From Cardinal Hermite Splines to Multiwavelets

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ABSTRACT This paper traces the origin of multiwavelets. It surveys the evolution of multiwavelets from cardinal Hermite splines and reviews the recent developments.

1 Cardinal splines and wavelets: a historical perspective

1.1 *Forward B-splines and wavelets*

As the theory of wavelets and shift-invariant subspaces unfolds, it becomes increasingly transparent that they belong to the extended family of cardinal spline functions. Spline functions were introduced by Schoenberg in 1946 [60] and further developed by him in a series of papers in the seventies (see [62] and the references therein). A popular approach in the construction of wavelets is via the multiresolution analysis of Meyer and Mallat [53]. Particularly relevant to our discussion are the B-splines. The forward B-spline, $Q_n(x)$ of degree n, is refinable and it is the solution of the refinement equation

$$Q_n(x) = \sum_{k=0}^{n+1} \frac{1}{2^n} \binom{n+1}{k} Q_n(2x-k), \quad x \in \mathbb{R}. \tag{1.1}$$

The polynomial $\Pi_n(z) := \sum_{j=0}^{n-1} Q_n(j+1)z^j$ is called the *Euler-Frobenius polynomial* of degree $n-1$. There is evidence to suggest that the use of piecewise polynomials for numerical approximation was known to Euler (see [62]). The coefficients of $\Pi_n(z)$ are known as the *Eulerian numbers* (see [5]).

For a fixed integer n, Q_n is a refinable function with impulse response $H_n(z) = \left(\frac{1+z}{2}\right)^{n+1}$. The first encounter with splines as wavelets was made by Lemarié [51] who constructed an infinite linear combination of B-splines with orthonormal integer shifts. Recognizing that the integer shifts of Q_n form a Riesz basis of their closed linear span in $L^2(\mathbb{R})$, and that it is

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practically more advantageous to work with B-splines than with the Lemaré wavelets, Chui and Wang [9] showed that Q_n generates a multiresolution of $L^2(\mathbb{R})$, and they went on to construct the corresponding spline wavelets and their duals which are now commonly known as Chui and Wang's wavelets. Slightly earlier, without recourse to theory of spline functions, Daubechies [14] constructed a class of orthonormal compactly supported wavelets. Such a wavelet is constructed from a scaling function which is a convolution of a uniform B-spline with a distribution that is chosen such that the resulting scaling function has orthonormal integer shifts.

1.2 Cardinal Hermite splines and multiwavelets

The forward B-splines Q_n are associated with Lagrange interpolation at an infinite number of points by piecewise polynomials with simple integer knots (see [61] and [62] and the references therein). Historically, Lagrange (1736–1813) performed interpolation using polynomials which take the values of a given function at a finite number of points. About one hundred years later, Hermite (1822–1901) extended Lagrange's idea by including the values of the function and its consecutive derivatives. This evolution of the interpolation process prompted Schoenberg (1974) to introduce the cardinal Hermite interpolation problem and the cardinal Hermite B-splines [63], a brief description of which follows.

Take a fixed positive integer n. For simplicity, we assume that n is odd. For a positive integer r satisfying $r \leq (n+1)/2$, let $\mathcal{S}_{n,r}$ be the space of spline functions of degree n defined on $\mathbb{R}$ with integer knots of multiplicity r, i.e. $f \in \mathcal{S}_{n,r}$ if and only if $f \in C^{n-r}(\mathbb{R})$ and is a polynomial of degree n on each of the intervals $[\nu, \nu+1],\ \nu \in \mathbb{Z}$. The space $\mathcal{S}_{n,r}$ has a spline basis comprising functions $N_\ell,\ \ell = 1, \dots, r$, with compact supports. These functions are uniquely determined by the requirement that they vanish outside the interval $[0, n-2r+3]$ and that they satisfy the Hermite interpolating conditions

$$N_\ell^{(k-1)}(\nu) = c(\nu)\delta_\ell(k), \quad \nu = 1, \dots, n-2r+2, \tag{1.2}$$

for $k, \ell = 1, \dots, r$, where $c(\nu),\ \nu = 1, \dots, n-2r+2$, are the coefficients of the generalized Euler-Frobenius polynomial

$$\Pi_{n,r}(z) = \sum_{\nu=0}^{n-2r+1} c(\nu+1)z^\nu, \quad z \in \mathbb{C} \setminus \{0\}.$$

The generalized Euler-Frobenius polynomial $\Pi_{n,r}(z)$ is defined as the minor of order $(n-r+1) \times (n-r+1)$ obtained by deleting the first r rows and the last r columns of the matrix

$$P_n(z) := \left(\binom{k}{\ell} - z\delta_\ell(k) \right)_{k,\ell=0}^{n} .$$

The functions $N_\ell,\ \ell = 1,\dots,r$, are called *cardinal Hermite B-splines.* They are nice symmetric/antisymmetric functions. These B-splines were introduced by Schoenberg and Sharma [63], and it was shown in [49] that their integer shifts $N_\ell(\cdot - k),\ \ell = 1,\dots,r,\ k \in \mathbb{Z}$, indeed form a spline basis of $\mathcal{S}_{n,r}$, i.e. every $f \in \mathcal{S}_{n,r}$ is uniquely expressible in the form $f(x) = \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} c_\ell(k)N_\ell(x-k)$.

Example 1: With $n = 3,\ r = 2$, there are two B-splines $N_1,\ N_2$ supported on $[0,2]$. They are given explicitly by

$$N_1(x) = \begin{cases} 3x^2 - 2x^3, & 0 \le x \le 1, \\ -4 + 12x - 9x^2 + 2x^3, & 1 \le x \le 2, \end{cases}$$

$$N_2(x) = \begin{cases} -x^2 + x^3, & 0 \le x \le 1, \\ -4 + 8x - 5x^2 + x^3, & 1 \le x \le 2. \end{cases}$$

