PAIRS OF OBLIQUE DUALS IN SPACES OF PERIODIC FUNCTIONS

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Abstract. We construct non-tight frames in finite-dimensional spaces consisting of periodic functions. In order for these frames to be useful in practice one needs to calculate a dual frame; while the canonical dual frame might be cumbersome to work with, the setup presented here enables us to obtain explicit constructions of some particularly convenient oblique duals. We also provide explicit oblique duals belonging to prescribed spaces different from the space where we obtain the expansion. In particular this leads to oblique duals that are trigonometric polynomials.

1. Introduction

During the last few years, frames in finite-dimensional Hilbert spaces have become increasingly popular as well in mathematics as in engineering. The focus has been on frames in $\mathbb{C}^n$, especially the construction of tight frames; see, e.g., [1, 2].

The present paper is a contribution to the theory for frames in finite-dimensional spaces, but from a different angle. We construct non-tight frames in finite-dimensional spaces consisting of periodic functions, i.e., in subspaces of $L^2(0, 2\pi)$. The fact that the frames are non-tight makes it cumbersome to find the canonical dual frame, but we show that by allowing generators outside the space where we have the frame expansion, we have considerable freedom in the choice of *oblique duals*. We use this freedom to provide explicit constructions of some particularly convenient oblique duals, as well as oblique duals belonging to prescribed spaces different from the space where we obtain the expansion.

In the rest of this introduction we give a more technical description of our aims, and provide a short presentation of some basic facts about frames and polyphase splines.

Let $N \in \mathbb{N}$ be given. Define the translation operator

$$T : L^2(0, 2\pi) \to L^2(0, 2\pi), \quad (Tf)(\cdot) := f(\cdot - \frac{2\pi}{N}),$$

and let $\mathcal{R} := \{0, 1, \ldots, N-1\}$. Given functions $\phi_1, \ldots, \phi_r \in L^2(0, 2\pi)$, we consider the vector space

$$V := \operatorname{span}\{T^\ell \phi_m : \ m = 1, \ldots, r; \ \ell \in \mathcal{R}\}; \tag{1.1}$$

here $T^\ell = TT \cdots T$ with ℓ factors. The functions $\phi_1, \ldots, \phi_r$ are known as *generators* of V. The multiresolution and wavelet subspaces studied in periodic wavelet analysis are typical examples of vector spaces of the form (1.1). In periodic wavelet analysis, the number N takes consecutive nonnegative powers of 2, or more generally, consecutive nonnegative powers of the dilation factor of

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the underlying multiresolution analysis. The multiple generator setup of (1.1) reflects the situations encountered with multiple refinable functions, multiwavelets and wavelet frames; see, e.g., [5, 6, 7].

Our purpose here is to obtain expansions of the functions in V of the form

$$f = \sum_{m=1}^{r} \sum_{\ell \in \mathcal{R}} \langle f, T^{\ell} \tilde{\phi}_m \rangle T^{\ell} \phi_m \tag{1.2}$$

for suitable functions $\tilde{\phi}_1, \ldots, \tilde{\phi}_r \in L^2(0, 2\pi)$. An advantage of our approach is that it is easy to calculate the expansion coefficients, compared to the quite tedious computations with the canonical frame expansion; see the end of Section 4 for an illustration of this in a concrete case. Classical frame theory amounts to choose the functions $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ belonging to V; such functions which satisfy (1.2) are called *dual generators*. However, as we will see, considerable freedom is obtained by removing this restriction and consider arbitrary functions $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ in $L^2(0, 2\pi)$ satisfying (1.2); such functions are called *oblique dual generators*. We will use this freedom to obtain explicit constructions of oblique dual generators with prescribed properties, respectively, belonging to prescribed subspaces.

This paper is organized as follows. We end this section with an introduction to the basic notations and results used in the paper. In Section 2, we formulate the construction of oblique dual generators as an eigenvalue problem involving a mixed Gramian matrix. In a special case, this leads to a construction of oblique dual generators, which differ from prescribed desired functions only by trigonometric polynomials. Section 3 deals with construction of oblique dual generators that belong to prescribed subspaces of $L^2(0, 2\pi)$. In addition, it turns out that for any generators of V satisfying a certain technical condition, it is always possible to find oblique dual generators which are trigonometric polynomials. In Section 4, we re-examine all these results for the situation where V is generated by a single function. The results then become simpler, and in particular, the technical condition present in some of them holds automatically. Finally, in Section 5, we obtain characterizations for the invertibility of mixed Gramians, which plays an important role in the paper.

1.1. **Basic frame theory.** For convenience, we introduce frames in the general setting of arbitrary separable Hilbert spaces $\mathcal{H}$ and subspaces hereof. For more information we refer to the book [3]. A countable sequence of elements $\{f_k\}_{k \in I}$ in $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}.$$

Suitable constants A, B are called lower and upper frame bounds, respectively. If we can choose $A = B$, the frame is said to be *tight*.

Given a frame $\{f_k\}_{k \in I}$, the *frame operator*

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k \in I} \langle f, f_k \rangle f_k$$

is a bounded, invertible, and self-adjoint operator on $\mathcal{H}$; this leads to the frame decomposition

$$f = SS^{-1} f = \sum_{k \in I} \langle f, S^{-1} f_k \rangle f_k, \ \forall f \in \mathcal{H}. \tag{1.3}$$

A frame for $\mathcal{H}$ is automatically complete in $\mathcal{H}$. If a sequence $\{f_k\}_{k\in I}$ satisfies the frame condition for all $f \in \overline{\text{span}}\{f_k\}_{k\in I}$ – which, in general, is a pure subspace of $\mathcal{H}$ – then $\{f_k\}_{k\in I}$ is called a *frame sequence*. All finite sequences are frame sequences. If $\{f_k\}_{k\in I}$ is a frame for a subspace $V \subset \mathcal{H}$, then a *dual frame* is a frame $\{g_k\}_{k\in I}$ for V, for which

$$f = \sum_{k\in I} \langle f, g_k\rangle f_k, \ \forall f \in V. \tag{1.4}$$

In particular, (1.3) shows that $\{S^{-1}f_k\}_{k\in I}$ is a dual frame, the so-called *canonical dual frame.* Note that (1.4) very well can hold for sequences $\{g_k\}_{k\in I}$ which do not belong to V but just to the larger space $\mathcal{H}$; every such sequence, which forms a frame sequence for the space it spans, is called an *oblique dual*. Frames and their oblique duals are a special case of the theory for pseudo-dual frames in [9]. For a more detailed discussion of oblique duals in general Hilbert spaces we refer to [4].

1.2. **Polyphase splines.** It turns out to be convenient to express the functions ϕ_m generating the vector space V in (1.1) in terms of the so-called *polyphase splines* introduced in [5] and further studied in [7]. Polyphase splines are generalizations of the *orthogonal splines* formulated in [8] for the situation of V having a single generating function, i.e., $r = 1$.

Given a function $\phi \in L^2(0, 2\pi)$, we consider the Fourier series expansion

$$\phi(\cdot) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(n) e^{in(\cdot)},$$

where the Fourier coefficients are given by

$$\widehat{\phi}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)e^{-inx}\,dx, \ n \in \mathbb{Z}.$$

The polyphase splines associated to ϕ are obtained by splitting the Fourier series into N series, corresponding to contributions with different frequencies. To be precise, for $j \in \mathcal{R} = \{0, 1, \dots, N - 1\}$, we let

$$v_j(\cdot) := \sum_{p\in\mathbb{Z}} \widehat{\phi}(j + Np)e^{i(j+Np)(\cdot)}.$$

We will need several relations between the function ϕ and its associated polyphase splines. For later reference we collect them in a lemma:

Lemma 1.1. *Let $\phi \in L^2(0, 2\pi)$ be a function and $v_j, j \in \mathcal{R}$, its associated polyphase splines. Then the following hold:*

(i) $\phi = \sum_{j\in\mathcal{R}} v_j$.
(ii) $T^\ell\phi = \sum_{j\in\mathcal{R}} e^{-\frac{2\pi i\ell j}{N}} v_j$ *for each* $\ell \in \mathcal{R}$.
(iii) $v_j = \frac{1}{N}\sum_{\ell\in\mathcal{R}} e^{\frac{2\pi i\ell j}{N}} T^\ell\phi$ *for each* $j \in \mathcal{R}$.
(iv) *If $\theta \in L^2(0, 2\pi)$ has the associated polyphase splines $w_j, j \in \mathcal{R}$, then*

$$\langle v_j, w_\ell\rangle = 0 \ \textit{if}\ j \neq \ell$$

and

$$\sum_{\ell\in\mathcal{R}} \langle f, T^\ell\theta\rangle T^\ell\phi = N \sum_{j\in\mathcal{R}} \langle f, w_j\rangle v_j, \ \forall f \in L^2(0, 2\pi). \tag{1.5}$$

Proof. Most statements are easy consequences of the definition and were also observed in earlier papers (see, e.g., [5, 7, 8]), so we only prove (1.5). Given $f \in L^2(0,2\pi)$, we have by (ii) that

$$\begin{aligned} \sum_{\ell\in\mathcal{R}} \langle f, T^\ell\theta\rangle T^\ell\phi &= \sum_{\ell\in\mathcal{R}} \langle f, \sum_{\nu\in\mathcal{R}} e^{-\frac{2\pi i\ell\nu}{N}} w_\nu\rangle \sum_{j\in\mathcal{R}} e^{-\frac{2\pi i\ell j}{N}} v_j \\ &= \sum_{\nu\in\mathcal{R}}\sum_{j\in\mathcal{R}} \langle f, w_\nu\rangle v_j \sum_{\ell\in\mathcal{R}} e^{-\frac{2\pi i\ell(j-\nu)}{N}} = N \sum_{j\in\mathcal{R}} \langle f, w_j\rangle v_j, \end{aligned}$$

where we have used that for $j \neq \nu$,

$$\sum_{\ell\in\mathcal{R}} e^{-\frac{2\pi i\ell(j-\nu)}{N}} = \frac{1-e^{-\frac{2\pi i(j-\nu)}{N}N}}{1-e^{-\frac{2\pi i(j-\nu)}{N}}} = 0.$$

□

For given functions $\phi_1, \dots, \phi_r \in L^2(0,2\pi)$, our strategy is to use polyphase splines to construct explicitly oblique dual generators with convenient expressions. This is motivated by the fact that even for the simplest case of one generator, the corresponding canonical dual generator might have a complicated expression.

