Uncertainty Products of Local Periodic Wavelets

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Abstract

This paper is on the angle-frequency localization of periodic scaling functions and wavelets. It is shown that the uncertainty products of uniformly local, uniformly regular and uniformly stable scaling functions and wavelets are uniformly bounded from above by a constant. Results for the construction of such scaling functions and wavelets are also obtained. As an illustration, scaling functions and wavelets associated with a family of generalized periodic splines are studied. This family is generated by periodic weighted convolutions, and it includes the well-known periodic B-splines and trigonometric B-splines.

Keywords: scaling functions, wavelets, uncertainty products, uniformly local, uniformly regular and uniformly stable functions, periodic weighted convolutions

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1 Introduction

Let $L^2_{2\pi}$ denote the space of all 2π-periodic square-integrable complex-valued functions with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ given by

$$\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx, \qquad \|f\| := \langle f,f\rangle^{1/2}.$$

The Fourier series of a function $f\in L^2_{2\pi}$ is defined by $\sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{in\cdot}$, where $\widehat{f}(n) := \langle f, e^{in\cdot}\rangle$, $n\in\mathbb{Z}$, are the Fourier coefficients of f. Fixing notations, we define $\mathbb{N} := \{0,1,\dots\}$ and $\mathbb{E}_k := \{0,1,\dots,2^k-1\}$, $k\in\mathbb{N}$. For a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$, the function $f_k(\cdot - \frac{2\pi\ell}{2^k})$, where $k\in\mathbb{N}$ and $\ell\in\mathbb{E}_k$, is denoted by f_k^ℓ, and thus $f_k^0 := f_k$. For $k\in\mathbb{N}$, let $\mathcal{S}_k$ be the space of all complex 2^k-periodic sequences. The finite Fourier transform, which is invertible, on an element $c\in\mathcal{S}_k$ is defined by

$$\widehat{c}(j) := \sum_{\ell\in\mathbb{E}_k} c(\ell)e^{-i\frac{2\pi j}{2^k}\ell}, \quad j\in\mathbb{E}_k.$$

Recall (see, for instance, [4]) that a sequence of subspaces $\{V_k\}_{k\in\mathbb{N}}$ of $L^2_{2\pi}$ forms a *multiresolution* if it satisfies the following conditions:

MR1. For $k\in\mathbb{N}$, $\dim V_k = 2^k$ and there exists $\phi_k\in V_k$ such that $\{\phi_k^\ell : \ell\in\mathbb{E}_k\}$ forms a basis of V_k.

MR2. $V_k\subset V_{k+1}$ for all $k\in\mathbb{N}$.

MR3. $\overline{\bigcup_{k\in\mathbb{N}} V_k} = L^2_{2\pi}$.

The function ϕ_k is called a *scaling function.* The condition MR2 holds if and only if for every $k\in\mathbb{N}$, there exists $h_{k+1}\in\mathcal{S}_{k+1}$ such that

$$\phi_k = \sum_{\ell\in\mathbb{E}_{k+1}} h_{k+1}(\ell)\phi_{k+1}^\ell. \tag{1.1}$$

This is also equivalent to

$$\widehat{\phi}_k(n) = \widehat{h}_{k+1}(n)\widehat{\phi}_{k+1}(n), \quad n\in\mathbb{Z}. \tag{1.2}$$

For $k\in\mathbb{N}$, let W_k be the orthogonal complement of V_k in V_{k+1}. A function ψ_k in V_{k+1} is said to be a *wavelet* if $\{\psi_k^\ell : \ell\in\mathbb{E}_k\}$ forms a basis of W_k. Since $\psi_k\in V_{k+1}$, there exists $g_{k+1} \in \mathcal{S}_{k+1}$ such that

$$\psi_k = \sum_{\ell \in \mathbb{E}_{k+1}} g_{k+1}(\ell) \phi_{k+1}^{\ell}, \tag{1.3}$$

which is equivalent to

$$\widehat{\psi}_k(n) = \widehat{g}_{k+1}(n) \widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}. \tag{1.4}$$

The references [2], [4]–[6], [8]–[9] contain studies of periodic wavelets.

