INEQUALITIES ON TIME-CONCENTRATED OR FREQUENCY-CONCENTRATED FUNCTIONS

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In commemoration of the sixtieth birthday of Charles A. Micchelli

Abstract.
We obtain an inequality on a measure of the spread in time of periodic functions that are ϵ-concentrated in frequency, i.e. all but a fixed finite number of Fourier coefficients vanish with mean-squared error up to ϵ. We characterize an extremal function and give an asymptotic formula for the measure of spread of this extremal function as ϵ approaches 0. We also consider the corresponding problem for functions on the real line that are ϵ-concentrated in time or frequency. When $\epsilon = 0$, the above reduce to inequalities on time-limited or band-limited functions and these are discussed in more detail.

1. Introduction

The famous Heisenberg uncertainty principle [13] states that for suitable f in $\mathcal{L}^2(\mathbb{R})$,

$$\Delta(f)\,\Delta(\hat{f}) \geq \frac{1}{2},$$

where

$$\Delta(f) := \frac{\left\{ \int_{-\infty}^{\infty} x^2 |f(x)|^2\,dx\, \|f\|^2 - \left| \int_{-\infty}^{\infty} x |f(x)|^2\,dx \right|^2 \right\}^{1/2}}{\|f\|^2},$$

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and $\hat{f}$ is the Fourier transform of f. The quantity $\Delta(f)^2$ is the variance of the function f. Thus the Heisenberg uncertainty principle gives the lower bound for the product of the variance of f and the variance of $\hat{f}$. It is well known that this lower bound is attained by the Gaussian function $G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$.

In [2], an analog for periodic functions was introduced. To describe this, consider $\mathcal{L}^2([0, 2\pi])$ with the inner product $\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt$. For an absolutely continuous nonzero f in $\mathcal{L}^2([0, 2\pi])$ with f' in $\mathcal{L}^2([0, 2\pi])$, define $\tau(f) := \frac{1}{2\pi} \int_0^{2\pi} e^{it}|f(t)|^2\, dt$,

$$\Delta_\theta(f) := \frac{\{\|f\|^4 - |\tau(f)|^2\}^{1/2}}{|\tau(f)|}, \qquad \Delta_m(f) := \frac{\{\|f'\|^2\|f\|^2 - |\langle f', f\rangle|^2\}^{1/2}}{\|f\|^2}.$$

Then the principle states that

$$\Delta_\theta(f)\, \Delta_m(f) > \frac{1}{2}, \tag{1.1}$$

where the constant is the best possible though it is not attainable [10]. The expressions $\Delta_\theta(f)$ and $\Delta_m(f)$ were denoted in [8] by $\Delta\theta$ and Δm (corresponding to variations in angle and angular momentum) and that is why we have chosen the subscripts θ and m.

Inequality (1.1) has been much studied [4], [5], [6], [8], [10], [11] and extended to 2-spheres [9], radial functions in general n-spheres [12] and general functions on n-spheres [4]. If we express $f \in \mathcal{L}^2([0, 2\pi])$ in its Fourier series $f(t) = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$, $t \in \mathbb{R}$, then

$$\Delta_\theta(f)^2 = \frac{\left(\sum_{j=-\infty}^{\infty} |a_j|^2\right)^2 - \left|\sum_{j=-\infty}^{\infty} a_j\overline{a_{j+1}}\right|^2}{\left|\sum_{j=-\infty}^{\infty} a_j\overline{a_{j+1}}\right|^2}, \tag{1.2}$$

$$\Delta_m(f)^2 = \frac{\left(\sum_{j=-\infty}^{\infty} j^2|a_j|^2\right)\left(\sum_{j=-\infty}^{\infty} |a_j|^2\right) - \left(\sum_{j=-\infty}^{\infty} j|a_j|^2\right)^2}{\left(\sum_{j=-\infty}^{\infty} |a_j|^2\right)^2}.$$

We can think of $\Delta_\theta(f)$ and $\Delta_m(f)$ as measures of 'spread' of f and $a = (a_j)_{-\infty}^{\infty}$ respectively. Indeed $\Delta_m(f)^2$ is simply the variance of a.

An alternative measure of spread of f is the entropy

$$H(|f|^2) := -\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \log|f(t)|^2\, dt,$$

and similarly the spread of a can be measured by

$$H(|a|^2) := -\sum_{j=-\infty}^{\infty} |a_j|^2 \log|a_j|^2.$$

It is known [7] that for f in $\mathcal{L}^2([0, 2\pi])$ with $\|f\| = 1$,

$$H(|f|^2) + H(|a|^2) \geq 0, \tag{1.3}$$

with equality if $f(t) = 1$, $0 \leq t \leq 2\pi$.

Now in many applications it is important to have band-limited functions, i.e. f in $\mathcal{L}^2([0,2\pi])$ such that a has finite support. Suppose that $|\operatorname{supp} a| = N$. Then $H(|a|^2)$ attains its maximum when $|a_j|$ is constant in its support, and so

$$H(|a|^2) \leq -N\left(\frac{1}{N}\log\left(\frac{1}{N}\right)\right) = \log N. \tag{1.4}$$

Then (1.3) and (1.4) give

$$H(|f|^2) + \log N \geq 0,$$

i.e. for all f in $\mathcal{L}^2([0,2\pi])$ such that $\|f\| = 1$ and $|\operatorname{supp} a| = N$,

$$Ne^{H(|f|^2)} \geq 1. \tag{1.5}$$

Inequality (1.5) can be thought of as an uncertainty principle, where N gives a measure of the spread of a.

Correspondingly, suppose that f has compact support in $[0,2\pi]$ with measure L. Then $H(|f|^2)$ attains its maximum when $|f|$ is constant on its support, and so

$$H(|f|^2) \leq -\frac{1}{2\pi}\int_{\operatorname{supp} f} \frac{2\pi}{L}\log\left(\frac{2\pi}{L}\right) = -\log\left(\frac{2\pi}{L}\right). \tag{1.6}$$

Then (1.3) and (1.6) give

$$H(|a|^2) - \log\left(\frac{2\pi}{L}\right) \geq 0,$$

i.e. for all f in $\mathcal{L}^2([0,2\pi])$ such that $\|f\| = 1$ and of compact support in $[0,2\pi]$ with measure L,

$$Le^{H(|a|^2)} \geq 2\pi. \tag{1.7}$$

Inequality (1.7) can also be viewed as an uncertainty principle, where L gives a measure of the spread of f.

Combining both (1.4) and (1.6) with (1.3) gives

$$\log N - \log\left(\frac{2\pi}{L}\right) \geq 0,$$

i.e.

$$NL \geq 2\pi. \tag{1.8}$$

This inequality is in fact trivial since when N is finite, then f must have support on all of $[0,2\pi]$, i.e. $L = 2\pi$. However based on ideas in [3], one can prove a stronger version of (1.8), as follows.

