satisfied with $a_0, a_5 > 0$ and $P(z) = \sum_{\ell=0}^{5} a_\ell z^\ell$ has all its zeros in $\{z : \operatorname{Re} z \leq 0\}$. Since $(z + \alpha)^3$ is a Hurwitz polynomial, it follows from [9] that the matrix $(\kappa(2i - \ell))_{i=1,\ell=0}^{4,5}$ has full rank. So if P is a Hurwitz polynomial, we may make a perturbation so that (3.10) still holds, P is still a Hurwitz polynomial and $\kappa(i) > 0$, $i = 1, \ldots, 4$, but $\sum_{i=1}^{4} \kappa(i) z^{i-1}$ has some complex zeros, which contradicts all the roots being negative. So we can write $P(z) = (z^2 + \beta) R(z)$, for some $\beta > 0$ and a cubic polynomial R. Now by (3.5),

$$2z(z^2 + \alpha)^3 = P(z)(z + \alpha)^3 - P(-z)(-z + \alpha)^3 = (z^2 + \beta)\{R(z)(z + \alpha)^3 - R(-z)(-z + \alpha)^3\},$$

and so $\beta = \alpha$. Thus

$$2z(z^2 + \alpha)^2 = R(z)(z + \alpha)^3 - R(-z)(-z + \alpha)^3.$$

By a similar argument to that above, we can show that R must have zeros on the imaginary axis and thus $R(z) = (z^2 + \alpha) S(z)$ for a linear polynomial S. Hence

$$2z(z^2 + \alpha) = S(z)(z + \alpha)^3 - S(-z)(-z + \alpha)^3.$$

Putting $S(z) = az + b$ gives

$$3\alpha a + b = 1, \qquad \alpha^2 a + 3\alpha b = 1,$$

and so $a = (3\alpha - 1)/(8\alpha^2)$, $b = \alpha(3 - \alpha)/(8\alpha^2)$. So if $\alpha < \frac{1}{3}$ or $\alpha > 3$, P has a positive zero, which is a contradiction.

Another possibility is that $n = 4$ and $(z + \alpha)^3 = \sum_{i=0}^{3} \kappa(i) z^i$. A similar argument to that above shows that $P(z) = \sum_{\ell=0}^{4} a_\ell z^\ell$ cannot have all zeros in the left half-plane unless $\alpha = \frac{1}{3}$ or 3. The case $n = 4$, $(z + \alpha)^3 = \sum_{i=1}^{4} \kappa(i) z^{i-1}$ follows similarly. By Lemma 3.1, the only other possibility is $n = 3$ and $(z + \alpha)^3 = \sum_{i=0}^{3} \kappa(i) z^i$. In this case $P(z) = z^3 + az^2 + bz + 1$, where a simple calculation shows that

$$\alpha^2 a + 3\alpha b = 3(\alpha - 1), \qquad 3\alpha a + b = 3\alpha(1 - \alpha).$$

Suppose P has all zeros in the left half-plane. Then $a, b \geq 0$ and so $3\alpha(1 - \alpha) \geq 0$, $3(\alpha - 1) \geq 0$. This implies $\alpha = 1$ and hence $a = b = 0$, giving $P(z) = z^3 + 1$, which is a contradiction.

References

[1] M. Aissen, A. Edrei, I.J. Schoenberg, and A. Whitney, On the generating functions of totally positive sequences, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 303–307.

[2] J. Babaud, A.P. Witkin, M. Baudin, and R.O. Duda, Uniqueness of the Gaussian kernel for scale-space filtering, IEEE Trans. Pattern Anal. Machine Intelligence 8 (1986), 26–33.

[3] L.H.Y. Chen, T.N.T. Goodman, and S.L. Lee, Asymptotic normality of scaling functions, SIAM J. Math. Anal. 36 (2004), 323–346.

[4] H.B. Curry and I.J. Schoenberg, On Pólya frequency functions IV: The fundamental splines and their limits, J. d'Analyse Math. 17 (1966), 71–107.

[5] F.R. Gantmakher, *Applications of the Theory of Matrices*, Interscience Publishers, New York, 1959.

[6] T.N.T. Goodman and S.L. Lee, Asymptotic optimality of wavelets, preprint.

[7] T.N.T. Goodman and C.A. Micchelli, On refinement equations determined by Pólya frequency sequences, SIAM J. Math. Anal. 23 (1992), 766–784.

[8] S. Karlin, *Total Positivity*, Stanford University Press, Stanford, 1968.

[9] J.H.B. Kemperman, A Hurwitz matrix is totally positive, SIAM J. Math. Anal. 13 (1982), 331–341.

[10] I.J. Schoenberg, On variation-diminishing integral operators of the convolution type, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 164–169.

[11] M. Unser, A. Aldroubi, and M. Eden, On the asymptotic convergence of B-spline wavelets to Gabor functions, IEEE Trans. Information Theory 38 (1992), 864–872.

[12] Y.-P. Wang and S.L. Lee, Scale-space derived from B-splines, IEEE Trans. Pattern Anal. Machine Intelligence 20 (1998), 1040–1055.

[13] A.L. Yuille and T.A. Poggio, Scaling theorems for zero-crossings, IEEE Trans. Pattern Anal. Machine Intelligence 8 (1986), 15–25.