The focus of this paper is on the angle-frequency localization of periodic scaling functions and wavelets. First, let us review (see, for instance, [1] or [5]) how the angle-frequency localization of periodic functions can be measured. For any function $f \in L^2_{2\pi}$, we define the *first trigonometric moment* of f by

$$\tau(f) := \frac{1}{2\pi} \int_0^{2\pi} e^{ix} |f(x)|^2 \, dx. \tag{1.5}$$

For $f \in L^2_{2\pi}$ such that $\tau(f) \neq 0$, the angle localization of f is measured by the quantity

$$\triangle_\theta(f) := \sqrt{\frac{\|f\|^4 - |\tau(f)|^2}{|\tau(f)|^2}}. \tag{1.6}$$

For a nontrivial function f in $L^2_{2\pi}$ with Fourier coefficients $\widehat{f}(n)$, $n \in \mathbb{Z}$, satisfying $\sum_{n \in \mathbb{Z}} n^2 |\widehat{f}(n)|^2 < \infty$, we define the *variance in the frequency* of f by

$$\mathrm{Var}_m(f) := \frac{\sum_{n \in \mathbb{Z}} n^2 |\widehat{f}(n)|^2}{\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2} - \left(\frac{\sum_{n \in \mathbb{Z}} n |\widehat{f}(n)|^2}{\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2} \right)^2.$$

The frequency localization of f is represented by the quantity

$$\triangle_m(f) := \sqrt{\mathrm{Var}_m(f)}. \tag{1.7}$$

The quantities $\triangle_\theta(f)$ and $\triangle_m(f)$ are related by the uncertainty principle for periodic functions, which says that

$$\triangle_m(f) \triangle_\theta(f) \geq \frac{1}{2}. \tag{1.8}$$

This uncertainty relation was proposed by Breitenberger in [1], and the product $\triangle_m(f)\triangle_\theta(f)$ in (1.8) is known as the *uncertainty product* of f. The uncertainty product of f measures the angle-frequency localization of f, and in practice, one desires it to be as small as possible.

The growth of uncertainty products of scaling functions and wavelets was first investigated by Narcowich and Ward. In [5], Narcowich and Ward constructed orthonormal scaling functions $\{\phi_k\}_{k\in\mathbb{N}}$ and wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ from periodic basis functions which have the property that

$$\triangle_m(\phi_k)\triangle_\theta(\phi_k) = O(\sqrt{2^k}), \qquad \triangle_m(\psi_k)\triangle_\theta(\psi_k) = O(\sqrt{2^k}),$$

as $k \to \infty$. Other interesting studies on the growth of uncertainty products are contained in [7]–[9]. In [9], by considering the de la Vallée Poussin means of the Dirichlet kernel, Selig constructed orthonormal trigonometric polynomial scaling functions and wavelets with uncertainty products that are uniformly bounded from above by a constant. This growth order of $O(1)$ is the best possible due to the constant lower bound of $1/2$ in the uncertainty principle (1.8).

In this paper, we introduce notions of uniformly local, uniformly regular and uniformly stable sequences of 2π-periodic functions. The concepts of uniform locality and uniform regularity are discussed in Section 2 with emphasis on localization in angle and frequency. Uniformly local functions can be viewed as periodic analogues of compactly supported functions over the real line. In Section 3, we derive results for the construction of uniformly stable, uniformly local and uniformly regular scaling functions and wavelets. These sequences of functions, which are not trigonometric polynomials, are shown to have the optimal rate of growth of $O(1)$. In Section 4, we introduce a periodic weighted convolution operator to obtain a rich family of such scaling functions. As an illustration, we construct local periodic scaling functions which are periodic spline functions. The well-known periodic B-splines in [4] and trigonometric B-splines in [3] are included in our construction via suitable choices of weights in the convolution operator.