Take $\epsilon, \eta > 0$, $S \subset \mathbb{Z}$ with $|S| = N$ and $T \subset [0,2\pi]$ with measure L. Consider f in $\mathcal{L}^2([0,2\pi])$ with Fourier coefficients a as before. Suppose that a satisfies

$$\sum_{j\notin S} |a_j|^2 \leq \epsilon^2 \sum_{j=-\infty}^{\infty} |a_j|^2. \tag{1.9}$$

Following [3] we say that a is 'ϵ-concentrated' on S. Similarly we suppose that f is 'η-concentrated' on T, i.e.

$$\|f - f|_T\| \leq \eta \|f\|,$$

where $f|_T$ denotes the function which coincides with f on T and is zero elsewhere. Then

$$NL \geq 2\pi(1 - \epsilon - \eta)^2. \tag{1.10}$$

However, in certain circumstances, the measure of the support of f may not be an appropriate measure of the spread of f, as is illustrated by inequality (1.10) reducing to the trivial inequality (1.8) when $\epsilon = \eta = 0$. So in considering such functions f it may be useful to measure the spread of f by $\Delta_\theta(f)$ rather than L. In Section 2 we show that for all functions f satisfying (1.9) for some S with $|S| = N$, the minimum of $\Delta_\theta(f)$ is attained by a particular function g which we characterize. We do not express the minimum $\Delta_\theta(g)$ explicitly in terms of N and ϵ but we do give an asymptotic formula for $\Delta_\theta(g)$ when $\epsilon \to 0$. When $\epsilon = 0$, this reduces to $\Delta_\theta(g) = \tan \frac{\pi}{N+1}$. In particular this implies uncertainty principles of form similar to (1.5), e.g. $N\Delta_\theta(f) \geq \frac{5}{\sqrt{3}}$, where equality is attained for a function g with $N = 5$.

In Section 3 we consider the corresponding problem for a function f in $\mathcal{L}^2(\mathbb{R})$. We suppose that $\hat{f}$ is 'ϵ-concentrated' on an interval I of length L, i.e.

$$\int_{\mathbb{R}\setminus I} |\hat{f}|^2 \leq \epsilon^2 \int_{-\infty}^{\infty} |\hat{f}|^2.$$

Then we show that under suitable decay conditions on $\hat{f}$, $\hat{f}'$, $\hat{f}''$,

$$L\Delta(f) \geq \pi\Delta(q), \tag{1.11}$$

for a function q whose Fourier transform $\hat{q}$ is ϵ-concentrated on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. In fact we find it more convenient to replace functions by their Fourier transforms and derive (1.11) by establishing

$$L\Delta(\hat{f}) \geq \pi\Delta(\hat{g}) \tag{1.12}$$

for functions f in $\mathcal{L}^2(\mathbb{R})$ with appropriate decay which are ϵ-concentrated on an interval I of length L, where $\hat{g} = q$ and g is ϵ-concentrated on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We characterize the function g and provide an asymptotic formula for $\Delta(\hat{g})$ as $\epsilon \to 0$. When $\epsilon = 0$, (1.12) becomes $L\Delta(\hat{f}) \geq \pi$ which gives the well-known [1] Wirtinger's inequality $L\,\|f'\| \geq \pi\,\|f\|$.

2. Frequency-concentrated periodic functions

As before, we take $f(t) = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$, $t \in \mathbb{R}$, and define $\Delta_\theta(f) > 0$ by (1.2).

Theorem 2.1. *Take $N \geq 2$, $0 \leq \epsilon < 1$, and suppose that for some $S \subset \mathbb{Z}$ with $|S| = N$, $a = (a_j)_{-\infty}^{\infty}$ satisfies $\sum_{j \notin S} |a_j|^2 \leq \epsilon^2 \sum_{j=-\infty}^{\infty} |a_j|^2$. Then*

$$\Delta_\theta(f) \geq \Delta_\theta(g),$$

where $g(t) = \sum_{j=-\infty}^{\infty} b_j e^{ijt}$ and $b = (b_j)_{-\infty}^{\infty}$ is the unique nonnegative sequence satisfying

$$\sum_{j=-\infty}^{\infty} b_j^2 = 1, \qquad \sum_{j=1}^{N} b_j^2 = 1 - \epsilon^2, \tag{2.1}$$

$$b_j = A \cos\left(j - \tfrac{N+1}{2}\right)\gamma, \quad j = 0, 1, \ldots, N+1, \tag{2.2}$$

$$b_j = B\alpha^{j-N},\ j \geq N, \qquad b_j = B\alpha^{1-j},\ j \leq 1, \tag{2.3}$$

for some numbers $A > 0$, $B \geq 0$, $0 < \gamma \leq \frac{\pi}{N+1}$, $0 < \alpha < 1$.

The sequence $b = (b_j)_{-\infty}^{\infty}$ is symmetric about $\frac{N+1}{2}$. Indeed, by (2.2) and (2.3), we have $b_j = b_{N+1-j}$ for all $j \in \mathbb{Z}$.

Before proving Theorem 2.1, we illustrate the result with the case $\epsilon = 0$. By (2.1), $b_j = 0$ for $j \leq 0$ and $j \geq N+1$. Then by (2.2),

$$0 = b_{N+1} = A \cos\left(\tfrac{N+1}{2}\right)\gamma$$

and so $\gamma = \frac{\pi}{N+1}$. Thus for $j = 0, \ldots, N+1$,

$$b_j = A \cos\left(\frac{j\pi}{N+1} - \frac{\pi}{2}\right) = A \sin \frac{j\pi}{N+1}.$$

Then by (2.1),

$$1 = \sum_{j=-\infty}^{\infty} b_j^2 = A^2 \sum_{j=0}^{N} \sin^2 \frac{j\pi}{N+1}$$

and

$$\begin{aligned}\sum_{j=-\infty}^{\infty} b_j b_{j+1} &= \frac{\sum_{j=0}^{N} \sin \frac{j\pi}{N+1} \sin \frac{(j+1)\pi}{N+1}}{\sum_{j=0}^{N} \sin^2 \frac{j\pi}{N+1}} \\ &= \frac{\sum_{j=0}^{N} \left(\cos \frac{\pi}{N+1} - \cos \frac{(2j+1)\pi}{N+1}\right)}{\sum_{j=0}^{N} \left(1 - \cos \frac{2j\pi}{N+1}\right)} = \cos \frac{\pi}{N+1},\end{aligned}$$

since $\sum_{j=0}^{N} \cos \frac{(2j+1)\pi}{N+1} = \sum_{j=0}^{N} \cos \frac{2j\pi}{N+1} = 0$. Hence we have the following.

Corollary 2.1. *Take $N \geq 2$ and suppose $a = (a_j)_{-\infty}^{\infty}$ has support on a set of N integers. Then*

$$\Delta_\theta(f) \geq \tan \frac{\pi}{N+1}, \tag{2.4}$$

and equality is attained for the sequence a with

$$a_j = \sin \frac{j\pi}{N+1}, \quad j = 1, \ldots, N. \tag{2.5}$$

Inequality (2.4) for the special case of real-valued even trigonometric polynomials of degree at most n, i.e. $N = 2n+1$, can also be obtained from the Gauss quadrature formula, as indicated in [5]. Our approach here is different and we do not assume the support of a to comprise consecutive integers and allow N to be any integer at least 2.