They are C^1 functions and satisfy the interpolating conditions

$$N_1(\nu+1) = \delta_0(\nu), \qquad N_1'(\nu) = 0, \quad \nu \in \mathbb{Z},$$

$$N_2'(\nu+1) = \delta_0(\nu), \qquad N_2(\nu) = 0, \quad \nu \in \mathbb{Z}.$$

Example 2: For $n = 5,\ r = 2$, the generalized Euler-Frobenius polynomial is given by $\Pi_{5,2}(z) = -1 + 6z - z^2$. There are two B-splines $N_1,\ N_2$ which are supported on $[0,4]$; N_1 is symmetric while N_2 is antisymmetric about 2. To express N_1 and N_2 explicitly, note that $N_j(\cdot+2),\ j = 1,2$, are supported on $[-2,2]$, so that $N_1(\cdot+2)$ is an even function while $N_2(\cdot+2)$ is odd. Therefore, they are completely described by

$$N_1(x+2) = 8(1-x)_+^5 - 50(1-x)_+^4 + 4(2-x)_+^5 - 5(2-x)_+^4, \quad x \ge 0,$$

$$N_2(x+2) = 10(1-x)_+^5 - 26(1-x)_+^4 + (2-x)_+^5 - (2-x)_+^4, \quad x \ge 0.$$

They are C^3 functions satisfying the interpolating conditions:

$$N_1(1) = -1, N_1(2) = 6, N_1(3) = -1, \qquad N_1'(\nu) = 0, \quad \nu \in \mathbb{Z},$$

$$N_2'(1) = -1, N_2'(2) = 6, N_2'(3) = -1, \qquad N_2(\nu) = 0, \quad \nu \in \mathbb{Z}.$$

When $r = 1$, there is only one such B-spline, N_1, which is equal to the forward B-spline, Q_n, up to a constant multiple. Since wavelets are closely related to the forward B-splines which are associated with cardinal Lagrange interpolation, it is natural to follow the course of history to search for the corresponding 'wavelets' related to the cardinal Hermite B-splines which are associated with cardinal Hermite interpolation. This has led us ([27], [25]) to introduce the idea of multiresolution of multiplicity r and to construct spline wavelets of multiplicity r, the precursor to multiwavelets.

Let $V_0 \subset L^2(\mathbb{R})$ be the shift invariant subspace generated by the integer shifts of $N_\ell,\ \ell = 1, \dots, r$, and let $V_m := \{f(2^m\cdot) : f \in V_0\},\ m \in \mathbb{Z}$. It was shown in [27] that the sequence of subspaces V_m satisfies
(MR1) $V_m \subset V_{m+1},\ m \in \mathbb{Z}$,
(MR2) $\cap_{m\in\mathbb{Z}} V_m = \{0\}$,
(MR3) $\cup_{m\in\mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R})$,
(MR4) $\in V_m \iff f(2\cdot) \in V_{m+1}$,
(MR5) the integer shifts of N_ℓ, $\ell = 1, \dots, r$, form a Riesz basis of V_0.
We say that $N_\ell,\ \ell = 1, \dots, r$, generate a *multiresolution of* $L^2(\mathbb{R})$ *of multiplicity* r.

Let W_0 be the orthogonal complement of V_0 in V_1. If $\psi_1, \dots, \psi_r \in W_0$, and their integer shifts form a Riesz basis of W_0, then $\psi_1, \dots, \psi_r$ will be called *multiwavelets.* If their integer shifts form an orthonormal basis of W_0, then they will be called *orthonormal multiwavelets.* For convenience we shall also call the vector function $\Psi = (\psi_1, \dots, \psi_r)^T$ multiwavelets. Spline multiwavelets with minimal support analogous to Chui and Wang's wavelets were constructed in [25].

We now form from the B-splines $N_\ell,\ \ell = 1, \dots, r$, the vector function $N := (N_1, \dots, N_r)^T$. Since $\{V_m\}_{m\in\mathbb{Z}}$ forms a multiresolution of $L^2(\mathbb{R})$, the condition (MR1) implies that the vector function N satisfies a *matrix refinement equation*

$$N(x) = \sum_{k\in\mathbb{Z}} 2h(k)N(2x-k), \quad x \in \mathbb{R}, \tag{1.3}$$

where $h(k),\ k \in \mathbb{Z}$, are $r \times r$ matrices such that the entries $h_{i,j}(k),\ k \in \mathbb{Z}$, are sequences in $\ell^2(\mathbb{Z})$. The vector function N is called a *scaling vector.* The sequence $(h(k))_{k\in\mathbb{Z}}$ is called the *mask* for N. In general, the mask h is an infinite sequence, even though N has compact support. However, it can be shown that $h(k) \to 0$ exponentially as $k \to \infty$.

Further, since $\psi_j \in V_1$, $j = 1, \dots, r$, there is a matrix sequence $(g(k))_{k\in\mathbb{Z}}$ with entries $g_{i,j} \in \ell^2(\mathbb{Z})$ such that

$$\Psi(x) = \sum_{k\in\mathbb{Z}} 2g(k)N(2x-k), \quad x \in \mathbb{R}. \tag{1.4}$$

The sequence $(g(k))_{k\in\mathbb{Z}}$ is called a *high pass multifilter.* In this context, $(h(k))_{k\in\mathbb{Z}}$ is also called a *low pass multifilter.*

2 Multiwavelets in wandering subspaces

It turns out that the above process of finding multiwavelets from a multiresolution of $L^2(\mathbb{R})$ (generated by N_ℓ, $\ell = 1, \dots, r$) can be generalized to a Hilbert space setting, and there is a close connection between orthonormal multiwavelets and wandering subspaces for unitary operators.

Let H be a complex Hilbert space, and let $U = (U_1, ..., U_d)$ be an ordered d-tuple of distinct unitary operators on H such that $U_k U_j = U_j U_k$, $k, j = 1, \ldots, d$. We shall use the multi-index notation $U^n = U_1^{n_1} \cdots U_d^{n_d}$ for $n = (n_1, ..., n_d) \in \mathbb{Z}^d$, with the convention that U_j^0 is the identity operator on H, $j = 1, ..., d$. We also assume that U^n is the identity operator only if $n = 0$. For a subset S of H, let $\langle S \rangle$ denote the closed linear span of S in H, and write $U^{\mathbb{Z}^d}(S) := \{U^n s : n \in \mathbb{Z}^d, s \in S\}$. We say that a closed linear subspace V of H is a *wandering subspace* for U if $U^p(V) \perp U^n(V)$ for all $p, n \in \mathbb{Z}^d$, $p \neq n$. Further, if $W = \sum_{n \in \mathbb{Z}^d} U^n(V)$, then we say that V is a complete wandering subspace of W for U.

The following theorem was proved in [28, Theorem 3.1], by first extending the results of Robertson [58] on wandering subspaces for unitary operators. (The case $d = 1$ was previously considered in [27, 50].)

