EXTENSION PRINCIPLES FOR TIGHT WAVELET FRAMES OF PERIODIC FUNCTIONS

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Abstract

A unitary extension principle for constructing normalized tight wavelet frames of periodic functions of one or higher dimensions is established. While the wavelets are nonstationary, the method much simplifies their construction by reducing it to a matrix extension problem that involves finite rows of complex numbers. Further flexibility is achieved by reformulating the result as an oblique extension principle. In addition, with a constructive proof, necessary and sufficient conditions for a solution of the matrix extension problem are obtained. A complete characterization of all possible solutions is also provided. As illustration, parametric families of trigonometric polynomial tight wavelet frames are constructed.

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1. Introduction

Let s be a positive integer and $L^2([0,2\pi)^s)$ the space of all s-dimensional 2π-periodic square-integrable complex-valued functions over $\mathbb{R}^s$. Consider the inner product $\langle\,\cdot,\cdot\,\rangle$ of $L^2([0,2\pi)^s)$ given by $\langle\, f,g\,\rangle := \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} f(x)\overline{g(x)}\,dx$, where $f,g\in L^2([0,2\pi)^s)$, and denote its corresponding norm by $\|\cdot\| := \langle\,\cdot,\cdot\,\rangle^{1/2}$. For a function $f\in L^2([0,2\pi)^s)$, we express its Fourier series as $\sum_{n\in\mathbb{Z}^s}\widehat{f}(n)e^{in\cdot}$, where $\widehat{f}(n) := \langle\, f, e^{in\cdot}\,\rangle$, $n\in\mathbb{Z}^s$, are its Fourier coefficients.

Let A be an $s\times s$ matrix with integer entries such that all its eigenvalues lie outside the unit circle. The matrix A is known as a *dilation matrix*, and we set

$$D := A^T, \qquad d := |\det(A)| = |\det(D)|.$$

For $k\geq 0$, let $\mathcal{L}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/A^k\mathbb{Z}^s$ and $\mathcal{R}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/D^k\mathbb{Z}^s$. Then $d^k = |\mathcal{L}_k| = |\mathcal{R}_k|$. For $\ell\in\mathbb{Z}^s$, we define the $2\pi A^{-k}\ell$*-shift operator* $T_k^\ell : L^2([0,2\pi)^s)\longrightarrow L^2([0,2\pi)^s)$ by

$$T_k^\ell f := f(\cdot - 2\pi A^{-k}\ell), \quad f\in L^2([0,2\pi)^s).$$

Our aim is to construct functions ϕ_0^m, $m=1,2,\ldots,r$, and ψ_k^m, $k\geq 0$, $m=1,2,\ldots,\rho_k$, in $L^2([0,2\pi)^s)$, where r and ρ_k are positive integers, such that the collection $\{\,\phi_0^m : m=1,2,\ldots,r\,\}\cup\{\,T_k^\ell\psi_k^m : k\geq 0,\ m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{L}_k\,\}$ forms a *normalized tight wavelet frame* for the space $L^2([0,2\pi)^s)$, that is,

$$\|f\|^2 = \sum_{m=1}^{r} |\langle\, f,\phi_0^m\,\rangle|^2 + \sum_{k=0}^{\infty}\sum_{m=1}^{\rho_k}\sum_{\ell\in\mathcal{L}_k} |\langle\, f, T_k^\ell\psi_k^m\,\rangle|^2 \tag{1.1}$$

for all $f\in L^2([0,2\pi)^s)$. A normalized tight wavelet frame is also referred to as a *tight wavelet frame with bound* 1. It is a generalization of an orthonormal wavelet basis. The functions ψ_k^m are called *periodic wavelets*, or simply *wavelets*.

For positive integers ρ and r, we use the notation $\mathcal{S}(A^k)^{\rho\times r}$ to denote the class of all periodic sequences H_k of $\rho\times r$ complex matrices of period A^k, that is, $H_k(\ell + A^k p) = H_k(\ell)$ for all $\ell,p\in\mathbb{Z}^s$. Our construction of wavelets is based on *refinable functions* $\phi_k^m\in L^2([0,2\pi)^s)$, $k\geq 0$, $m=1,2,\ldots,r$, which satisfy the *periodic refinement equation*

$$\phi_k = \sum_{\ell\in\mathcal{L}_{k+1}} H_{k+1}(\ell)T_{k+1}^\ell\phi_{k+1} \tag{1.2}$$

for some $H_{k+1}\in\mathcal{S}(A^{k+1})^{r\times r}$, where $\phi_k := (\phi_k^1,\ldots,\phi_k^r)^T$, $k\geq 0$. Note that (1.2) is equivalent to

$$\widehat{\phi}_k(n) = \widehat{H}_{k+1}(n)\widehat{\phi}_{k+1}(n), \quad n\in\mathbb{Z}^s, \tag{1.3}$$

where $\widehat{\phi}_k(n) := (\widehat{\phi}_k^1(n),\ldots,\widehat{\phi}_k^r(n))^T$ and $\widehat{H}_{k+1}\in\mathcal{S}(D^{k+1})^{r\times r}$ denotes the discrete Fourier transform of H_{k+1} given by

$$\widehat{H}_{k+1}(j) := \sum_{\ell\in\mathcal{L}_{k+1}} H_{k+1}(\ell)e^{-ij\cdot(2\pi A^{-(k+1)}\ell)}, \quad j\in\mathcal{R}_{k+1}.$$

For every $k \geq 0$ and some positive integer ρ_k, the wavelets $\psi_k^m \in L^2([0,2\pi)^s)$, $m = 1, 2, \ldots, \rho_k$, are given by the *periodic wavelet equation*

$$\psi_k := \sum_{\ell \in \mathcal{L}_{k+1}} G_{k+1}(\ell) T_{k+1}^{\ell} \phi_{k+1}, \tag{1.4}$$

where $G_{k+1} \in \mathcal{S}(A^{k+1})^{\rho_k \times r}$, $\psi_k := (\psi_k^1, \ldots, \psi_k^{\rho_k})^T$. Analogous to (1.3), the equation (1.4) is equivalent to

$$\widehat{\psi}_k(n) = \widehat{G}_{k+1}(n) \widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}^s, \tag{1.5}$$

where $\widehat{\psi}_k(n) := (\widehat{\psi}_k^1(n), \ldots, \widehat{\psi}_k^{\rho_k}(n))^T$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ denotes the discrete Fourier transform of G_{k+1}. It is clear that there exist $H_{k+1} \in \mathcal{S}(A^{k+1})^{r \times r}$ and $G_{k+1} \in \mathcal{S}(A^{k+1})^{\rho_k \times r}$ for which (1.2) and (1.4) hold, if and only if

$$\phi_k^m, \psi_k^\mu \in \langle \{ T_{k+1}^{\ell} \phi_{k+1}^q : q = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_{k+1} \} \rangle \tag{1.6}$$

for all $m = 1, 2, \ldots, r$, $\mu = 1, 2, \ldots, \rho_k$.

The concept of frames first introduced in [11] for a general Hilbert space is gaining significant importance in wavelet analysis, especially during the past decade. In particular, the unitary extension principle obtained in [15] provides an elegant method for constructing tight wavelet frames, also known as *framelets*, for the space $L^2(\mathbb{R}^s)$ of all s-dimensional square-integrable complex-valued functions over $\mathbb{R}^s$. It assumes a matrix condition on the masks from the respective refinement and wavelet equations. Since its introduction, the unitary extension principle has generated much interest in the area (see for instance [2], [5], [6], [14], [16]) and led to the construction of compactly supported tight wavelet frames with desired properties such as symmetry and high approximation orders. The unitary extension principle also gave rise to the oblique extension principle, which is an even more flexible method of obtaining tight wavelet frames for $L^2(\mathbb{R}^s)$. This was introduced in [10] and also independently obtained in [7]. With these two extension principles, useful approaches are in place for constructing tight wavelet frames for $L^2(\mathbb{R}^s)$. Various authors have applied them to derive new examples of interest, some of which already found practical applications in signal and image processing (see for instance [3], [4]).

In [13], motivated by the fact that signals in practice are often periodic, tight wavelet frames for $L^2([0,2\pi)^s)$ were constructed. The underlying setting there is that for each $k \geq 0$, the refinable functions ϕ_k^m, $m = 1, 2, \ldots, r$, are orthogonal to the wavelets ψ_k^m, $m = 1, 2, \ldots, \rho_k$, where $r(d-1) \leq \rho_k \leq rd$. (When $s = 1$, $A = D = 2$ and $r = \rho_k = 1$, the results are periodic analogs of those in [1] for $L^2(\mathbb{R})$.) The approach is general, but further flexibility could be achieved if orthogonality between refinable functions and wavelets is relaxed. This is exactly the case for the unitary and oblique extension principles for $L^2(\mathbb{R}^s)$. In this paper, our objective is to develop analogous extension principles for $L^2([0,2\pi)^s)$, which will provide helpful tools for advancing periodic wavelet analysis in both theory and applications.

The matrices $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r \times r}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$, $k \geq 0$, in (1.3) and (1.5) play a central role in our extension principles for constructing tight wavelet frames of periodic functions. More specifically, for every $k \geq 0$ and some $\rho_k \geq r(d-1)$, we are concerned with the $(r + \rho_k) \times rd$

matrices

$$P_k(j) := \begin{pmatrix} \widehat{H}_{k+1}(j + D^k\ell_1) & \dots & \widehat{H}_{k+1}(j + D^k\ell_d) \\ \widehat{G}_{k+1}(j + D^k\ell_1) & \dots & \widehat{G}_{k+1}(j + D^k\ell_d) \end{pmatrix}, \quad j \in \mathcal{R}_k, \tag{1.7}$$

where $\ell_1, \dots, \ell_d$ denote all the elements of $\mathcal{R}_1$. Our focus is the condition

$$P_k(j)^* P_k(j) = dI_{rd}, \quad j \in \mathcal{R}_k, \tag{1.8}$$

which means that the columns of each of the matrices $\frac{1}{\sqrt{d}} P_k(j)$, $j \in \mathcal{R}_k$, are orthonormal. In contrast to the extension principles on $L^2(\mathbb{R}^s)$, the periodic situation is generally nonstationary, in the sense that different refinable functions and wavelets are involved for different levels k. (See [8] and [9] for recent studies of nonstationary wavelet frames of other function spaces.) As such, (1.8) has to be satisfied for every value of k. However, this periodic formulation also presents the nice simplification of handling (1.8) point by point from the finite set $\mathcal{R}_k$. This will enable easy construction of trigonometric polynomial frames which are well localized in the frequency domain.

The paper is organized as follows. In Section 2, we first provide insight to (1.8) by establishing some of its equivalent conditions. Then we show that under some mild additional condition, if (1.8) holds, the resulting collection $\{\, \phi_0^m : m = 1, 2, \dots, r \,\} \cup \{\, T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \dots, \rho_k,\ \ell \in \mathcal{L}_k \,\}$ forms a normalized tight wavelet frame for $L^2([0, 2\pi)^s)$. We name this result (Theorem 2.2) the unitary extension principle for $L^2([0, 2\pi)^s)$. Further flexibility for the above-mentioned collection to be a normalized tight wavelet frame is achievable if the matrix sequence $\widehat{G}_{k+1}$ for the entries of $P_k(j)$, $j \in \mathcal{R}_k$, in (1.7) can be appropriately modified. The oblique extension principle for $L^2([0, 2\pi)^s)$ (Theorem 3.1), as derived in Section 3, describes how $\widehat{G}_{k+1}$ should be suitably modified. Given the importance of the condition (1.8), Section 4 is devoted to finding $\widehat{G}_{k+1}$'s that are solutions to it. Necessary and sufficient conditions for a solution via a constructive proof are established, together with a complete characterization of all possible solutions. The case when $r = 1$ is further analyzed, with examples of parametric families of trigonometric polynomial tight wavelet frames constructed.

2. Unitary extension principle

Let r be a fixed positive integer. For each $k \geq 0$, let ϕ_k^m, $m = 1, 2, \dots, r$, be refinable functions in $L^2([0, 2\pi)^s)$. For $k \geq 0$, we define *polyphase splines* $v_{k,j}^m$ in $L^2([0, 2\pi)^s)$ by

$$v_{k,j}^m(x) := \sum_{p \in \mathbb{Z}^s} \widehat{\phi}_k^m(j + D^k p) e^{i(j + D^k p)\cdot x}, \quad x \in \mathbb{R}^s, \tag{2.1}$$

for $m = 1, 2, \dots, r$, $j \in \mathcal{R}_k$, where $\widehat{\phi}_k^m(n)$, $n \in \mathbb{Z}^s$, are the Fourier coefficients of ϕ_k^m. Recall from [12] and [13] that

$$\widehat{v}_{k,j}^m(n) = \begin{cases} \widehat{\phi}_k^m(j + D^k p), & \text{if } n = j + D^k p \text{ for some } p \in \mathbb{Z}^s, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

and

$$\langle\, v_{k,j}^m, v_{k,\nu}^\mu \,\rangle = \sum_{n \in \mathbb{Z}^s} \widehat{v}_{k,j}^m(n) \overline{\widehat{v}_{k,\nu}^\mu(n)} = 0 \quad \text{if } j \neq \nu,$$

for $m, \mu = 1, 2, \ldots, r$ and $j, \nu \in \mathcal{R}_k$. Further, the periodic refinement equation (1.2) is also equivalent to

$$v_{k,j} = \sum_{\ell \in \mathcal{R}_1} \widehat{H}_{k+1}(j + D^k \ell) v_{k+1, j+D^k\ell}\,, \quad j \in \mathcal{R}_k, \tag{2.3}$$

for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$, where $v_{k,j} := (v^1_{k,j}, \ldots, v^r_{k,j})^T$, $k \geq 0$.