2 Uniformly local and uniformly regular functions

A sequence of nontrivial functions $\{f_k\}_{k\in\mathbb{N}}$ in $L^2_{2\pi}$ is said to be *uniformly local* if the support of f_k, $\mathrm{supp}(f_k)$, lies in an interval of length $O(\frac{1}{2^k})$ as $k \to \infty$. We refer to periodic scaling functions and wavelets that are uniformly local as *local periodic scaling functions* and *local periodic wavelets* respectively. Examples of a uniformly local sequence of functions in $L^2_{2\pi}$ can be obtained by periodizing compactly supported functions in $L^2(\mathbb{R})$. Indeed, given a compactly supported function $f \in L^2(\mathbb{R})$, the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ given by

$$f_k(x) := \sqrt{2^k}\sum_{n\in\mathbb{Z}} f\left(\frac{2^k}{2\pi}(x-2\pi n)\right), \quad k\in\mathbb{N},\ x\in\mathbb{R}, \tag{2.1}$$

is uniformly local.

Theorem 2.1 *Let $\{f_k\}_{k\in\mathbb{N}}$ be a uniformly local sequence of functions in $L^2_{2\pi}$. Then as $k\to\infty$,*

$$\triangle_\theta(f_k) = O(\frac{1}{2^k}).$$

Proof. First, observe that for any $f\in L^2_{2\pi}$ and $a\in\mathbb{R}$,

$$\tau(f(\cdot+a)) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix}|f(x+a)|^2\,dx = e^{-ia}\tau(f).$$

Thus the quantity $|\tau(f)|$ is translation invariant. We note from (1.6) that $\triangle_\theta(f)$ only depends on $\|f\|$ and $|\tau(f)|$. Since both $\|f\|$ and $|\tau(f)|$ are translation invariant, it suffices to assume that $\mathrm{supp}(f_k)\subseteq[0,\frac{c}{2^k}]\subseteq[0,\frac{\pi}{2})$, $k\ge k_0$, for some positive constant c and nonnegative integer k_0.

Using the definition of $\tau(f)$ in (1.5), for $k\ge k_0$, we have

$$\tau(f_k) = \frac{1}{2\pi}\int_0^{\frac{c}{2^k}}\cos x|f_k(x)|^2\,dx + \frac{i}{2\pi}\int_0^{\frac{c}{2^k}}\sin x|f_k(x)|^2\,dx.$$

As a result, this yields

$$|\tau(f_k)|^2 \ge (\mathrm{Re}\{\tau(f_k)\})^2 = \left(\frac{1}{2\pi}\int_0^{\frac{c}{2^k}}\cos x|f_k(x)|^2\,dx\right)^2 \ge \cos^2\frac{c}{2^k}\|f_k\|^4 > 0.$$

Therefore, we have

$$\triangle_\theta(f_k)^2 = \frac{\|f_k\|^4 - |\tau(f_k)|^2}{|\tau(f_k)|^2} \le \frac{1-\cos^2\frac{c}{2^k}}{\cos^2\frac{c}{2^k}} = \tan^2\frac{c}{2^k},$$

giving the inequality

$$\triangle_\theta(f_k) \le \tan\frac{c}{2^k}.$$

Now, $\tan x = x + \dfrac{x^3}{3} + O(x^5)$ as $x \to 0$. Thus as $k \to \infty$, we conclude that

$$\triangle_\theta(f_k) = O(\frac{1}{2^k}). \quad \blacksquare$$

We say that a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in $L^2_{2\pi}$ is *uniformly* (α,β)*-regular*, where $\alpha > \dfrac{1}{2}$ and $\beta \in \mathbb{R}$, if there exist a positive constant C and a nonnegative integer L such that for $k \in \mathbb{N}$ and $|n| > 2^k L$,

$$|\widehat{f}_k(n)| \le \frac{C2^{k(\beta-\frac{1}{2})}}{|n|^\alpha}. \tag{2.2}$$