Theorem 2.1 *Let $X := \{x_1, ..., x_r\}$ be a finite subset of H, and $V_0 = \langle U^{\mathbb{Z}^d}(X) \rangle$. Suppose that there is a unitary operator D on H such that*

$$V_0 \subset DV_0 =: V_1 \tag{2.1}$$

and

$$U^n D = DU^{Mn}, \quad n \in \mathbb{Z}^d, \tag{2.2}$$

where M is a $d \times d$ matrix with integer entries and $m := |\det(M)| \geq 2$. Let W_0 be the orthogonal complement of V_0 in V_1.
(a) *If $U^{\mathbb{Z}^d}(X)$ is a Riesz basis of V_0, then there exists a finite subset Γ of W_0, with $r(m-1)$ elements, such that $U^{\mathbb{Z}^d}(\Gamma)$ is a Riesz basis of W_0, and $U^{\mathbb{Z}^d}(X \cup \Gamma)$ is a Riesz basis of V_1.*
(b) *If $U^{\mathbb{Z}^d}(X)$ is an orthonormal basis of V_0, then there exists a finite subset Γ of W_0, with $r(m-1)$ elements, such that $U^{\mathbb{Z}^d}(\Gamma)$ is an orthonormal basis of W_0, and $U^{\mathbb{Z}^d}(X \cup \Gamma)$ is an orthonormal basis of V_1.*

Part (b) of Theorem 2.1 can be rephrased as follows: If $\langle X \rangle$ is a complete wandering subspace of V_0 for U, then there exists a finite subset Γ of W_0, with $r(m-1)$ elements, such that $\langle \Gamma \rangle$ is a complete wandering subspace of W_0 for U, and $\langle X \cup \Gamma \rangle$ is a complete wandering subspace of V_1 for U. This connection between orthonormal multiwavelets and wandering subspaces was first observed in [27], where part (b) of Theorem 2.1 for the case $d = 1$ was derived as a consequence of [58, Theorem 2].

Let us apply Theorem 2.1 to the concrete setting of $L^2(\mathbb{R}^d)$. If $H = L^2(\mathbb{R}^d)$,

$$(U_k f)(x) = f(x - e_k),$$

where $e_k = (\delta_k(j))_{j=1}^d$, $k = 1, \ldots, d$, and

$$(Df)(x) = |\det(M)|^{\frac{1}{2}} f(Mx),$$

for x in $\mathbb{R}^d$ and f in $L^2(\mathbb{R}^d)$, then

$$D^j U^n f = |\det(M)|^{\frac{j}{2}} f(M^j \cdot -n), \quad j \in \mathbb{Z},\, n \in \mathbb{Z}^d,$$

and (2.2) is satisfied. If $\{\phi_\ell(\cdot - n) : n \in \mathbb{Z}^d, \ell = 1, \dots, r\}$ is a Riesz basis of V_0 and $V_0 \subset V_1 := \{f(M\cdot) : f \in V_0\}$, then the above theorem gives the existence of $r(m-1)$ multiwavelets in W_0. In particular, if $M = 2I_d$, where I_d is the $d \times d$ identity matrix, then we have $r(2^d - 1)$ multiwavelets. Our previous discussion in Subsection 1.2 involves the special case $d = 1$.

Recently, Dai and Larson [13] have obtained generalizations of part (b) of Theorem 2.1 to other systems of unitary operators. Part (a) of Theorem 2.1 has also been extended to the biorthogonal setting in [70] and [75].

3 Matrix refinement equations with finite mask

Multifilters are the most important ingredients in the applications of multiwavelets. Therefore, the design of 'good multifilters' is an important aspect in the study of multiwavelets. Good scaling vectors and multiwavelets are associated with good multifilters. Therefore, the problem of multifilter design reduces to the problem of constructing good scaling vectors and hence multiwavelets. Thus to design good multifilters we need to know the properties for good scaling vectors in terms of the mask. In practice, we require the multifilters to be finite.

The most general matrix refinement equation (MRE) with finite mask that is being actively studied at present is of the form

$$\Phi(x) = \sum_{k \in \mathbb{Z}^d} \det(M) h(k) \Phi(Mx - k), \quad x \in \mathbb{R}^d, \tag{3.1}$$

where the mask $(h(k))_{k\in\mathbb{Z}^d}$ is a finitely supported sequence of $r \times r$ matrices, $\Phi = (\phi_1, \dots, \phi_r)^T$ is a vector of tempered distributions on $\mathbb{R}^d$, and M is a $d \times d$ dilation matrix with integer entries and all its eigenvalues are of modulus greater than 1.

In the Fourier domain, (3.1) becomes

$$\widehat{\Phi}(u) = H((M^{-1})^T u) \widehat{\Phi}((M^{-1})^T u), \quad u \in \mathbb{R}^d, \tag{3.2}$$

where $\widehat{\Phi}(u) := (\hat{\phi}_1(u), \dots, \hat{\phi}_r(u))^T$ and $H(u) := \sum_k h(k) e^{-ik^T u}$. A solution of the MRE (3.1), or equivalently (3.2), is called an (M, h)-*scaling vector*.

The problems addressed in connection with the matrix refinement equation with finite mask include (1) existence of solutions of matrix refinement equations, (2) convergence of the cascade algorithm, (3) stability, orthonormality, and biorthogonality of (M, h)-refinable vectors, (4) approximation order of the shift invariant subspaces generated by (M, h)-refinable vectors, (5) regularity of (M, h)-refinable vectors, (6) construction of (M, h)-refinable vectors, (7) multiwavelet construction, (8) applications.

Problems 1, 2, 3, 4 and 5 were recently addressed in [30], [31], [12], [56], [64], [43], [54], [39], [38], [36], [37]. The theory of multiwavelets mushroomed in the last 5 years, but few examples of multiwavelets are known. In one dimension, symmetric/antisymmetric spline multiwavelets or spline wavelets of multiplicity r were constructed in [25]. The method of construction in [25] exploits the properties of spline functions, and it does not seem to work for other multiwavelets. A general construction of orthonormal multiwavelets from a given orthonormal scaling vector in one dimension can be found in [46]. The results in [46] have been extended in [23] to the biorthogonal setting. Not-a-spline scaling functions with $r = 2$ were first constructed by Geronimo, Hardin and Massopust [21]. Subsequently the corresponding wavelets were constructed in [18], [68] and [46]. They are commonly called GHM multiwavelets. They are orthonormal symmetric pair with supports in $[0, 2]$. This example is simple, but it has provided much inspiration for more constructions ([8], [18]). Recently, Jiang has constructed a large class of scaling vectors and multiwavelets with optimum time-frequency localization in a series of papers [40], [41], [42]. Multiwavelets with 'good filter characteristics' have also been constructed recently in [71]. They are symmetric/antisymmetric pairs of orthonormal multiwavelets. The corresponding multifilters are proven to have high energy concentration, a feature useful for image and video compression [71]. In higher dimensions, very few examples of multiwavelets are known. An example of linear spline multiwavelets ($r = 2$) in two dimensions can be found in [28], and pairs of refinable splines in $\mathbb{R}^2$ were constructed in [24]. Recently, another example in two dimensions has been constructed in [19] These are the only examples known in more than one dimension. Much more has to be done.