For every $k \geq 0$ and some positive integer ρ_k, the desired wavelets $\psi^m_k \in L^2([0, 2\pi)^s)$, $m = 1, 2, \ldots, \rho_k$, are defined by the periodic wavelet equation (1.4). As shown in [13], (1.4) is also equivalent to

$$u_{k,j} = \sum_{\ell \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k \ell) v_{k+1, j+D^k\ell}\,, \quad j \in \mathcal{R}_k, \tag{2.4}$$

for some $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$, where $u_{k,j} := (u^1_{k,j}, \ldots, u^{\rho_k}_{k,j})^T$, and the $u^m_{k,j}$'s are the corresponding polyphase splines defined by

$$u^m_{k,j}(x) := \sum_{p \in \mathbb{Z}^s} \widehat{\psi}^m_k (j + D^k p) e^{i(j+D^k p)\cdot x}, \quad x \in \mathbb{R}^s, \tag{2.5}$$

for $m = 1, 2, \ldots, \rho_k$, $j \in \mathcal{R}_k$. Note that by (2.5), we have

$$\psi^m_k = \sum_{j \in \mathcal{R}_k} u^m_{k,j}, \quad m = 1, 2, \ldots, \rho_k. \tag{2.6}$$

Contrary to the basis case considered in [12], the matrices $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ need not be unique (see [13]). However it is possible to provide additional information about the values of these matrices over $\mathcal{R}_{k+1}$. To this end, for each $\nu \in \mathcal{R}_{k+1}$, note that the matrix

$$M_{k+1}(\nu) := (\, \langle\, v^m_{k+1,\nu}, v^\mu_{k+1,\nu} \,\rangle\,)^r_{m,\mu=1}$$

is Hermitian and positive semi-definite, and so there exists an $r \times r$ unitary matrix $U_{k+1}(\nu)$ such that

$$U_{k+1}(\nu) M_{k+1}(\nu) U_{k+1}(\nu)^* = \operatorname{diag}(\, \lambda^1_{k+1}(\nu), \ldots, \lambda^r_{k+1}(\nu)\,), \tag{2.7}$$

where $\lambda^m_{k+1}(\nu) \geq 0$, $m = 1, 2, \ldots, r$. Defining the vector $w_{k+1,\nu} = (w^1_{k+1,\nu}, \ldots, w^r_{k+1,\nu})^T$ by

$$w_{k+1,\nu} := U_{k+1}(\nu) v_{k+1,\nu}, \tag{2.8}$$

it follows from (2.7) that

$$(\, \langle\, w^m_{k+1,\nu}, w^\mu_{k+1,\nu} \,\rangle\,)^r_{m,\mu=1} = U_{k+1}(\nu) M_{k+1}(\nu) U_{k+1}(\nu)^* = \operatorname{diag}(\, \lambda^1_{k+1}(\nu), \ldots, \lambda^r_{k+1}(\nu)\,), \tag{2.9}$$

and thus

$$\lambda^m_{k+1}(\nu) = \|w^m_{k+1,\nu}\|^2, \quad m = 1, 2, \ldots, r. \tag{2.10}$$

Further, we may rewrite (2.3) and (2.4) as

$$v_{k,j} = \sum_{\ell \in \mathcal{R}_1} \widehat{H}_{k+1}(j + D^k \ell) U_{k+1}(j + D^k \ell)^* w_{k+1, j+D^k\ell}\,, \quad j \in \mathcal{R}_k, \tag{2.11}$$

and

$$u_{k,j} = \sum_{\ell \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k \ell) U_{k+1}(j + D^k \ell)^* w_{k+1, j+D^k\ell}\,, \quad j \in \mathcal{R}_k. \tag{2.12}$$

Proposition 2.1. *For $k \geq 0$, $j \in \mathcal{R}_k$ and $\ell \in \mathcal{R}_1$, let $U_{k+1}(j+D^k\ell)$ be an $r\times r$ unitary matrix as in* (2.7)*. For $m = 1, 2, \dots, r$, whenever $w^m_{k+1,j+D^k\ell} \neq 0$, the m-th columns of $\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ and $\widehat{G}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ are uniquely determined by the vectors*

$$\left(\frac{\langle\, v^1_{k,j}, w^m_{k+1,j+D^k\ell} \,\rangle}{\|w^m_{k+1,j+D^k\ell}\|^2}, \dots, \frac{\langle\, v^r_{k,j}, w^m_{k+1,j+D^k\ell} \,\rangle}{\|w^m_{k+1,j+D^k\ell}\|^2} \right)^T \tag{2.13}$$

and

$$\left(\frac{\langle\, u^1_{k,j}, w^m_{k+1,j+D^k\ell} \,\rangle}{\|w^m_{k+1,j+D^k\ell}\|^2}, \dots, \frac{\langle\, u^{\rho_k}_{k,j}, w^m_{k+1,j+D^k\ell} \,\rangle}{\|w^m_{k+1,j+D^k\ell}\|^2} \right)^T \tag{2.14}$$

respectively. On the other hand, if $w^m_{k+1,j+D^k\ell} = 0$, then the m-th columns of $\widehat{H}_{k+1}(j+D^k\ell)U_{k+1}(j+D^k\ell)^*$ and $\widehat{G}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ can be arbitrary vectors in $\mathbb{C}^r$ and $\mathbb{C}^{\rho_k}$ respectively.

Proof. Fix $k \geq 0$ and $j \in \mathcal{R}_k$. It follows from (2.11) that for $\mu = 1, 2, \dots, r$,

$$v^\mu_{k,j} = \sum_{\ell\in\mathcal{R}_1} \sum_{\zeta=1}^{r} \left(\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*\right)_{\mu,\zeta} w^\zeta_{k+1,j+D^k\ell}, \tag{2.15}$$

where $B_{\mu,\zeta}$ denotes the (μ, ζ)-entry of the matrix B. This shows that for $\ell \in \mathcal{R}_1$ and $m = 1, 2, \dots, r$, when $w^m_{k+1,j+D^k\ell} = 0$, the values $\left(\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*\right)_{\mu,m}$, $\mu = 1, 2, \dots, r$, can take any complex numbers, which means that the m-th column of $\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ may be arbitrarily chosen. However when $w^m_{k+1,j+D^k\ell} \neq 0$, taking inner product with $w^m_{k+1,j+D^k\ell}$, (2.15) gives

$$\langle\, v^\mu_{k,j}, w^m_{k+1,j+D^k\ell} \,\rangle = \left(\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*\right)_{\mu,m} \|w^m_{k+1,j+D^k\ell}\|^2$$

for $\mu = 1, 2, \dots, r$. Thus the m-th column of $\widehat{H}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ is exactly the vector in (2.13).

The respective vector in (2.14) for the matrix $\widehat{G}_{k+1}(j + D^k\ell)U_{k+1}(j + D^k\ell)^*$ follows from the same arguments when applied to (2.12). ■

In view of (2.10), it follows from Proposition 2.1 that for each $\nu \in \mathcal{R}_{k+1}$, the eigenvalues of $M_{k+1}(\nu)$ provide crucial information on which columns of $\widehat{H}_{k+1}(\nu)U_{k+1}(\nu)^*$ and $\widehat{G}_{k+1}(\nu)U_{k+1}(\nu)^*$ are uniquely determined and which columns could be arbitrarily chosen. This information is used in proving the following theorem which is the key to our desired unitary extension principle for $L^2([0, 2\pi)^s)$.

Theorem 2.1. *For $k \geq 0$, suppose that $\phi^m_k, \phi^m_{k+1}, \psi^\mu_k \in L^2([0, 2\pi)^s)$, $m = 1, 2, \dots, r$, $\mu = 1, 2, \dots, \rho_k$, satisfy* (1.6)*. Then the following are equivalent.*

(i) *There exist $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ such that* (1.3) *and* (1.5) *hold, and for each $j \in \mathcal{R}_k$, the $(r+\rho_k)\times rd$ matrix $P_k(j)$ as defined in* (1.7) *satisfies $P_k(j)^*P_k(j) = dI_{rd}$.*

(ii) *There holds*

$$\sum_{m=1}^{r} \sum_{\ell\in\mathcal{L}_{k+1}} |\,\langle\, f, T^\ell_{k+1}\phi^m_{k+1} \,\rangle\,|^2 = \sum_{m=1}^{r} \sum_{\ell\in\mathcal{L}_k} |\,\langle\, f, T^\ell_k\phi^m_k \,\rangle\,|^2 + \sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{L}_k} |\,\langle\, f, T^\ell_k\psi^m_k \,\rangle\,|^2 \tag{2.16}$$

for all $f \in L^2([0,2\pi)^s)$, *where* $\rho_k \geq r(d-1)$.

(iii) *The decomposition*

$$\sum_{m=1}^{r} \sum_{\ell \in \mathcal{L}_{k+1}} \langle f, T_{k+1}^{\ell} \phi_{k+1}^{m} \rangle \, T_{k+1}^{\ell} \phi_{k+1}^{m} = \sum_{m=1}^{r} \sum_{\ell \in \mathcal{L}_{k}} \langle f, T_{k}^{\ell} \phi_{k}^{m} \rangle \, T_{k}^{\ell} \phi_{k}^{m} + \sum_{m=1}^{\rho_k} \sum_{\ell \in \mathcal{L}_{k}} \langle f, T_{k}^{\ell} \psi_{k}^{m} \rangle \, T_{k}^{\ell} \psi_{k}^{m} \tag{2.17}$$

is valid for all $f \in L^2([0,2\pi)^s)$, *where* $\rho_k \geq r(d-1)$.

Proof. To simplify expressions in the proof, for $k \geq 0$, define linear operators S_k and E_k on $L^2([0,2\pi)^s)$ by

$$S_k f := \sum_{m=1}^{r} \sum_{\ell \in \mathcal{L}_k} \langle f, T_k^{\ell} \phi_k^m \rangle \, T_k^{\ell} \phi_k^m, \qquad E_k f := \sum_{m=1}^{\rho_k} \sum_{\ell \in \mathcal{L}_k} \langle f, T_k^{\ell} \psi_k^m \rangle \, T_k^{\ell} \psi_k^m, \quad f \in L^2([0,2\pi)^s). \tag{2.18}$$

Fix $k \geq 0$ and take arbitrary $f, g \in L^2([0,2\pi)^s)$. For $m = 1, 2, \ldots, r$, since $\phi_k^m = \sum_{j \in \mathcal{R}_k} v_{k,j}^m$, we have

$$T_k^{\ell} \phi_k^m = \sum_{j \in \mathcal{R}_k} e^{-ij \cdot (2\pi A^{-k} \ell)} \, v_{k,j}^m, \quad \ell \in \mathcal{L}_k.$$

Therefore, using the relation $\sum_{\ell \in \mathcal{L}_k} e^{i(j-\nu)\cdot(2\pi A^{-k}\ell)} = |\mathcal{L}_k| \, \delta_{j,\nu}$ for $j, \nu \in \mathcal{R}_k$, we see that

$$\langle S_k f, g \rangle = d^k \sum_{m=1}^{r} \sum_{j \in \mathcal{R}_k} \langle f, v_{k,j}^m \rangle \overline{\langle g, v_{k,j}^m \rangle} \tag{2.19}$$

since $|\mathcal{L}_k| = d^k$. Similarly, we have

$$\langle E_k f, g \rangle = d^k \sum_{m=1}^{\rho_k} \sum_{j \in \mathcal{R}_k} \langle f, u_{k,j}^m \rangle \overline{\langle g, u_{k,j}^m \rangle}. \tag{2.20}$$

For $f \in L^2([0,2\pi)^s)$ and $j \in \mathcal{R}_k$, we define the row vectors

$$\begin{aligned} a_{k,j}(f;\ell) &:= (\langle f, v^1_{k+1,j+D^k\ell} \rangle, \ldots, \langle f, v^r_{k+1,j+D^k\ell} \rangle), \quad \ell \in \mathcal{R}_1, \\ b_{k,j}(f) &:= (\langle f, v^1_{k,j} \rangle, \ldots, \langle f, v^r_{k,j} \rangle), \\ c_{k,j}(f) &:= (\langle f, u^1_{k,j} \rangle, \ldots, \langle f, u^{\rho_k}_{k,j} \rangle). \end{aligned}$$

In view of (1.6), there exist $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r \times r}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ such that (2.3) and (2.4) hold. For all such matrices,

$$b_{k,j}(f) = \sum_{\ell \in \mathcal{R}_1} a_{k,j}(f;\ell) \, \widehat{H}_{k+1}(j + D^k \ell)^*, \qquad c_{k,j}(f) = \sum_{\ell \in \mathcal{R}_1} a_{k,j}(f;\ell) \, \widehat{G}_{k+1}(j + D^k \ell)^*. \tag{2.21}$$

Now define the column vectors

$$\beta_{k,j}(f) := (b_{k,j}(f), c_{k,j}(f))^T, \qquad \alpha_{k,j}(f) := (a_{k,j}(f;\ell_1), \ldots, a_{k,j}(f;\ell_d))^T,$$

where $\mathcal{R}_1 = \{ \ell_1, \ldots, \ell_d \}$. Then the equations in (2.21) imply that

$$\beta_{k,j}(f) = \overline{P_k(j)} \, \alpha_{k,j}(f), \tag{2.22}$$

where $\overline{B}$ denotes the conjugate of the matrix B.