Since $\tan x > x$, $0 < x < \frac{\pi}{2}$, it follows from (2.4) that

$$(N+1)\Delta_\theta(f) > \pi. \tag{2.6}$$

The inequality (2.6) is of form similar to (1.5) and it is best possible in the sense that if a is of the form (2.5) for each N, then

$$\lim_{N\to\infty}(N+1)\Delta_\theta(f) = \lim_{N\to\infty}(N+1)\tan\tfrac{\pi}{N+1} = \pi.$$

Comparing with (1.5) it would seem more natural to consider instead the quantity $N\Delta_\theta(f)$. It can be checked that $N\tan\frac{\pi}{N+1}$ obtains its minimum for $N = 5$ and thus we deduce from (2.4) that

$$N\Delta_\theta(f) \geq 5\tan\tfrac{\pi}{6} = \tfrac{5}{\sqrt{3}},$$

where equality is attained for $N = 5$ and the sequence a with

$$a_j = \sin\tfrac{j\pi}{6}, \quad j = 1, \ldots, 5.$$

Proof of Theorem 2.1: Without loss of generality we may assume $\sum_{j=-\infty}^{\infty}|a_j|^2 = 1$. Then by (1.2),

$$\Delta_\theta(f)^2 = \frac{1 - \left|\sum_{j=-\infty}^{\infty} a_j\overline{a_{j+1}}\right|^2}{\left|\sum_{j=-\infty}^{\infty} a_j\overline{a_{j+1}}\right|^2}$$

and so it is sufficient to prove

$$\left|\sum_{j=-\infty}^{\infty} a_j\overline{a_{j+1}}\right| \leq \sum_{j=-\infty}^{\infty} b_j b_{j+1}. \tag{2.7}$$

By replacing $(a_j)_{-\infty}^{\infty}$ by $(|a_j|)_{-\infty}^{\infty}$, there is no loss of generality in assuming $a_j \geq 0$, $j \in \mathbb{Z}$.

We first consider the case when S comprises consecutive integers and without loss of generality we assume $S = \{1, \ldots, N\}$. Define

$$c_j = \sqrt{\frac{a_j^2 + a_{N+1-j}^2}{2}}, \quad j \in \mathbb{Z}.$$

Then

$$\sum_{j=-\infty}^{\infty} c_j^2 = \sum_{j=-\infty}^{\infty} a_j^2 = 1, \qquad \sum_{j\notin S} c_j^2 = \sum_{j\notin S} a_j^2 \leq \epsilon^2.$$

By the Cauchy-Schwarz inequality, for $j \in \mathbb{Z}$,

$$a_j a_{j+1} + a_{N-j}a_{N-j+1} \leq \sqrt{a_j^2 + a_{N+1-j}^2}\sqrt{a_{j+1}^2 + a_{N-j}^2}$$

and so

$$a_j a_{j+1} + a_{N-j} a_{N-j+1} \le 2c_j c_{j+1} = c_j c_{j+1} + c_{N-j} c_{N-j+1},$$

which gives

$$\sum_{j=-\infty}^{\infty} a_j a_{j+1} \le \sum_{j=-\infty}^{\infty} c_j c_{j+1}.$$

Thus we may assume that $a_j = a_{N+1-j}$ for all $j \in \mathbb{Z}$.

First consider the case $\sum_{j \notin S} a_j^2 = \epsilon^2$. We now take $B > 0$ and maximize $\sum_{j=N}^{\infty} a_j a_{j+1}$ subject to $a_N = B$ and $\sum_{j=N+1}^{\infty} a_j^2 = \frac{1}{2}\epsilon^2$. If $\epsilon = 0$, then $a_j = 0$, $j \ge N+1$, and so we may assume $\epsilon > 0$. Suppose that the maximum is attained. We use a Lagrange multiplier and consider the expression

$$\sum_{j=N}^{\infty} a_j a_{j+1} - \lambda \left(\sum_{j=N+1}^{\infty} a_j^2 - \frac{1}{2}\epsilon^2 \right).$$

Differentiating with respect to a_j, $j \ge N+1$, gives

$$a_{j-1} + a_{j+1} - 2\lambda a_j = 0, \quad j \ge N+1,$$

and so

$$a_j = B\alpha^{j-N}, \quad j \ge N, \tag{2.8}$$

where α, $0 < \alpha < 1$, satisfies $\alpha^2 - 2\lambda\alpha + 1 = 0$. Then $\dfrac{B^2\alpha^2}{1-\alpha^2} = \dfrac{1}{2}\epsilon^2$ and so $\alpha^2 = \dfrac{\epsilon^2}{2B^2 + \epsilon^2}$ and

$$\sum_{j=N}^{\infty} a_j a_{j+1} = \frac{B^2\alpha}{1-\alpha^2} = \frac{1}{2}\epsilon\sqrt{2B^2 + \epsilon^2}.$$

If the maximum is not attained, then we can construct a sequence of sequences for which the supremum is approached, and which converges element-wise to a sequence c_j, $j \ge N$, with $c_N = B$, $\sum_{j=N+1}^{\infty} c_j^2 = \frac{1}{2}\delta^2 < \frac{1}{2}\epsilon^2$. For any $M > N+1$, $\sum_{j=M}^{\infty} a_j a_{j+1} \le \sum_{j=M}^{\infty} a_j^2$ and thus the supremum is bounded by

$$\sum_{j=N}^{\infty} c_j c_{j+1} + \frac{1}{2}\epsilon^2 - \frac{1}{2}\delta^2. \tag{2.9}$$

Now for any $M > N+1$, consider the sequence

$$a_j = \begin{cases} c_j, & j = N, \dots, M, \\ \frac{1}{\sqrt{M}} \left(\frac{1}{2}\epsilon^2 - \sum_{\nu=N+1}^{M} c_\nu^2 \right)^{1/2}, & j = M+1, \dots, 2M, \\ 0, & j \ge 2M+1. \end{cases}$$

Letting $M \to \infty$, we see that the supremum equals (2.9). Then the sequence c_j, $j \ge N$, maximizes $\sum_{j=N}^{\infty} c_j c_{j+1}$ over all sequences d_j, $j \ge N$, with $d_N = B$, $\sum_{j=N+1}^{\infty} d_j^2 = \frac{1}{2}\delta^2$ and so, from above, $\sum_{j=N}^{\infty} c_j c_{j+1} = \frac{1}{2}\delta\sqrt{2B^2 + \delta^2}$, where $0 < \delta < \epsilon$. But

$$\frac{1}{2}\delta\sqrt{2B^2 + \delta^2} + \frac{1}{2}\epsilon^2 - \frac{1}{2}\delta^2 < \frac{1}{2}\epsilon\sqrt{2B^2 + \epsilon^2},$$

which contradicts (2.9) being the supremum. Thus the maximum is attained by the sequence (2.8).

