ON THE DESIGN OF PERFECT RECONSTRUCTION MULTIWAVELET FILTER BANKS

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ABSTRACT

The lifting scheme is a flexible method of designing perfect reconstruction multiwavelet filter banks for signal and image processing. This paper is on the usage of the lifting scheme in constructing such filter banks. It is shown that the constructed filter banks can be implemented very efficiently. Two algorithms on the design of multiwavelet filter banks using the lifting scheme are proposed. One is for the optimization of desirable properties based on the application concerned, and the other is for the reduction of computational complexity in implementation. Two examples are provided to illustrate the main ideas of the paper.

1. INTRODUCTION

Let

$$H(z) := \sum_{k\in\mathbb{Z}} h(k)z^{-k}, \quad \widetilde{H}(z) := \sum_{k\in\mathbb{Z}} \widetilde{h}(k)z^{-k},$$

$$G(z) := \sum_{k\in\mathbb{Z}} g(k)z^{-k}, \quad \widetilde{G}(z) := \sum_{k\in\mathbb{Z}} \widetilde{g}(k)z^{-k}$$

be FIR $r\times r$ matrix filters with real matrix coefficients. Then H, $\widetilde{H}$, G, $\widetilde{G}$ are said to form a **perfect reconstruction multiwavelet filter bank (PRMFB)** if

$$\begin{cases} H(z)\widetilde{H}(z)^* + H(-z)\widetilde{H}(-z)^* = 2I_r, \\ H(z)\widetilde{G}(z)^* + H(-z)\widetilde{G}(-z)^* = 0_r, \\ G(z)\widetilde{H}(z)^* + G(-z)\widetilde{H}(-z)^* = 0_r, \\ G(z)\widetilde{G}(z)^* + G(-z)\widetilde{G}(-z)^* = 2I_r, \quad |z|=1, \end{cases} \tag{1}$$

where I_r and 0_r are the $r\times r$ identity and zero matrices respectively, and the notation B^* denotes the conjugate transpose of the matrix B. Under suitable conditions, based on the PRMFB H, $\widetilde{H}$, G, $\widetilde{G}$, vector-valued functions

$$\Phi = (\phi_1,\ldots,\phi_r)^T, \quad \widetilde{\Phi} = (\widetilde{\phi}_1,\ldots,\widetilde{\phi}_r)^T,$$

$$\Psi = (\psi_1,\ldots,\psi_r)^T, \quad \widetilde{\Psi} = (\widetilde{\psi}_1,\ldots,\widetilde{\psi}_r)^T$$

can be obtained from the equations

$$\Phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h(k)\Phi(2x-k),$$
$$\widetilde{\Phi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \widetilde{h}(k)\widetilde{\Phi}(2x-k),$$
$$\Psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g(k)\Phi(2x-k),$$
$$\widetilde{\Psi}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \widetilde{g}(k)\widetilde{\Phi}(2x-k).$$

The functions Φ and $\widetilde{\Phi}$ are called **scaling vectors**, and Ψ and $\widetilde{\Psi}$ are called **multiwavelets**.

The theory of multiwavelets is a recent topic of active research, and application of multiwavelets to signal and image processing is gaining interest as well (see, for instance, [4]). The PRMFB H, $\widetilde{H}$, G, $\widetilde{G}$ plays a central role in applying multiwavelets to practical situations. Indeed, given a $r\times 1$ vector signal $\{c^1(k)\}$, decompose it into two $r\times 1$ vector signals $\{c^0(k)\}$ and $\{d^0(k)\}$ by

$$\begin{cases} c^0(k) = \sum_{n\in\mathbb{Z}} \widetilde{h}(n-2k)c^1(n), \\ d^0(k) = \sum_{n\in\mathbb{Z}} \widetilde{g}(n-2k)c^1(n). \end{cases} \tag{2}$$

(The notation B^T denotes the transpose of the matrix B.) The original signal $\{c^1(k)\}$ can be reconstructed from the

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decomposed signals $\{c^0(k)\}$ and $\{d^0(k)\}$ by

$$c^1(k) = \sum_{n\in\mathbb{Z}} \left(h(k-2n)^T c^0(n) + g(k-2n)^T d^0(n)\right). \tag{3}$$

The formulas (2) and (3) are known as the **decomposition and reconstruction multiwavelet algorithms**. The property (1) ensures perfect reconstruction of the original signal $\{c^1(k)\}$ from $\{c^0(k)\}$ and $\{d^0(k)\}$.