To illustrate this, we consider the periodic B-spline $B_{N,k}$ of order k, $k \in \mathbb{N}$, studied in [8]. Fixing the number $N \in \mathbb{N}$, they are defined recursively by

$$B_{N,1}(x) := \begin{cases} N, & \text{if } x \in (0, \frac{\pi}{N}] \cup [2\pi - \frac{\pi}{N}, 2\pi), \\ 0, & \text{if } x \in (\frac{\pi}{N}, 2\pi - \frac{\pi}{N}), \end{cases} \tag{1.6}$$

and, for $\nu \geq 1$,

$$B_{N,\nu+1}(x) := (B_{N,\nu} * B_{N,1})(x) = \frac{1}{2\pi}\int_0^{2\pi} B_{N,\nu}(x-t)B_{N,1}(t)\,dt. \tag{1.7}$$

All these functions belong to $L^2(0,2\pi)$. Letting

$$\phi := B_{N,k}, \tag{1.8}$$

we obtain the Fourier coefficients

$$\widehat{\phi}(n) = \left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^k, \; n \in \mathbb{Z}. \tag{1.9}$$

Its associated polyphase splines v_j, $j \in \mathcal{R}$, satisfy

$$\begin{aligned} \|v_j\|^2 &= \sum_{p\in\mathbb{Z}} \left(\frac{\sin(\pi(j+Np)/N)}{\pi(j+Np)/N}\right)^{2k} \\ &= \begin{cases} 1, & \text{if } j = 0, \\ N^{2k}\sin^{2k}(\pi j/N)\sum_{p\in\mathbb{Z}}(\pi(j+Np))^{-2k}, & \text{if } j \in \mathcal{R}\setminus\{0\}. \end{cases} \end{aligned} \tag{1.10}$$

With V as in (1.1), it follows from (1.5) that the frame operator $S : V \to V$ is given by

$$Sf = \sum_{\ell\in\mathcal{R}} \langle f, T^\ell\phi\rangle T^\ell\phi = N\sum_{j\in\mathcal{R}} \langle f, v_j\rangle v_j, \; \forall f \in V.$$

By (iii) and (iv) of Lemma 1.1, for every $j \in \mathcal{R}$, the polyphase spline v_j lies in V and $Sv_j = N\|v_j\|^2 v_j$, implying that $S^{-1}v_j = \frac{v_j}{N\|v_j\|^2}$. Using Lemma 1.1(i), the canonical dual generator $\tilde{\phi}$ is given by

$$\tilde{\phi} = S^{-1}\phi = \sum_{j\in\mathcal{R}} S^{-1}v_j = \sum_{j\in\mathcal{R}} \frac{v_j}{N\|v_j\|^2}.$$

In terms of Fourier coefficients, we have

$$\widehat{\tilde{\phi}}(n) = \frac{1}{N\|v_{n \bmod N}\|^2}\left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^k, \ n \in \mathbb{Z},$$

where the values $\|v_j\|^2$, $j \in \mathcal{R}$, are as in (1.10). Although the expressions for $\widehat{\tilde{\phi}}(n)$, $n \in \mathbb{Z}$, and hence $\tilde{\phi}$, are explicit, they are nevertheless complicated. This motivates us to construct oblique dual generators which provide much simpler formulas.

2. Oblique duals of finitely-generated frames

We now return to the setting where we are given functions $\phi_1, \ldots, \phi_r \in L^2(0, 2\pi)$ and want to obtain convenient expansions in the space V given by (1.1). Our goal is to construct oblique dual generators for the frame sequence $\{T^\ell \phi_m\}_{m=1,\ldots,r;\ell\in\mathcal{R}}$, i.e., to find functions $\{T^\ell \tilde{\phi}_m\}_{m=1,\ldots,r;\ell\in\mathcal{R}}$ such that

$$f = \sum_{m=1}^{r}\sum_{\ell\in\mathcal{R}} \langle f, T^\ell \tilde{\phi}_m\rangle T^\ell \phi_m, \ \forall f \in V. \tag{2.1}$$

We will reformulate (2.1) in terms of polyphase splines. Letting v_j^m and $\tilde{v}_j^m$, $j \in \mathcal{R}$, denote the polyphase splines associated to the functions ϕ_m and $\tilde{\phi}_m$, it follows from Lemma 1.1(iv) that (2.1) is equivalent to

$$f = N\sum_{m=1}^{r}\sum_{j\in\mathcal{R}} \langle f, \tilde{v}_j^m\rangle v_j^m, \ \forall f \in V. \tag{2.2}$$

The vector space V itself can also be expressed in terms of the polyphase splines. In fact, as shown in [7],

$$V = \operatorname{span}\{v_j^m : \ m = 1, \ldots, r; \ j \in \mathcal{R}\}. \tag{2.3}$$

For $j \in \mathcal{R}$, define the column vector

$$\mathbf{v}_j := \left(v_j^1, \ldots, v_j^r\right)^T$$

and the matrix

$$M(j) := \left(\langle v_j^m, \tilde{v}_j^\mu\rangle\right)_{m,\mu=1}^{r}. \tag{2.4}$$

Note that for a fixed $j \in \mathcal{R}$, $M(j)$ is the mixed Gramian matrix associated with the vectors $v_j^m, m = 1, \ldots r$, and $\tilde{v}_j^\mu, \mu = 1, \ldots, r$. We now prove that (2.1) can be formulated as an eigenvalue problem for this matrix. Most of the results in the paper will be derived as a consequence of this result.

Theorem 2.1. *Given the above setup,* (2.1) *holds if and only if*

$$\frac{1}{N}\mathbf{v}_j = M(j)\mathbf{v}_j, \ \forall j \in \mathcal{R}. \tag{2.5}$$

Proof. Assume that (2.1) holds. Using (2.2) with $f = v_\nu^\mu$ for some $\mu = 1, \dots, r$ and $\nu \in \mathcal{R}$, it follows from Lemma 1.1(iv) that

$$v_\nu^\mu = N \sum_{m=1}^{r} \sum_{j \in \mathcal{R}} \langle v_\nu^\mu, \tilde{v}_j^m \rangle v_j^m = N \sum_{m=1}^{r} \langle v_\nu^\mu, \tilde{v}_\nu^m \rangle v_\nu^m.$$

So $\mathbf{v}_\nu = N\, M(\nu)\mathbf{v}_\nu, \ \nu \in \mathcal{R}$.

Now assume that (2.5) is satisfied. Then, for each $m = 1, \dots, r, \ j \in \mathcal{R}$,

$$v_j^m = N \sum_{\mu=1}^{r} \langle v_j^m, \tilde{v}_j^\mu \rangle v_j^\mu.$$

By (2.3), every $f \in V$ can be written as

$$f = \sum_{m=1}^{r} \sum_{j \in \mathcal{R}} c_m(j) v_j^m.$$

Thus, again applying Lemma 1.1(iv),

$$\begin{aligned} N \sum_{\mu=1}^{r} \sum_{\nu \in \mathcal{R}} \langle f, \tilde{v}_\nu^\mu \rangle v_\nu^\mu &= N \sum_{\mu=1}^{r} \sum_{\nu \in \mathcal{R}} \langle \sum_{m=1}^{r} \sum_{j \in \mathcal{R}} c_m(j) v_j^m, \tilde{v}_\nu^\mu \rangle v_\nu^\mu \\ &= N \sum_{m=1}^{r} \sum_{j \in \mathcal{R}} c_m(j) \sum_{\mu=1}^{r} \langle v_j^m, \tilde{v}_j^\mu \rangle v_j^\mu \\ &= \sum_{m=1}^{r} \sum_{j \in \mathcal{R}} c_m(j) v_j^m = f. \end{aligned}$$

Hence (2.2) is satisfied, i.e., we have obtained (2.1). □

We will now use Theorem 2.1 to obtain particularly convenient sufficient conditions for (2.1) to hold, via polyphase splines. A sufficient condition for (2.5) is

$$M(j) = \frac{1}{N} I, \ \forall j \in \mathcal{R}. \tag{2.6}$$

As shown in [6], (2.6) is equivalent to biorthogonality of $\{T^\ell \phi_m\}_{m=1,\dots,r;\ell \in \mathcal{R}}$ and $\{T^\ell \tilde{\phi}_m\}_{m=1,\dots,r;\ell \in \mathcal{R}}$. It then follows that the functions in $\{T^\ell \phi_m\}_{m=1,\dots,r;\ell \in \mathcal{R}}$ are linearly independent. This is not assumed here where we work with $\{T^\ell \phi_m\}_{m=1,\dots,r;\ell \in \mathcal{R}}$ being a frame sequence. The eigenvalue condition (2.5) is clearly weaker than (2.6). Indeed, a matrix $M(j)$ satisfying (2.5) need not be diagonal. In addition, even if we make the convenient assumption that $M(j)$ is diagonal, it might happen that certain frequencies do not appear in the Fourier series for some of the functions $\phi_1, \dots, \phi_r$ and result in some of the polyphase splines v_j^m to vanish identically. For $j \in \mathcal{R}$, let

$$\mathcal{N}_j := \{m \in \{1, \dots, r\} : \ v_j^m = 0\}, \tag{2.7}$$

$$\mathcal{N}_j^{\mathrm{c}} := \{m \in \{1, \dots, r\} : \ v_j^m \neq 0\}. \tag{2.8}$$

The set $\mathcal{N}_j^{\mathrm{c}}$ is useful in formulating a more suitable sufficient condition for (2.1) with regards to the situation on hand.