Hence for any $f, g \in L^2([0,2\pi)^s)$, it follows from (2.19), (2.20) and (2.22) that

$$\begin{aligned} \langle S_k f, g\rangle + \langle E_k f, g\rangle &= d^k \sum_{j\in\mathcal{R}_k} \Big(\sum_{m=1}^{r} \langle f, v_{k,j}^m\rangle \overline{\langle g, v_{k,j}^m\rangle} + \sum_{m=1}^{\rho_k} \langle f, u_{k,j}^m\rangle \overline{\langle g, u_{k,j}^m\rangle} \Big) \\ &= d^k \sum_{j\in\mathcal{R}_k} \beta_{k,j}(g)^* \beta_{k,j}(f) \\ &= d^k \sum_{j\in\mathcal{R}_k} \alpha_{k,j}(g)^* \overline{P_k(j)^*}\, \overline{P_k(j)} \alpha_{k,j}(f). \end{aligned} \tag{2.23}$$

We also have

$$\begin{aligned} \langle S_{k+1} f, g\rangle &= d^{k+1} \sum_{m=1}^{r} \sum_{j\in\mathcal{R}_{k+1}} \langle f, v_{k+1,j}^m\rangle \overline{\langle g, v_{k+1,j}^m\rangle} \\ &= d^{k+1} \sum_{m=1}^{r} \sum_{j\in\mathcal{R}_k} \sum_{\ell\in\mathcal{R}_1} \langle f, v_{k+1,j+D^k\ell}^m\rangle \overline{\langle g, v_{k+1,j+D^k\ell}^m\rangle} \\ &= d^{k+1} \sum_{j\in\mathcal{R}_k} \alpha_{k,j}(g)^* \alpha_{k,j}(f). \end{aligned} \tag{2.24}$$

We are now ready to establish the equivalence of (i) and (ii). If $P_k(j)^* P_k(j) = dI_{rd}$ for all $j \in \mathcal{R}_k$, then by (2.23) and (2.24), for any $f, g \in L^2([0,2\pi)^s)$,

$$\langle S_{k+1} f, g\rangle = \langle S_k f, g\rangle + \langle E_k f, g\rangle. \tag{2.25}$$

Choosing $f = g$ gives (2.16). Further, for each $j \in \mathcal{R}_k$, since the $(r+\rho_k)\times rd$ matrix $\frac{1}{\sqrt{d}} P_k(j)$ has orthonormal columns, $r + \rho_k \ge rd$ and so $\rho_k \ge r(d-1)$.

Conversely, suppose that (2.16) holds for all $f \in L^2([0,2\pi)^s)$ with $\rho_k \ge r(d-1)$. For any $f, g \in L^2([0,2\pi)^s)$, applying (2.16) to f, g and then $f+g$, we obtain after a straightforward calculation

$$\mathrm{Re}\{\langle S_{k+1} f, g\rangle\} = \mathrm{Re}\{\langle S_k f, g\rangle\} + \mathrm{Re}\{\langle E_k f, g\rangle\}. \tag{2.26}$$

Replacing g by ig in (2.26), and then combining with (2.26) yields (2.25) which, in terms of (2.23) and (2.24), implies that

$$d \sum_{j\in\mathcal{R}_k} \alpha_{k,j}(g)^* \alpha_{k,j}(f) = \sum_{j\in\mathcal{R}_k} \alpha_{k,j}(g)^* \overline{P_k(j)^*}\, \overline{P_k(j)} \alpha_{k,j}(f) \tag{2.27}$$

for all $f, g \in L^2([0,2\pi)^s)$.

Let $U_{k+1}(\nu)$, $\nu \in \mathcal{R}_{k+1}$, be $r\times r$ unitary matrices as in (2.7). For $j \in \mathcal{R}_k$ and $\ell \in \mathcal{R}_1$, define $w_{k+1,j+D^k\ell} = (w_{k+1,j+D^k\ell}^1, \ldots, w_{k+1,j+D^k\ell}^r)^T$ by (2.8). Then for any $f \in L^2([0,2\pi)^s)$, we obtain

$$d_{k,j}(f;\ell) := (\langle f, w_{k+1,j+D^k\ell}^1\rangle, \ldots, \langle f, w_{k+1,j+D^k\ell}^r\rangle) = a_{k,j}(f;\ell) U_{k+1}(j + D^k\ell)^*. \tag{2.28}$$

Thus setting

$$\gamma_{k,j}(f) := (d_{k,j}(f;\ell_1), \ldots, d_{k,j}(f;\ell_d))^T, \tag{2.29}$$

it follows that

$$\gamma_{k,j}(f) = \mathrm{diag}\big(\overline{U_{k+1}(j + D^k\ell_1)}, \ldots, \overline{U_{k+1}(j + D^k\ell_d)}\big)\, \alpha_{k,j}(f).$$

Consequently, since $U_{k+1}(\nu)$, $\nu \in \mathcal{R}_{k+1}$, are unitary matrices, (2.27) can be rewritten as

$$
\begin{aligned}
d \sum_{j\in\mathcal{R}_k} \gamma_{k,j}(g)^* \gamma_{k,j}(f) &= \sum_{j\in\mathcal{R}_k} \gamma_{k,j}(g)^* \operatorname{diag}\big(\overline{U_{k+1}(j+D^k\ell_1)},\dots,\overline{U_{k+1}(j+D^k\ell_d)}\,\big)\,\overline{P_k(j)^*} \\
&\qquad\qquad \times \overline{P_k(j)}\operatorname{diag}\big(\overline{U_{k+1}(j+D^k\ell_1)}^*,\dots,\overline{U_{k+1}(j+D^k\ell_d)}^*\,\big)\,\gamma_{k,j}(f) \\
&= \sum_{j\in\mathcal{R}_k} \gamma_{k,j}(g)^* \overline{C_k(j)^*}\,\overline{C_k(j)}\gamma_{k,j}(f) \qquad (2.30)
\end{aligned}
$$

for all $f, g \in L^2([0,2\pi)^s)$, where for $j \in \mathcal{R}_k$, $C_k(j)$ is the $(r+\rho_k)\times rd$ matrix

$$
\begin{aligned}
C_k(j) &:= P_k(j)\operatorname{diag}\big(U_{k+1}(j+D^k\ell_1)^*,\dots,U_{k+1}(j+D^k\ell_d)^*\,\big) \qquad (2.31)\\
&= \begin{pmatrix} \widehat{H}_{k+1}(j+D^k\ell_1)U_{k+1}(j+D^k\ell_1)^* & \dots & \widehat{H}_{k+1}(j+D^k\ell_d)U_{k+1}(j+D^k\ell_d)^* \\ \widehat{G}_{k+1}(j+D^k\ell_1)U_{k+1}(j+D^k\ell_1)^* & \dots & \widehat{G}_{k+1}(j+D^k\ell_d)U_{k+1}(j+D^k\ell_d)^* \end{pmatrix}.
\end{aligned}
$$

For each $j \in \mathcal{R}_k$ and $\ell \in \mathcal{R}_1$, writing as column vectors, we have

$$
\begin{pmatrix} \widehat{H}_{k+1}(j+D^k\ell)U_{k+1}(j+D^k\ell)^* \\ \widehat{G}_{k+1}(j+D^k\ell)U_{k+1}(j+D^k\ell)^* \end{pmatrix} = \big(C^1_{k+1}(j,\ell)\ \dots\ C^r_{k+1}(j,\ell)\big),
$$

where $C^m_{k+1}(j,\ell) \in \mathbb{C}^{r+\rho_k}$ for $m = 1,2,\dots,r$. Thus this enables us to enumerate

$$
C_k(j) = \big(C^1_{k+1}(j,\ell_1)\ \dots\ C^r_{k+1}(j,\ell_1)\,|\ \dots\ |\,C^1_{k+1}(j,\ell_d)\ \dots\ C^r_{k+1}(j,\ell_d)\big). \qquad (2.32)
$$

Now fix $j \in \mathcal{R}_k$, and take $g = w^m_{k+1,j+D^k\ell}$ and $f = w^\mu_{k+1,j+D^k\eta}$, where $m,\mu = 1,2,\dots,r$ and $\ell,\eta \in \mathcal{R}_1$. Then it follows from (2.9), (2.28) and (2.29) that each of $\gamma_{k,j}(g)$ and $\gamma_{k,j}(f)$ is a column vector with at most one nonzero entry. Thus (2.30) gives

$$
d\,\|w^m_{k+1,j+D^k\ell}\|^4\,\delta_{\ell,\eta}\delta_{m,\mu} = \|w^m_{k+1,j+D^k\ell}\|^2\,\overline{C^m_{k+1}(j,\ell)^*}\,\overline{C^\mu_{k+1}(j,\eta)}\,\|w^\mu_{k+1,j+D^k\eta}\|^2 \qquad (2.33)
$$

for all $m,\mu = 1,2,\dots,r$ and $\ell,\eta \in \mathcal{R}_1$. By Proposition 2.1, for $\ell \in \mathcal{R}_1$ and $m = 1,2,\dots,r$, if $w^m_{k+1,j+D^k\ell} \neq 0$, the column vector $C^m_{k+1}(j,\ell)$ is uniquely determined, whereas otherwise it could be any vector in $\mathbb{C}^{r+\rho_k}$.

Let us examine closer those columns of $C_k(j)$ as defined in (2.32) that are uniquely determined. Consider $m,\mu \in \{1,2,\dots,r\}$ and $\ell,\eta \in \mathcal{R}_1$ such that $w^m_{k+1,j+D^k\ell} \neq 0$ and $w^\mu_{k+1,j+D^k\eta} \neq 0$. Then by (2.33), $C^m_{k+1}(j,\ell)^* C^\mu_{k+1}(j,\eta) = d\,\delta_{\ell,\eta}\delta_{m,\mu}$. This implies that all the uniquely determined columns of $\frac{1}{\sqrt{d}}C_k(j)$ are orthonormal. The remaining columns of $\frac{1}{\sqrt{d}}C_k(j)$, which can be arbitrarily chosen, are selected to ensure that the entire matrix $\frac{1}{\sqrt{d}}C_k(j)$ has orthonormal columns. This is possible since $r + \rho_k \geq rd$.

With this choice of the columns of $C_k(j)$, we obtain the matrix values $\widehat{H}_{k+1}(j+D^k\ell)$ and $\widehat{G}_{k+1}(j+D^k\ell)$, where $\ell \in \mathcal{R}_1$, via (2.31), that is

$$
\begin{pmatrix} \widehat{H}_{k+1}(j+D^k\ell_1) & \dots & \widehat{H}_{k+1}(j+D^k\ell_d) \\ \widehat{G}_{k+1}(j+D^k\ell_1) & \dots & \widehat{G}_{k+1}(j+D^k\ell_d) \end{pmatrix} = C_k(j)\operatorname{diag}\big(U_{k+1}(j+D^k\ell_1),\dots,U_{k+1}(j+D^k\ell_d)\big).
$$

Since $C_k(j)^*C_k(j) = dI_{rd}$, it also follows from (2.31) that $P_k(j)^*P_k(j) = dI_{rd}$.

As for the equivalence of (ii) and (iii), if (2.16) holds for all $f \in L^2([0,2\pi)^s)$, as seen earlier, this implies that (2.25) is valid for any $f, g \in L^2([0,2\pi)^s)$. Rewriting (2.25) and then choosing $g = S_{k+1}f - S_kf - E_kf$, we obtain (2.17). On the other hand, taking inner product of both sides of (2.17) with $f \in L^2([0,2\pi)^s)$ yields (2.16). This completes the proof of the theorem. ■

For the special setting of $s = r = 1$ and $A = D = 2$, at first glance, the equivalence of (i) and (ii) in Theorem 2.1 seems to be the periodic analog of [6, Lemma 1]. This is not exactly the case as [6, Lemma 1] for $L^2(\mathbb{R})$ assumes and uses the hypothesis that the refinable function concerned is compactly supported. A periodic analog of this assumption is not made in Theorem 2.1, and instead we deal with the more general situation where some of the $\lambda^m_{k+1}(\nu)$'s in (2.7) could be zero. This subtlety is reflected in the nonunique values of the matrices $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$. Theorem 2.1 implies that whenever (2.16) holds, it is always possible to find such a pair of $\widehat{H}_{k+1}$ and $\widehat{G}_{k+1}$ for (1.8).