We now claim that the maximum of $\sum_{j=-\infty}^{\infty} a_j a_{j+1}$ over the sequences a is attained. For suppose this is not the case. Then we can construct a sequence $(c_j)_{-\infty}^{\infty}$ with $\sum_{j=1}^{N} c_j^2 = 1-\epsilon^2$, $c_j = c_{N-j}$, $j \in \mathbb{Z}$, $\sum_{j=N+1}^{\infty} c_j^2 = \frac{1}{2}\delta^2 < \frac{1}{2}\epsilon^2$, and the supremum of $\sum_{j=-\infty}^{\infty} a_j a_{j+1}$ equals

$$\sum_{j=-\infty}^{\infty} c_j c_{j+1} + \epsilon^2 - \delta^2. \tag{2.10}$$

From our above argument, there is a sequence d_j, $j \geq N$, such that $d_N = c_N$, $\sum_{j=N+1}^{\infty} d_j^2 = \frac{1}{2}\epsilon^2$ and

$$\sum_{j=N}^{\infty} d_j d_{j+1} > \sum_{j=N}^{\infty} c_j c_{j+1} + \frac{1}{2}\epsilon^2 - \frac{1}{2}\delta^2.$$

Thus replacing c_j and c_{N+1-j} by d_j, for $j \geq N+1$, contradicts (2.10) being the supremum.

So we may suppose (2.7) for some sequence $b = (b_j)_{-\infty}^{\infty}$. From our earlier argument, see (2.8), we know that (2.3) holds for some α, $0 < \alpha < 1$, where $B = 0$ if $\epsilon = 0$ and $B > 0$ if $\epsilon > 0$.

Now the sequence $b_0, \dots, b_{N+1}$ maximizes $\sum_{j=0}^{N} a_j a_{j+1}$ over all sequences $a_0, \dots, a_{N+1}$ with $a_0 = a_{N+1} = b_{N+1}$ and $\sum_{j=1}^{N} a_j^2 = 1-\epsilon^2$. As in the previous case we find that

$$b_{j-1} + b_{j+1} - 2\mu b_j = 0, \quad 1 \leq j \leq N,$$

where $\mu > 0$ and so, by symmetry,

$$b_j = \frac{1}{2} A \left(\zeta^{j-\frac{N+1}{2}} + \zeta^{\frac{N+1}{2}-j} \right), \quad 0 \leq j \leq N+1,$$

where $A > 0$ and ζ satisfies $\zeta^2 - 2\mu\zeta + 1 = 0$. If $\mu \geq 1$, then $(b_j)_0^{N+1}$ would be convex and since from (2.3), $b_N > b_{N+1}$, it follows that $(b_j)_0^{N+1}$ would be strictly decreasing, which contradicts $b_0 = b_{N+1}$. Thus $0 < \mu < 1$ and putting $\zeta = e^{i\gamma}$ shows that b satisfies (2.2), where $0 < \gamma \leq \frac{\pi}{N+1}$, since $b_j \geq 0$, $j = 0, \dots, N+1$.

We now turn to the uniqueness of the function g in the theorem. We have already seen this for the case $\epsilon = 0$ and so we assume that $\epsilon > 0$. We shall show that there is a unique choice of $A > 0$, $B \geq 0$, $0 < \gamma \leq \frac{\pi}{N+1}$, $0 < \alpha < 1$, for which (2.1)–(2.3) hold. From (2.2) and (2.3) we have

$$A \cos\left(j - \tfrac{N+1}{2}\right)\gamma = B\alpha^{j-N}, \quad j = N, N+1. \tag{2.11}$$

Since $\sum_{j=N+1}^{\infty} b_j^2 = \frac{1}{2}\epsilon^2$, (2.3) gives

$$\frac{B^2\alpha^2}{1-\alpha^2} = \frac{1}{2}\epsilon^2. \tag{2.12}$$

Also since $\sum_{j=1}^{N} b_j^2 = 1-\epsilon^2$, (2.2) gives

$$1-\epsilon^2 = A^2 \sum_{j=1}^{N} \cos^2\left(j - \tfrac{N+1}{2}\right)\gamma = \frac{1}{2}A^2 \sum_{j=1}^{N} (1 + \cos(2j-N-1)\gamma)$$

and so

$$\frac{1}{2}A^2\left(N+\frac{\sin N\gamma}{\sin\gamma}\right)=1-\epsilon^2. \tag{2.13}$$

From (2.11) with $j=N+1$ we have

$$A^2=\frac{B^2\alpha^2}{\cos^2\left(\frac{N+1}{2}\right)\gamma}=\frac{\epsilon^2(1-\alpha^2)}{2\cos^2\left(\frac{N+1}{2}\right)\gamma}, \tag{2.14}$$

by (2.12). Also from (2.11),

$$\alpha=\frac{\cos\left(\frac{N+1}{2}\right)\gamma}{\cos\left(\frac{N-1}{2}\right)\gamma} \tag{2.15}$$

and so

$$A^2=\frac{\epsilon^2\left(\cos^2\left(\frac{N-1}{2}\right)\gamma-\cos^2\left(\frac{N+1}{2}\right)\gamma\right)}{2\cos^2\left(\frac{N-1}{2}\right)\gamma\,\cos^2\left(\frac{N+1}{2}\right)\gamma}=\frac{\epsilon^2\sin N\gamma\sin\gamma}{2\cos^2\left(\frac{N-1}{2}\right)\gamma\,\cos^2\left(\frac{N+1}{2}\right)\gamma}.$$

Substituting into (2.13) gives

$$F(\gamma)=\frac{4(1-\epsilon^2)}{\epsilon^2}, \tag{2.16}$$

where

$$\begin{aligned} F(\gamma) &:= \frac{\sin N\gamma\,(N\sin\gamma+\sin N\gamma)}{\cos^2\left(\frac{N-1}{2}\right)\gamma\,\cos^2\left(\frac{N+1}{2}\right)\gamma} \\ &= \left(\tan\left(\tfrac{N-1}{2}\right)\gamma+\tan\left(\tfrac{N+1}{2}\right)\gamma\right)\left(\frac{N\sin\gamma}{\cos\left(\frac{N-1}{2}\right)\gamma\,\cos\left(\frac{N+1}{2}\right)\gamma}+\tan\left(\tfrac{N-1}{2}\right)\gamma+\tan\left(\tfrac{N+1}{2}\right)\gamma\right). \end{aligned} \tag{2.17}$$

From the last expression we see that F is strictly increasing on $[0,\frac{\pi}{N+1})$ with $F(0)=0$ and $\lim_{\gamma\to\frac{\pi}{N+1}}F(\gamma)=\infty$. Thus (2.16) has a unique solution γ in $(0,\frac{\pi}{N+1})$.

We now show that $\sum_{j=-\infty}^{\infty}b_jb_{j+1}$ is an increasing function of ϵ, $0<\epsilon<1$. Consider

$$\varphi=b_{N-1}x+xy+yb_{N+2},$$

where $x^2+y^2=b_N^2+b_{N+1}^2$. Then when $x=b_N$, $y=b_{N+1}$,

$$\frac{d\varphi}{dy}=-\frac{(b_{N-1}+b_{N+1})b_{N+1}}{b_N}+b_N+b_{N+2}>0,$$

since $b_{N-1}+b_{N+1}<2b_N$ and $b_N+b_{N+2}>2b_{N+1}$. Thus we can increase $\sum_{j=-\infty}^{\infty}b_jb_{j+1}$ by increasing b_{N+1}, and hence ϵ, while keeping $\sum_{j=-\infty}^{\infty}b_j^2$ fixed.