In practice, one needs PRMFBs H, $\widetilde{H}$, G, $\widetilde{G}$ with desirable properties to facilitate processing and analysis. It is a challenging problem to construct FIR matrix filters H, $\widetilde{H}$, G, $\widetilde{G}$ that satisfy (1). The task is even more difficult when certain properties, depending on the practical setting concerned, need to be factored into the matrix filters. Extending the results in [1], [2], [5] on the lifting scheme for the scalar case $r = 1$, the lifting scheme for multiwavelet filter banks was introduced in [3] to provide a flexible platform for designing desirable PRMFBs. The objective of this paper is to highlight the usage of the lifting scheme for constructing good PRMFBs, and to demonstrate how such PRMFBs lead to reduction of computational complexity in the implementation of the multiwavelet algorithms (2) and (3).

2. THE LIFTING SCHEME FOR MULTIWAVELET FILTER BANKS

We shall now review some of the recent results on the lifting scheme established in [3]. Let $H^{(0)}$, $\widetilde{H}^{(0)}$, $G^{(0)}$, $\widetilde{G}^{(0)}$ be a known PRMFB. A popular example is given by the **lazy matrix filters**

$$H^{(0)}(z) = \widetilde{H}^{(0)}(z) = I_r, \quad G^{(0)}(z) = \widetilde{G}^{(0)}(z) = z^{-1} I_r. \tag{4}$$

For $1 \le \ell \le L$, let $S^{(\ell)}$ and $\widetilde{S}^{(\ell)}$ be FIR $r \times r$ matrix filters, and $T_1^{(\ell)}$ and $T_2^{(\ell)}$ be FIR $r \times r$ upper or lower triangular matrix filters whose diagonal entries are entirely 1's. Define

$$\begin{cases} H^{(\ell)}(z) := T_1^{(\ell)}(z^2)\left(H^{(\ell-1)}(z) + S^{(\ell)}(z^2)G^{(\ell-1)}(z)\right), \\ G^{(\ell)}(z) := T_2^{(\ell)}(z^2)G^{(\ell-1)}(z) - \widetilde{S}^{(\ell)}(z^2)^* H^{(\ell)}(z), \end{cases} \tag{5}$$

and

$$\begin{cases} \widetilde{G}^{(\ell)}(z) := (T_2^{(\ell)}(z^2)^*)^{-1} \\ \qquad\qquad \left(\widetilde{G}^{(\ell-1)}(z) - S^{(\ell)}(z^2)^* \widetilde{H}^{(\ell-1)}(z)\right), \\ \widetilde{H}^{(\ell)}(z) := (T_1^{(\ell)}(z^2)^*)^{-1} \widetilde{H}^{(\ell-1)}(z) + \widetilde{S}^{(\ell)}(z^2)\widetilde{G}^{(\ell)}(z), \end{cases} \tag{6}$$

for $|z| = 1$. The process described by (5) and (6) is called the **lifting scheme**.

As shown in [3], if $H^{(\ell-1)}$, $\widetilde{H}^{(\ell-1)}$, $G^{(\ell-1)}$, $\widetilde{G}^{(\ell-1)}$ form a PRMFB, then $H^{(\ell)}$, $\widetilde{H}^{(\ell)}$, $G^{(\ell)}$, $\widetilde{G}^{(\ell)}$ also form a PRMFB. Since $H^{(0)}$, $\widetilde{H}^{(0)}$, $G^{(0)}$, $\widetilde{G}^{(0)}$ form a PRMFB, it follows that $H^{(\ell)}$, $\widetilde{H}^{(\ell)}$, $G^{(\ell)}$, $\widetilde{G}^{(\ell)}$ form a PRMFB for $0 \le \ell \le L$. In particular, at the final step L, we obtain a PRMFB

$$H = H^{(L)}, \quad \widetilde{H} = \widetilde{H}^{(L)}, \quad G = G^{(L)}, \quad \widetilde{G} = \widetilde{G}^{(L)}.$$

In this construction, flexibility is provided by the FIR matrix filters $S^{(\ell)}$, $\widetilde{S}^{(\ell)}$, $T_1^{(\ell)}$, $T_2^{(\ell)}$ for $1 \le \ell \le L$.