Corollary 2.1. *The condition* (2.1) *is satisfied if*

$$\sum_{p\in\mathbb{Z}} \widehat{\phi_m}(j+Np)\overline{\widehat{\tilde{\phi}_\mu}(j+Np)} = \frac{1}{N}\delta_{m,\mu} \quad \textit{for all } m,\mu \in \mathcal{N}_j^c, j\in\mathcal{R}. \tag{2.9}$$

Proof. Via Theorem 2.1 it is enough to verify that for all $m = 1,\dots,r$ and all $j \in \mathcal{R}$,

$$v_j^m = N\sum_{\mu=1}^{r}\langle v_j^m, \tilde{v}_j^\mu\rangle v_j^\mu.$$

In the above expression, we can replace the sum over $\mu = 1,\dots,r$ with a sum over $\mu \in \mathcal{N}_j^{\mathrm{c}}$. The resulting condition automatically holds for $m \in \mathcal{N}_j$, so it is enough to check that

$$v_j^m = N\sum_{\mu\in\mathcal{N}_j^{\mathrm{c}}}\langle v_j^m, \tilde{v}_j^\mu\rangle v_j^\mu, \ \forall m\in\mathcal{N}_j^{\mathrm{c}}, j\in\mathcal{R}.$$

This condition is clearly satisfied if

$$\langle v_j^m, \tilde{v}_j^\mu\rangle = \frac{1}{N}\delta_{m,\mu} \ \text{ for all } m,\mu\in\mathcal{N}_j^{\mathrm{c}}, j\in\mathcal{R}.$$

Writing this condition in terms of the Fourier coefficients of the functions ϕ_m and $\tilde{\phi}_m$ yields (2.9). □

We now use Corollary 2.1 to show that under certain technical conditions, basically every collection of functions $\tilde{\phi}_1,\dots,\tilde{\phi}_r$ can be used as oblique dual generators – up to certain trigonometric polynomials.

Corollary 2.2. *Let $\phi_1,\dots,\phi_r \in L^2(0,2\pi)$. Suppose that for each $j\in\mathcal{R}$, there exist numbers $p_j^\mu \in \mathbb{Z}$, $\mu\in\mathcal{N}_j^c$, such that the matrix*

$$Q_j := \left(\widehat{\phi_m}(j+Np_j^\mu)\right)_{m,\mu\in\mathcal{N}_j^c} \tag{2.10}$$

is invertible. For any desired functions $\theta_1,\dots,\theta_r \in L^2(0,2\pi)$, there exist functions $\tilde{\phi}_1,\dots,\tilde{\phi}_r$ satisfying (2.1) *of the form*

$$\tilde{\phi}_m = \theta_m + \tau_m, \ m = 1,\dots,r, \tag{2.11}$$

where each τ_m is a trigonometric polynomial with at most $\displaystyle\sum_{j\in\mathcal{R} \text{ with } m\in\mathcal{N}_j^c} \left|\mathcal{N}_j^c\right|$ nonzero terms.

Proof. We will define suitable oblique dual generator $\tilde{\phi}_1,\dots,\tilde{\phi}_r$ explicitly in terms of their Fourier coefficients. Fix $j\in\mathcal{R}$. For all $m\in\mathcal{N}_j$ and $p\in\mathbb{Z}$, we define

$$\widehat{\tilde{\phi}_m}(j+Np) := \widehat{\theta_m}(j+Np).$$

Now, choose the numbers $p_j^\mu, \mu\in\mathcal{N}_j^{\mathrm{c}}$, such that the matrix Q_j in (2.10) is invertible, and define the set

$$P_j := \{p_j^\mu : \ \mu\in\mathcal{N}_j^{\mathrm{c}}\}.$$

For all $m \in \mathcal{N}_j^c$ and $p \in \mathbb{Z} \setminus P_j$, let

$$\widehat{\tilde{\phi}_m}(j+Np) := \widehat{\theta_m}(j+Np).$$

It remains to define the values of $\widehat{\tilde{\phi}_m}(j+Np)$ for $m \in \mathcal{N}_j^c$ and $p \in P_j$. To do this, we use the condition (2.9) in Corollary 2.1. Splitting the sum in (2.9) into two parts, we see that this condition will be satisfied if our construction here gives

$$\sum_{p \in P_j} \widehat{\phi_m}(j+Np)\overline{\widehat{\tilde{\phi}_\mu}(j+Np)} = \frac{1}{N}\delta_{m,\mu} - \sum_{p \in \mathbb{Z}\setminus P_j} \widehat{\phi_m}(j+Np)\overline{\widehat{\tilde{\phi}_\mu}(j+Np)}$$

for all $m, \mu \in \mathcal{N}_j^c$; inserting the already defined values for $\widehat{\tilde{\phi}_m}$ and taking complex conjugates, this amounts to

$$\sum_{p \in P_j} \overline{\widehat{\phi_m}(j+Np)}\widehat{\tilde{\phi}_\mu}(j+Np) = \frac{1}{N}\delta_{m,\mu} - \sum_{p \in \mathbb{Z}\setminus P_j} \overline{\widehat{\phi_m}(j+Np)}\widehat{\theta}_\mu(j+Np), \ m, \mu \in \mathcal{N}_j^c.$$

In matrix form, this set of equations is given by

$$\begin{aligned} &\left(\overline{\widehat{\phi_m}(j+Np_j^\nu)}\right)_{m,\nu \in \mathcal{N}_j^c} \left(\widehat{\tilde{\phi}_\mu}(j+Np_j^\nu)\right)_{\nu,\mu \in \mathcal{N}_j^c} \\ = &\left(\frac{1}{N}\delta_{m,\mu} - \sum_{p \in \mathbb{Z}\setminus P_j} \overline{\widehat{\phi_m}(j+Np)}\widehat{\theta}_\mu(j+Np)\right)_{m,\mu \in \mathcal{N}_j^c}. \end{aligned}$$

Due to the invertibility of the matrix Q_j, this implies that

$$\left(\widehat{\tilde{\phi}_\mu}(j+Np_j^\nu)\right)_{\nu,\mu \in \mathcal{N}_j^c} = \left(\overline{Q_j}\right)^{-1} \left(\frac{1}{N}\delta_{m,\mu} - \sum_{p \in \mathbb{Z}\setminus P_j} \overline{\widehat{\phi_m}(j+Np)}\widehat{\theta}_\mu(j+Np)\right)_{m,\mu \in \mathcal{N}_j^c}. \tag{2.12}$$

Repeating the above procedure for all $j \in \mathcal{R}$ shows that (2.9) holds.

As for the form of $\tilde{\phi}_1, \dots, \tilde{\phi}_r$, notice that for $m = 1, \dots r$, the Fourier coefficients $\widehat{\tilde{\phi}_m}(n)$ agree with $\widehat{\theta_m}(n)$ for all $n \in \mathbb{Z}$, except possibly on the set

$$\Gamma_m := \bigcup_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^c} \{j + Np_j^\nu : \ \nu \in \mathcal{N}_j^c\}.$$

Thus, for $m = 1, \dots, r$,

$$\begin{aligned} \tilde{\phi}_m(\cdot) &= \sum_{n \in \mathbb{Z}\setminus\Gamma_m} \widehat{\tilde{\phi}_m}(n)e^{in(\cdot)} + \sum_{n \in \Gamma_m} \widehat{\tilde{\phi}_m}(n)e^{in(\cdot)} \\ &= \sum_{n \in \mathbb{Z}\setminus\Gamma_m} \widehat{\theta_m}(n)e^{in(\cdot)} + \sum_{n \in \Gamma_m} \widehat{\tilde{\phi}_m}(n)e^{in(\cdot)} \\ &= \theta_m(\cdot) - \sum_{n \in \Gamma_m} \widehat{\theta_m}(n)e^{in(\cdot)} + \sum_{n \in \Gamma_m} \widehat{\tilde{\phi}_m}(n)e^{in(\cdot)} \\ &= \theta_m(\cdot) + \sum_{n \in \Gamma_m} \left(\widehat{\tilde{\phi}_m}(n) - \widehat{\theta_m}(n)\right) e^{in(\cdot)}. \end{aligned}$$

Writing

$$\tau_m(\cdot) := \sum_{n\in\Gamma_m} \left(\widehat{\tilde{\phi}_m}(n) - \widehat{\theta_m}(n)\right) e^{in(\cdot)}, \tag{2.13}$$

we see that τ_m is a trigonometric polynomial with at most $|\Gamma_m|$ nonzero coefficients. Notice that

$$|\Gamma_m| = \sum_{j\in\mathcal{R} \text{ with } m\in\mathcal{N}_j^{\mathrm{c}}} |\mathcal{N}_j^{\mathrm{c}}| \le \sum_{j\in\mathcal{R}} |\mathcal{N}_j^{\mathrm{c}}| \le Nr,$$

which is finite. □

We remark that one can easily construct functions $\phi_1, \dots, \phi_r \in L^2(0, 2\pi)$ which satisfy the technical condition on (2.10). More precisely, for each $j \in \mathcal{R}$, let $\mathcal{N}_j$ and $\mathcal{N}_j^{\mathrm{c}}$ be any two sets that form a disjoint union of $\{1, \dots, r\}$. Starting with distinct prescribed integers p_j^μ, $\mu \in \mathcal{N}_j^{\mathrm{c}}$, and an invertible $|\mathcal{N}_j^{\mathrm{c}}|$ by $|\mathcal{N}_j^{\mathrm{c}}|$ matrix Q_j, we define the values of the Fourier coefficients $\widehat{\phi_m}(j + Np_j^\mu)$, $m, \mu \in \mathcal{N}_j^{\mathrm{c}}$, by setting the matrix $\left(\widehat{\phi_m}(j + Np_j^\mu)\right)_{m,\mu\in\mathcal{N}_j^{\mathrm{c}}}$ to Q_j. The rest of the Fourier coefficients of the functions $\phi_1, \dots, \phi_r$ may be arbitrarily selected. This not only gives functions that ensure the invertibility of (2.10) but also provides the flexibility of allowing their Fourier coefficients $\widehat{\phi_m}(n)$ to take desired values for $n \in \mathbb{Z} \setminus \bigcup_{j\in\mathcal{R}}\{j + Np_j^\mu : \mu \in \mathcal{N}_j^{\mathrm{c}}\}$.

For the case of having one generator ϕ for V in (1.1), we will see in Section 4 that the invertibility condition on (2.10) is automatically satisfied and the dual generator $\tilde{\phi}$ in (2.11) can be described even more explicitly.

3. Oblique duals in prescribed subspaces

We continue with the setting in Section 2, where we are given the functions $\phi_1, \dots, \phi_r \in L^2(0, 2\pi)$ and want to obtain expansions of the type (2.1) in the space V given by (1.1). Our purpose is to find oblique dual generators $\tilde{\phi}_1, \dots, \tilde{\phi}_r$ belonging to a prescribed subspace, spanned by translates of some functions $\theta_m, m = 1, \dots, r$.