It follows that if S comprises consecutive integers, then for any sequence $a=(a_j)_{-\infty}^{\infty}$ with $\sum_{j=-\infty}^{\infty}|a_j|^2=1$, $\sum_{j\notin S}|a_j|^2\le\epsilon^2$, inequality (2.7) is satisfied for $b=(b_j)_{-\infty}^{\infty}$ given by (2.1)–(2.3).

Finally we consider the case when S does not comprise consecutive integers. Take a sequence $a=(a_j)_{-\infty}^{\infty}$ of nonnegative numbers with $\sum_{j=-\infty}^{\infty}a_j^2=1$, $\sum_{j\notin S}a_j^2\le\epsilon^2$. It is sufficient to construct a set $\widetilde{S}$ of consecutive integers with $|\widetilde{S}|=|S|$, and a sequence $\widetilde{a}=(\widetilde{a}_j)_{-\infty}^{\infty}$ of nonnegative numbers with $\sum_{j=-\infty}^{\infty}\widetilde{a}_j^2=1$, $\sum_{j\notin\widetilde{S}}\widetilde{a}_j^2\le\epsilon^2$ and $\sum_{j=-\infty}^{\infty}a_ja_{j+1}\le\sum_{j=-\infty}^{\infty}\widetilde{a}_j\widetilde{a}_{j+1}$.

Let $k = \min S$, $\ell = \max S$. Suppose there is some $i \notin S$, $k < i < \ell$, with $a_i \geq a_m$ for some $m \in S$. Then we may replace S by $\widehat{S} = S \cup \{i\} \backslash \{m\}$. Thus

$$\sum_{j \notin \widehat{S}} a_j^2 = \sum_{j \notin S} a_j^2 - a_i^2 + a_m^2 \leq \epsilon^2.$$

By repeating this procedure we may assume that for all $i \notin S$ with $k < i < \ell$, $a_i \leq a_j$ for all $j \in S$.

Now choose $m \notin S$, $k < m < \ell$, with $a_m = \min\{a_i : k \leq i \leq \ell\}$. We first suppose $a_m = 0$. In this case we can just remove the term a_m. To be precise, we replace S by

$$\widehat{S} = \{j \in S : j \leq m-1\} \cup \{j-1 : j \in S,\, j \geq m+1\}, \tag{2.18}$$

and replace a by $\widehat{a}$ where

$$\widehat{a}_j = \begin{cases} a_j, & j \leq m-1, \\ a_{j+1}, & j \geq m. \end{cases}$$

Then $\sum_{j=-\infty}^{\infty} \widehat{a}_j^2 = \sum_{j=-\infty}^{\infty} a_j^2$, $\sum_{j \in \widehat{S}} \widehat{a}_j^2 = \sum_{j \in S} a_j^2$ and

$$\sum_{j=-\infty}^{\infty} \widehat{a}_j \widehat{a}_{j+1} - \sum_{j=-\infty}^{\infty} a_j a_{j+1} = a_{m-1} a_{m+1} \geq 0.$$

Next, suppose $a_m > 0$. Since $a_m \leq a_\ell$ and $\lim_{j \to \infty} a_j = 0$, we may choose $p \geq \ell$ with $a_p \geq a_m > a_{p+1}$. Then we replace the term a_m between a_p and a_{p+1}. To be precise, we replace S by $\widehat{S}$ as in (2.18) and replace a by $\widehat{a}$ where

$$\widehat{a}_j = \begin{cases} a_j, & j \leq m-1 \text{ and } j \geq p+1, \\ a_{j+1}, & m \leq j \leq p-1, \\ a_m, & j = p. \end{cases}$$

Then $\sum_{j=-\infty}^{\infty} \widehat{a}_j^2 = \sum_{j=-\infty}^{\infty} a_j^2$, $\sum_{j \in \widehat{S}} \widehat{a}_j^2 = \sum_{j \in S} a_j^2$ and

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \widehat{a}_j \widehat{a}_{j+1} - \sum_{j=-\infty}^{\infty} a_j a_{j+1} &= a_{m-1}a_{m+1} - a_{m-1}a_m - a_m a_{m+1} + a_p a_m + a_m a_{p+1} - a_p a_{p+1} \\ &= (a_{m-1} - a_m)(a_{m+1} - a_m) + (a_p - a_m)(a_m - a_{p+1}) \geq 0. \end{aligned}$$

Since each step reduces by 1 the cardinality of $\{\min S < i < \max S : i \notin S\}$, we eventually construct the desired $\widetilde{S}$ and $\widetilde{a}$. ■

Consider the simplest case $N = 2$ of Theorem 2.1. Since $b_1 = b_2 = B$, $B^2 = \frac{1}{2}(b_1^2 + b_2^2) = \frac{1}{2}(1 - \epsilon^2)$. Also

$$\frac{1}{2}\epsilon^2 = \sum_{j=3}^{\infty} b_j^2 = \sum_{j=1}^{\infty} B^2 \alpha^{2j} = \frac{1}{2}(1-\epsilon^2)\frac{\alpha^2}{1-\alpha^2}$$

and so $\alpha = \epsilon$. Thus

$$b_j = b_{3-j} = \frac{1}{\sqrt{2}}(1-\epsilon^2)^{1/2}\epsilon^{j-2}, \quad j \geq 2,$$

$$\sum_{j=-\infty}^{\infty} b_j b_{j+1} = \frac{1}{2}(1 + 2\epsilon - \epsilon^2).$$

In general the quantities A, B, α can be expressed in terms of γ and ϵ, but γ is known only as the root of equation (2.16). So we shall now derive asymptotic formulas for A, B, α, γ as $\epsilon \to 0$.

Put $x = \frac{\pi}{N+1} - \gamma$. Then by continuity, $x \to 0$ as $\epsilon \to 0$. Now from (2.17) we can see that

$$\frac{1}{F(\gamma)} = \frac{(N+1)x^2}{4}(1 + O(x)).$$

Then by (2.16),

$$\epsilon = \sqrt{N+1}\, x\,(1 + O(x))$$

and so

$$\gamma = \tfrac{\pi}{N+1} - \tfrac{\epsilon}{\sqrt{N+1}} + O(\epsilon^2). \tag{2.19}$$

From (2.15), (2.14) and (2.11) with $j = N$, we can derive

$$\alpha = \frac{\sqrt{N+1}\,\epsilon}{2\sin\frac{\pi}{N+1}} + O(\epsilon^2), \qquad A = \sqrt{\tfrac{2}{N+1}} + O(\epsilon), \qquad B = \sqrt{\tfrac{2}{N+1}}\sin\tfrac{\pi}{N+1} + O(\epsilon).$$

We note that $b_0 = B\alpha = \frac{\epsilon}{\sqrt{2}} + O(\epsilon^2)$.