This definition is also motivated by the periodization formula (2.1). Indeed, if f is a compactly supported function in $L^2(\mathbb{R})$ that has ν continuous derivatives, then the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ defined by (2.1) satisfies (2.2) with $\alpha = \beta = \nu$ and $L = 0$. Here in (2.2), we allow the greater generality of $\alpha > \dfrac{1}{2}$, $\beta \in \mathbb{R}$ and $L \ge 0$. The assumption $\alpha > \dfrac{1}{2}$ is essential to ensure that each function f_k lies in the space $L^2_{2\pi}$. However, for the purpose of considering $\dfrac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f}_k(n)|^2$ in our study of frequency localization, we further require α to be greater than $\dfrac{3}{2}$. It should be mentioned that the periodic scaling functions and wavelets constructed in [5] from periodic basis functions satisfy (2.2) with $\beta \ge \alpha > \dfrac{3}{2}$ and $L = 0$.

Theorem 2.2 *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of uniformly (α,β)-regular functions in $L^2_{2\pi}$, where $\alpha > \dfrac{3}{2}$ and $\beta \ge \dfrac{1}{2}$. Suppose that there exists a positive constant A such that for every $k \in \mathbb{N}$, $A \le \|f_k\|^2$. Then as $k \to \infty$,*

$$\frac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f}_k(n)|^2 = \begin{cases} O(2^{2k})^{\beta-\alpha+1}, & \text{if } \beta > \alpha, \\ O(2^{2k}), & \text{if } \beta \le \alpha, \end{cases}$$

and

$$\triangle_m(f_k) = \begin{cases} O(2^k)^{\beta-\alpha+1}, & \textit{if } \beta > \alpha, \\ O(2^k), & \textit{if } \beta \le \alpha. \end{cases} \tag{2.3}$$

Proof. Since the sequence $\{f_k\}_{k\in\mathbb{N}}$ is uniformly (α,β)-regular, there exist a positive constant C and a positive integer L such that for $k \in \mathbb{N}$ and $|n| > 2^k L$, (2.2) holds. Using the bound for $\widehat{f_k}(n)$ in (2.2), we have

$$\begin{aligned} \frac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f_k}(n)|^2 &= \frac{1}{\|f_k\|^2}\sum_{|n|\le 2^k L} n^2|\widehat{f_k}(n)|^2 + \frac{1}{\|f_k\|^2}\sum_{|n|>2^k L} n^2|\widehat{f_k}(n)|^2 \\ &\le \frac{2^{2k}L^2}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} |\widehat{f_k}(n)|^2 + \frac{2C^2 2^{2k(\beta-\frac{1}{2})}}{\|f_k\|^2}\sum_{n=2^k L+1}^{\infty} |n|^{2-2\alpha}. \end{aligned}$$

Further estimations using our hypothesis lead to

$$\begin{aligned} \frac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f_k}(n)|^2 &\le 2^{2k}L^2 + \frac{2C^2 2^{2k(\beta-\frac{1}{2})}}{A}\int_{2^k L}^{\infty}\frac{1}{x^{2\alpha-2}}\,dx \\ &\le 2^{2k}L^2 + \frac{2C^2}{(2\alpha-3)L^{2\alpha-3}A}2^{2k(\beta-\alpha+1)}. \end{aligned}$$

In conclusion, as $k \to \infty$,

$$\frac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f_k}(n)|^2 = \begin{cases} O(2^{2k})^{\beta-\alpha+1}, & \text{if } \beta > \alpha, \\ O(2^{2k}), & \text{if } \beta \le \alpha, \end{cases}$$

and the resulting order of $\triangle_m(f_k)$ is a consequence of the estimate $\mathrm{Var}_m(f_k) \le \frac{1}{\|f_k\|^2}\sum_{n\in\mathbb{Z}} n^2|\widehat{f_k}(n)|^2$ and the definition (1.7). ■

In Theorem 2.2, we have assumed the condition that $\beta \ge \frac{1}{2}$. This is necessary due to the hypothesis of $A \le \|f_k\|^2$ for every $k \in \mathbb{N}$. The above hypothesis will be automatically satisfied when we consider sequences of uniformly stable scaling functions $\{\phi_k\}_{k\in\mathbb{N}}$ and wavelets $\{\psi_k\}_{k\in\mathbb{N}}$. This concept of uniform stability will be introduced in the beginning of the next section.