The lifting scheme provides a strategy for constructing PRMFBs that optimize certain desirable properties. The entries of the FIR matrix filters $S^{(\ell)}$, $\widetilde{S}^{(\ell)}$, $T_1^{(\ell)}$, $T_2^{(\ell)}$ for $1 \le \ell \le L$ are Laurent polynomials in z whose coefficients can be viewed as parameters to be selected. Thus each lifting step provides many free parameters which can be used to adapt the resulting PRMFB to the problem concerned. In practice, some of these parameters are used to satisfy certain basic properties of the matrix filters H, $\widetilde{H}$, G, $\widetilde{G}$. These properties include symmetry and antisymmetry of the matrix filters. The rest of the parameters are used to optimize certain desirable properties such as time-frequency localization of the resulting scaling vectors and multiwavelets, and the performance in image compression on a class of training images.

The above strategy can be formulated as an algorithm for the design of PRMFBs.

Algorithm 1

Step 1: *Use some of the free parameters in* $S^{(\ell)}$, $\widetilde{S}^{(\ell)}$, $T_1^{(\ell)}$, $T_2^{(\ell)}$ *for* $1 \le \ell \le L$ *to ensure that the resulting PRMFB satisfies certain basic properties.*

Step 2: *Based on desirable properties, identify an object function* $\mathcal{F}$ *in terms of the remaining parameters in* $S^{(\ell)}$, $\widetilde{S}^{(\ell)}$, $T_1^{(\ell)}$, $T_2^{(\ell)}$ *for* $1 \le \ell \le L$.

Step 3: *Search for the values of the free parameters in Step 2 that optimize the object function* $\mathcal{F}$.

3. EFFICIENT IMPLEMENTATION OF MULTIWAVELET ALGORITHMS

In this section, we shall show that PRMFBs constructed using the lifting scheme can be implemented efficiently. To simplify the discussion, for $1 \le \ell \le L$, let the matrix filters $T_1^{(\ell)}$ and $T_2^{(\ell)}$ be the identity matrix I_r. Then the equations in (6) reduces to

$$\begin{cases} \widetilde{G}^{(\ell)}(z) = \widetilde{G}^{(\ell-1)}(z) - S^{(\ell)}(z^2)^* \widetilde{H}^{(\ell-1)}(z), \\ \widetilde{H}^{(\ell)}(z) = \widetilde{H}^{(\ell-1)}(z) + \widetilde{S}^{(\ell)}(z^2)\widetilde{G}^{(\ell)}(z), \end{cases} \tag{7}$$

for $|z|=1$, which is equivalent to

$$\begin{cases} \widetilde{g}^{(\ell)}(k) = \widetilde{g}^{(\ell-1)}(k) - \sum_{n\in\mathbb{Z}} s^{(\ell)}(n)^T \widetilde{h}^{(\ell-1)}(k+2n), \\ \widetilde{h}^{(\ell)}(k) = \widetilde{h}^{(\ell-1)}(k) + \sum_{n\in\mathbb{Z}} \widetilde{s}^{(\ell)}(n)\widetilde{g}^{(\ell)}(k-2n), \end{cases} \tag{8}$$

where

$$S^{(\ell)}(z) = \sum_{k\in\mathbb{Z}} s^{(\ell)}(k)z^{-k}, \quad \widetilde{S}^{(\ell)}(z) = \sum_{k\in\mathbb{Z}} \widetilde{s}^{(\ell)}(k)z^{-k}.$$

Now, based on a given vector signal $\{c^1(k)\}$, set

$$\begin{cases} c^{0,(0)}(k) := \sum_{n\in\mathbb{Z}} \widetilde{h}^{(0)}(n-2k)c^1(n), \\ d^{0,(0)}(k) := \sum_{n\in\mathbb{Z}} \widetilde{g}^{(0)}(n-2k)c^1(n). \end{cases} \tag{9}$$

For $1 \le \ell \le L$, define

$$\begin{cases} d^{0,(\ell)}(k) := d^{0,(\ell-1)}(k) - \sum_{n\in\mathbb{Z}} s^{(\ell)}(k-n)^T c^{0,(\ell-1)}(n), \\ c^{0,(\ell)}(k) := c^{0,(\ell-1)}(k) + \sum_{n\in\mathbb{Z}} \widetilde{s}^{(\ell)}(n-k)d^{0,(\ell)}(n). \end{cases} \tag{10}$$