Next we derive an asymptotic formula for $\sum_{j=-\infty}^{\infty} b_j b_{j+1}$ as $\epsilon \to 0$. Now

$$\begin{aligned}
1 - \sum_{j=-\infty}^{\infty} b_j b_{j+1} &= \sum_{j=-\infty}^{\infty} b_j^2 - \sum_{j=-\infty}^{\infty} b_j b_{j+1} = \frac{1}{2}\sum_{j=-\infty}^{\infty}(b_{j+1} - b_j)^2 \\
&= \frac{1}{2}\sum_{j=0}^{N} A^2\left(\cos\left(j+1-\tfrac{N+1}{2}\right)\gamma - \cos\left(j-\tfrac{N+1}{2}\right)\gamma\right)^2 + \sum_{j=N+1}^{\infty} B^2(\alpha^{j+1-N} - \alpha^{j-N})^2 \\
&= 2A^2\sum_{j=0}^{N}\sin^2\tfrac{\gamma}{2}\sin^2\left(j-\tfrac{N}{2}\right)\gamma + B^2\sum_{j=N+1}^{\infty}\alpha^{2j-2N}(\alpha-1)^2 \\
&= A^2\sin^2\tfrac{\gamma}{2}\sum_{j=0}^{N}(1-\cos(2j-N)\gamma) + B^2(\alpha-1)^2\frac{\alpha^2}{1-\alpha^2} \\
&= A^2\sin^2\tfrac{\gamma}{2}\left(N+1-\frac{\sin(N+1)\gamma}{\sin\gamma}\right) + B^2(\alpha-1)^2\frac{\alpha^2}{1-\alpha^2} \\
&= 2(1-\epsilon^2)\sin^2\tfrac{\gamma}{2}\frac{(N+1)\sin\gamma - \sin(N+1)\gamma}{N\sin\gamma + \sin N\gamma} + \frac{1}{2}(\alpha-1)^2\epsilon^2,
\end{aligned} \tag{2.20}$$

by (2.13) and (2.12). Thus

$$\begin{aligned}
1 - \sum_{j=-\infty}^{\infty} b_j b_{j+1} &= 2\sin^2\tfrac{\gamma}{2}\frac{(N+1)\sin\gamma - \sin(N+1)\gamma}{N\sin\gamma + \sin N\gamma} + O(\epsilon^2) \\
&= 1 - \cos\tfrac{\pi}{N+1} - \tfrac{2\epsilon}{\sqrt{N+1}}\sin\tfrac{\pi}{N+1} + O(\epsilon^2),
\end{aligned}$$

on applying (2.19) and performing some simplification. Hence

$$\sum_{j=-\infty}^{\infty} b_j b_{j+1} = \cos\tfrac{\pi}{N+1} + \tfrac{2\epsilon}{\sqrt{N+1}} \sin\tfrac{\pi}{N+1} + O(\epsilon^2).$$

So some further calculation gives

$$\Delta_\theta(g)^2 = \frac{1-\left(\sum_{j=-\infty}^{\infty} b_j b_{j+1}\right)^2}{\left(\sum_{j=-\infty}^{\infty} b_j b_{j+1}\right)^2} = \tan^2\tfrac{\pi}{N+1}\left[1-\tfrac{4\epsilon}{\sqrt{N+1}}\left(\tan\tfrac{\pi}{N+1}+\cot\tfrac{\pi}{N+1}\right)\right] + O(\epsilon^2).$$

Thus we have proved the following.

Corollary 2.2. *Under the conditions of Theorem* 2.1*, for fixed* N*, as* $\epsilon \to 0$*,*

$$\Delta_\theta(f) \geq \tan\tfrac{\pi}{N+1}\left[1-\tfrac{2\epsilon}{\sqrt{N+1}}\left(\tan\tfrac{\pi}{N+1}+\cot\tfrac{\pi}{N+1}\right)\right] + O(\epsilon^2).$$

3. Time- or frequency-concentrated functions on the real line

Recall that for suitable f in $\mathcal{L}^2(\mathbb{R})$, $\Delta(\hat{f})^2 = \dfrac{\|f'\|^2\|f\|^2 - |\langle f', f\rangle|^2}{\|f\|^4}$. If f is real-valued, then $\langle f', f\rangle = \frac{1}{2}\int_{-\infty}^{\infty}(f^2)' = 0$ and so $\Delta(\hat{f}) = \dfrac{\|f'\|}{\|f\|}$.

Theorem 3.1. *Suppose* f *is in* $\mathcal{C}^2(\mathbb{R})$ *and* $|f(x)|$, $|f'(x)|$ *and* $|f''(x)|$ *are bounded by* $C(1+|x|)^{-\alpha}$ *for some constants* $C>0$, $\alpha > \frac{1}{2}$*. Take* $0 \leq \epsilon < 1$*, let* I *be an interval of length* $L>0$*, and suppose* $\int_{\mathbb{R}\setminus I} |f|^2 \leq \epsilon^2 \int_{-\infty}^{\infty} |f|^2$*. Then*

$$L\Delta(\hat{f}) \geq \pi\Delta(\hat{g}), \tag{3.1}$$

where g *is the unique nonnegative function satisfying*

$$\int_{-\infty}^{\infty} g^2 = 1, \qquad \int_{-\pi/2}^{\pi/2} g^2 = 1-\epsilon^2, \tag{3.2}$$

$$g(x) = \begin{cases} A\cos(\gamma x), & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \\ Be^{-\beta|x|}, & \textit{otherwise}, \end{cases} \tag{3.3}$$

for some numbers $A>0$, $B \geq 0$, $0<\gamma\leq 1$, $\beta>0$*, which is continuous and, if* $\epsilon>0$*, is also* $\mathcal{C}^1$*.*

Proof: Let $I = [a-\frac{L}{2}, a+\frac{L}{2}]$ and define $\widetilde{f}(x) = f(\frac{Lx}{\pi}+a)$. Then $\int_{|x|\geq\pi/2} |\widetilde{f}|^2 \leq \epsilon^2 \int_{-\infty}^{\infty} |\widetilde{f}|^2$ and $\Delta(\widehat{\widetilde{f}}) = \frac{L}{\pi}\Delta(\hat{f})$. Thus (3.1) becomes $\Delta(\widehat{\widetilde{f}}) \geq \Delta(\hat{g})$, i.e. we may assume $L = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and we need to prove $\Delta(\hat{f}) \geq \Delta(\hat{g})$. We may also assume $\int_{-\infty}^{\infty}|f|^2 = 1$.