In the rest of this section, we shall review some recent developments of multiwavelets in more detail. For simplicity, we shall restrict to the case $d = 1$ and $M = (2)$. The general case is more elaborate but the results are similar.

3.1 Existence of solution of MRE

For $d = 1$ and $M = (2)$, the MRE becomes

$$\Phi(x) = \sum_{k=0}^{N} 2h(k)\Phi(2x - k), \quad x \in \mathbb{R}, \tag{3.3}$$

where $h(k)$, $k = 0, \ldots, N$, is a finite sequence of $r \times r$ matrices, and $\Phi = (\phi_1, \ldots, \phi_r)^T$ is a vector of tempered distributions. In the Fourier domain, (3.3) is equivalent to

$$\widehat{\Phi}(u) = H(u/2)\widehat{\Phi}(u/2), \quad u \in \mathbb{R}. \tag{3.4}$$

Iterating (3.4) leads to

$$\widehat{\Phi}(u) = \prod_{j=1}^{n} H(u/2^j)\widehat{\Phi}(u/2^n), \tag{3.5}$$

for any positive integer n.

The existence of $\widehat{\Phi}$, and hence the distributional solution of the MRE (3.3), depends on the convergence of the infinite product $\prod_{j=1}^{\infty} H(u/2^j)$. This problem was first considered by Heil and Colella [30] in one dimension and substantially extended by Jiang and Shen [43] to higher dimensions. We first introduce some terminologies. A square matrix A is said to satisfy *Condition* E^{**} if it has unit spectral radius and all its eigenvalues on the unit circle are nondegenerate. If in addition, 1 is the only eigenvalue on the unit circle, then A is said to satisfy *Condition* E^*. We say that A satisfies *Condition* E if it satisfies Condition E^* and 1 is a simple eigenvalue.

The following result gives a necessary and sufficient condition for the convergence of infinite product of matrices of continuous functions. It can be deduced from Theorem 2.1 in [22].

Proposition 3.1 *Let $A(u)$ be a square matrix of continuous functions on $\mathbb{R}$, satisfying*

$$\sum_{j=1}^{\infty} ||A(u/2^j) - A(0)|| < \infty \quad \textit{for all } u \in \mathbb{R}. \tag{3.6}$$

Then $\lim_{n\to\infty} \prod_{j=1}^{n} A(u/2^j)$ *exists, and the convergence is uniform on compact sets if and only if $A(0)$ satisfies Condition E^*.*

For a MRE with finite mask, the frequency response $H(u)$ satisfies (3.6). It follows that the infinite product $\prod_{j=1}^{\infty} H(u/2^j)$ converges locally uniformly if and only if $H(0)$ satisfies Condition E^*.

Example: Consider the sequence of matrices $h(k)$ with 3 nonzero terms

$$h(0) = \frac{1}{16}\begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}, \quad h(1) = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad h(2) = \frac{1}{16}\begin{pmatrix} 4 & -6 \\ 1 & -1 \end{pmatrix}.$$

Then $H(u) = h(0) + h(1)^{-iu} + h(2)e^{-i2u}$, and

$$H(0) = h(0) + h(1) + h(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1/8 \end{pmatrix},$$

which satisfies Condition E^*. In fact $H(0)$ satisfies Condition E.

Suppose that $H(0)$ satisfies Condition E^* and its eigenvalue 1 has multiplicity s with linearly independent eigenvectors $v_1, \dots, v_s$. Then $P(u) := \prod_{j=1}^{\infty} H(u/2^j),\ u \in \mathbb{R}$, defines an $r \times r$ matrix of continuous functions. By (3.5),

$$\widehat{\Phi}(u) = P(u)\widehat{\Phi}(0). \tag{3.7}$$

The matrix $P(u)$ has polynomial growth as $|u| \to \infty$. The proof is standard and can be found in [15] for the scalar case and [30] for the vector case. Consider $P(u)v_k = \prod_{j=1}^{\infty} H(u/2^j)v_k,\ k = 1, \dots, s$, which is an r-vector whose components are continuous functions with polynomial growth at infinity. There is a vector $\Phi_k = (\phi_{k,1}, \dots, \phi_{k,r})^T$ of (tempered) distributions whose Fourier transform satisfies $\widehat{\Phi}_k(u) = P(u)v_k,\ u \in \mathbb{R}$. This follows from the fact that the Fourier transform is an isomorphism on the space of tempered distributions $\mathcal{S}'$, and that continuous functions with polynomial growth belong to $\mathcal{S}'$.

For each $k = 1, \dots, s$, Φ_k is a distributional solution of the MRE (3.3) since $\widehat{\Phi}_k(u) = H(u/2)\widehat{\Phi}_k(u/2)$. Now, take any v in $\mathbb{C}^r$. By the same argument as above, there is a vector Φ_v of tempered distributions whose Fourier transform satisfies $\widehat{\Phi}_v(u) = P(u)v,\ u \in \mathbb{R}$, which gives the relation $\widehat{\Phi}_v(u) = H(u/2)\widehat{\Phi}_v(u/2)$. This means that Φ_v is also a distributional solution of the MRE (3.3). It is easy to see that $\widehat{\Phi}_v$ belongs to the linear span of $\{\widehat{\Phi}_1, \dots, \widehat{\Phi}_s\}$.

If $H(0)$ satisfies Condition E^*, then the solutions of the MRE are, in general, vectors of tempered distributions. It is therefore natural to search for simple conditions for the solutions to belong to $L^2(\mathbb{R})^r$. As far as we know, necessary and sufficient conditions in terms of the spectrum of transition operators, are not available, but a simple sufficient condition may be deduced from the characterization of weak convergence of the cascade algorithm.