Then a straightforward application of (8) shows that if

$$\begin{cases} c^{0,(\ell-1)}(k) = \sum_{n\in\mathbb{Z}} \widetilde{h}^{(\ell-1)}(n-2k)c^1(n), \\ d^{0,(\ell-1)}(k) = \sum_{n\in\mathbb{Z}} \widetilde{g}^{(\ell-1)}(n-2k)c^1(n), \end{cases}$$

then

$$\begin{cases} c^{0,(\ell)}(k) = \sum_{n\in\mathbb{Z}} \widetilde{h}^{(\ell)}(n-2k)c^1(n), \\ d^{0,(\ell)}(k) = \sum_{n\in\mathbb{Z}} \widetilde{g}^{(\ell)}(n-2k)c^1(n). \end{cases}$$

Since $\widetilde{H} = \widetilde{H}^{(L)}$ and $\widetilde{G} = \widetilde{G}^{(L)}$, it follows that $c^0(k) = c^{0,(L)}(k)$ and $d^0(k) = d^{0,(L)}(k)$, where $\{c^{0,(L)}(k)\}$ and $\{d^{0,(L)}(k)\}$ are given by (10). Thus the decomposition algorithm (2) can be expressed as a recursive application of (10). We shall call (10) the **lifted decomposition algorithm**.

For a FIR $r\times r$ matrix filter $A(z) = \sum_{k=M}^{N} a(k)z^{-k}$, with $M \le N$, $a(M) \ne 0_r$ and $a(N) \ne 0_r$, we define the **support** and **length** of A as

$$\text{supp}\,(A) := \{M,\dots,N\}, \quad |A| := N-M+1$$

respectively. Assume that for $1 \le \ell \le L$, $S^{(\ell)}$ and $\widetilde{S}^{(\ell)}$ are chosen such that

$$\text{supp}\,(\widetilde{G}^{(\ell)}) = \text{supp}\,(S^{(\ell)}_{\widetilde{H}}), \quad \text{supp}\,(\widetilde{H}^{(\ell)}) = \text{supp}\,(\widetilde{S}^{(\ell)}_{\widetilde{G}}), \tag{11}$$

where $S^{(\ell)}_{\widetilde{H}}$ and $\widetilde{S}^{(\ell)}_{\widetilde{G}}$ are FIR $r\times r$ matrix filters defined by

$$S^{(\ell)}_{\widetilde{H}}(z) := S^{(\ell)}(z^2)^*\widetilde{H}^{(\ell-1)}(z),$$
$$\widetilde{S}^{(\ell)}_{\widetilde{G}}(z) := \widetilde{S}^{(\ell)}(z^2)\widetilde{G}^{(\ell)}(z),$$

for $|z|=1$. Then $|\widetilde{G}^{(\ell)}| = |\widetilde{H}^{(\ell-1)}| + 2|S^{(\ell)}| - 2$ and $|\widetilde{H}^{(\ell)}| = |\widetilde{H}^{(\ell-1)}| + 2(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|) - 4$ for $1 \le \ell \le L$. Consequently, the final matrix filters $\widetilde{G} = \widetilde{G}^{(L)}$ and $\widetilde{H} = \widetilde{H}^{(L)}$ have lengths

$$|\widetilde{G}| = |\widetilde{H}^{(0)}| + 2\sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|) - 2|\widetilde{S}^{(L)}| - 4L + 2,$$

$$|\widetilde{H}| = |\widetilde{H}^{(0)}| + 2\sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|) - 4L.$$

Let m be the computational effort of left multiplying a $r\times r$ matrix to a $r\times 1$ vector, and a be that of adding two $r\times 1$ vectors. Suppose that the decomposition algorithm (2) is used to compute the decomposed signals $c^0(k)$ and $d^0(k)$ at a value k. Then the computational effort required for evaluating $c^0(k)$ is $|\widetilde{H}|m + (|\widetilde{H}|-1)a$, and that for $d^0(k)$ is $|\widetilde{G}|m + (|\widetilde{G}|-1)a$. Hence, the total computational effort required is given by

$$\begin{aligned}\mathcal{C} =& \Big(2|\widetilde{H}^{(0)}| + 4\sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|) - 2|\widetilde{S}^{(L)}| \\ &- 8L + 2\Big)(m+a) - 2a.\end{aligned} \tag{12}$$