Now suppose $f = u+iv$, where u and v are real-valued. Then $|f|^2 = u^2+v^2$ and so $|f|\,|f'| = uu'+vv'$. Thus

$$\int_{-\infty}^{\infty}(|f|')^2 = \int_{-\infty}^{\infty}\frac{(uu'+vv')^2}{u^2+v^2} = \int_{-\infty}^{\infty}(u')^2 + \int_{-\infty}^{\infty}(v')^2 - \int_{-\infty}^{\infty}\frac{(uv'-u'v)^2}{u^2+v^2}.$$

Now by the Cauchy-Schwarz inequality,

$$\left|\int_{-\infty}^{\infty}(uv'-u'v)\right|^2 \leq \int_{-\infty}^{\infty}\frac{(uv'-u'v)^2}{u^2+v^2}\int_{-\infty}^{\infty}(u^2+v^2),$$

and since $\int_{-\infty}^{\infty}(u^2+v^2)=\int_{-\infty}^{\infty}|f|^2=1$,

$$\int_{-\infty}^{\infty}(|f|')^2 \leq \int_{-\infty}^{\infty}(u')^2+\int_{-\infty}^{\infty}(v')^2-\left|\int_{-\infty}^{\infty}(uv'-u'v)\right|^2,$$

i.e. $\Delta(\widehat{|f|})\leq\Delta(\hat{f})$. Thus we may assume that f is real-valued and nonnegative, and we need to prove $\|f'\|\geq\|g'\|$.

Take $h>0$. By using Riemann sums, we can show as in [4] that

$$\lim_{h\to 0} h\sum_{j=-\infty}^{\infty} f(jh)^2=\int_{-\infty}^{\infty} f^2=1.$$

Write $a_{h,j}=f(jh)$ and choose h small enough so that $\sum_{j=-\infty}^{\infty}a_{h,j}^2<\infty$. Define N_h by

$$N_h:=\min\left\{N:\sum_{j=-N}^{N}a_{h,j}^2\geq\int_{-\pi/2}^{\pi/2}f^2\sum_{j=-\infty}^{\infty}a_{h,j}^2\right\}.$$

Then

$$\lim_{h\to 0}N_h h=\frac{\pi}{2}, \tag{3.4}$$

$$\lim_{h\to 0} h\sum_{j=-N_h}^{N_h}a_{h,j}^2=\int_{-\pi/2}^{\pi/2}f^2, \qquad \sum_{|j|\geq N_h+1}a_{h,j}^2\leq\epsilon^2\sum_{j=-\infty}^{\infty}a_{h,j}^2.$$

By Theorem 2.1 with $S=\{-N_h,\ldots,N_h\}$,

$$\Delta_h^2:=\frac{\left(\sum_{j=-\infty}^{\infty}a_{h,j}^2\right)^2-\left(\sum_{j=-\infty}^{\infty}a_{h,j}a_{h,j+1}\right)^2}{\left(\sum_{j=-\infty}^{\infty}a_{h,j}a_{h,j+1}\right)^2}\geq\Delta_\theta(g_h)^2,$$

where $g_h(t)=\sum_{j=-\infty}^{\infty}b_{h,j}e^{ijt}$ is as in Theorem 2.1 with $N=2N_h+1$. Now it can be shown as in [4] that

$$\lim_{h\to 0}h^{-2}\Delta_h^2=\|f'\|^2,$$

and so it is sufficient to show that

$$\lim_{h\to 0}h^{-2}\Delta_\theta(g_h)^2=\|g'\|^2. \tag{3.5}$$

First let $\epsilon=0$. Then by Corollary 2.1, $\Delta_\theta(g_h)=\tan\frac{\pi}{2(N_h+1)}$. So by (3.4),

$$\lim_{h\to 0}h^{-1}\Delta_\theta(g_h)=\lim_{N_h\to\infty}\tfrac{2N_h}{\pi}\tan\tfrac{\pi}{2(N_h+1)}=1. \tag{3.6}$$

On the other hand, by (3.2), g has support in $[-\frac{\pi}{2},\frac{\pi}{2}]$ and so

$$0=g\left(\tfrac{\pi}{2}\right)=A\cos\left(\gamma\tfrac{\pi}{2}\right),$$

by (3.3). Thus $\gamma = 1$ and by (3.2),

$$1 = \int_{-\pi/2}^{\pi/2} A^2 \cos^2 x \, dx = A^2 \frac{\pi}{2}.$$

Therefore $A = \sqrt{\frac{2}{\pi}}$ and

$$\|g'\|^2 = \int_{-\pi/2}^{\pi/2} \frac{2}{\pi} \sin^2 x \, dx = 1, \tag{3.7}$$

which, with (3.6), gives (3.5).

Now assume $\epsilon > 0$. Let A_h, B_h, γ_h, α_h equal A, B, γ, α as in Theorem 2.1 with $N = 2N_h + 1$. Then by (2.16), γ_h is the root of $F_h(\gamma_h) = 4(1-\epsilon^2)/\epsilon^2$, where F_h equals F as in (2.17), with N replaced by $2N_h + 1$. Suppose that $\{x_h : h > 0\}$ satisfies $\lim_{h\to 0} N_h x_h = \delta$, $0 \le \delta < \frac{\pi}{2}$. Then by (2.17),

$$\lim_{h\to 0} F_h(x_h) = \frac{\sin 2\delta\,(2\delta + \sin 2\delta)}{\cos^4 \delta} = U(\delta), \quad \text{say.} \tag{3.8}$$

Since U is increasing, $U(0) = 0$, $\lim_{\delta \to \frac{\pi}{2}-} U(\delta) = \infty$, there is a unique δ in $(0, \frac{\pi}{2})$ with

$$U(\delta) = \frac{4(1-\epsilon^2)}{\epsilon^2}. \tag{3.9}$$

Since from Theorem 2.1, $0 < 2(N_h+1)\gamma_h < \pi$, there is a sequence (h_n) with $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} N_{h_n}\gamma_{h_n} = \widetilde{\delta}$, for some $\widetilde{\delta} \in [0, \frac{\pi}{2}]$. Then $U(\widetilde{\delta}) = \lim_{n\to\infty} F_{h_n}(\gamma_{h_n}) = 4(1-\epsilon^2)/\epsilon^2$ and so $\widetilde{\delta} = \delta$ as in (3.9). Thus $\lim_{h\to 0} N_h \gamma_h = \delta$.

Now from (2.15),

$$\begin{aligned} \lim_{h\to 0} N_h(1-\alpha_h) &= \lim_{h\to 0} N_h \frac{\cos N_h\gamma_h - \cos(N_h+1)\gamma_h}{\cos N_h \gamma_h} \\ &= \lim_{h\to 0} \frac{2N_h \sin\frac{\gamma_h}{2} \sin(N_h + \frac{1}{2})\gamma_h}{\cos N_h\gamma_h} = \delta \tan\delta. \end{aligned}$$

Also, denoting $\tau_h := \sum_{j=-\infty}^{\infty} b_{h,j} b_{h,j+1}$, we have $\Delta_\theta(g_h)^2 = (1-\tau_h^2)/\tau_h^2$. By (2.20),

$$\begin{aligned} \lim_{h\to 0} N_h^2(1-\tau_h) &= 2(1-\epsilon^2) \lim_{h\to 0} N_h^2 \sin^2 \tfrac{\gamma_h}{2} \frac{2(N_h+1)\sin\gamma_h - \sin 2(N_h+1)\gamma_h}{(2N_h+1)\sin\gamma_h + \sin(2N_h+1)\gamma_h} \\ &\quad + \frac{1}{2}\epsilon^2 \lim_{h\to 0} N_h^2 (1-\alpha_h)^2 \\ &= (1-\epsilon^2)\frac{\delta^2}{2}\frac{2\delta - \sin 2\delta}{2\delta + \sin 2\delta} + \frac{1}{2}\epsilon^2\delta^2\tan^2\delta = V(\delta), \quad \text{say.} \end{aligned}$$