On a different perspective, the condition (1.8) will be our starting point of constructing tight wavelet frames for $L^2([0,2\pi)^s)$. Its characterizations provided by Theorem 2.1 will be instrumental in the derivation and also give insight to the wavelet frames constructed by our main result, which we call the *unitary extension principle* (UEP) for $L^2([0,2\pi)^s)$.

Theorem 2.2. *Suppose that $\phi^m_k \in L^2([0,2\pi)^s)$, $k \geq 0$, $m = 1,2,\ldots,r$, satisfy* (1.3) *for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$, and*

$$\lim_{k\to\infty} d^k \sum_{m=1}^{r} |\,\widehat{\phi}^m_k(n)\,|^2 = 1, \quad n \in \mathbb{Z}^s. \tag{2.34}$$

For every $k \geq 0$ and some positive integer $\rho_k \geq r(d-1)$, let $\psi^m_k \in L^2([0,2\pi)^s)$, $m = 1,2,\ldots,\rho_k$, be as defined in (1.5)*, where $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$. Suppose that for each $j \in \mathcal{R}_k$, the $(r+\rho_k)\times rd$ matrix $P_k(j)$ as defined in* (1.7) *satisfies $P_k(j)^*P_k(j) = dI_{rd}$. Then the collection $\{\,\phi^m_0 : m = 1,2,\ldots,r\,\} \cup \{\,T^\ell_k\psi^m_k : k \geq 0,\ m = 1,2,\ldots,\rho_k,\ \ell \in \mathcal{L}_k\,\}$ forms a normalized tight wavelet frame for $L^2([0,2\pi)^s)$.*

For the proof of Theorem 2.2, in addition to Theorem 2.1, we need the following lemma on the implication of the condition (2.34).

Lemma 2.1. *Suppose that $\phi^m_k \in L^2([0,2\pi)^s)$, $k \geq 0$, $m = 1,2,\ldots,r$, satisfy* (2.34)*. Let f be an arbitrary trigonometric polynomial. Then for any given $\epsilon > 0$, there exists $K \geq 0$ such that for all $k \geq K$,*

$$(1-\epsilon)\|f\|^2 \leq \sum_{m=1}^{r}\sum_{\ell\in\mathcal{L}_k} |\,\langle\, f, T^\ell_k\phi^m_k\,\rangle\,|^2 \leq (1+\epsilon)\|f\|^2. \tag{2.35}$$

Proof. Let S denote the support of the Fourier coefficients $\{\widehat{f}(n)\}_{n\in\mathbb{Z}^s}$ of f. Note that $S \subset \mathbb{Z}^s$ is a finite set since f is a trigonometric polynomial. Consequently, it follows from (2.34) that there exists $K_0 \geq 0$ such that for all $k \geq K_0$,

$$1-\epsilon \leq d^k \sum_{m=1}^{r} |\,\widehat{\phi}^m_k(n)\,|^2 \leq 1+\epsilon \tag{2.36}$$

for all $n \in S$.

Now by Parseval's identity and (2.2), we have

$$\langle f, v_{k,j}^m \rangle = \sum_{n\in\mathbb{Z}^s} \widehat{f}(n)\overline{\widehat{v_{k,j}^m}(n)} = \sum_{p\in\mathbb{Z}^s} \widehat{f}(j+D^k p)\overline{\widehat{\phi}_k^m(j+D^k p)}, \quad j\in\mathcal{R}_k. \tag{2.37}$$

As S is a finite set, there exists a positive number N such that $S \subseteq B(N) := \{ q\in\mathbb{Z}^s : \|q\| \le N \}$. The proof of [12, Theorem 2.3] shows that there exists $K_1 \ge 0$ such that for all $k \ge K_1$, the elements of $B(N)$ lie in different cosets of $\mathbb{Z}^s/D^k\mathbb{Z}^s$. Thus the cardinality of $S\cap(j+D^k\mathbb{Z}^s)$ is at most 1 for each $k\ge K_1$, $j\in\mathcal{R}_k$. Consequently, (2.37) implies that

$$\sum_{j\in\mathcal{R}_k} |\langle f, v_{k,j}^m \rangle|^2 = \sum_{n\in S} |\widehat{f}(n)|^2 |\widehat{\phi}_k^m(n)|^2.$$

Hence for all $k\ge K_1$, we have by (2.19) that

$$\sum_{m=1}^{r}\sum_{\ell\in\mathcal{L}_k} |\langle f, T_k^\ell \phi_k^m \rangle|^2 = d^k \sum_{m=1}^{r}\sum_{j\in\mathcal{R}_k} |\langle f, v_{k,j}^m \rangle|^2 = \sum_{n\in S}\Big(|\widehat{f}(n)|^2\, d^k \sum_{m=1}^{r} |\widehat{\phi}_k^m(n)|^2 \Big). \tag{2.38}$$

Let $K=\max\{K_0,K_1\}$. Then by (2.36) and (2.38), we obtain

$$(1-\epsilon)\sum_{n\in S}|\widehat{f}(n)|^2 \le \sum_{m=1}^{r}\sum_{\ell\in\mathcal{L}_k} |\langle f, T_k^\ell\phi_k^m\rangle|^2 \le (1+\epsilon)\sum_{n\in S}|\widehat{f}(n)|^2$$

for all $k\ge K$. As $\|f\|^2 = \sum_{n\in\mathbb{Z}^s}|\widehat{f}(n)|^2 = \sum_{n\in S}|\widehat{f}(n)|^2$, this leads to (2.35). ■

Proof of Theorem 2.2. We need to show that the collection $\{ \phi_0^m : m=1,2,\ldots,r \} \cup \{ T_k^\ell \psi_k^m : k\ge 0,\ m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{L}_k \}$ satisfies (1.1) for all $f\in L^2([0,2\pi)^s)$. Since the set of all trigonometric polynomials is dense in $L^2([0,2\pi)^s)$, it suffices to show that (1.1) holds for all trigonometric polynomials f.

Fix an arbitrary trigonometric polynomial f. Applying Theorem 2.1 repeatedly, we obtain

$$\sum_{m=1}^{r}\sum_{\ell\in\mathcal{L}_k} |\langle f, T_k^\ell\phi_k^m\rangle|^2 = \sum_{m=1}^{r} |\langle f, \phi_0^m\rangle|^2 + \sum_{\nu=0}^{k-1}\sum_{m=1}^{\rho_\nu}\sum_{\ell\in\mathcal{L}_\nu} |\langle f, T_\nu^\ell\psi_\nu^m\rangle|^2.$$

Thus by (2.35) in Lemma 2.1, we see that for all $k\ge K$,

$$(1-\epsilon)\|f\|^2 \le \sum_{m=1}^{r} |\langle f, \phi_0^m\rangle|^2 + \sum_{\nu=0}^{k-1}\sum_{m=1}^{\rho_\nu}\sum_{\ell\in\mathcal{L}_\nu} |\langle f, T_\nu^\ell\psi_\nu^m\rangle|^2 \le (1+\epsilon)\|f\|^2.$$

Hence by letting $k\to\infty$, we get

$$(1-\epsilon)\|f\|^2 \le \sum_{m=1}^{r} |\langle f, \phi_0^m\rangle|^2 + \sum_{\nu=0}^{\infty}\sum_{m=1}^{\rho_\nu}\sum_{\ell\in\mathcal{L}_\nu} |\langle f, T_\nu^\ell\psi_\nu^m\rangle|^2 \le (1+\epsilon)\|f\|^2.$$

Since ϵ is arbitrary, this implies that (1.1) holds for every trigonometric polynomial f, and the result follows. ■

For $k\ge 0$, let ϕ_k^m, $m=1,2,\ldots,r$, and ψ_k^m, $m=1,2,\ldots,\rho_k$, be the refinable functions and wavelets in Theorem 2.2. Define

$$V_k := \langle \{ T_k^\ell\phi_k^m : m=1,2,\ldots,r,\ \ell\in\mathcal{L}_k \} \rangle, \qquad W_k := \langle \{ T_k^\ell\psi_k^m : m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{L}_k \} \rangle. \tag{2.39}$$

The sequence of subspaces $\{V_k\}_{k\geq 0}$ of $L^2([0,2\pi)^s)$ then forms a *multiresolution analysis* (MRA) *of* $L^2([0,2\pi)^s)$ *with multiplicity* r *and dilation matrix* A, that is, $V_k \subseteq V_{k+1}$ for $k \geq 0$ and $\overline{\bigcup_{k\geq 0} V_k} = L^2([0,2\pi)^s)$. The former holds because of the periodic refinement equation (1.2). The latter follows from the proof of Theorem 2.2 via (2.16) and (1.1).

Note that the subspaces as defined in (2.39) satisfy

$$V_{k+1} = V_k + W_k, \quad k \geq 0. \tag{2.40}$$

Indeed, for $k \geq 0$, recall from [13] that

$$V_k = \langle \{ v^m_{k,j} : m = 1,2,\dots,r,\ j \in \mathcal{R}_k \} \rangle, \qquad W_k = \langle \{ u^m_{k,j} : m = 1,2,\dots,\rho_k,\ j \in \mathcal{R}_k \} \rangle. \tag{2.41}$$

By (2.3) and (2.4), it is clear that $V_k + W_k \subseteq V_{k+1}$. In addition, after some straightforward manipulation, they lead to

$$(v^T_{k,j}, u^T_{k,j})^T = P_k(j) (v^T_{k+1,j+D^k\ell_1}, \dots, v^T_{k+1,j+D^k\ell_d})^T, \quad j \in \mathcal{R}_k,$$

where $P_k(j)$, $j \in \mathcal{R}_k$, are as in (1.7) and $\mathcal{R}_1 = \{ \ell_1, \dots, \ell_d \}$. Then it follows from (1.8) that

$$d(v^T_{k+1,j+D^k\ell_1}, \dots, v^T_{k+1,j+D^k\ell_d})^T = P_k(j)^* (v^T_{k,j}, u^T_{k,j})^T, \quad j \in \mathcal{R}_k.$$

This together with (2.41) give $V_{k+1} \subseteq V_k + W_k$, and hence (2.40) holds.

The subspace decomposition (2.40) presents a more general setting than in [13] where for each $k \geq 0$, V_{k+1} is the orthogonal direct sum of V_k and W_k. This allows us to have greater flexibility in constructing tight wavelet frames for $L^2([0,2\pi)^s)$. As noted in Example 4.1 of Section 4, the sum in (2.40) need not even be a direct sum and $\dim(V_k \cap W_k)$ could be nonzero.

In view of Theorem 2.1, a tight wavelet frame $\{ \phi^m_0 : m = 1,2,\dots,r \} \cup \{ T^\ell_k \psi^m_k : k \geq 0,\ m = 1,2,\dots,\rho_k,\ \ell \in \mathcal{L}_k \}$ constructed from Theorem 2.2 possesses the property that (2.17) holds for all $k \geq 0$ and $f \in L^2([0,2\pi)^s)$. Following [6], we say that such a frame is a *minimum-energy wavelet frame* associated with the MRA $\{V_k\}_{k\geq 0}$. The motivation of this is as follows. For $k \geq 0$, let S_k be the linear operator from $L^2([0,2\pi)^s)$ into V_k as defined in (2.18). For each $f \in L^2([0,2\pi)^s)$, (2.17) becomes

$$S_{k+1}f - S_k f = \sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{L}_k} \langle f, T^\ell_k \psi^m_k \rangle T^\ell_k \psi^m_k,$$

which is an expansion in W_k of the difference between the images of the respective operators into V_{k+1} and V_k. Suppose that this difference is represented by another expansion in W_k, say

$$S_{k+1}f - S_k f = \sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{L}_k} c^m_k(\ell)\, T^\ell_k \psi^m_k$$

for some $c^m_k(\ell) \in \mathbb{C}$, $m = 1,2,\dots,\rho_k$, $\ell \in \mathcal{L}_k$. Then proceeding as in [6] for the $L^2(\mathbb{R})$-case, we obtain

$$\sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{L}_k} | \langle f, T^\ell_k \psi^m_k \rangle |^2 \leq \sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{L}_k} |c^m_k(\ell)|^2,$$

demonstrating that (2.17) gives a minimum-energy decomposition.