On the other hand, if the lifted decomposition algorithm (10) is used to find $c^0(k)$ and $d^0(k)$, then in the ℓth step, the computational effort required for $d^{0,(\ell)}(k)$ is $|S^{(\ell)}|(m+a)$ and that for $c^{0,(\ell)}(k)$ is $|\widetilde{S}^{(\ell)}|(m+a)$. In addition, the computational effort for evaluating $c^{0,(0)}(k)$ and $d^{0,(0)}(k)$ in (9) is $(|\widetilde{H}^{(0)}| + |\widetilde{G}^{(0)}|)(m+a) - 2a$. Thus the total computational effort needed to find $c^0(k)$ and $d^0(k)$ is

$$\begin{aligned}\mathcal{D} =& \Big(|\widetilde{H}^{(0)}| + |\widetilde{G}^{(0)}| + \sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|)\Big)(m+a) \\ &- 2a.\end{aligned} \tag{13}$$

In general, $3\sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|)$ is significantly larger than $2|S^{(L)}| + 8L$. Thus it follows from (12) and (13) that $\mathcal{D}$

is much smaller than $\mathcal{C}$. We have just proved the following theorem.

Theorem 1 *Suppose that for* $1 \le \ell \le L$, $S^{(\ell)}$ *and* $\widetilde{S}^{(\ell)}$ *are FIR* $r \times r$ *matrix filters satisfying* (11). *Then the computational efforts for the decomposition algorithm* (2) *and the lifted decomposition algorithm* (10) *are given by* (12) *and* (13) *respectively. Furthermore, if* $3\sum_{\ell=1}^{L}(|S^{(\ell)}| + |\widetilde{S}^{(\ell)}|)$ *is much larger than* $2|S^{(L)}| + 8L$, *then the lifted decomposition algorithm reduces the computational complexity of the decomposition algorithm significantly.*

A natural question one would ask is how can the matrix filters $S^{(\ell)}$ and $\widetilde{S}^{(\ell)}$, $1 \le \ell \le L$, be chosen so that (11) holds? In this connection, let $\operatorname{supp}(S^{(\ell)}) = \{M_\ell, \dots, N_\ell\}$, $\operatorname{supp}(\widetilde{S}^{(\ell)}) = \{\widetilde{M}_\ell, \dots, \widetilde{N}_\ell\}$, $\operatorname{supp}(\widetilde{G}^{(\ell)}) = \{P_\ell, \dots, Q_\ell\}$ and $\operatorname{supp}(\widetilde{H}^{(\ell)}) = \{J_\ell, \dots, K_\ell\}$. Then (11) is equivalent to

$$\begin{gathered} P_\ell = J_{\ell-1} - 2N_\ell, \quad Q_\ell = K_{\ell-1} - 2M_\ell, \\ J_\ell = J_{\ell-1} - 2(N_\ell - \widetilde{M}_\ell), \quad K_\ell = K_{\ell-1} + 2(\widetilde{N}_\ell - M_\ell). \end{gathered} \tag{14}$$

Suppose that $M_1, N_1, \widetilde{M}_1, \widetilde{N}_1$ are integers satisfying

$$\begin{gathered} M_1 < (K_0 - Q_0)/2, \quad N_1 > (J_0 - P_0)/2, \\ \widetilde{M}_1 < N_1 \quad \widetilde{N}_1 > M_1. \end{gathered}$$

Then straightforward calculations show that

$$J_0 - 2N_1 < P_0, \quad Q_0 < K_0 - 2M_1,$$

which, together with (7), yield the first two equations in (14) for $\ell = 1$. Consequently,

$$P_1 + 2\widetilde{M}_1 < J_0, \quad K_0 < Q_1 + 2\widetilde{N}_1,$$

and we have the other two equations in (14) for $\ell = 1$.

For $\ell \ge 2$, let $M_\ell, N_\ell, \widetilde{M}_\ell, \widetilde{N}_\ell$ be integers satisfying

$$\begin{gathered} M_\ell < \widetilde{N}_{\ell-1}, \quad N_\ell > \widetilde{M}_{\ell-1}, \\ \widetilde{M}_\ell < N_\ell, \quad \widetilde{N}_\ell > M_\ell. \end{gathered}$$

Assuming (14) for the $(\ell-1)$th step, we have

$$J_{\ell-1} - 2N_\ell < P_{\ell-1}, \quad Q_{\ell-1} < K_{\ell-1} - 2M_\ell,$$

which give the first two equations in (14) for the ℓth step. Thus

$$P_\ell + 2\widetilde{M}_\ell < J_{\ell-1}, \quad K_{\ell-1} < Q_\ell + 2\widetilde{N}_\ell,$$

and so the other two equations in (14) also hold for the ℓth step.