So by (3.4), $\lim_{h\to 0} \tau_h = 1$ and

$$\lim_{h\to 0} h^{-2}\Delta_\theta(g_h)^2 = \lim_{h\to 0} \frac{4}{\pi^2} N_h^2 \frac{(1-\tau_h^2)}{\tau_h^2} = \frac{8}{\pi^2} V(\delta).$$

Thus from (3.5) it remains to show that

$$\|g'\|^2 = \frac{8}{\pi^2} V(\delta).$$

From (3.3),

$$\int_{-\pi/2}^{\pi/2} g^2 = A^2 \int_{-\pi/2}^{\pi/2} \cos^2(\gamma x)\, dx = \frac{1}{2} A^2 \left(\pi + \frac{1}{\gamma}\sin(\gamma\pi)\right)$$

and so from (3.2),

$$A^2 \left(\pi + \frac{1}{\gamma}\sin(\gamma\pi)\right) = 2(1-\epsilon^2). \tag{3.10}$$

Also from (3.3),

$$\int_{\pi/2}^{\infty} g^2 = B^2 \int_{\pi/2}^{\infty} e^{-2\beta x}\, dx = \frac{B^2}{2\beta} e^{-\beta\pi}.$$

So from (3.2),

$$B^2 e^{-\beta\pi} = \beta\epsilon^2. \tag{3.11}$$

Since g is $\mathcal{C}^1$, (3.3) gives

$$A\cos\left(\gamma\tfrac{\pi}{2}\right) = Be^{-\beta\frac{\pi}{2}}, \qquad \gamma A \sin\left(\gamma\tfrac{\pi}{2}\right) = B\beta e^{-\beta\frac{\pi}{2}},$$

and hence

$$\gamma\tan\left(\gamma\tfrac{\pi}{2}\right) = \beta, \tag{3.12}$$

$$A^2 = B^2 \sec^2\left(\gamma\tfrac{\pi}{2}\right) e^{-\beta\pi} = \beta\epsilon^2 \sec^2\left(\gamma\tfrac{\pi}{2}\right),$$

by (3.11). So by (3.10) and (3.12),

$$\gamma\tan\left(\gamma\tfrac{\pi}{2}\right)\sec^2\left(\gamma\tfrac{\pi}{2}\right)\left(\pi + \frac{1}{\gamma}\sin(\gamma\pi)\right) = \frac{2(1-\epsilon^2)}{\epsilon^2},$$

which can be written as

$$U\left(\tfrac{\gamma\pi}{2}\right) = \frac{4(1-\epsilon^2)}{\epsilon^2},$$

with U as in (3.8). Hence it follows from (3.9) that $\delta = \frac{\gamma\pi}{2}$. From (3.3),

$$\begin{aligned} \|g'\|^2 &= A^2\gamma^2 \int_{-\pi/2}^{\pi/2} \sin^2(\gamma x)\, dx + 2B^2\beta^2 \int_{\pi/2}^{\infty} e^{-2\beta x}\, dx \\ &= \frac{1}{2}A^2\gamma^2\left(\pi - \frac{1}{\gamma}\sin(\gamma\pi)\right) + B^2\beta e^{-\beta\pi} = (1-\epsilon^2)\gamma^2 \frac{\pi - \frac{1}{\gamma}\sin(\gamma\pi)}{\pi + \frac{1}{\gamma}\sin(\gamma\pi)} + \beta^2\epsilon^2, \end{aligned}$$

by (3.10) and (3.11). So putting $\delta = \frac{\gamma\pi}{2}$ and using (3.12), we have

$$\|g'\|^2 = (1-\epsilon^2)\frac{4\delta^2}{\pi^2}\frac{2\delta - \sin 2\delta}{2\delta + \sin 2\delta} + \epsilon^2 \frac{4\delta^2}{\pi^2}\tan^2\delta = \frac{8}{\pi^2}V(\delta), \tag{3.13}$$

as desired. ■

Putting $\epsilon = 0$ and recalling (3.7) gives the following.

Corollary 3.1. *If f in $\mathcal{L}^2(\mathbb{R})$ is absolutely continuous with support an interval of length L and f' is in $\mathcal{L}^2(\mathbb{R})$, then*

$$L\,\Delta(\hat{f}) \geq \pi. \tag{3.14}$$

Since $\dfrac{\|f'\|}{\|f\|} \geq \Delta(\hat{f})$, (3.14) implies that

$$L\,\|f'\| \geq \pi\,\|f\|,$$

which is the well-known Wirtinger's inequality, see [1].

In practice we are often interested in functions which, corresponding to Section 2, are close to being band-limited, i.e. ϵ-concentrated in frequency. By replacing f in Theorem 3.1 by $\hat{f}$, we see that for f in $\mathcal{L}^2(\mathbb{R})$ such that $\hat{f}$ is in $\mathcal{C}^2(\mathbb{R})$ and $\hat{f}$, $\hat{f}'$, $\hat{f}''$ obey the appropriate decay conditions, then if $\int_{\mathbb{R}\setminus I} |\hat{f}|^2 \leq \epsilon^2 \int_{-\infty}^{\infty} |f|^2$ for an interval I of length $L > 0$, it follows that

$$L\,\Delta(f) \geq \pi\,\Delta(\hat{g}),$$

where g is as in Theorem 3.1.

Finally we shall derive an asymptotic formula for $\Delta(\hat{g})$ as $\epsilon \to 0$. Put $x = \frac{\pi}{2} - \delta$. Then as $\epsilon \to 0$, $x \to 0$ and

$$\frac{1}{U(\delta)} = \frac{\sin^4 x}{\sin 2x\,(\pi - 2x + \sin 2x)} = \frac{x^3}{2\pi}(1 + O(x^2)).$$

Thus by (3.9),

$$x^3(1 + O(x^2)) = \frac{\pi\epsilon^2}{2(1-\epsilon^2)}$$

and so

$$\epsilon^2 = \frac{2x^3}{\pi}(1 + O(x^2)),$$

giving

$$x = \left(\frac{\pi}{2}\right)^{1/3} \epsilon^{2/3}(1 + O(\epsilon^{4/3})).$$

Recalling $\delta = \frac{\pi}{2} - x$, (3.13) yields

$$\|g'\|^2 = 1 - 3\left(\frac{2\epsilon}{\pi}\right)^{2/3} + O(\epsilon^{4/3}).$$

Since $\Delta(\hat{g}) = \|g'\|$, we have the following.

Corollary 3.2. *Under the conditions of Theorem* 3.1, *as* $\epsilon \to 0$,

$$L\,\Delta(\hat{f}) \geq \pi\left(1 - \frac{3}{2}\left(\frac{2\epsilon}{\pi}\right)^{2/3}\right) + O(\epsilon^{4/3}).$$

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