3. Oblique extension principle

For $k \geq 0$, let $\phi_k^m \in L^2([0,2\pi)^s)$, $m = 1,2,\ldots,r$, be refinable functions satisfying (1.3) for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$, and $\psi_k^m \in L^2([0,2\pi)^s)$, $m = 1,2,\ldots,\rho_k$, be defined by (1.5) for some $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$, where $\rho_k \geq r(d-1)$. In order to apply the UEP, we need to show that the matrices $P_k(j)$, $j \in \mathcal{R}_k$, in (1.7) satisfy (1.8). However, there are other sufficient conditions of similar form as (1.8) that ensure the collection $\{\, \phi_0^m : m = 1,2,\ldots,r \,\} \cup \{\, T_k^\ell \psi_k^m : k \geq 0,\ m = 1,2,\ldots,\rho_k,\ \ell \in \mathcal{L}_k \,\}$ to be a normalized tight wavelet frame for $L^2([0,2\pi)^s)$. These conditions involve matrices obtained from pre- and post-multiplying appropriate matrices to $\widehat{H}_{k+1}$ and $\widehat{G}_{k+1}$. This forms the basis of our next result, which we call the *oblique extension principle* (OEP) for $L^2([0,2\pi)^s)$. Specializing it to $r = 1$ gives the periodic analog of the extension principle for $L^2(\mathbb{R}^s)$ introduced independently in both [7] and [10].

Theorem 3.1. *Suppose that $\phi_k^m \in L^2([0,2\pi)^s)$, $k \geq 0$, $m = 1,2,\ldots,r$, satisfy* (1.3) *for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$, and* (2.34) *holds. For every $k \geq 0$ and some positive integer $\rho_k \geq r(d-1)$, let $\psi_k^m \in L^2([0,2\pi)^s)$, $m = 1,2,\ldots,\rho_k$, be as defined in* (1.5)*, where $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$. Assume that $\widehat{\Theta}_k \in \mathcal{S}(D^k)^{r\times r}$, $k \geq 0$, satisfy the conditions: $\widehat{\Theta}_k(j)$ is invertible for every $k \geq 0$ and $j \in \mathcal{R}_k$;*

$$\lim_{k\to\infty} \widehat{\Theta}_k(n)^* \widehat{\Theta}_k(n) = I_r, \quad n \in \mathbb{Z}^s; \tag{3.1}$$

and $\widehat{\Theta}_0(0) = I_r$. For $k \geq 0$, let $\widehat{A}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ and $\widehat{B}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ be defined by

$$\widehat{A}_{k+1}(j) := \widehat{\Theta}_k(j)\widehat{H}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1}, \qquad \widehat{B}_{k+1}(j) := \widehat{G}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1}, \quad j \in \mathcal{R}_{k+1}. \tag{3.2}$$

If the $(r+\rho_k)\times rd$ matrices

$$Q_k(j) := \begin{pmatrix} \widehat{A}_{k+1}(j+D^k\ell_1) & \ldots & \widehat{A}_{k+1}(j+D^k\ell_d) \\ \widehat{B}_{k+1}(j+D^k\ell_1) & \ldots & \widehat{B}_{k+1}(j+D^k\ell_d) \end{pmatrix}, \quad j \in \mathcal{R}_k, \tag{3.3}$$

where $\ell_1,\ldots,\ell_d$ denote all the elements of $\mathcal{R}_1$, satisfy

$$Q_k(j)^* Q_k(j) = dI_{rd}, \quad j \in \mathcal{R}_k, \tag{3.4}$$

for all $k \geq 0$, then the collection $\{\, \phi_0^m : m = 1,2,\ldots,r \,\} \cup \{\, T_k^\ell \psi_k^m : k \geq 0,\ m = 1,2,\ldots,\rho_k,\ \ell \in \mathcal{L}_k \,\}$ forms a normalized tight wavelet frame for $L^2([0,2\pi)^s)$.

Proof. For each $k \geq 0$ and $m = 1,2,\ldots,r$, define $\varphi_k^m \in L^2([0,2\pi)^s)$ by

$$\widehat{\varphi}_k(n) := \widehat{\Theta}_k(n)\widehat{\phi}_k(n), \quad n \in \mathbb{Z}^s, \tag{3.5}$$

where $\widehat{\varphi}_k(n) := (\widehat{\varphi}_k^1(n),\ldots,\widehat{\varphi}_k^r(n))^T$. Note that by (1.3), (3.2) and (3.5), for every $n \in \mathbb{Z}^s$,

$$\begin{aligned} \widehat{\varphi}_k(n) &= \widehat{\Theta}_k(n)\widehat{H}_{k+1}(n)\widehat{\phi}_{k+1}(n) \\ &= \widehat{\Theta}_k(n)\widehat{H}_{k+1}(n)\widehat{\Theta}_{k+1}(n)^{-1}\widehat{\Theta}_{k+1}(n)\widehat{\phi}_{k+1}(n) = \widehat{A}_{k+1}(n)\widehat{\varphi}_{k+1}(n). \end{aligned} \tag{3.6}$$

Fix $n \in \mathbb{Z}^s$ and let $\epsilon > 0$. It follows from (2.34) that there exists $K_0 > 0$ such that for all $k \geq K_0$,

$$|d^k \widehat{\phi}_k(n)^* \widehat{\phi}_k(n) - 1| < \epsilon. \tag{3.7}$$

By (3.1), there exists $K_1 > 0$ such that if $k \geq K_1$,

$$|(\widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n) - I_r)_{m,\mu}| = |(\widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n))_{m,\mu} - \delta_{m,\mu}| < \epsilon \tag{3.8}$$

for all $m, \mu = 1, 2, \ldots, r$. Observe from (3.5) that

$$|d^k\widehat{\varphi}_k(n)^*\widehat{\varphi}_k(n) - 1| \leq |d^k\widehat{\phi}_k(n)^*(\widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n) - I_r)\widehat{\phi}_k(n)| + |d^k\widehat{\phi}_k(n)^*\widehat{\phi}_k(n) - 1|. \tag{3.9}$$

Now using (3.8) and the Cauchy-Schwarz inequality, for all $k \geq \max\{K_0, K_1\}$, we have

$$\begin{aligned}|d^k\widehat{\phi}_k(n)^*(\widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n) - I_r)\widehat{\phi}_k(n)| &\leq d^k \sum_{m=1}^{r}\sum_{\mu=1}^{r} |\widehat{\phi}_k^m(n)||(\widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n) - I_r)_{m,\mu}||\widehat{\phi}_k^\mu(n)| \\ &< \epsilon d^k\Big(\sum_{m=1}^{r}|\widehat{\phi}_k^m(n)|\Big)^2 \leq r\epsilon d^k\,\widehat{\phi}_k(n)^*\widehat{\phi}_k(n).\end{aligned}$$

Hence it follows from (3.7) and (3.9) that $|d^k\widehat{\varphi}_k(n)^*\widehat{\varphi}_k(n) - 1| < \epsilon(r(1+\epsilon)+1)$ for all $k \geq \max\{K_0, K_1\}$. This shows that

$$\lim_{k\to\infty} d^k \sum_{m=1}^{r} |\,\widehat{\varphi}_k^m(n)\,|^2 = \lim_{k\to\infty} d^k\widehat{\varphi}_k(n)^*\widehat{\varphi}_k(n) = 1, \quad n \in \mathbb{Z}^s.$$

Thus $\widehat{\varphi}_k$, $k \geq 0$, satisfy the hypotheses of Theorem 2.2.

Now define $\omega_k^m \in L^2([0,2\pi)^s)$, $k \geq 0$, $m = 1, 2, \ldots, \rho_k$, by

$$\widehat{\omega}_k(n) := \widehat{B}_{k+1}(n)\widehat{\varphi}_{k+1}(n), \quad n \in \mathbb{Z}^s, \tag{3.10}$$

where $\widehat{\omega}_k(n) := (\widehat{\omega}_k^1(n), \ldots, \widehat{\omega}_k^{\rho_k}(n))^T$. Since the matrices $Q_k(j)$, $j \in \mathcal{R}_k$, defined by (3.3) satisfy (3.4), we conclude from Theorem 2.2 that the collection $\{\,\varphi_0^m : m = 1, 2, \ldots, r\,\} \cup \{\,T_k^\ell\omega_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{L}_k\,\}$ forms a normalized tight wavelet frame for $L^2([0,2\pi)^s)$.

As $\widehat{\Theta}_0 \in \mathcal{S}(D^0)^{r\times r}$, we have $\widehat{\Theta}_0(n) = \widehat{\Theta}_0(0) = I_r$ for all $n \in \mathbb{Z}^s$. Therefore by (3.5), we obtain $\widehat{\varphi}_0(n) = \widehat{\Theta}_0(n)\widehat{\phi}_0(n) = \widehat{\phi}_0(n)$ for all $n \in \mathbb{Z}^s$; that is, $\varphi_0 = \phi_0$. Further, by (1.5), (3.2), (3.5) and (3.10), we see that for $k \geq 0$ and $n \in \mathbb{Z}^s$,

$$\widehat{\omega}_k(n) = \widehat{G}_{k+1}(n)\widehat{\Theta}_{k+1}(n)^{-1}\widehat{\Theta}_{k+1}(n)\widehat{\phi}_{k+1}(n) = \widehat{G}_{k+1}(n)\widehat{\phi}_{k+1}(n) = \widehat{\psi}_k(n).$$

Thus $\omega_k = \psi_k$ for all $k \geq 0$. Hence the result follows. ■

As seen from its proof, the OEP is obtained as a consequence of the UEP. On the other hand, if the matrices $\widehat{\Theta}_k$, $k \geq 0$, in the OEP are taken to be the $r \times r$ identity matrix, then the OEP reduces to exactly the UEP. In other words, the UEP and OEP are mathematically equivalent results. Nevertheless, the OEP is still a useful reformulation of the UEP. This is because for a given collection of functions, the OEP provides an alternative, and in fact more general, sufficient condition for the collection to form a tight wavelet frame.

The upshot in the proof of the OEP is that despite the change of matrix sequences in the refinement and wavelet equations as given by (3.6) and (3.10), the resulting wavelets are identical to the original wavelets. However, as noted from (3.5), the refinable functions change in the process, but the subspaces they generated remain the same. More precisely, for $k \geq 0$, let φ_k^m, $m = 1, 2, \ldots, r$, be the new refinable functions considered in the proof of the OEP. Then

$$\langle \{ T_k^\ell \varphi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \} \rangle = \langle \{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \} \rangle. \tag{3.11}$$

To see this, observe that (3.5) is equivalent to

$$\varphi_k = \sum_{\ell \in \mathcal{L}_k} \Theta_k(\ell) T_k^\ell \phi_k,$$

where $\Theta_k \in \mathcal{S}(A^k)^{r \times r}$ is the inverse discrete Fourier transform of $\widehat{\Theta}_k$. Thus for $\eta \in \mathcal{L}_k$,

$$T_k^\eta \varphi_k = \sum_{\ell \in \mathcal{L}_k} \Theta_k(\ell) T_k^{\eta+\ell} \phi_k.$$

Since $\widehat{\Theta}_k(j)$ is invertible for every $j \in \mathcal{R}_k$, each of $T_k^\ell \phi_k$, $\ell \in \mathcal{L}_k$, can also be obtained in terms of $T_k^\eta \varphi_k$, $\eta \in \mathcal{L}_k$. Hence (3.11) holds.

4. Solutions to extension principles

In both the UEP (Theorem 2.2) and OEP (Theorem 3.1), we essentially assume solutions to the following matrix extension problem. For each $k \geq 0$, starting from $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r \times r}$, we seek $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$, where $\rho_k \geq r(d-1)$, for which the $(r + \rho_k) \times rd$ matrices $P_k(j)$, $j \in \mathcal{R}_k$, as defined in (1.7) satisfy (1.8). We shall show that whenever $\rho_k \geq rd$, it is always possible to find such a solution $\widehat{G}_{k+1}$. Depending on $\widehat{H}_{k+1}$, the total number of wavelets ρ_k could even be less than rd but at least $r(d-1)$.

Fixing notations, define the $r \times rd$ matrices

$$\mathbb{H}_k(j) := \frac{1}{\sqrt{d}} \begin{pmatrix} \widehat{H}_{k+1}(j + D^k \ell_1) & \ldots & \widehat{H}_{k+1}(j + D^k \ell_d) \end{pmatrix}, \quad j \in \mathcal{R}_k, \tag{4.1}$$

and the $\rho_k \times rd$ matrices

$$\mathbb{G}_k(j) := \frac{1}{\sqrt{d}} \begin{pmatrix} \widehat{G}_{k+1}(j + D^k \ell_1) & \ldots & \widehat{G}_{k+1}(j + D^k \ell_d) \end{pmatrix}, \quad j \in \mathcal{R}_k, \tag{4.2}$$

where $\mathcal{R}_1 = \{ \ell_1, \ldots, \ell_d \}$. Then (1.8) is equivalent to

$$\mathbb{H}_k(j)^* \mathbb{H}_k(j) + \mathbb{G}_k(j)^* \mathbb{G}_k(j) = I_{rd}, \quad j \in \mathcal{R}_k. \tag{4.3}$$

For each $j \in \mathcal{R}_k$, $\mathbb{H}_k(j)^* \mathbb{H}_k(j)$ is an $rd \times rd$ positive semi-definite Hermitian matrix and so all its eigenvalues are nonnegative. Renaming if necessary, let $\lambda_k^m(j)$, $m = 1, 2, \ldots, rd$, be the eigenvalues of $\mathbb{H}_k(j)^* \mathbb{H}_k(j)$ such that

$$\lambda_k^1(j) \geq \lambda_k^2(j) \geq \cdots \geq \lambda_k^{rd}(j) \geq 0. \tag{4.4}$$

Since $\operatorname{rank}(\mathbb{H}_k(j)) \leq r$, it follows that $\lambda_k^m(j) = 0$ for $m = r+1, \ldots, rd$, and we are concerned with only the remaining first r eigenvalues.