By induction on ℓ, we obtain the following algorithm for ensuring the condition (11) for $1 \le \ell \le L$. (The notation $\#(E)$ denotes the number of elements in the set E.)

Algorithm 2 *Suppose that* $\operatorname{supp}(\widetilde{G}^{(0)}) = \{P_0, \dots, Q_0\}$ *and* $\operatorname{supp}(\widetilde{H}^{(0)}) = \{J_0, \dots, K_0\}$. *The following steps ensure that* (11) *holds for* $1 \le \ell \le L$.

Step 1: *Choose first* M_1 *and* N_1 *such that*

$$M_1 < (K_0 - Q_0)/2, \quad N_1 > (J_0 - P_0)/2,$$

and then $\widetilde{M}_1$ *and* $\widetilde{N}_1$ *such that*

$$\#(\{\widetilde{M}_1, \dots, \widetilde{N}_1\} \cap \{M_1, \dots, N_1\}) > 1.$$

Step 2: *For* $2 \le \ell \le L$, *choose first* M_ℓ *and* N_ℓ *such that*

$$\#(\{M_\ell, \dots, N_\ell\} \cap \{\widetilde{M}_{\ell-1}, \dots, \widetilde{N}_{\ell-1}\}) > 1,$$

and then $\widetilde{M}_\ell$ *and* $\widetilde{N}_\ell$ *such that*

$$\#(\{\widetilde{M}_\ell, \dots, \widetilde{N}_\ell\} \cap \{M_\ell, \dots, N_\ell\}) > 1.$$

Step 3: *For* $1 \le \ell \le L$, *define* $S^{(\ell)}$ *and* $\widetilde{S}^{(\ell)}$ *to be FIR* $r \times r$ *matrix filters with* $\operatorname{supp}(S^{(\ell)}) = \{M_\ell, \dots, N_\ell\}$ *and* $\operatorname{supp}(\widetilde{S}^{(\ell)}) = \{\widetilde{M}_\ell, \dots, \widetilde{N}_\ell\}$.

Example 1 Let $H^{(0)}$, $\widetilde{H}^{(0)}$, $G^{(0)}$, $\widetilde{G}^{(0)}$ be the lazy matrix filters (4). Consider the case when $|S^{(\ell)}| = |\widetilde{S}^{(\ell)}| = \lambda \ge 2$ and (11) holds for $1 \le \ell \le L$. Then $\mathcal{C}$ and $\mathcal{D}$ in (12) and (13) reduces to

$$\mathcal{C} = (8\lambda L - 2\lambda - 8L + 4)(m + a) - 2a$$

and

$$\mathcal{D} = 2\lambda L(m + a).$$

The above expression for $\mathcal{D}$ is slightly smaller than that in (13) because in this case, $\widetilde{H}^{(0)}$ and $\widetilde{G}^{(0)}$ are the lazy matrix filters, and thus (9) involves no multiplications and additions.

We shall compare the number of left multiplication of $r \times r$ matrices to $r \times 1$ vectors. Let $\mathcal{C}_m$ and $\mathcal{D}_m$ be the number of such multiplications in $\mathcal{C}$ and $\mathcal{D}$ respectively. Then $\mathcal{C}_m = 8\lambda L - 2\lambda - 8L + 4$ and $\mathcal{D}_m = 2\lambda L$. Table 1 shows the reduction in computational complexity for all possible pairs of matrix filters $\widetilde{H}$ and $\widetilde{G}$, whose filter length is less than 20, that can be constructed under this setting. The percentage reduction in computational effort

$$r := \frac{\mathcal{C}_m - \mathcal{D}_m}{\mathcal{C}_m} \times 100\% \tag{15}$$

for each pair of matrix filters is also computed. As can be seen from Table 1, the reduction in computational complexity for all these pairs of matrix filters is at least 50%. $\square$