Remark 2.1 The notions of uniform locality and uniform regularity can also be defined with respect to a general increasing sequence of positive integers $\{\gamma_k\}_{k\in\mathbb{N}}$. In this case, we replace 2^k by γ_k in their definitions. In addition, similar analysis shows that Theorems 2.1 and 2.2 still hold with γ_k in place of 2^k.

3 Uniformly stable scaling functions and wavelets

We say that a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in $L^2_{2\pi}$ is *uniformly stable* if there exist positive constants A and B such that for every $k \in \mathbb{N}$,

$$A\sum_{\ell\in\mathbb{E}_k}|c(\ell)|^2 \le \|\sum_{\ell\in\mathbb{E}_k}c(\ell)f_k^\ell\|^2 \le B\sum_{\ell\in\mathbb{E}_k}|c(\ell)|^2 \qquad \text{for all } c\in\mathcal{S}_k. \tag{3.1}$$

A direct consequence of (3.1) is that for every $k\in\mathbb{N}$, $A \le \|f_k\|^2 \le B$. The following proposition contains a characterization of uniform stability.

Proposition 3.1 *A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in $L^2_{2\pi}$ is uniformly stable if and only if there exist positive constants A and B such that for every $k\in\mathbb{N}$,*

$$\frac{A}{2^k} \le \|w_{k,j}\|^2 \le \frac{B}{2^k}, \quad j\in\mathbb{E}_k, \tag{3.2}$$

where

$$w_{k,j} := \sum_{q\in\mathbb{Z}} \widehat{f}_k(j+2^kq)e^{i(j+2^kq)\cdot}. \tag{3.3}$$

Proof. For $k\in\mathbb{N}$ and $c\in\mathcal{S}_k$, by the Parseval identity, we have

$$\|\sum_{\ell\in\mathbb{E}_k}c(\ell)f_k^\ell\|^2 = \sum_{n\in\mathbb{Z}}|\widehat{f}_k(n)|^2|\widehat{c}(n)|^2 = \sum_{j\in\mathbb{E}_k}\sum_{q\in\mathbb{Z}}|\widehat{f}_k(j+2^kq)|^2|\widehat{c}(j+2^kq)|^2.$$

Since $\widehat{c}\in\mathcal{S}_k$, this gives

$$\|\sum_{\ell\in\mathbb{E}_k}c(\ell)f_k^\ell\|^2 = \sum_{j\in\mathbb{E}_k}|\widehat{c}(j)|^2\|w_{k,j}\|^2,$$

where $w_{k,j}$ is as in (3.3). Consequently, (3.1) holds if and only if

$$\frac{A}{2^k}\sum_{j\in\mathbb{E}_k}|\widehat{c}(j)|^2 \le \sum_{j\in\mathbb{E}_k}|\widehat{c}(j)|^2\|w_{k,j}\|^2 \le \frac{B}{2^k}\sum_{j\in\mathbb{E}_k}|\widehat{c}(j)|^2 \qquad \text{for all } \widehat{c}\in\mathcal{S}_k.$$

Choose $\widehat{c}^\nu(j) = \delta_{j,\nu}$ for $j,\nu\in\mathbb{E}_k$. Then we obtain

$$\frac{A}{2^k} \le \|w_{k,\nu}\|^2 \le \frac{B}{2^k}, \quad \nu\in\mathbb{E}_k.$$

The converse is straightforward and the proof is complete. ■

We shall now provide sufficient conditions for uniformly stable scaling functions.