Given functions $\theta_1, \dots, \theta_r \in L^2(0, 2\pi)$, let

$$W := \operatorname{span}\{T^\ell\theta_m : \ m = 1, \dots, r; \ \ell \in \mathcal{R}\}.$$

For $m = 1, \dots, r$, let $w_j^m, j \in \mathcal{R}$, denote the polyphase splines associated with θ_m. Then, as we saw already in (2.3),

$$W = \operatorname{span}\{w_j^m : \ m = 1, \dots, r; \ j \in \mathcal{R}\}.$$

Thus, any candidates for $\tilde{\phi}_m \in W$ can be written as finite linear combinations

$$\tilde{\phi}_m = \sum_{j\in\mathcal{R}} \sum_{\mu=1}^{r} c(j)_{m,\mu} w_j^\mu. \tag{3.1}$$

Looking at the polyphase spline representation stated in Lemma 1.1, i.e.,

$$\tilde{\phi}_m = \sum_{j\in\mathcal{R}} \tilde{v}_j^m,$$

we see that (3.1) is equivalent to

$$\tilde{v}_j^m = \sum_{\mu=1}^{r} c(j)_{m,\mu} w_j^{\mu},\ j \in \mathcal{R},\ m = 1, \ldots, r. \tag{3.2}$$

Let

$$\tilde{\mathbf{v}}_j := \left(\tilde{v}_j^1, \ldots, \tilde{v}_j^r\right)^T, \quad \mathbf{w}_j := \left(w_j^1, \ldots, w_j^r\right)^T$$

and define the $r \times r$ matrices $C(j),\ j \in \mathcal{R}$, by

$$C(j) := (c(j)_{m,\mu})_{m,\mu=1}^{r};$$

then the set of equations in (3.2) takes the form

$$\tilde{\mathbf{v}}_j = C(j)\mathbf{w}_j,\ j \in \mathcal{R}.$$

Our purpose is to determine the matrix $C(j)$ such that (2.1) holds. To this purpose we again define the sets $\mathcal{N}_j$ and $\mathcal{N}_j^c$ by (2.7) and (2.8), and consider the mixed Gramians

$$P(j) := \left(\langle v_j^m, w_j^{\mu}\rangle\right)_{m,\mu=1}^{r},\ j \in \mathcal{R},$$

and

$$P'(j) := \left(\langle v_j^m, w_j^{\mu}\rangle\right)_{m,\mu\in\mathcal{N}_j^c},\ j \in \mathcal{R}. \tag{3.3}$$

We now prove that if $P'(j)$ is invertible for all $j \in \mathcal{R}$, then we can find oblique dual generators belonging to W.

Theorem 3.1. *Let $\phi_m, \theta_m \in L^2(0, 2\pi)$ for $m = 1, \ldots, r$. If*

$$\det(P'(j)) \neq 0,\ \forall j \in \mathcal{R}, \tag{3.4}$$

then there exist $\tilde{\phi}_m \in W$, $m = 1, \ldots, r$, such that (2.1) *holds. In particular, one can define $\tilde{\phi}_m$ by* (3.1)*, where*

$$C'(j) := (c(j)_{m,\mu})_{m,\mu\in\mathcal{N}_j^c} = \frac{1}{N}((P'(j))^{-1})^*, \tag{3.5}$$

$$c(j)_{m,\mu} = 0 \quad \textit{otherwise}. \tag{3.6}$$

Proof. Consider the matrices

$$M(j) = \left(\langle v_j^m, \tilde{v}_j^{\mu}\rangle\right)_{m,\mu=1}^{r},\ j \in \mathcal{R},$$

defined already in (2.4). We first prove that

$$M(j) = P(j)C(j)^*. \tag{3.7}$$

In fact, for $m, \mu = 1, \ldots, r$,

$$(P(j)C(j)^*)_{m,\mu} = \sum_{\nu=1}^{r} P(j)_{m,\nu} C(j)^*_{\nu,\mu} = \sum_{\nu=1}^{r} \langle v_j^m, w_j^{\nu}\rangle \overline{c(j)_{\mu,\nu}} = \langle v_j^m, \sum_{\nu=1}^{r} c(j)_{\mu,\nu} w_j^{\nu}\rangle;$$

via (3.2), this expression equals $\langle v_j^m, \tilde{v}_j^{\mu}\rangle = M(j)_{m,\mu}$.

We now aim at determining the entries in the matrix $C(j)$ such that $M(j)$ satisfies (2.5) in Theorem 2.1. By (3.7), we see that (2.5) is equivalent to

$$(I - NP(j)C(j)^*)\,\mathbf{v}_j = \mathbf{0},\ j \in \mathcal{R}. \tag{3.8}$$

Writing (3.8) coordinatewise yields the conditions

$$\sum_{\mu=1}^{r} (I - NP(j)C(j)^*)_{m,\mu}\, v_j^\mu = 0,\ m = 1, \dots, r,\ j \in \mathcal{R},$$

which are equivalent to

$$\sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} (I - NP(j)C(j)^*)_{m,\mu}\, v_j^\mu = 0,\ m = 1, \dots, r,\ j \in \mathcal{R}. \tag{3.9}$$

To establish (3.9), we first consider $m \in \mathcal{N}_j$. Then $I_{m,\mu} = 0$ for all $\mu \in \mathcal{N}_j^{\mathrm{c}}$; via (3.7) and (2.4),

$$\sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} (I - NP(j)C(j)^*)_{m,\mu}\, v_j^\mu = -N \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} \langle v_j^m, \tilde{v}_j^\mu \rangle v_j^\mu = 0$$

because $v_j^m = 0$. Consequently, we only need to verify (3.9) for $m \in \mathcal{N}_j^{\mathrm{c}}, j \in \mathcal{R}$. Now, fixing $j \in \mathcal{R}$, for $m, \mu \in \mathcal{N}_j^{\mathrm{c}}$,

$$(P(j)C(j)^*)_{m,\mu} = \langle v_j^m, \sum_{\nu=1}^{r} c(j)_{\mu,\nu} w_j^\nu \rangle = \langle v_j^m, \sum_{\nu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{\mu,\nu} w_j^\nu \rangle = \left(P'(j)C'(j)^*\right)_{m,\mu},$$

where we have assumed (3.6) to ensure $c(j)_{\mu,\nu} = 0$ for $\mu \in \mathcal{N}_j^{\mathrm{c}}$, $\nu \in \mathcal{N}_j$. Thus, in matrix form and considering only $m \in \mathcal{N}_j^{\mathrm{c}}$, (3.9) takes the form

$$\left(I - NP'(j)C'(j)^*\right) \mathbf{v}'_j = \mathbf{0} \tag{3.10}$$

where $\mathbf{v}'_j := (v_j^m)_{m \in \mathcal{N}_j^{\mathrm{c}}}$. Since $\mathbf{v}'_j \neq \mathbf{0}$, (3.10) is equivalent to

$$\det\left(I - NP'(j)C'(j)^*\right) = 0;$$

this is clearly satisfied if we choose $C'(j)$ such that

$$P'(j)C'(j)^* = \frac{1}{N} I,$$

i.e., as in (3.5). □

Remark 3.1. The proof of Theorem 3.1 actually shows more freedom than stated in the result. In fact, instead of (3.6), we can define

$$\begin{aligned} c(j)_{m,\mu} &= 0 \text{ for } m \in \mathcal{N}_j^{\mathrm{c}}, \mu \in \mathcal{N}_j, \\ c(j)_{m,\mu} &\quad \text{arbitrary for } m \in \mathcal{N}_j, \mu = 1, \dots, r. \end{aligned}$$

As an application of Theorem 3.1, we now see that for any functions $\phi_1, \dots, \phi_r \in L^2(0, 2\pi)$, under the same technical conditions as in Corollary 2.2, it is *always* possible to construct oblique dual generators which are trigonometric polynomials. Interestingly, this result holds regardless of whether the functions $\phi_1, \dots, \phi_r$ themselves are trigonometric polynomials.

Corollary 3.1. *Let $\phi_1, \dots, \phi_r \in L^2(0, 2\pi)$. Suppose that for each $j \in \mathcal{R}$, there exist numbers $p_j^\mu \in \mathbb{Z}$, $\mu \in \mathcal{N}_j^c$, such that the matrix Q_j in (2.10) is invertible. Then there exist trigonometric polynomials $\tilde{\phi}_1, \dots, \tilde{\phi}_r$ satisfying (2.1); and for every $m = 1, \dots, r$, the trigonometric polynomial $\tilde{\phi}_m$ has at most $\displaystyle\sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^c} \left|\mathcal{N}_j^c\right|$ nonzero terms.*

Proof. To identify the functions $\theta_1, \dots, \theta_r$ for the application of Theorem 3.1, we proceed by first defining their polyphase splines w_j^m, $m = 1, \dots, r$, $j \in \mathcal{R}$. For $j \in \mathcal{R}$, let p_j^μ, $\mu \in \mathcal{N}_j^{\mathrm{c}}$, be some numbers for which the matrix Q_j in (2.10) is invertible. Then we set

$$w_j^\mu(\cdot) := \begin{cases} e^{i(j+Np_j^\mu)(\cdot)}, & \text{if } \mu \in \mathcal{N}_j^{\mathrm{c}}, \\ 0, & \text{if } \mu \in \mathcal{N}_j. \end{cases} \tag{3.11}$$

The prescribed functions $\theta_1, \dots, \theta_r$ in Theorem 3.1 are given by

$$\theta_m = \sum_{j \in \mathcal{R}} w_j^m, \ m = 1, \dots, r.$$

Now consider the matrices $P'(j)$, $j \in \mathcal{R}$, as defined in (3.3). For $j \in \mathcal{R}$ and $m, \mu \in \mathcal{N}_j^{\mathrm{c}}$, it follows from (3.11) that

$$\langle v_j^m, w_j^\mu \rangle = \sum_{n \in \mathbb{Z}} \widehat{v_j^m}(n) \overline{\widehat{w_j^\mu}(n)} = \widehat{v_j^m}(j + Np_j^\mu) \cdot 1 = \widehat{\phi_m}(j + Np_j^\mu).$$

Therefore

$$\det(P'(j)) = \det(Q_j) \neq 0, \ \forall j \in \mathcal{R},$$

via (2.10).