3.2 Cascade algorithms and transition operators

The cascade algorithm for the MRE (3.3) is a means to establish the existence of its solutions and to compute them. It is very much like the Picard iteration for the solution of differential equations. To define the cascade algorithm, we choose a starting vector function $\Phi_0 = (\phi_{0,1}, \dots, \phi_{0,r})^T$ in $L^2(\mathbb{R})^r$ with support in $[0, N]$, and define the cascade sequence Φ_n by

$$\Phi_n(x) = \sum_{k=0}^{N} 2h(k)\Phi_{n-1}(2x - k), \quad x \in \mathbb{R},\ n = 1, 2, \dots . \tag{3.8}$$

Each Φ_n belongs to $L^2(\mathbb{R})^r$, and has support in $[0, N]$. Note that if $\Phi_n \to \Phi$, then Φ is a solution of the MRE (3.3). Consider the $r \times r$ matrix

$$f_n(t) = \int_{\mathbb{R}} \Phi_n(x)\Phi_n(x-t)^* dx \ . \tag{3.9}$$

Equations (3.8) and (3.9) lead to

$$f_n(t) = \sum_{k=0}^{N} \sum_{\ell=k-N}^{k} 2h(k) f_{n-1}(2t-\ell) h(k-\ell)^* \ . \tag{3.10}$$

Setting $t = \nu$ in (3.10) gives

$$f_n(\nu) = \sum_{k=0}^{N} \sum_{\ell=k-N}^{k} 2h(k) f_{n-1}(2\nu-\ell) h(k-\ell)^* \ . \tag{3.11}$$

In order to simplify (3.11), we define the *transition operator* T_h for all sequences $(b(n))_{n\in[-N+1,N-1]}$ of $r \times r$ matrices by

$$(T_h b)(\nu) = \sum_{k=0}^{N} \sum_{\ell=k-N}^{k} 2h(k) b(2\nu-\ell) h(k-\ell)^* \ . \tag{3.12}$$

Then (3.11) simplifies to

$$f_n = T_h f_{n-1}, \quad n = 1, 2 \ldots \ . \tag{3.13}$$

To describe the transition operator T_h in the Fourier domain which is easier to deal with at times, we introduce the space $\mathcal{V}_N$ of all $r \times r$ matrices with trigonometric polynomial entries whose Fourier coefficients are real and supported in $[-N+1, N-1]$. Let

$$T_H B(u) := \sum_{k=-N+1}^{N-1} T_h b(k) e^{-iku},$$

where $B(u) := \sum_{k=-N+1}^{N-1} b(k) e^{-iku}$. Then T_H is a linear operator on $\mathcal{V}_N$ given by

$$T_H B(u) = H(\frac{u}{2}) B(\frac{u}{2}) H^*(\frac{u}{2}) + H(\frac{u}{2}+\pi) B(\frac{u}{2}+\pi) H^*(\frac{u}{2}+\pi), \quad B \in \mathcal{V}_N. \tag{3.14}$$

Equation (3.13) becomes

$$F_n = T_H F_{n-1}, \quad n = 1, 2 \ldots , \tag{3.15}$$

where F_n is the Fourier series with Fourier coefficients f_n.

We shall say that T_h satisfies Condition E^{**} (Condition E^* or Condition E) if its representation matrix satisfies Condition E^{**} (Condition E^* or Condition E respectively). Theorems 3.1 and 3.2 below are special cases of convergence of nonstationary vector cascade algorithms considered in [26].

Theorem 3.1 *The cascade sequence Φ_n converges weakly in $L^2(\mathbb{R})^r$ for any starting vector Φ_0 if and only if T_h satisfies Condition E^{**}.*

The following corollary gives a sufficient condition for the solutions of the MRE to belong to $L^2(\mathbb{R})^r$.

Corollary 3.1 *If T_h satisfies Condition E^{**}, then the solutions of the MRE* (3.3) *belong to $L^2(\mathbb{R})^r$.*

Stronger conditions are required for strong convergence in $L^2(\mathbb{R})^r$. We say that h is *fundamental* with respect to a nonzero column vector $v \in \mathbb{R}^r$ if

$$v^T \sum_{j\in\mathbb{Z}} h(2j) = v^T \sum_{j\in\mathbb{Z}} h(2j+1) = \frac{v^T}{2}.$$

Theorem 3.2 *Suppose that*
(a) *T_h satisfies Condition E and h is fundamental with respect to v.*
Then
(b) *for any initial vector Φ_0 satisfying*

$$v^T \sum_{j\in\mathbb{Z}} \widehat{\Phi}_0(\,\cdot\, - j) = 1 \text{ almost everywhere,} \tag{3.16}$$

the cascade sequence Φ_n converges strongly in $L^2(\mathbb{R})^r$ to a solution of the MRE (3.3).

Conversely, if $H(0)$ satisfies Condition E, then (b) *implies* (a).

Remark 1 *In* [26], *Theorems* 3.1 *and* 3.2 *were proved in its full generality for multidimensional nonstationary cascade algorithms with a general dilation matrix M.*

Remark 2 *If $H(0)$ satisfies Condition E, then the MRE* (3.3) *has a unique solution. Under this assumption the statements* (a) *and* (b) *in Theorem* 3.2 *are equivalent, a result obtained in* [64]. *Shen's result in* [64] *generalizes corresponding results in* [66] *for the scalar case in one dimension and in* [48] *for the scalar case in higher dimensions.*

3.3 Stability and orthonormality of scaling vectors

Let $G_\Phi(u)$ denote the Gram matrix of the $(2,h)$-refinable vector $\Phi \in L^2(\mathbb{R})^r$ defined by

$$G_\Phi(u) := \sum_{k\in\mathbb{Z}} \widehat{\Phi}(u+2\pi k)\widehat{\Phi}^*(u+2\pi k).$$

Then Φ is stable, i.e. the integer shifts of $\phi_j,\ j=1,\ldots,r$, form a Riesz basis of their closed linear span in $L^2(\mathbb{R})$, if and only if $G_\Phi(u)$ is positive definite for all $u \in \mathbb{T}$; and Φ is orthonormal, i.e. the integer shifts of $\phi_j,\ j=1,\ldots,r$, form an orthonormal set, if and only if $G_\Phi(u) = I_r$ for all $u \in \mathbb{T}$, where $\mathbb{T}$

is the unit circle and I_r denotes the $r \times r$ identity matrix (see [28]). If Φ is orthonormal, then h satisfies

$$h(u)h^*(u) + h(u+\pi)h^*(u+\pi) = I_r, \quad u \in \mathbb{T}. \tag{3.17}$$

A filter sequence h that satisfies (3.17) is called a matrix *conjugate quadrature filter* (CQF).