Theorem 4.1. *For $k \geq 0$ and $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r \times r}$, suppose that for every $j \in \mathcal{R}_k$,*

$$\lambda_k^m(j) \leq 1, \quad m = 1, 2, \ldots, r, \tag{4.5}$$

where $\lambda_k^m(j)$, $m = 1, 2, \ldots, r$, *are the* r *largest eigenvalues of* $\mathbb{H}_k(j)^*\mathbb{H}_k(j)$ *with* $\mathbb{H}_k(j)$ *as in* (4.1). *Define*

$$q_k(j) := |\{ m \in \{1, 2, \ldots, r\} : \lambda_k^m(j) < 1 \}|, \quad j \in \mathcal{R}_k, \tag{4.6}$$

and set

$$q_k := \max_{j \in \mathcal{R}_k} \{ q_k(j) \}. \tag{4.7}$$

Then for any $\rho_k \geq r(d-1) + q_k$, *there exists* $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ *such that the matrices* $P_k(j)$, $j \in \mathcal{R}_k$, *as defined in* (1.7) *satisfy* (1.8). *Conversely, for* $k \geq 0$ *and a given* $\widehat{H}_{k+1}$, *if such a matrix* $\widehat{G}_{k+1}$ *exists for some* $\rho_k \geq 1$, *then* (4.5) *holds for every* $j \in \mathcal{R}_k$, *and* $\rho_k \geq r(d-1) + q_k$.

Proof. Suppose that (4.5) holds and that $\rho_k \geq r(d-1) + q_k$. Fix $k \geq 0$, $j \in \mathcal{R}_k$. To simplify notation, set $H := \mathbb{H}_k(j)$. Applying the singular value decomposition (see for example [17, Theorem 6.1]) to H, we express H as a product

$$H = USV^*,$$

where U and V are $r \times r$ and $rd \times rd$ unitary matrices respectively, and S is the $r \times rd$ matrix

$$S := (\mathrm{diag}(\sigma_1, \ldots, \sigma_r)|\, 0).$$

Here $\sigma_m := \sqrt{\lambda_k^m(j)}$, $m = 1, 2, \ldots, r$. Since $\lambda_k^1(j) \geq \lambda_k^2(j) \geq \cdots \geq \lambda_k^r(j) \geq 0$ by (4.4), it follows from (4.5)–(4.7) that $\sigma_m = \sqrt{\lambda_k^m(j)} = 1$ for $m = 1, 2, \ldots, r - q_k$. Let $p := r - q_k$. Then S can be written as

$$S = \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & D & 0 \end{array}\right),$$

where I_p denotes the $p \times p$ identity matrix and $D := \mathrm{diag}(\sigma_{p+1}, \ldots, \sigma_r)$. Note that if $q_k = r$, then I_p does not exist and $S = (D|\, 0)$.

Let D' be the diagonal matrix

$$D' := \mathrm{diag}(\alpha_{p+1}, \ldots, \alpha_r),$$

where $\alpha_m := \sqrt{1 - \sigma_m^2}$, $m = p+1, \ldots, r$. Note that if $q_k = 0$, then D does not exist and we skip the construction of D'. As $\rho_k - q_k \geq r(d-1)$, there exists an $(\rho_k - q_k) \times r(d-1)$ matrix A such that the columns of A are orthonormal. (For example, we could choose the columns of A to be any $r(d-1)$ vectors from the standard basis of $\mathbb{C}^{\rho_k - q_k}$.) Now let Q be the $(r + \rho_k) \times rd$ matrix

$$Q := \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & D & 0 \\ \hline 0 & D' & 0 \\ \hline 0 & 0 & A \end{array}\right).$$

It is clear that the columns of Q are orthonormal. Thus $Q^*Q = I_{rd}$. Note that Q is an extension of S, so we write

$$Q = \left(\frac{S}{B}\right),$$

where B is the $\rho_k \times rd$ matrix formed by the last ρ_k rows of Q. Let W be any $\rho_k \times \rho_k$ unitary matrix, and set $G := WBV^*$. Define the $\rho_k \times r$ matrices $\widehat{G}_{k+1}(j + D^k\ell)$, $\ell \in \mathcal{R}_1$, by

$$G = \frac{1}{\sqrt{d}} \begin{pmatrix} \widehat{G}_{k+1}(j + D^k\ell_1) & \dots & \widehat{G}_{k+1}(j + D^k\ell_d) \end{pmatrix} = \mathbb{G}_k(j),$$

where $\mathcal{R}_1 = \{ \ell_1, \dots, \ell_d \}$. Let U' be the $(r + \rho_k) \times (r + \rho_k)$ unitary matrix

$$U' := \left(\begin{array}{c|c} U & 0 \\ \hline 0 & W \end{array} \right).$$

Observe that

$$U'QV^* = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & W \end{array} \right) \left(\frac{S}{B} \right) V^* = \left(\frac{USV^*}{WBV^*} \right) = \left(\frac{H}{G} \right).$$

Since U' and V are unitary matrices and $Q^*Q = I_{rd}$, it follows that

$$\left(\frac{H}{G} \right)^* \left(\frac{H}{G} \right) = I_{rd}.$$

This shows that $H^*H + G^*G = I_{rd}$. Therefore we conclude that (4.3) holds. In other words, the matrices $P_k(j)$ as defined in (1.7) satisfy (1.8). By extending periodically the values of the matrices $\widehat{G}_{k+1}(j + D^k\ell)$, $j \in \mathcal{R}_k$, $\ell \in \mathcal{R}_1$, we obtain $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$.

Conversely, suppose that for $k \geq 0$, there exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ for some $\rho_k \geq 1$ such that the matrices $P_k(j)$, $j \in \mathcal{R}_k$, as defined in (1.7) satisfy (1.8). For $j \in \mathcal{R}_k$, let $\mathbb{H}_k(j)$ and $\mathbb{G}_k(j)$ be defined as in (4.1) and (4.2) respectively. Then (4.3) holds by assumption. As $\lambda_k^m(j)$, $m = 1, 2, \dots, rd$, are all the eigenvalues (not necessarily distinct) of $\mathbb{H}_k(j)^*\mathbb{H}_k(j)$, it is easy to deduce from (4.3) that $1 - \lambda_k^m(j)$, $m = 1, 2, \dots, rd$, are all the eigenvalues of $\mathbb{G}_k(j)^*\mathbb{G}_k(j)$. Consequently, since $\mathbb{G}_k(j)^*\mathbb{G}_k(j)$ is Hermitian and positive semi-definite, we have $1 - \lambda_k^m(j) \geq 0$ for all $j \in \mathcal{R}_k$, $m = 1, 2, \dots, rd$. In particular, (4.5) holds for every $j \in \mathcal{R}_k$.

Now by (4.4) and (4.6), we see that $\lambda_k^m(j) = 1$ for $m = 1, 2, \dots, r - q_k(j)$, and $\lambda_k^m(j) < 1$ for $m = r - q_k(j) + 1, \dots, rd$ (note that $\lambda_k^m(j) = 0$ for $m = r + 1, \dots, rd$). Therefore exactly $rd - (r - q_k(j))$ of the eigenvalues $1 - \lambda_k^m(j)$ of $\mathbb{G}_k(j)^*\mathbb{G}_k(j)$ are nonzero. This implies that

$$\operatorname{rank}(\mathbb{G}_k(j)) = \operatorname{rank}(\mathbb{G}_k(j)^*\mathbb{G}_k(j)) = rd - r + q_k(j).$$

On the other hand, since $\mathbb{G}_k(j)$ is an $\rho_k \times rd$ matrix, we have

$$\operatorname{rank}(\mathbb{G}_k(j)) \leq \min\{ \rho_k, rd \} \leq \rho_k.$$

Hence we obtain $\rho_k \geq r(d - 1) + q_k(j)$. Since this inequality holds for all $j \in \mathcal{R}_k$ and ρ_k is independent of j, we conclude that

$$\rho_k \geq r(d - 1) + \max_{j \in \mathcal{R}_k} \{ q_k(j) \} = r(d - 1) + q_k.$$

This completes the proof of the theorem. ∎

Theorem 4.1 gives necessary and sufficient conditions for (1.8). The minimum value of ρ_k required depends on q_k in (4.7). Nevertheless, for $k \geq 0$, starting from any $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r \times r}$ for which (4.5) holds, a solution $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times r}$ to (1.8) always exists when we take ρ_k to be at least

rd. Indeed, it follows from (4.6) and (4.7) that $q_k \le r$ and so $\rho_k \ge rd \ge r(d-1)+q_k$. The solution is then guaranteed by Theorem 4.1.

Let us next consider the case when $\rho_k = r(d-1)$ which is the number of wavelets for the basis setting in [12]. Here $q_k = 0$ which implies that $\lambda_k^m(j) = 1$ for all $m = 1, 2, \ldots, r$ and $j \in \mathcal{R}_k$. Consequently, for every $j \in \mathcal{R}_k$, all the eigenvalues (counting multiplicity) of the $r \times r$ matrix $\mathbb{H}_k(j)\mathbb{H}_k(j)^*$ are 1. This means that $\mathbb{H}_k(j)\mathbb{H}_k(j)^* = I_r$ which translates to

$$\sum_{\ell\in\mathcal{R}_1} \widehat{H}_{k+1}(j + D^k\ell)\widehat{H}_{k+1}(j + D^k\ell)^* = dI_r. \tag{4.8}$$

Note that (4.8) is satisfied for all $k \ge 0$ and $j \in \mathcal{R}_k$ when $\{\, T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \,\}$ forms an orthonormal basis for V_k in (2.39) for every $k \ge 0$ (see [12]).

The other extreme to the case when $q_k = 0$ is the situation when $q_k = r$. In this instance, the minimum value of ρ_k needed for a solution to (1.8) is $r(d-1) + r = rd$.

The proof of Theorem 4.1 is constructive and it provides a solution $\widehat{G}_{k+1}$ to (1.8). The next theorem shows that armed with a particular solution, we can always find all the solutions to (1.8).

Theorem 4.2. *For $k \ge 0$, $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ and some $\rho_k \ge 1$, let $\widehat{G}^0_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ be such that the matrices $P_k^0(j)$, $j \in \mathcal{R}_k$, defined as in (1.7) satisfy (1.8). Then for any $U_k \in \mathcal{S}(D^k)^{\rho_k\times\rho_k}$ where $U_k(j)$, $j \in \mathcal{R}_k$, are unitary matrices, the matrix $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ given by*

$$\begin{aligned}&\left(\widehat{G}_{k+1}(j + D^k\ell_1) \quad \ldots \quad \widehat{G}_{k+1}(j + D^k\ell_d)\right) := \\ &\qquad U_k(j)\left(\widehat{G}^0_{k+1}(j + D^k\ell_1) \quad \ldots \quad \widehat{G}^0_{k+1}(j + D^k\ell_d)\right), \quad j \in \mathcal{R}_k,\end{aligned} \tag{4.9}$$

with $\mathcal{R}_1 = \{\, \ell_1, \ldots, \ell_d \,\}$, is also a solution to (1.8). Conversely, for any solution $\widehat{G}_{k+1}$ to (1.8), there exists such a matrix U_k for which (4.9) holds.

Proof. Fix $k \ge 0$, $j \in \mathcal{R}_k$. To simplify notations, set $H := \mathbb{H}_k(j)$, where $\mathbb{H}_k(j)$ is defined as in (4.1), and

$$G^0 := \frac{1}{\sqrt{d}}\left(\widehat{G}^0_{k+1}(j + D^k\ell_1) \quad \ldots \quad \widehat{G}^0_{k+1}(j + D^k\ell_d)\right).$$

Since $P_k^0(j)$ satisfies (1.8), we have $\begin{pmatrix} H \\ G^0 \end{pmatrix}^* \begin{pmatrix} H \\ G^0 \end{pmatrix} = I_{rd}$, that is,

$$H^*H + (G^0)^*G^0 = I_{rd}. \tag{4.10}$$

Now suppose that the matrix $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ is given by (4.9) for some unitary matrix $U_k(j)$. Let $G := \mathbb{G}_k(j)$, where $\mathbb{G}_k(j)$ is defined as in (4.2). Then we see from (4.9) that $G = U_k(j)G^0$. As $U_k(j)$ is unitary, this implies that $G^*G = (G^0)^*G^0$. Therefore by (4.10),

$$H^*H + G^*G = H^*H + (G^0)^*G^0 = I_{rd}.$$

In other words, $P_k(j)^*P_k(j) = dI_{rd}$. Since $j \in \mathcal{R}_k$ is arbitrary, this shows that $\widehat{G}_{k+1}$ is a solution to (1.8).