Applying Theorem 3.1, the functions $\tilde{\phi}_m$, $m = 1, \dots, r$, given by (3.1), are oblique dual generators satisfying (2.1), where for every $j \in \mathcal{R}$, the matrix entries $c(j)_{m,\mu}$, $m, \mu = 1, \dots, r$, are as follows:

$$\begin{aligned} (c(j)_{m,\mu})_{m,\mu \in \mathcal{N}_j^{\mathrm{c}}} &= \frac{1}{N}(Q_j^{-1})^*, \\ c(j)_{m,\mu} &= 0 \ \text{ otherwise.} \end{aligned}$$

Hence by (3.1), for $m = 1, \dots, r$,

$$\begin{aligned} \tilde{\phi}_m &= \sum_{j \in \mathcal{R}} \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{m,\mu} w_j^\mu \\ &= \sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^{\mathrm{c}}} \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{m,\mu} w_j^\mu + \sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j} \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{m,\mu} w_j^\mu \\ &= \sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^{\mathrm{c}}} \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{m,\mu} w_j^\mu. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\tilde{\phi}_m(\cdot) = \sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^{\mathrm{c}}} \sum_{\mu \in \mathcal{N}_j^{\mathrm{c}}} c(j)_{m,\mu} e^{i(j+Np_j^\mu)(\cdot)},$$

which is a trigonometric polynomial with at most $\displaystyle\sum_{j \in \mathcal{R} \text{ with } m \in \mathcal{N}_j^{\mathrm{c}}} \left|\mathcal{N}_j^{\mathrm{c}}\right|$ nonzero terms. $\square$

It is interesting to compare the oblique dual generators constructed in Corollaries 2.2 and 3.1, as both results have the same assumption (i.e., invertibility of the matrices Q_j in (2.10)). For $j \in \mathcal{R}$, let p_j^μ, $\mu \in \mathcal{N}_j^{\rm c}$, be the numbers ensuring the invertibility of the matrix Q_j. Consider the subspace of trigonometric polynomials defined by

$$T := \operatorname{span}\{e^{i(j+Np_j^\mu)(\cdot)} : \ j \in \mathcal{R}; \ \mu \in \mathcal{N}_j^{\rm c}\}.$$

In Corollary 2.2, we construct oblique dual generators based on certain desired functions. The difference between an oblique dual generator and its corresponding desired function is a trigonometric polynomial in the subspace T. On the other hand, in Corollary 3.1, we construct oblique dual generators that lie inside the subspace T. Both approaches are made possible by the same standing assumption on Q_j.

The theory of periodic wavelets via polyphase splines in [5, 6, 7] was developed for the more general multidimensional setting of $L^2((0,2\pi)^s)$, where $s \in \mathbb{N}$, as minimal additional efforts compared to the one-dimensional case were needed. Likewise, all the results here on oblique duals for finite-dimensional spaces in $L^2(0,2\pi)$ could be easily extended to the multidimensional setting. We will outline this briefly.

Let D be an $s \times s$ invertible matrix with integer entries. Let $\mathcal{R}$ and $\mathcal{L}$ denote full collections of coset representatives of $\mathbb{Z}^s/D\mathbb{Z}^s$ and $\mathbb{Z}^s/(D^T)\mathbb{Z}^s$ respectively. Then $|\mathcal{R}| = |\mathcal{L}| = |\det(D)|$. For functions $\phi_1, \ldots, \phi_r \in L^2((0,2\pi)^s)$, we consider the vector space

$$V := \operatorname{span}\{T^\ell \phi_m : \ m = 1, \ldots, r; \ \ell \in \mathcal{L}\},$$

where for $\ell \in \mathcal{L}$,

$$T^\ell : L^2((0,2\pi)^s) \to L^2((0,2\pi)^s), \quad (T^\ell f)(\cdot) := f(\cdot - 2\pi(D^T)^{-1}\ell). \tag{3.13}$$

For a function $\phi \in L^2((0,2\pi)^s)$ with Fourier coefficients $\widehat{\phi}(n)$, $n \in \mathbb{Z}^s$, its associated polyphase splines v_j, $j \in \mathcal{R}$, now take the form

$$v_j(\cdot) := \sum_{p \in \mathbb{Z}^s} \widehat{\phi}(j + Dp) e^{i(j+Dp)\cdot(\cdot)}. \tag{3.14}$$

Equipped with the identity

$$\sum_{\ell \in \mathcal{L}} e^{2\pi i \ell \cdot D^{-1}(j-\nu)} = \begin{cases} |\det(D)|, & \text{if } j = \nu, \\ 0, & \text{if } j \neq \nu, \end{cases}$$

for $j, \nu \in \mathcal{R}$, we can derive the multidimensional analog of Lemma 1.1. For instance, (1.5) becomes

$$\sum_{\ell \in \mathcal{L}} \langle f, T^\ell \theta \rangle T^\ell \phi = |\det(D)| \sum_{j \in \mathcal{R}} \langle f, w_j \rangle v_j, \ \forall f \in L^2((0,2\pi)^s), \tag{3.15}$$

where w_j, $j \in \mathcal{R}$, are the associated polyphase splines of $\theta \in L^2((0,2\pi)^s)$.

The derivations in all our results on oblique duals for finite-dimensional spaces in $L^2(0,2\pi)$ go through readily for this multidimensional setup. The formulations in (3.13)–(3.15) give the essential idea in how results for the $L^2((0,2\pi)^s)$-case would appear. In fact, as one proceeds from $L^2(0,2\pi)$ to $L^2((0,2\pi)^s)$, the positive integer N in the original results are appropriately replaced by the $s \times s$ invertible integer matrix D, its transpose D^T, or the absolute value of its determinant.

4. The single generator case

Let us investigate further the situation where the vector space V in (1.1) has only one generating function $\phi \in L^2(0,2\pi)$, i.e., $r=1$ and

$$V = \text{span}\{T^\ell \phi : \ \ell \in \mathcal{R}\}.$$

We will see that in this case, many of the results obtained take simpler forms. In particular, the technical condition on (2.10) is always satisfied, and we can find explicit and convenient expressions for the various oblique duals. We will derive the results for this special setting from the general results in Sections 2 and 3 as illustration of our method, but they can of course also be derived directly.

For $\phi \in L^2(0,2\pi)$, our objective is to find an oblique dual generator $\tilde{\phi} \in L^2(0,2\pi)$ such that

$$f = \sum_{\ell \in \mathcal{R}} \langle f, T^\ell \tilde{\phi} \rangle T^\ell \phi, \ \forall f \in V. \tag{4.1}$$

Let v_j and $\tilde{v}_j$, $j \in \mathcal{R}$, be the polyphase splines associated to ϕ and $\tilde{\phi}$ respectively. Then the matrices $M(j)$, $j \in \mathcal{R}$, in (2.4) are simply scalars:

$$M(j) = \langle v_j, \tilde{v}_j \rangle, \ j \in \mathcal{R}.$$

As some of the v_j's may vanish identically, consolidating the sets in (2.7) and (2.8), we define

$$\begin{aligned} \mathcal{N} &:= \{j \in \mathcal{R} : \ v_j = 0\}, \\ \mathcal{N}^{\mathrm{c}} &:= \{j \in \mathcal{R} : \ v_j \neq 0\}. \end{aligned}$$

Using Theorem 2.1 and proceeding as in the proof of Corollary 2.1, we obtain the following characterization of (4.1). Note that Corollary 2.1 only provides a sufficient condition for (4.1), but $M(j)$, $j \in \mathcal{R}$, now being scalars, enable the converse to hold as well.

Corollary 4.1. *For $\phi, \tilde{\phi} \in L^2(0,2\pi)$, (4.1) holds if and only if*

$$\sum_{p \in \mathbb{Z}} \widehat{\phi}(j+Np)\overline{\widehat{\tilde{\phi}}(j+Np)} = \frac{1}{N} \ \text{ for all } j \in \mathcal{N}^c.$$

In the single-generator setting, the technical assumption on (2.10) can also be removed from Corollary 2.2.

Corollary 4.2. *For $\phi \in L^2(0,2\pi)$, given any desired function $\theta \in L^2(0,2\pi)$, there exists a function $\tilde{\phi}$ satisfying (4.1) of the form*

$$\tilde{\phi} = \theta + \tau, \tag{4.2}$$

where τ is a trigonometric polynomial with at most $|\mathcal{N}^c|$ nonzero terms.

Proof. The result follows from Corollary 2.2 once we establish the invertibility of Q_j in (2.10). Indeed, for $j \in \mathcal{N}^{\mathrm{c}}$, $v_j \neq 0$ implies that some of its Fourier coefficients must be nonzero, i.e., there exists $p_j \in \mathbb{Z}$ for which $\widehat{\phi}(j+Np_j) \neq 0$. Thus we may define the scalar Q_j in (2.10) to be $\widehat{\phi}(j+Np_j)$, which completes the proof. $\square$

Examining the constructive proof of Corollary 2.2, it follows from (2.13) that the trigonometric polynomial τ in (4.2) is given explicitly by

$$\tau(\cdot) = \sum_{j\in\mathcal{N}^c} \left(\widehat{\tilde{\phi}}(j+Np_j) - \widehat{\theta}(j+Np_j)\right) e^{i(j+Np_j)(\cdot)}.$$

Using (2.12), for $j \in \mathcal{N}^c$,

$$\begin{aligned}
\widehat{\tau}(j+Np_j) &= \frac{1}{\overline{\widehat{\phi}(j+Np_j)}} \left(\frac{1}{N} - \sum_{p\in\mathbb{Z}\setminus\{p_j\}} \overline{\widehat{\phi}(j+Np)}\widehat{\theta}(j+Np)\right) - \widehat{\theta}(j+Np_j) \\
&= \frac{1}{\overline{\widehat{\phi}(j+Np_j)}} \left(\frac{1}{N} - \sum_{p\in\mathbb{Z}} \overline{\widehat{\phi}(j+Np)}\widehat{\theta}(j+Np)\right) \\
&= \frac{1}{\overline{\widehat{\phi}(j+Np_j)}} \left(\frac{1}{N} - \langle w_j, v_j\rangle\right),
\end{aligned}$$

where v_j and w_j, $j \in \mathcal{R}$, are the polyphase splines associated to ϕ and θ respectively. Hence, letting $\Gamma := \{j + Np_j : \ j \in \mathcal{N}^c\}$,

$$\widehat{\tau}(n) := \begin{cases} \frac{1}{\overline{\widehat{\phi}(n)}} \left(\frac{1}{N} - \langle w_n, v_n\rangle\right), & \text{if } n \in \Gamma, \\ 0, & \text{if } n \in \mathbb{Z}\setminus\Gamma, \end{cases}$$

as $\langle w_n, v_n\rangle = \langle w_{n \bmod N}, v_{n \bmod N}\rangle$ for $n \in \mathbb{Z}$. By (4.2),

$$\widehat{\tilde{\phi}}(n) := \begin{cases} \widehat{\theta}(n) + \frac{1}{\overline{\widehat{\phi}(n)}} \left(\frac{1}{N} - \langle w_n, v_n\rangle\right), & \text{if } n \in \Gamma, \\ \widehat{\theta}(n), & \text{if } n \in \mathbb{Z}\setminus\Gamma. \end{cases}$$

This implies that in both low and high frequencies, the constructed oblique dual generator $\tilde{\phi}$ agrees exactly with the desired function θ. More precisely, if $|n| < \min\{|\nu| : \ \nu \in \Gamma\}$ or $|n| > \max\{|\nu| : \nu \in \Gamma\}$, we have $\widehat{\tilde{\phi}}(n) = \widehat{\theta}(n)$.