Example 2 Assume the setting of Example 1 except $|S^{(\ell)}|$ and $|\widetilde{S}^{(\ell)}|$ are allowed to vary for different values of ℓ. Let $\widetilde{H} = \widetilde{H}^{(L)}$ and $\widetilde{G} = \widetilde{G}^{(L)}$. Fixing the total length $|\widetilde{H}|+|\widetilde{G}|$, we seek the values L, $|S^{(1)}|$, $|\widetilde{S}^{(1)}|$, $\ldots$, $|S^{(L)}|$, $|\widetilde{S}^{(L)}|$, where $|S^{(\ell)}| \geq 2$ and $|\widetilde{S}^{(\ell)}| \geq 2$ for $1 \leq \ell \leq L$, that maximize the percentage reduction in computational effort r in (15). In this way, for a fixed value of $|\widetilde{H}| + |\widetilde{G}|$, an optimal structure of $\widetilde{H}$ and $\widetilde{G}$ via lifting that gives the most efficient implementation of the multiwavelet algorithms can be obtained. Based on this optimal structure for implementation, one can apply Algorithm 1 to the free parameters in $S^{(\ell)}$ and $\widetilde{S}^{(\ell)}$ for $1 \leq \ell \leq L$ to optimize the object function identified for the problem concerned.

As an illustration, consider $|\widetilde{H}| + |\widetilde{G}| = 24$. It turns out that $L = 1$, $|S^{(1)}| = 6$, $|\widetilde{S}^{(1)}| = 2$ give the optimal structure for implementation. In this case, $|\widetilde{H}| = 13$ and $|\widetilde{G}| = 11$. The corresponding reduction in computational complexity is 67%. Note that this is higher than the 50% stated in Table 1, where $L = 3$ and $|S^{(\ell)}| = |\widetilde{S}^{(\ell)}| = 2$ for $\ell = 1, 2, 3$. □

The steps in the lifted decomposition algorithm (10) are easily reversible. Indeed, based on the decomposed vector signals $\{c^0(k)\}$ and $\{d^0(k)\}$, set $c^{0,(L)}(k) := c^0(k)$ and $d^{0,(L)}(k) := d^0(k)$. For $L \geq \ell \geq 1$, define

$$\begin{cases} c^{0,(\ell-1)}(k) := c^{0,(\ell)}(k) - \sum_{n\in\mathbb{Z}} \widetilde{s}^{(\ell)}(n-k) d^{0,(\ell)}(n), \\ d^{0,(\ell-1)}(k) := d^{0,(\ell)}(k) + \sum_{n\in\mathbb{Z}} s^{(\ell)}(k-n)^T c^{0,(\ell-1)}(n). \end{cases} \tag{16}$$

Then it follows that

$$c^1(k) = \sum_{n\in\mathbb{Z}} \big(h^{(0)}(k-2n)^T c^{0,(0)}(n) + g^{(0)}(k-2n)^T d^{0,(0)}(n)\big).$$

We shall call (16) the **lifted reconstruction algorithm**. Using similar arguments as before, one can also show that the lifted reconstruction algorithm reduces the computational complexity of the reconstruction algorithm (3).

As a final note, although we assume that $T_1^{(\ell)} = T_2^{(\ell)} = I_r$, and $S^{(\ell)}$ and $\widetilde{S}^{(\ell)}$ satisfy (11) throughout this section, similar analysis can be used to establish the reduction in computational complexity for more general situations.

λ	L	$\lvert\widetilde{G}\rvert$	$\lvert\widetilde{H}\rvert$	$\mathcal{C}_m$	$\mathcal{D}_m$	r
2	1	3	5	8	4	50%
2	2	7	9	16	8	50%
2	3	11	13	24	12	50%
2	4	15	17	32	16	50%
3	1	5	9	14	6	57%
3	2	13	17	30	12	60%
4	1	7	13	20	8	60%
5	1	9	17	26	10	62%

TABLE 1. Reduction in computational complexity.

4. CONCLUSION

We have reviewed some of the results on the lifting scheme for the design of multiwavelet filter banks established in [3]. We have highlighted how the lifting scheme can be used to construct optimal PRMFBs for a specific application. We have also shown that PRMFBs obtained from the lifting scheme can be implemented very efficiently in practice with much reduction in computational complexity.

The paper [3] contains other applications of the lifting scheme. The applications include multiwavelet transforms that map integers to integers, construction of multiwavelets with optimum time-frequency localization, and image compression based on these optimal multiwavelets.

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