Conversely, suppose that $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k\times r}$ is any solution to (1.8). Then

$$H^*H + G^*G = I_{rd},$$

where $G := \mathbb{G}_k(j)$. This and (4.10) imply that $G^*G = (G^0)^*G^0$. Since the $rd \times rd$ matrix G^*G is Hermitian and positive semi-definite, there exists an $rd \times rd$ unitary matrix V such that

$$V^*(G^*G)V = V^*((G^0)^*G^0)V = \operatorname{diag}(\lambda_1, \ldots, \lambda_{rd}), \tag{4.11}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{rd}$ are the eigenvalues of G^*G which are all nonnegative. Applying the singular value decomposition to G and G^0, we obtain

$$G = USV^*, \qquad G^0 = U^0SV^*, \tag{4.12}$$

where U and U^0 are $\rho_k \times \rho_k$ unitary matrices, V is the same matrix as that appearing in (4.11), and $S = [s_{pq}]$ is an $\rho_k \times rd$ matrix with $s_{pq} = 0$ for all $p \neq q$ (see the proof of [17, Theorem 6.1]). Then it follows from (4.12) that $G = U(U^0)^*G^0$. Now if we define $U_k(j) := U(U^0)^*$, which is an $\rho_k \times \rho_k$ unitary matrix, and extend periodically the values of $U_k(j)$, $j \in \mathcal{R}_k$, we obtain $U_k \in \mathcal{S}(D^k)^{\rho_k \times \rho_k}$ for which (4.9) holds. ∎

We shall now focus on the special case of $r = 1$, that is, the number of refinable functions for each $k \geq 0$ is one. For a start, the sufficient condition (4.5) in Theorem 4.1 is much simpler. Indeed, for fixed $k \geq 0$ and $j \in \mathcal{R}_k$, the matrix $\mathbb{H}_k(j)$ as defined in (4.1) is a $1 \times d$ row vector and the corresponding $d \times d$ matrix $\mathbb{H}_k(j)^*\mathbb{H}_k(j)$ has at most one nonzero eigenvalue. Since $\mathbb{H}_k(j)^*\mathbb{H}_k(j)$ and $\mathbb{H}_k(j)\mathbb{H}_k(j)^*$ have the same nonzero eigenvalues (counting multiplicity) and $\mathbb{H}_k(j)\mathbb{H}_k(j)^*$ is the scalar $\frac{1}{d}\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j+D^k\ell)|^2$, it follows that $\lambda_k^1(j)$ in (4.4) is given by $\lambda_k^1(j) = \frac{1}{d}\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j+D^k\ell)|^2$. Hence the sufficient condition (4.5) reduces to

$$\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j + D^k\ell)|^2 \leq d. \tag{4.13}$$

When $s = 1$ and $A = D = 2$, this gives the periodic analog of the condition in [6, Theorem 1] and [14, Theorem 2.2] for $L^2(\mathbb{R})$.

While the proofs of both Theorems 4.1 and 4.2 are constructive and use the singular value decomposition, for the special case of $r = 1$, it is possible to obtain more directly a parametric family of solutions $\widehat{G}_{k+1}$ to (1.8) with explicit expressions via Householder matrices.

Proposition 4.1. *For fixed $k \geq 0$ and $j \in \mathcal{R}_k$, suppose that* (4.13) *holds and consider $\rho_k \geq d$. Define the column vector $z = (z_1, \ldots, z_{\rho_k+1})^T \in \mathbb{C}^{\rho_k+1}$ by*

$$z_m := \begin{cases} \frac{1}{\sqrt{d}}\overline{\widehat{H}_{k+1}(j + D^k\ell_m)}, & \text{if } m = 1, \ldots, d, \\ t_k^{m-d}(j), & \text{if } m = d+1, \ldots, \rho_k + 1, \end{cases} \tag{4.14}$$

where $\mathcal{R}_1 = \{\ell_1, \ldots, \ell_d\}$, and $t_k^1(j), \ldots, t_k^{\rho_k-d+1}(j)$ are arbitrary in $\mathbb{C}$ with

$$\sum_{m=1}^{\rho_k-d+1} |t_k^m(j)|^2 = 1 - \frac{1}{d}\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j + D^k\ell)|^2. \tag{4.15}$$

If $\widehat{H}_{k+1}(j+D^k\ell_1)\neq 0$*, we set*

$$\mathbb{G}_k(j)=\begin{pmatrix} \frac{\bar z_1 z_2}{z_1} & \frac{|z_1|}{z_1}\left(\frac{|z_2|^2}{1+|z_1|}-1\right) & \frac{|z_1|}{z_1}\left(\frac{\bar z_3 z_2}{1+|z_1|}\right) & \dots & \frac{|z_1|}{z_1}\left(\frac{\bar z_d z_2}{1+|z_1|}\right)\\ \frac{\bar z_1 z_3}{z_1} & \frac{|z_1|}{z_1}\left(\frac{\bar z_2 z_3}{1+|z_1|}\right) & \frac{|z_1|}{z_1}\left(\frac{|z_3|^2}{1+|z_1|}-1\right) & \dots & \frac{|z_1|}{z_1}\left(\frac{\bar z_d z_3}{1+|z_1|}\right)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{\bar z_1 z_d}{z_1} & \frac{|z_1|}{z_1}\left(\frac{\bar z_2 z_d}{1+|z_1|}\right) & \frac{|z_1|}{z_1}\left(\frac{\bar z_3 z_d}{1+|z_1|}\right) & \dots & \frac{|z_1|}{z_1}\left(\frac{|z_d|^2}{1+|z_1|}-1\right)\\ \frac{\bar z_1 z_{d+1}}{z_1} & \frac{|z_1|}{z_1}\left(\frac{\bar z_2 z_{d+1}}{1+|z_1|}\right) & \frac{|z_1|}{z_1}\left(\frac{\bar z_3 z_{d+1}}{1+|z_1|}\right) & \dots & \frac{|z_1|}{z_1}\left(\frac{\bar z_d z_{d+1}}{1+|z_1|}\right)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{\bar z_1 z_{\rho_k+1}}{z_1} & \frac{|z_1|}{z_1}\left(\frac{\bar z_2 z_{\rho_k+1}}{1+|z_1|}\right) & \frac{|z_1|}{z_1}\left(\frac{\bar z_3 z_{\rho_k+1}}{1+|z_1|}\right) & \dots & \frac{|z_1|}{z_1}\left(\frac{\bar z_d z_{\rho_k+1}}{1+|z_1|}\right) \end{pmatrix}; \tag{4.16}$$

otherwise, we take

$$\mathbb{G}_k(j)=\begin{pmatrix} z_2 & |z_2|^2-1 & \bar z_3 z_2 & \dots & \bar z_d z_2\\ z_3 & \bar z_2 z_3 & |z_3|^2-1 & \dots & \bar z_d z_3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ z_d & \bar z_2 z_d & \bar z_3 z_d & \dots & |z_d|^2-1\\ z_{d+1} & \bar z_2 z_{d+1} & \bar z_3 z_{d+1} & \dots & \bar z_d z_{d+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ z_{\rho_k+1} & \bar z_2 z_{\rho_k+1} & \bar z_3 z_{\rho_k+1} & \dots & \bar z_d z_{\rho_k+1} \end{pmatrix}. \tag{4.17}$$

Then with $\mathbb{H}_k(j)$ *as in* (4.1), $\mathbb{H}_k(j)^*\mathbb{H}_k(j)+\mathbb{G}_k(j)^*\mathbb{G}_k(j)=I_d$.

Proof. With the assumption (4.13), it follows from (4.14) and (4.15) that $z=(z_1,\dots,z_{\rho_k+1})^T$ is a column vector of Euclidean norm 1. Define the scalar

$$\tau:=\begin{cases}\frac{z_1}{|z_1|}, & \text{if } z_1\neq 0,\\ 1, & \text{if } z_1=0,\end{cases}$$

and the $(\rho_k+1)\times 1$ column vector $v:=z+(\tau,0,\dots,0)^T$. By the theory of Householder matrices, the $(\rho_k+1)\times(\rho_k+1)$ matrix

$$Q:=\overline{\tau}\left(\frac{2}{v^*v}vv^*-I_{\rho_k+1}\right) \tag{4.18}$$

is unitary and satisfies $Qz=(1,0,\dots,0)^T$.

Let Q' be the $(\rho_k+1)\times(\rho_k+1)$ matrix whose first row is z^* and remaining rows are the corresponding ρ_k rows of Q. Thus Q' has orthonormal rows and hence orthonormal columns. Observe that we may equate $\begin{pmatrix}\mathbb{H}_k(j)\\ \mathbb{G}_k(j)\end{pmatrix}$ to the first d columns of Q'. This gives $\mathbb{H}_k(j)^*\mathbb{H}_k(j)+$

$\mathbb{G}_k(j)^*\mathbb{G}_k(j) = I_d$. Further, a direct calculation based on (4.18) leads to (4.16) and (4.17). ∎

For $k \geq 0$, by defining $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho_k \times 1}$ via $\mathbb{G}_k(j)$, $j \in \mathcal{R}_k$, in (4.2), Proposition 4.1 gives a parametric family of solutions to (1.8). The parameters in the construction are provided by the scalars $t_k^m(j)$, $m = 1, 2, \ldots, \rho_k - d + 1$, $j \in \mathcal{R}_k$.

The quantity q_k in (4.7) is either 0 or 1. The former takes place if and only if $\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j + D^k\ell)|^2 = d$ for every $j \in \mathcal{R}_k$. By (4.15), this forces $t_k^m(j) = 0$ for all $m = 1, 2, \ldots, \rho_k - d + 1$ and $j \in \mathcal{R}_k$. It then follows from (4.16) and (4.17) that the last $\rho_k - d + 1$ rows of $\mathbb{G}_k(j)$ are entirely zeros. Using (2.6), we see that $\psi_k^m = 0$ for $m = d, d+1, \ldots, \rho_k$. In other words, the construction in Proposition 4.1 automatically reduces to $d - 1$ wavelets when $q_k = 0$.

To better illustrate the results of this paper, we now further specialize our discussion to the case when $s = 1$ and $A = D = 2$, which is the most commonly studied one-dimensional dyadic setting in wavelet analysis. In this case, for $k \geq 0$, $\mathcal{L}_k = \mathcal{R}_k = \{0, 1, \ldots, 2^k - 1\}$. We write $\widehat{H}_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$ and $\widehat{G}_{k+1} \in \mathcal{S}(2^{k+1})^{\rho_k \times 1}$ in (1.3) and (1.5) as

$$\widehat{H}_{k+1}(j) = \widehat{h}_{k+1}(j), \qquad \widehat{G}_{k+1}(j) = (\widehat{g}_{k+1}^1(j), \ldots, \widehat{g}_{k+1}^{\rho_k}(j))^T, \quad j \in \mathcal{R}_{k+1}.$$

Under the sufficient condition (4.13) which now takes the form

$$|\widehat{h}_{k+1}(j)|^2 + |\widehat{h}_{k+1}(j + 2^k)|^2 \leq 2, \quad j \in \mathcal{R}_k, \tag{4.19}$$

it follows from (4.2), (4.16) and (4.17) that for $j \in \mathcal{R}_k$, if $\widehat{h}_{k+1}(j) \neq 0$,

$$\begin{cases} \widehat{g}_{k+1}^1(j) = \frac{\widehat{h}_{k+1}(j)\,\overline{\widehat{h}_{k+1}(j+2^k)}}{\overline{\widehat{h}_{k+1}(j)}}, & \widehat{g}_{k+1}^1(j + 2^k) = \frac{\sqrt{2}\,|\widehat{h}_{k+1}(j)|}{\overline{\widehat{h}_{k+1}(j)}} \left(\frac{|\widehat{h}_{k+1}(j+2^k)|^2}{2+\sqrt{2}\,|\widehat{h}_{k+1}(j)|} - 1 \right), \\ \widehat{g}_{k+1}^{m+1}(j) = \frac{\sqrt{2}\,\widehat{h}_{k+1}(j)\, t_k^m(j)}{\overline{\widehat{h}_{k+1}(j)}}, & \widehat{g}_{k+1}^{m+1}(j + 2^k) = \frac{\sqrt{2}\,|\widehat{h}_{k+1}(j)|\,\widehat{h}_{k+1}(j+2^k)\, t_k^m(j)}{\overline{\widehat{h}_{k+1}(j)}\,(\sqrt{2}+|\widehat{h}_{k+1}(j)|)}, \quad m = 1, 2, \ldots, \rho_k - 1, \end{cases} \tag{4.20}$$

and if $\widehat{h}_{k+1}(j) = 0$,

$$\begin{cases} \widehat{g}_{k+1}^1(j) = \overline{\widehat{h}_{k+1}(j + 2^k)}, & \widehat{g}_{k+1}^1(j + 2^k) = \frac{1}{\sqrt{2}}\,|\widehat{h}_{k+1}(j + 2^k)|^2 - \sqrt{2}, \\ \widehat{g}_{k+1}^{m+1}(j) = \sqrt{2}\, t_k^m(j), & \widehat{g}_{k+1}^{m+1}(j + 2^k) = \widehat{h}_{k+1}(j + 2^k)\, t_k^m(j), \quad m = 1, 2, \ldots, \rho_k - 1. \end{cases} \tag{4.21}$$

We shall use (4.20) and (4.21) to construct a parametric family of tight trigonometric polynomial wavelet frames.