Next, we consider the problem of finding oblique dual generators $\tilde{\phi}$ belonging to a prescribed subspace

$$W := \operatorname{span}\{T^\ell\theta : \ \ell \in \mathcal{R}\},$$

where θ is a function in $L^2(0, 2\pi)$. The following result is a consequence of Theorem 3.1.

Corollary 4.3. *For $\phi, \theta \in L^2(0, 2\pi)$, let v_j and w_j, $j \in \mathcal{R}$, be their associated polyphase splines. If*

$$\langle v_j, w_j\rangle \neq 0, \ \forall j \in \mathcal{N}^c, \tag{4.3}$$

then there exists $\tilde{\phi} \in W$ such that (4.1) holds. In particular, one may define $\tilde{\phi}$ by

$$\tilde{\phi} = \frac{1}{N} \sum_{j\in\mathcal{N}^c} \frac{1}{\overline{\langle v_j, w_j\rangle}} w_j. \tag{4.4}$$

Proof. Since $r = 1$, the condition (3.4) in Theorem 3.1 is the same as (4.3). By Theorem 3.1, (4.4) follows from (3.1), (3.5) and (3.6). $\square$

With Corollaries 3.1 and 4.3 in place, we are ready to construct oblique dual generators which are trigonometric polynomials with convenient expressions. We emphasize again that the construction is independent of whether ϕ itself is a trigonometric polynomial.

Corollary 4.4. *For any function $\phi \in L^2(0,2\pi)$, there exists a trigonometric polynomial $\tilde{\phi}$ which satisfies* (4.1) *and is of the form*

$$\tilde{\phi}(\cdot) = \frac{1}{N} \sum_{j \in \mathcal{N}^c} \frac{1}{\overline{\widehat{\phi}(j+Np_j)}} e^{i(j+Np_j)(\cdot)}, \tag{4.5}$$

where each p_j is a number such that $\widehat{\phi}(j+Np_j) \neq 0$.

Proof. As in the proof of Corollary 4.2, we note that for every $j \in \mathcal{N}^{\mathrm{c}}$, there exists $p_j \in \mathbb{Z}$ such that $\widehat{\phi}(j+Np_j) \neq 0$. This guarantees the invertibility assumption on Q_j in (2.10), and Corollary 3.1 implies the existence of a trigonometric polynomial $\tilde{\phi}$ satisfying (4.1). The proof of Corollary 3.1 also shows that $\tilde{\phi}$ can be of the form (4.4) and $w_j(\cdot) = e^{i(j+Np_j)(\cdot)}$ for all $j \in \mathcal{N}^{\mathrm{c}}$. Since $\langle v_j, w_j \rangle = \widehat{\phi}(j+Np_j)$ for $j \in \mathcal{N}^{\mathrm{c}}$, (4.5) follows from (4.4). □

The expression (4.5) gives very convenient oblique dual generators. For instance, if $\phi \in L^2(0,2\pi)$ is a function for which $\widehat{\phi}(j) \neq 0$ for all $j \in \mathcal{N}^{\mathrm{c}}$, then we may set $p_j := 0$ for all $j \in \mathcal{N}^{\mathrm{c}}$. This implies that the trigonometric polynomial

$$\tilde{\phi}(\cdot) = \frac{1}{N} \sum_{j \in \mathcal{N}^c} \frac{1}{\overline{\widehat{\phi}(j)}} e^{ij(\cdot)} \tag{4.6}$$

is an oblique dual generator.

An advantage of having oblique dual generators that are trigonometric polynomials of the form (4.5) is the ease of computing coefficients in the frame expansion (4.1). To elaborate on this, recall from Lemma 1.1(iv) that for every $f \in V$,

$$f = \sum_{\ell \in \mathcal{R}} \langle f, T^\ell \tilde{\phi} \rangle T^\ell \phi = N \sum_{j \in \mathcal{R}} \langle f, \tilde{v}_j \rangle v_j.$$

Further, by Lemma 1.1(ii), for $\ell \in \mathcal{R}$,

$$\langle f, T^\ell \tilde{\phi} \rangle = \langle f, \sum_{j \in \mathcal{R}} e^{-\frac{2\pi i \ell j}{N}} \tilde{v}_j \rangle = \sum_{j \in \mathcal{R}} \langle f, \tilde{v}_j \rangle e^{\frac{2\pi i \ell j}{N}}.$$

Thus the frame coefficients $\langle f, T^\ell \tilde{\phi} \rangle$, $\ell \in \mathcal{R}$, can be easily obtained once the values $\langle f, \tilde{v}_j \rangle$, $j \in \mathcal{R}$, are found. In general, we have to evaluate

$$\langle f, \tilde{v}_j \rangle = \sum_{p \in \mathbb{Z}} \widehat{f}(j+Np) \overline{\widehat{\tilde{\phi}}(j+Np)}, \; j \in \mathcal{R}, \tag{4.7}$$

which could be computationally intensive. However for $\tilde{\phi}$ of the form (4.5), the expression (4.7) reduces to simply

$$\langle f, \tilde{v}_j \rangle = \begin{cases} \frac{1}{N} \frac{\widehat{f}(j+Np_j)}{\widehat{\phi}(j+Np_j)}, & \text{if } j \in \mathcal{N}^{\mathrm{c}}, \\ 0 & \text{if } j \in \mathcal{N}, \end{cases} \tag{4.8}$$

and this makes the computation of $\langle f, T^\ell \tilde{\phi}\rangle$, $\ell \in \mathcal{R}$, much more manageable. Further, (4.8) shows that the Fourier coefficients $\widehat{f}(j + Np_j)$, $j \in \mathcal{N}^c$, contain sufficient information to recover the expansion coefficients of f in (4.1). More generally, for any oblique dual generator $\tilde{\phi}$ that is a trigonometric polynomial, each of the series in (4.7) is a finite sum and the expansion coefficients of f can be obtained from finitely many appropriate Fourier coefficients of f.

Example 4.1. For $k \in \mathbb{N}$, let $\phi \in L^2(0, 2\pi)$ be the periodic B-spline $B_{N,k}$ of order k with Fourier coefficients given by (1.9). Since $\widehat{\phi}(j) \neq 0$ for all $j \in \mathcal{R}$, it follows that $\mathcal{N}^c = \mathcal{R}$ and by (4.6),

$$\tilde{\phi}(\cdot) = \frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{\pi n/N}{\sin(\pi n/N)} \right)^k e^{in(\cdot)} \tag{4.9}$$

is an oblique dual generator.

To obtain oblique dual generators that are real-valued and even, if N is a positive odd integer, it follows from (1.9) that $\widehat{\phi}(n) \neq 0$ for $n = -(N-1)/2, \dots, (N-1)/2$. So for $j \in \mathcal{R}$, we may set

$$p_j := \begin{cases} 0, & \text{if } j = 0, \dots, (N-1)/2, \\ -1, & \text{if } j = (N+1)/2, \dots, N-1, \end{cases}$$

to derive from (4.5) an oblique dual generator of the form

$$\tilde{\phi}(\cdot) = \frac{1}{N} \left(1 + 2 \sum_{n=1}^{(N-1)/2} \left(\frac{\pi n/N}{\sin(\pi n/N)} \right)^k \cos n(\cdot) \right). \tag{4.10}$$

On the other hand, if N is a positive even integer, we employ Corollary 4.3 instead with $\theta(\cdot) = \sum_{n=-N/2}^{N/2} e^{in(\cdot)}$. In this case, we obtain an oblique dual generator of the form

$$\tilde{\phi}(\cdot) = \frac{1}{N} \left(1 + (\pi/2)^k \cos(N(\cdot)/2) + 2 \sum_{n=1}^{(N/2)-1} \left(\frac{\pi n/N}{\sin(\pi n/N)} \right)^k \cos n(\cdot) \right). \tag{4.11}$$

While $\tilde{\phi}$ in (4.11) has $N+1$ nonzero Fourier coefficients in contrast to N nonzero Fourier coefficients in (4.9), it has the desired property of being both real-valued and even.

The explicit expressions of oblique dual generators in (4.9)–(4.11) are significantly simpler than the canonical dual generator observed in Section 1.2. This demonstrates the possibility of having a generator of V with good time-localization, such as a periodic B-spline, and simultaneously enjoying the benefit of an oblique dual generator that is a trigonometric polynomial with good frequency-localization. Figure 1 shows the plots of ϕ and $\widehat{\phi}$ defined by (1.8) and (1.9) for $k = 8$ and $N = 32$, and the corresponding $\tilde{\phi}$ and $\widehat{\tilde{\phi}}$ from (4.11).