If the compactly supported $(2,h)$-refinable vector Φ is *stable,* then it is well known that Φ generates a multiresolution of $L^2(\mathbb{R})$ (see [2]). It is also known that $H(0)$ satisfies Condition E and its left eigenvector e corresponding to the eigenvalue 1 satisfies $e\,H(\pi) = 0$ (see, for instance, [43]). In this case, the cascade sequence Φ_n converges strongly in $L^2(\mathbb{R})^r$, and $\lim_{n\to\infty} \Phi_n = \Phi$.

As in the scalar case, it is useful to characterize the stability and orthonormality of Φ in terms of h.

Theorem 3.3 *Suppose that $H(0)$ satisfies Condition E and its left eigenvector e corresponding to the eigenvalue* 1 *satisfies $eH(\pi) = 0$. Then*
(a) *Φ is stable if and only if the transition operator T_H satisfies Condition E and the eigenvector $V(u)$ corresponding to the eigenvalue 1 is nonsingular for all $u \in \mathbb{R}$,*
(b) *Φ is orthonormal if and only if h is a CQF and T_H satisfies Condition E.*

Remark 3 *The multidimensional vector case of Theorem* 3.3 *with $M = 2I$ was due to Shen* [64]. *Solutions of matrix refinement equations and their properties with dilation $M = 2I$ was also considered in* [52]. *The corresponding result with a general dilation matrix was established by Jiang* [39]. *These results generalize the corresponding multivariate results in* [47] *and* [34] *for the scalar case of $r = 1$. The search for a characterization of orthonormality of a scaling function in terms of its mask was started by Lawton* [44]. *The result in part* (b) *of Theorem* 3.3 *for the scalar case in one dimension was due to him, and the condition in* (b) *for the orthonormality of a scaling function is commonly known as* Lawton's condition.

Since the eigenvalues of a finite matrix can be computed easily, it is useful in practice to represent the operator T_H as a finite matrix. Such a representation can be found in [38], [29] and [57]. We shall describe it here. For any $r\times r$ matrix A, let $A(j)$ be its jth column, i.e. $A = (A(1), \dots, A(r))$, and define the $r^2 \times 1$ vector $\mathrm{vec}(A)$ by

$$\mathrm{vec}(A) := (A(1)^T, \dots, A(r)^T)^T. \tag{3.18}$$

For $B(u) = \sum_{k=-N+1}^{N-1} b(k)e^{-iku} \in \mathcal{V}_N$, let $\mathrm{vec}(B)$ be the $((2N-1)r^2) \times 1$ vector defined by

$$\mathrm{vec}(B) := \left(\mathrm{vec}(b(-N+1))^T, \dots, \mathrm{vec}(b(N-1))^T\right)^T.$$

For any two matrices $C = (c_{ij})$ and $D = (d_{ij})$, let $C \otimes D := (c_{ij} D)$ denote the Kronecker product of C and D. Then for any compatible matrices C, D, E, we have

$$\mathrm{vec}(CDE^T) = (E \otimes C)\mathrm{vec}(D). \tag{3.19}$$

It can be easily shown (see [38], [29] and [57]) that the matrix $\mathcal{T}_h$ representing the operator T_H is given by

$$\mathcal{T}_h := (2a(2i-j))_{i,j=-N+1}^{N-1}, \tag{3.20}$$

and that

$$\mathrm{vec}(T_H B) = \mathcal{T}_h \mathrm{vec}(B), \quad B \in \mathcal{V}_N,$$

where $a(j)$ is the $r^2 \times r^2$ matrix defined by $a(j) := \sum_{\ell=0}^{N} h(\ell - j) \otimes h(\ell)$.

3.4 Approximation order and vanishing moments

Let $\Phi = (\phi_1, \ldots, \phi_r)^T$ be a $(2, h)$-refinable vector and V_0 be the closed linear span of the integer shifts of ϕ_j, $j = 1, \ldots, r$. For the definition of approximation order for shift-invariant subspaces generated by Φ, see [3], [4], [32] and [33].

The multifilter h is said to have *vanishing moments of order* m if there exist real $1 \times r$ row vectors ℓ_0^k, $k = 0, \ldots, m-1$, with $\ell_0^0 \neq 0$, such that

$$\begin{cases} \displaystyle\sum_{j=0}^{k} \binom{k}{j} (2i)^{j-k} \ell_0^j D^{k-j} H(0) = 2^{-k} \ell_0^k, \\ \displaystyle\sum_{j=0}^{k} \binom{k}{j} (2i)^{j-k} \ell_0^j D^{k-j} H(\pi) = 0, \end{cases} \tag{3.21}$$

where $D^{k-j}H(u)$ denotes the matrix formed by the $(k-j)$th derivatives of the entries of $H(u)$.

The relationship between approximation order and vanishing moments has been studied by many researchers ([31], [56], [36]). The following results are due to Heil, Strang and Strela [31] and Plonka [56].

Theorem 3.4 *Suppose that the compactly supported* $(2, h)$*-refinable vector* $\Phi = (\phi_1, \ldots, \phi_r)^T$ *is stable. Then the following are equivalent:*
(a) Φ *has approximation of order* m,
(b) *polynomials of degree less than* m *lie in the closed linear span of the integer shifts of* ϕ_j, $j = 1, \ldots, r$,
(c) h *has vanishing moments of order* m.