Example 4.1. Let $\{N_k\}_{k\geq 0}$ and $\{L_k\}_{k\geq 0}$ be two strictly increasing sequence of nonnegative integers satisfying $N_k \leq L_k \leq 2^{k-1}$. Consider the trigonometric polynomials

$$\phi_k = \sum_{n=-L_k}^{L_k} \widehat{\phi}_k(n) e^{in\cdot}, \quad k \geq 0, \tag{4.22}$$

where for every $k \geq 0$, $\widehat{\phi}_k(n) = \frac{1}{\sqrt{2^k}}$ for $n \in \{-N_k, \ldots, N_k\}$, and $\widehat{\phi}_k(n) = \widehat{\phi}_k(-n) > 0$ with

$$0 < \frac{\widehat{\phi}_k(n)}{\widehat{\phi}_{k+1}(n)} < \sqrt{2} \tag{4.23}$$

for $n \in \{-L_k, \dots, L_k\} \backslash \{-N_k, \dots, N_k\}$. Then ϕ_k, $k \geq 0$, are refinable and for every $k \geq 0$, (1.3) holds for some $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$. It follows from Proposition 2.1 or [13, Proposition 4.1] that the values of $\widehat{h}_{k+1}$ on $\{-L_{k+1}, \dots, L_{k+1}\}$ are unique and given by

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2}, & \text{if } j \in \{-N_k, \dots, N_k\}, \\ \frac{\widehat{\phi}_k(j)}{\widehat{\phi}_{k+1}(j)}, & \text{if } j \in \{-L_k, \dots, L_k\} \backslash \{-N_k, \dots, N_k\}, \\ 0, & \text{if } j \in \{-L_{k+1}, \dots, L_{k+1}\} \backslash \{-L_k, \dots, L_k\}, \end{cases} \tag{4.24}$$

while we can set the values of $\widehat{h}_{k+1}$ on $\{-2^k, \dots, 2^k\} \backslash \{-L_{k+1}, \dots, L_{k+1}\}$ as

$$\widehat{h}_{k+1}(j) := 0, \quad j \in \{-2^k, \dots, 2^k\} \backslash \{-L_{k+1}, \dots, L_{k+1}\}. \tag{4.25}$$

Since $\widehat{\phi}_k(n) = \frac{1}{\sqrt{2^k}}$ for $n \in \{-N_k, \dots, N_k\}$, for any fixed $n \in \mathbb{Z}$, we have $\lim_{k\to\infty} 2^k |\widehat{\phi}_k(n)|^2 = 1$, as assumed in the UEP. Further, for every $k \geq 0$, it follows from (4.23)–(4.25) that (4.19) holds. For the sequences $\widehat{g}^m_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$, $m = 1, 2, \dots, \rho_k$, we define their values $\widehat{g}^m_{k+1}(j)$ and $\widehat{g}^m_{k+1}(j + 2^k)$ by (4.20) if $j \in \{0, \dots, L_k\}$, and by (4.21) if $j \in \{L_k + 1, \dots, 2^k - 1\}$.

Using (2.4) and (2.6), the wavelets ψ^m_k, $m = 1, 2, \dots, \rho_k$, are obtained from

$$\psi^m_k = \sum_{j \in \mathcal{R}_k} \left(\widehat{g}^m_{k+1}(j) v_{k+1,j} + \widehat{g}^m_{k+1}(j + 2^k) v_{k+1,j+2^k} \right), \quad m = 1, 2, \dots, \rho_k,$$

where $v_{k+1,j}$, $j \in \mathcal{R}_{k+1}$, are as in (2.1). The UEP then shows that the collection $\{\phi_0\} \cup \{T^\ell_k \psi^m_k : k \geq 0,\ m = 1, 2, \dots, \rho_k,\ \ell \in \mathcal{L}_k\}$ forms a normalized tight frame for $L^2[0, 2\pi)$.

Letting V_k and W_k, $k \geq 0$, be the subspaces as defined in (2.39), certain choices of the sequences $\{N_k\}_{k\geq 0}$ and $\{L_k\}_{k\geq 0}$ will lead to examples of which $\dim(V_k \cap W_k)$ is nonzero for some $k \geq 0$, in contrast to the orthogonal situation in [13]. For instance, this could arise from $N_k = L_k = k$ for $0 \leq k \leq 4$, and $N_k = 2^{k-2} - 1$, $L_k = 2^{k-2}$ for $k \geq 5$.

Returning to the more general setting of Proposition 4.1, as noted from (4.15), if $\sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j + D^k \ell)|^2 = d$ where $k \geq 0$ and $j \in \mathcal{R}_k$, then all of $t^m_k(j)$, $m = 1, 2, \dots, \rho_k - d + 1$, must be zero. When this occurs for every $k \geq 0$ and $j \in \mathcal{R}_k$, Proposition 4.1 only gives one solution to (1.8). However, the OEP under an additional assumption can provide an explicit family of parametric solutions similar to that from Proposition 4.1. Indeed, for $k \geq 0$, suppose that $\widehat{\Theta}_k$ in the OEP satisfies

$$\left| \frac{\widehat{\Theta}_k(j)}{\widehat{\Theta}_{k+1}(j)} \right| \leq 1 \tag{4.26}$$

for all $j \in \mathcal{R}_{k+1}$ for which $\widehat{H}_{k+1}(j) \neq 0$, with strict inequality at some of these j's. Then for $\widehat{A}_{k+1}$ as defined in (3.2), it follows from (4.26) that

$$\begin{aligned} \sum_{\ell \in \mathcal{R}_1} |\widehat{A}_{k+1}(j + D^k \ell)|^2 &= \sum_{\ell \in \mathcal{R}_1} \left| \frac{\widehat{\Theta}_k(j + D^k \ell)}{\widehat{\Theta}_{k+1}(j + D^k \ell)} \right|^2 |\widehat{H}_{k+1}(j + D^k \ell)|^2 \\ &\leq \sum_{\ell \in \mathcal{R}_1} |\widehat{H}_{k+1}(j + D^k \ell)|^2 = d, \quad j \in \mathcal{R}_k, \end{aligned}$$

with strict inequality for some $j \in \mathcal{R}_k$. Thus we may apply Proposition 4.1 to $\widehat{A}_{k+1}$ (in place of $\widehat{H}_{k+1}$) to obtain a parametric family of solutions.

The following trigonometric polynomial example on the typical one-dimensional dyadic case of $s = 1$ and $A = D = 2$ illustrates the above. For every $k \geq 0$, we write $\widehat{A}_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$ and $\widehat{B}_{k+1} \in \mathcal{S}(2^{k+1})^{\rho_k\times 1}$ in (3.2) of the OEP as

$$\widehat{A}_{k+1}(j) = \widehat{a}_{k+1}(j), \qquad \widehat{B}_{k+1}(j) = (\widehat{b}^1_{k+1}(j), \ldots, \widehat{b}^{\rho_k}_{k+1}(j))^T, \quad j \in \mathcal{R}_{k+1}.$$

Example 4.2. Consider the refinable trigonometric polynomials ϕ_k, $k \geq 0$, in (4.22) of Example 4.1 with $N_k = 2^{k-1} - 1$, $L_k = 2^{k-1}$ and

$$\widehat{\phi}_k(n) = \begin{cases} \frac{1}{\sqrt{2^k}}, & \text{if } n \in \{-2^{k-1}+1, \ldots, 2^{k-1}-1\}, \\ \frac{1}{\sqrt{2^{k+1}}}, & \text{if } n = \pm 2^{k-1}, \\ 0, & \text{otherwise}, \end{cases}$$

for all $k \geq 0$. For every $k \geq 0$, it follows from (4.24) that the values of $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$ are unique and given by

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2}, & \text{if } j \in \{-2^{k-1}+1, \ldots, 2^{k-1}-1\}, \\ 1, & \text{if } j = \pm 2^{k-1}, \\ 0, & \text{if } j \in \{-2^k, \ldots, 2^k\}\backslash\{-2^{k-1}, \ldots, 2^{k-1}\}. \end{cases} \tag{4.27}$$

This implies that

$$|\widehat{h}_{k+1}(j)|^2 + |\widehat{h}_{k+1}(j + 2^k)|^2 = 2, \quad j \in \mathcal{R}_k.$$

Let N be a positive constant. For $k \geq 0$, we construct a periodic sequence $\widehat{\Theta}_k \in \mathcal{S}(2^k)^{1\times 1}$ by setting

$$\widehat{\Theta}_k(j) := \begin{cases} \left(\frac{\sin(\pi j/2^k)}{\pi j/2^k}\right)^N, & \text{if } j \in \{-2^{k-1}, \ldots, 2^{k-1}\}\backslash\{0\}, \\ 1, & \text{if } j = 0. \end{cases} \tag{4.28}$$

Then $\widehat{\Theta}_k(j) \neq 0$ for every $j \in \mathcal{R}_k$. In addition, for any fixed $n \in \mathbb{Z}\backslash\{0\}$,

$$\lim_{k\to\infty} |\widehat{\Theta}_k(n)|^2 = \lim_{k\to\infty} \left(\frac{\sin(\pi n/2^k)}{\pi n/2^k}\right)^{2N} = 1,$$

and by definition, $\lim_{k\to\infty} |\widehat{\Theta}_k(0)|^2 = 1$. Now from (4.27), $\widehat{h}_{k+1}(j) \neq 0$ for $j \in \{-2^{k-1}, \ldots, 2^{k-1}\}$. Using (4.28), we see that for $j \in \{-2^{k-1}, \ldots, 2^{k-1}\}\backslash\{0\}$,

$$\left|\frac{\widehat{\Theta}_k(j)}{\widehat{\Theta}_{k+1}(j)}\right| = |\cos(\pi j/2^{k+1})|^N < 1.$$

Further, $\left|\frac{\widehat{\Theta}_k(0)}{\widehat{\Theta}_{k+1}(0)}\right| = 1$. In other words, (4.26) holds for all $j \in \{-2^{k-1}, \ldots, 2^{k-1}\}$, with strict inequality at $j \in \{-2^{k-1}, \ldots, 2^{k-1}\}\backslash\{0\}$.

Consequently, for every $k \geq 0$,

$$|\widehat{a}_{k+1}(j)|^2 + |\widehat{a}_{k+1}(j + 2^k)|^2 < 2, \quad j \in \mathcal{R}_k\backslash\{0\}, \tag{4.29}$$

and $|\widehat{a}_{k+1}(0)|^2 + |\widehat{a}_{k+1}(2^k)|^2 = 2$. Applying Proposition 4.1 to $\widehat{a}_{k+1}$, we obtain the sequences $\widehat{b}^m_{k+1} \in \mathcal{S}(2^{k+1})^{1\times 1}$ (in place of $\widehat{g}^m_{k+1}$), $m = 1, 2, \ldots, \rho_k$, where their values $\widehat{b}^m_{k+1}(j)$ and $\widehat{b}^m_{k+1}(j + 2^k)$ are given by (4.20) if $j \in \{ 0, \ldots, 2^{k-1} \}$, and by (4.21) if $j \in \{ 2^{k-1} + 1, \ldots, 2^k - 1 \}$. Since (4.29) holds, this results in a parametric family of $\widehat{b}^m_{k+1}$, $m = 1, 2, \ldots, \rho_k$.

Using (2.4), (2.6) and (3.2), we define the wavelets ψ^m_k, $m = 1, 2, \ldots, \rho_k$, by

$$\psi^m_k = \sum_{j\in\mathcal{R}_k} \left(\widehat{b}^m_{k+1}(j)\widehat{\Theta}_{k+1}(j)v_{k+1,j} + \widehat{b}^m_{k+1}(j + 2^k)\widehat{\Theta}_{k+1}(j + 2^k)v_{k+1,j+2^k} \right), \quad m = 1, 2, \ldots, \rho_k,$$

with $v_{k+1,j}$, $j \in \mathcal{R}_{k+1}$, as in (2.1). The OEP implies that the collection $\{ \phi_0 \} \cup \{ T^\ell_k \psi^m_k : k \geq 0, m = 1, 2, \ldots, \rho_k, \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for $L^2[0, 2\pi)$.

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