Example 4.2. Suppose that N is a positive even integer. For $k \in \mathbb{N}$, let $B_{N,k}$ and $B_{(N/2),k}$ be periodic B-splines of order k obtained as in (1.6) and (1.7). Define $\phi \in L^2(0, 2\pi)$ by setting

$$\phi := B_{N,k} - B_{(N/2),k}. \tag{4.12}$$

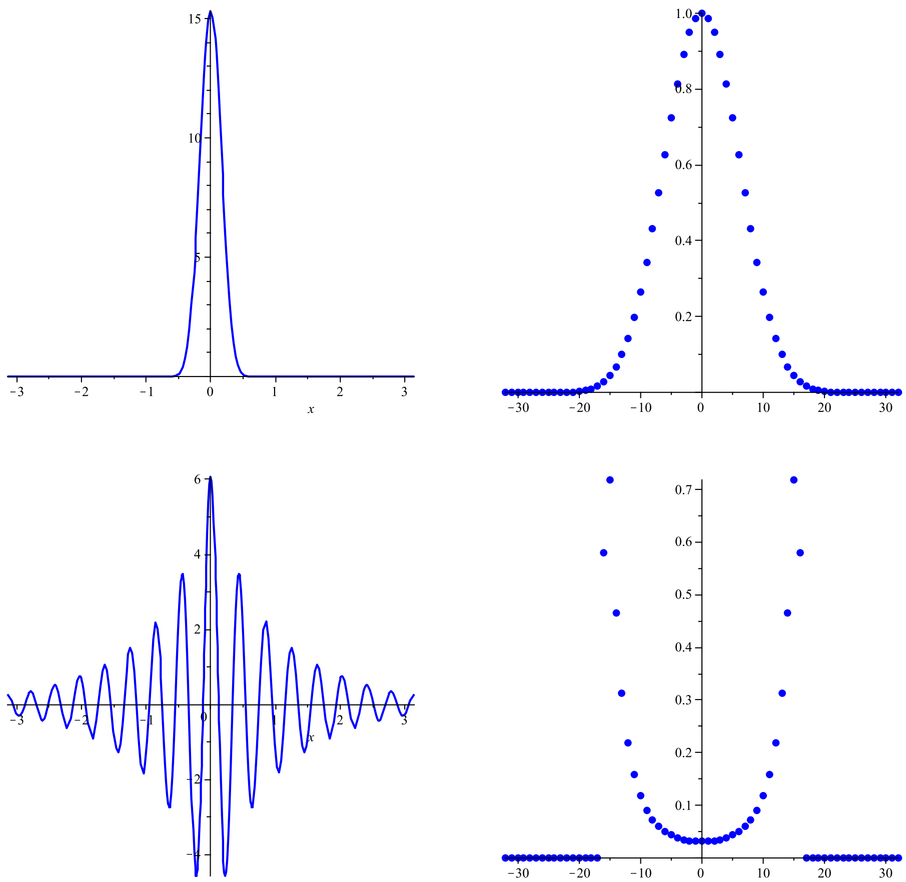

FIGURE 1. Plots of ϕ (top left) and $\widehat{\phi}$ (top right) defined by (1.8) and (1.9) for $k = 8$ and $N = 32$, and the corresponding $\tilde{\phi}$ (bottom left) and $\widehat{\tilde{\phi}}$ (bottom right) from (4.11) as in Example 4.1.

By (1.9), the Fourier coefficients of ϕ are

$$\begin{aligned} \widehat{\phi}(n) &= \left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^k - \left(\frac{\sin(2\pi n/N)}{2\pi n/N}\right)^k \\ &= \left(\frac{\sin(\pi n/N)}{\pi n/N}\right)^k \left(1 - \cos^k(\pi n/N)\right), \quad n \in \mathbb{Z}, \end{aligned} \tag{4.13}$$

from which we observe that $\widehat{\phi}(Np) = 0$ for all $p \in \mathbb{Z}$, and $\widehat{\phi}(n) = \widehat{\phi}(-n) \neq 0$ for every $n = 1, \ldots, N/2$. Therefore $\mathcal{N}^{\mathrm{c}} = \mathcal{R}\backslash\{0\}$ and it follows from Corollary 4.3 with $\theta(\cdot) = \sum_{n=-N/2}^{N/2} e^{in(\cdot)}$ that

$$\tilde{\phi}(\cdot) = \frac{1}{N}\left((\pi/2)^k \cos(N(\cdot)/2) + 2\sum_{n=1}^{(N/2)-1}\left(\frac{\pi n/N}{\sin(\pi n/N)}\right)^k \frac{\cos n(\cdot)}{1-\cos^k(\pi n/N)}\right) \tag{4.14}$$

is an oblique dual generator.

It is interesting to note that this case of $\mathcal{N} = \{0\} \neq \emptyset$ actually presents us with additional freedom to construct desired oblique dual generators. Indeed, again based on $\theta(\cdot) = \sum_{n=-N/2}^{N/2} e^{in(\cdot)}$, Remark

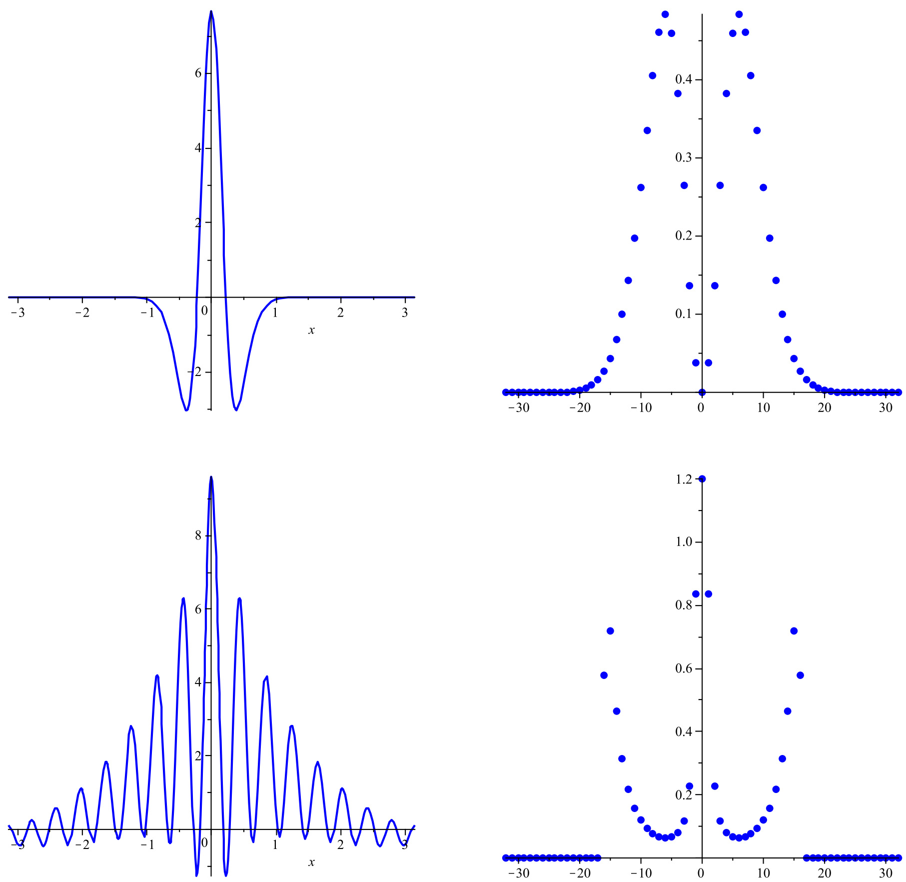

FIGURE 2. Plots of ϕ (top left) and $\widehat{\phi}$ (top right) defined by (4.12) and (4.13) for $k = 8$ and $N = 32$, and the corresponding $\tilde{\phi}$ (bottom left) and $\widehat{\tilde{\phi}}$ (bottom right) from (4.15) with $c = 1.2$ as in Example 4.2.

3.1 implies that for any constant $c \in \mathbb{C}$,

$$\tilde{\phi}(\cdot) = c + \frac{1}{N}\left((\pi/2)^k \cos(N(\cdot)/2) + 2\sum_{n=1}^{(N/2)-1}\left(\frac{\pi n/N}{\sin(\pi n/N)}\right)^k \frac{\cos n(\cdot)}{1-\cos^k(\pi n/N)}\right) \tag{4.15}$$

is an oblique dual generator. In contrast to (4.14), the flexibility provided by the parameter c can be exploited to obtain a trigonometric polynomial $\tilde{\phi}$ with better time-localization and a more natural Fourier coefficient at $n = 0$. Figure 2 shows the plots of ϕ and $\widehat{\phi}$ for $k = 8$ and $N = 32$, and the corresponding $\tilde{\phi}$ and $\widehat{\tilde{\phi}}$ from (4.15) with $c = 1.2$.

5. INVERTIBILITY OF MIXED GRAMIANS

The key assumption in Theorem 3.1 is the invertibility of the mixed Gramians $P'(j)$ in (3.3). We now examine this condition in more detail. For notational convenience we first consider mixed Gramians for finite sets of vectors in a general Hilbert space $\mathcal{H}$.

Theorem 5.1. *Let $\{f_k\}_{k=1}^M, \{g_k\}_{k=1}^M$ be sets of vectors in $\mathcal{H}$, and $G := (\langle f_j, g_k\rangle)_{j,k=1}^M$ the associated mixed Gramian. Then G is invertible if and only if the following three conditions are satisfied:*

(i) *$\{f_k\}_{k=1}^M$ is linearly independent;*
(ii) *$\{g_k\}_{k=1}^M$ is linearly independent;*
(iii) *If $g \in \operatorname{span}\{g_k\}_{k=1}^M$ and $\langle g, f\rangle = 0$ for all $f \in \operatorname{span}\{f_k\}_{k=1}^M$, then $g = 0$.*

Proof. If G is invertible, it is clear that (i) and (ii) are satisfied; in fact, if $\sum_{k=1}^M c_k f_k = 0$ for some $\{c_k\}_{k=1}^M \neq \mathbf{0}$, then a certain linear combination of the rows of G could produce a zero-row, which would imply that $\det(G) = 0$.

Now, G being invertible means that

$$\{\langle f_j, \sum_{k=1}^M c_k g_k\rangle\}_{j=1}^M = \mathbf{0} \Rightarrow c_k = 0, \ \forall k = 1, \dots, M;$$

this is equivalent to

$$\langle f, \sum_{k=1}^M c_k g_k\rangle = 0, \forall f \in \operatorname{span}\{f_j\}_{j=1}^M \Rightarrow c_k = 0, \ \forall k = 1, \dots, M,$$

which again is equivalent to

$$\begin{cases} \{g_k\}_{k=1}^M \text{ is linearly independent, and} \\ g \in \operatorname{span}\{g_k\}_{k=1}^M, \langle g, f\rangle = 0, \forall f \in \operatorname{span}\{f_k\}_{k=1}^M \Rightarrow g = 0. \end{cases}$$

□

Note that the condition (iii) is equivalent to

$$\left(\operatorname{span}\{f_k\}_{k=1}^M\right)^\perp \cap \operatorname{span}\{g_k\}_{k=1}^M = \{0\}. \tag{5.1}$$

Also observe that in order to conclude that G is invertible, we only need to assume (ii) and (iii). In fact, $\{f_k\}_{k=1}^M$ being linearly independent is a consequence of these two assumptions. This follows from the above proof (which did not use (i)), or from the fact that if $\operatorname{span}\{g_k\}_{k=1}^M$ is M-dimensional and $\operatorname{span}\{f_k\}_{k=1}^M$ has dimension less than M, then (5.1) cannot hold.