3.5 Regularity

In the one-dimensional scalar case, there is a simple characterization of regularity of a scaling function that provides a formula for computing the Sobolev exponent readily (see [15], [20], [73]). Interest in regularity therefore spilled over to refinable vectors (see [12], [64], [54], [38], [39], [37], [59]). This is a natural historical development. Various regularity estimates of a refinable vector Φ were given in [12], [64], [54], [38], [39], [37], [59]. The standard method motivated by the scalar case in one dimension is to estimate the decay of the Fourier transform $\widehat{\Phi}(u) = \prod_{j=1}^{\infty} H(u/2^j)\widehat{\Phi}(0)$. This requires a factorization of the mask $H(u)$ corresponding to the B-spline factor in the scalar case. Such a factorization for the vector case was obtained by Plonka [12], who was obviously also motivated by the cardinal Hermite splines [55]. In [55], she used the nonuniform B-splines with multiple knots of same multiplicity at the integers. Unlike the cardinal Hermite B-splines described here, these B-splines are refinable with finite masks which factorize completely into simple factors. Shen [64] avoided the use of factorization of the masks to obtain regularity estimates for scaling vectors. The results in [64] were subsequently extended by Jiang ([38], [39]) to cover general dilation matrices. Sharper regularity estimates were obtained in [38] and [39] in terms of the spectral radius of transition operators restricted to invariant subspaces that are smaller than those in [64]. For the scalar case $r = 1$, Jia [35] gave a characterization of regularity of (M, h)-refinable functions in $\mathbb{R}^d$ with isotropic dilation matrix M, i.e. M is similar to a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, and $|\lambda_1| = \cdots = |\lambda_d|$. Recently, Jia, Riemenschneider and Zhou [37] as well as Ron and Shen [59] gave a complete characterization of regularity of refinable vectors. The results in [59] also cover the case of infinite mask and provide an estimate for each component of the scaling vector. In what follows we shall describe the regularity estimates in [39] in more detail. This choice is just our personnel preference, because we find the results in [39] simpler and amenable to computation.

For $s \geq 0$, let $W^s(\mathbb{R})$ be the Sobolev space of functions f with $(1 + |u|^2)^{\frac{s}{2}} \widehat{f}(u) \in L^2(\mathbb{R})$. Let $\gamma = n + \lambda$ with $n \in \mathbb{Z}_+$ and $0 \leq \lambda < 1$. We define $f \in C^\gamma(\mathbb{R})$ if and only if $f \in C^{(n)}(\mathbb{R})$ and $f^{(n)}$ is uniformly Hölder continuous with exponent λ, i.e.

$$|D^n f(x+y) - D^n f(x)| \leq C|y|^\lambda,$$

for some constant C independent of $x, y \in \mathbb{R}$, where D denotes derivative. Then we have the well-known inclusion

$$W^s(\mathbb{R}) \subset C^\gamma(\mathbb{R}), \quad \text{for } s > \gamma + \frac{1}{2}.$$

Suppose that the finite mask h has vanishing moments of order m, i.e. its corresponding matrix H satisfies (3.21) for some $1 \times r$ vectors ℓ_0^k, $k = 0, \ldots, m-1$, with $\ell_0^0 \neq 0$. Let $m_0 \leq m$ be the largest nonnegative integer such that there exist $1 \times r$ vectors ℓ_0^k, $k = m, \ldots, m + m_0 - 1$, satisfying

$$\sum_{j=0}^{k} \binom{k}{j} (2i)^{j-k} \ell_0^j D^{k-j} H(0) = 2^{-k} \ell_0^k. \tag{3.22}$$

Note that if all the numbers 2^{-k}, $k = m, \ldots, m+m_0-1$, are not eigenvalues of $H(0)$ for some $m_0 \in \mathbb{Z}_+$, then the vectors ℓ_0^k, $k = m, \ldots, m + m_0 - 1$, can be chosen iteratively by

$$\ell_0^k \left(2^{-k} I_r - H(0)\right) = \sum_{j=0}^{k-1} \binom{k}{j} (2i)^{j-k} \ell_0^j D^{k-j} H(0).$$

For $\nu \in \mathbb{Z}$, define $1 \times r$ row vectors ℓ_ν^k by

$$\ell_\nu^k := \sum_{j=0}^{k} \binom{k}{j} (-\nu)^{k-j} \ell_0^k, \quad k = 0, \ldots, m + m_0 - 1. \tag{3.23}$$

Writing

$$\ell^k(\nu) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ell_{-\nu}^j \otimes \ell_0^{k-j}, \quad \nu \in \mathbb{Z},$$

let L_N^k be the $1 \times ((2N-1)r^2)$ vectors given by

$$L_N^k := (\ell^k(-N+1), \ldots, \ell^k(N-1)). \tag{3.24}$$

Then as shown in [39], $L_N^k \mathcal{T}_h = 2^{-k} L_N^k$. Therefore if $L_N^k \neq \mathbf{0}$, then L_N^k is a left eigenvector of $\mathcal{T}_h$ corresponding to the eigenvalue 2^{-k}.

Let $e_j := (\delta_j(k))_{k=1}^r$, $j = 1, \ldots, r$, be the standard unit row vectors in $\mathbb{R}^r$, and for $k < m$, let

$$p_j^k(\nu) := e_j \otimes \ell_\nu^k, \quad q_j^k(\nu) := \ell_{-\nu}^k \otimes e_j, \quad \nu \in \mathbb{Z}.$$

Then define the $1 \times ((2N-1)r^2)$ vectors $P_{j,N}^k$, $Q_{j,N}^k$, $k < m$, respectively by

$$P_{j,N}^k := (p_j^k(-N+1), \ldots, p_j^k(N-1)),$$
$$Q_{j,N}^k := (q_j^k(-N+1), \ldots, q_j^k(N-1)).$$

Let $\mathcal{L}_N$ be the $r^2(2N-1) \times (m + m_0)$ matrix defined by

$$\mathcal{L}_N := ((L_N^0)^T, \ldots, (L_N^{m+m_0-1})^T),$$

where L_N^k are the vectors in (3.24). For $j = 1, \ldots, r$, let $\mathcal{P}_{j,N}$ and $\mathcal{Q}_{j,N}$ be the $r^2(2N-1) \times m$ matrices defined respectively by

$$\mathcal{P}_{j,N} := ((P_{j,N}^0)^T, \ldots, (P_{j,N}^{m-1})^T), \quad \mathcal{Q}_{j,N} := ((Q_{j,N}^0)^T, \ldots, (Q_{j,N}^{m-1})^T).$$

Finally, we define the $r^2(2N-1) \times ((2r+1)m + m_0)$ matrix

$$\mathcal{M}_N := (\mathcal{L}_N, \mathcal{P}_{1,N}, \ldots, \mathcal{P}_{r,N}, \mathcal{Q}_{1,N}, \ldots, \mathcal{Q}_{r,N}).$$