Recall the definition of the *angle* $R(W,V) \in [0, \pi/2]$ from a closed subspace W of $\mathcal{H}$ to another closed subspace V, cf. [10]:

$$\cos R(W,V) := \inf_{w \in W, \|w\|=1} \|P_V w\|,$$

where P_V is the orthogonal projection of $\mathcal{H}$ on V.

Lemma 5.1. *Let V, W be closed subspaces of $\mathcal{H}$, and let $R(W,V)$ denote the angle from W to V. Then*

$$V^\perp \cap W = \{0\} \Leftrightarrow R(W,V) \in [0, \pi/2).$$

Proof. We first prove that the condition

$$V^\perp \cap W = \{0\} \tag{5.2}$$

is equivalent to

$$\sup_{v \in V^\perp, w \in W, \|v\|=\|w\|=1} |\langle v, w\rangle| < 1. \tag{5.3}$$

First, it is clear that (5.3) implies (5.2). To prove that (5.2) implies (5.3), assume that (5.3) does not hold. Then we can select sequences $\{v_k\}_{k=1}^\infty \subset V^\perp, \{w_k\}_{k=1}^\infty \subset W$ with $\|v_k\| = \|w_k\| = 1$ such that $\langle v_k, w_k\rangle \to 1$ as $k \to \infty$. But the unit sphere in W is compact, so we can select a subsequence of $\{w_k\}_{k=1}^\infty$ (call it again $\{w_k\}_{k=1}^\infty$) such that $w_k \to w$ for some $w \in W$. Now,

$$\|v_k - w\|^2 = \langle v_k - w, v_k - w\rangle = \|v_k\|^2 - \langle v_k, w\rangle - \langle w, v_k\rangle + \|w\|^2 \to 0$$

as $k \to \infty$; this implies that $w \in V^\perp$ because $V^\perp$ is closed. Thus $w \in W \cap V^\perp$, which implies that (5.2) does not hold.

To conclude the proof, note that by formula (2.2) in [10], (5.3) is equivalent to $R(W, V) \in [0, \pi/2)$. □

Corollary 5.1. *Let $\{f_k\}_{k=1}^M$ and $\{g_k\}_{k=1}^M$ in $\mathcal{H}$ be linearly independent, and set $V := span\{f_k\}_{k=1}^M$, $W := span\{g_k\}_{k=1}^M$. Then the mixed Gramian $G = (\langle f_j, g_k\rangle)_{j,k=1}^M$ is invertible if and only if $R(W, V) \in [0, \pi/2)$ (or if and only if $R(V, W) \in [0, \pi/2)$).*

Proof. If G is invertible, Theorem 5.1 shows that (5.1) holds. Via Lemma 5.1, this implies that $R(W, V) \in [0, \pi/2)$. The reverse implication follows from the same results. The equivalence to $R(V, W) \in [0, \pi/2)$ follows by looking at the conjugate transpose of G, i.e., reversing the roles of $\{f_k\}_{k=1}^M$ and $\{g_k\}_{k=1}^M$. □

Remark 5.1. By [10, Theorem 2.3], the conditions in Corollary 5.1 are equivalent to $\mathcal{H} = V \oplus W^\perp$.

Let us now return to the setup considered in this paper. Given a collection of functions $\phi_m \in L^2(0, 2\pi)$, $m = 1, \ldots, r$, we denote again the corresponding polyphase splines by $v_j^m, m = 1, \ldots, r$, $j \in \mathcal{R}$. For $m = 1, \ldots, r$ and $j \in \mathcal{R}$, consider the vector $\mathbf{\Phi}_j^m$ in $\ell^2(\mathbb{Z})$ defined by

$$\mathbf{\Phi}_j^m := \{\widehat{\phi_m}(j + Np)\}_{p \in \mathbb{Z}}.$$

Now, for any $j \in \mathcal{R}$ and any subset $\mathcal{F} \subseteq \{1, \ldots, r\}$,

$$\sum_{m \in \mathcal{F}} c_m v_j^m(\cdot) = \sum_{p \in \mathbb{Z}} \left(\sum_{m \in \mathcal{F}} c_m \widehat{\phi_m}(j + Np) \right) e^{i(j+Np)(\cdot)}.$$

Thus,

$$\begin{aligned} \sum_{m \in \mathcal{F}} c_m v_j^m = 0 \quad &\Leftrightarrow \quad \sum_{m \in \mathcal{F}} c_m \widehat{\phi_m}(j + Np) = 0, \ \forall p \in \mathbb{Z} \\ &\Leftrightarrow \quad \sum_{m \in \mathcal{F}} c_m \mathbf{\Phi}_j^m = \mathbf{0}; \end{aligned}$$

this implies that the vectors $\{v_j^m\}_{m \in \mathcal{F}}$ are linearly independent in $L^2(0, 2\pi)$ if and only if the vectors $\{\mathbf{\Phi}_j^m\}_{m \in \mathcal{F}}$ are linearly independent in $\ell^2(\mathbb{Z})$.

Suppose that we also have a collection of functions $\theta_m \in L^2(0,2\pi)$, $m=1,\dots,r$, with associated polyphase splines w_j^m, $m=1,\dots,r$, $j\in\mathcal{R}$. Let

$$\mathbf{\Theta}_j^m := \{\widehat{\theta_m}(j+Np)\}_{p\in\mathbb{Z}}. \tag{5.4}$$

For any $v\in \mathrm{span}\{v_j^m\}_{m\in\mathcal{F}}, w\in \mathrm{span}\{w_j^m\}_{m\in\mathcal{F}}$, write $v=\sum_{m\in\mathcal{F}} c_m v_j^m,\ w=\sum_{m\in\mathcal{F}} d_m w_j^m$; then

$$\begin{aligned}\langle v,w\rangle &= \langle\{\sum_{m\in\mathcal{F}} c_m\widehat{\phi_m}(j+Np)\}_{p\in\mathbb{Z}},\{\sum_{m\in\mathcal{F}} d_m\widehat{\theta_m}(j+Np)\}_{p\in\mathbb{Z}}\rangle_{\ell^2(\mathbb{Z})}\\ &= \langle\sum_{m\in\mathcal{F}} c_m\mathbf{\Phi}_j^m,\sum_{m\in\mathcal{F}} d_m\mathbf{\Theta}_j^m\rangle_{\ell^2(\mathbb{Z})}.\end{aligned}$$

These calculations show that all the conditions in Theorem 5.1 concerning invertibility of the mixed Gramian $\left(\langle v_j^m, w_j^\mu\rangle\right)_{m,\mu\in\mathcal{F}}$ can be turned into conditions concerning the vectors $\mathbf{\Phi}_j^m,\mathbf{\Theta}_j^m$:

Lemma 5.2. *Fix any $j\in\mathcal{R}$ and consider any subset $\mathcal{F}\subseteq\{1,\dots,r\}$. Then the mixed Gramian $\left(\langle v_j^m, w_j^\mu\rangle\right)_{m,\mu\in\mathcal{F}}$ is invertible if and only if the vectors $\{\mathbf{\Phi}_j^m\}_{m\in\mathcal{F}}$ are linearly independent in $\ell^2(\mathbb{Z})$, and*

$$g\in span\{\mathbf{\Phi}_j^m\}_{m\in\mathcal{F}},\ \langle g,f\rangle_{\ell^2(\mathbb{Z})}=0,\forall f\in span\{\mathbf{\Theta}_j^m\}_{m\in\mathcal{F}}\Rightarrow g=0. \tag{5.5}$$

Lemma 5.2 gives a procedure for finding functions $\theta^m, m=1,\dots,r$, such that the mixed Gramian $\left(\langle v_j^m, w_j^\mu\rangle\right)_{m,\mu\in\mathcal{N}_j^c}$ in Theorem 3.1 is invertible for each j. The procedure works backwards compared to what we have done so far. To elaborate on this, for each $j\in\mathcal{R}$, let $\mathcal{N}_j$ and $\mathcal{N}_j^c$ be any two sets that form a disjoint union of $\{1,\dots,r\}$. Fixing $j\in\mathcal{R}$, the first step is to check that the vectors $\{v_j^m\}_{m\in\mathcal{N}_j^c}$ (and equivalently, the vectors $\{\mathbf{\Phi}_j^m\}_{m\in\mathcal{N}_j^c}$) are linearly independent. After that, find vectors $\mathbf{\Theta}_j^m\in\ell^2(\mathbb{Z})$, $m\in\mathcal{N}_j^c$, which are linearly independent and satisfy (5.5). For $m\in\mathcal{N}_j$, let $\mathbf{\Theta}_j^m:=\mathbf{0}$. Now, for $m=1,\dots,r$, define the numbers $\widehat{\theta_m}(j+Np)$, $p\in\mathbb{Z}$, via (5.4) and

$$w_j^m(\cdot):=\sum_{p\in\mathbb{Z}}\widehat{\theta_m}(j+Np)e^{i(j+Np)(\cdot)}.$$

Then the choice

$$\theta_m:=\sum_{j\in\mathcal{R}} w_j^m,\ m=1,\dots,r,$$

implies that the mixed Gramian $\left(\langle v_j^m, w_j^\mu\rangle\right)_{m,\mu\in\mathcal{N}_j^c}$ is invertible for each $j\in\mathcal{R}$.

In practice, we are usually interested in the generators for the frame as well as the dual frame being trigonometric polynomials. This implies that the sum defining the polyphase splines v_j^m is really a finite sum over $p\in\{-P,\dots,P\}$ for some $P\in\mathbb{N}$, where P can be chosen independently of j,m. In this case, all conditions considered here actually take place in $\mathbb{C}^{2P+1}$ rather than $\ell^2(\mathbb{Z})$.

Remark 5.2. The condition (5.5) is obviously satisfied if we choose $\mathbf{\Theta}_j^m$ such that

$$\mathrm{span}\{\mathbf{\Phi}_j^m\}_{m\in\mathcal{F}}=\mathrm{span}\{\mathbf{\Theta}_j^m\}_{m\in\mathcal{F}}.$$

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