Let $\mathcal{V}_N^0$ denote the subspace of $\mathcal{V}_N$ which comprises $B \in \mathcal{V}_N$ satisfying $(\mathcal{M}_N)^T \mathrm{vec}(B) = 0$. Then $\mathcal{V}_N^0$ is invariant under T_H. Let $T_H|_{\mathcal{V}_N^0}$ denote the restriction of T_H to $\mathcal{V}_N^0$, and let

$$s_0 := -\log_4(\rho(T_H|_{\mathcal{V}_N^0})), \tag{3.25}$$

where $\rho(T_H|_{\mathcal{V}_N^0})$ is the spectral radius of $T_H|_{\mathcal{V}_N^0}$. Restricting Jiang's results in [39] to one dimension with dilation 2, we have

Theorem 3.5 *The components ϕ_j, $j = 1, \ldots, r$, of the $(2,h)$-refinable vector Φ belong to $W^{s_0-\epsilon}(\mathbb{R})$ for any $\epsilon > 0$.*

Corollary 3.2 *The components ϕ_j, $j = 1, \ldots, r$, of the $(2,h)$-refinable vector Φ belong to $C^{s_0-\frac{1}{2}-\epsilon}(\mathbb{R})$ for any $\epsilon > 0$.*

Note that the results in [39] cover multidimensional scaling vectors and multiwavelets with an arbitrary dilation matrix M whose eigenvalues are all nondegenerate.

4 Periodic multiwavelets

In many practical applications, one often deals with periodic functions and periodic filters. Thus the study of periodic multiwavelets of the space of 2π-periodic functions, $L^2([0,2\pi)^d)$, is of interest. Analogous to the construction of multiwavelets of $L^2(\mathbb{R}^d)$, periodic multiwavelets can be constructed via a periodic multiresolution $\{V_m\}_{m\geq 0}$ of $L^2([0,2\pi)^d)$ of multiplicity r with dilation matrix M generated by periodic scaling functions. In [22], a general theory of periodic multiresolutions and multiwavelets was derived. Here, we shall describe some of the results. For simplicity, we shall concentrate on the case $d = 1$ and $M = (2)$.

For $m \geq 0$, the multiresolution subspace V_m is generated by the $\frac{2\pi}{2^m}$-shifts of r scaling functions $\phi_1^m, \ldots, \phi_r^m$, i.e.

$$V_m = \langle \{\phi_j^m(\cdot - 2\pi k/2^m) : k = 0, \ldots, 2^m - 1, j = 1, \ldots, r\} \rangle.$$

For different levels m of the multiresolution, the scaling functions for the corresponding subspaces V_m need not be related by dilation. Thus the periodic multiresolution $\{V_m\}_{m\geq 0}$ is nonstationary. The sequence of scaling functions $\phi_1^m, \ldots, \phi_r^m$, $m \geq 0$, satisfies the *periodic matrix refinement equation*

$$\Phi^m(x) = \sum_{k=0}^{2^{m+1}-1} h_{m+1}(k)\Phi^{m+1}(x - 2\pi k/2^{m+1}), \quad x \in [0, 2\pi), \tag{4.1}$$

for every $m \geq 0$, where $h_{m+1}(k)$, $k = 0, \ldots, 2^{m+1}-1$, is a periodic sequence of $r \times r$ matrices of period 2^{m+1}, and $\Phi^m := (\phi_1^m, \ldots, \phi_r^m)^T$. By considering Fourier coefficients, we see that (4.1) is equivalent to

$$\widehat{\Phi}^m(n) = H_{m+1}(n)\widehat{\Phi}^{m+1}(n), \quad n \in \mathbb{Z}, \tag{4.2}$$

where $\widehat{\Phi}^m(n) := (\hat{\phi}_1^m(n), \ldots, \hat{\phi}_r^m(n))^T$, $n \in \mathbb{Z}$, and H_{m+1} denotes the finite Fourier transform (FFT) of h_{m+1}. The equations (4.1) and (4.2) are periodic analogues of (3.3) and (3.4) respectively.

The analysis of a periodic multiresolution is enriched by a class of linearly independent functions called *polyphase splines*, defined by

$$v_j^{m,k}(x) := \sum_{p \in \mathbb{Z}} \hat{\phi}_j^m(k + 2^m p)e^{i(k+2^m p)x}, \quad x \in [0, 2\pi), \tag{4.3}$$

for $m \geq 0$, $k = 0, \ldots, 2^m-1$, $j = 1, \ldots, r$. It was shown in [22] that $\{v_j^{m,k} : k = 0, \ldots, 2^m-1, j = 1, \ldots, r\}$ forms an alternative basis of the multiresolution subspace V_m. The polyphase spline vector $v^{m,k} := (v_1^{m,k}, \ldots, v_r^{m,k})^T$ largely facilitates the construction of periodic multiwavelets.

For $m \geq 0$, let W_m be the wavelet subspace defined by the orthogonal complement of V_m in V_{m+1}. We seek multiwavelets $\psi_1^m, \ldots, \psi_r^m$ such that their $\frac{2\pi}{2^m}$-shifts form a basis of W_m.

Theorem 4.1 *Let $\{V_m\}_{m\geq 0}$ be a periodic multiresolution of $L^2[0, 2\pi)$ of multiplicity r with dilation $M = 2$. Then there exists a sequence G_{m+1}, $m \geq 0$, of $r \times r$ periodic matrices of period 2^{m+1} such that for every $m \geq 0$, the $\frac{2\pi}{2^m}$-shifts of the functions $\psi_1^m, \ldots, \psi_r^m$, defined by*

$$\Psi^m := \sum_{k=0}^{2^m-1} u^{m,k}, \tag{4.4}$$

where $\Psi^m := (\psi_1^m, \ldots, \psi_r^m)^T$ and

$$u^{m,k} := G_{m+1}(k)v^{m+1,k} + G_{m+1}(k + 2^m)v^{m+1,k+2^m} \tag{4.5}$$

for $k = 0, \ldots, 2^m - 1$, form a basis of W_m.

The above theorem was proved in [22] under the most general multidimensional setting with an arbitrary dilation matrix M. Its proof is constructive which leads to an algorithmic approach of constructing multiwavelets once a periodic multiresolution is known. The polyphase spline basis $\{v_j^{m,k} : k = 0, \ldots, 2^m - 1, j = 1, \ldots, r\}$ of V_m, $m \geq 0$ enables one to reduce the multiwavelet construction problem to a tractable matrix extension problem, which gives the desired matrices G_{m+1}, $m \geq 0$.

5 References

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