Proposition 3.2 *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^2_{2\pi}$ satisfying* (1.2). *Suppose that there exist positive constants A and B such that for every $k \in \mathbb{N}$,*

$$|\widehat{\phi}_k(n)|^2 \geq \frac{A}{2^k}, \quad |n| \in \mathbb{E}_{k-1} \cup \{2^{k-1}\}, \tag{3.4}$$

and

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_k(j + 2^k q)|^2 \leq \frac{B}{2^k}, \quad j \in \mathbb{E}_k. \tag{3.5}$$

Then $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence of uniformly stable scaling functions.

Proof. In view of (3.2) and (3.5), to show that $\{\phi_k\}_{k\in\mathbb{N}}$ is uniformly stable, it suffices to establish the inequality

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_k(j + 2^k q)|^2 \geq \frac{A}{2^k}, \quad j \in \mathbb{E}_k,$$

for every $k \in \mathbb{N}$. Indeed, from (3.4), for $j \in \mathbb{E}_{k-1} \cup \{2^{k-1}\}$,

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_k(j + 2^k q)|^2 \geq |\widehat{\phi}_k(j)|^2 \geq \frac{A}{2^k}.$$

On the other hand, for $j \in \mathbb{E}_k \setminus (\mathbb{E}_{k-1} \cup \{2^{k-1}\})$,

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_k(j + 2^k q)|^2 \geq |\widehat{\phi}_k(j - 2^k)|^2 \geq \frac{A}{2^k}.$$

Next, we shall show that $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence of scaling functions. Now, an immediate consequence of (3.1) is that the functions ϕ_k^ℓ, $\ell \in \mathbb{E}_k$, are linearly independent. This gives the condition MR1. As mentioned in Section 1, the nesting property MR2 is equivalent to (1.2). Finally, recall (see [4, Corollary 4.1]) that the condition MR3 holds if and only if the set

$$\{n \in \mathbb{Z} : \widehat{\phi}_k(n) = 0 \text{ for all } k \in \mathbb{N}\}$$

is empty. Suppose that there exists $n_0 \in \mathbb{Z}$ such that $\widehat{\phi}_k(n_0) = 0$ for all $k \in \mathbb{N}$. If we choose k to be large enough for $|n_0| \in \mathbb{E}_{k-1} \cup \{2^{k-1}\}$, then we obtain from (3.4) that $|\widehat{\phi}_k(n_0)|^2 \geq \dfrac{A}{2^k}$, which is impossible. Hence, the sequence $\{\phi_k\}_{k\in\mathbb{N}}$ generates a multiresolution of $L^2_{2\pi}$. ■

As for the construction of wavelets from scaling functions, the following theorem in [4] is useful.

Theorem 3.1 *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence of scaling functions in $L^2_{2\pi}$ with $h_{k+1} \in \mathcal{S}_{k+1}$, $k \in \mathbb{N}$, as in* (1.1)*. For $k \in \mathbb{N}$, consider $\sigma_{k+1} \in \mathcal{S}_{k+1}$ that satisfies*

$$\sigma_{k+1}(j + 2^k) = -\sigma_{k+1}(j) \neq 0, \quad j \in \mathbb{E}_k,$$

and define $\widehat{g}_{k+1} \in \mathcal{S}_{k+1}$ by

$$\widehat{g}_{k+1}(j) := \sigma_{k+1}(j)\frac{\overline{\widehat{h}_{k+1}(j + 2^k)}}{\|v_{k+1,j}\|^2}, \quad j \in \mathbb{E}_{k+1},$$

where

$$v_{k+1,j} := \sum_{q\in\mathbb{Z}} \widehat{\phi}_{k+1}(j + 2^{k+1}q)e^{i(j+2^{k+1}q)\cdot}, \quad j \in \mathbb{E}_{k+1}.$$

Then the function ψ_k given by (1.3) *is a wavelet. Furthermore, if there exist positive constants A and B such that for every $k \in \mathbb{N}$,*

$$A \leq 2^k|\sigma_{k+1}(j)|^2 \left(\frac{|\widehat{h}_{k+1}(j)|^2}{\|v_{k+1,j+2^k}\|^2} + \frac{|\widehat{h}_{k+1}(j + 2^k)|^2}{\|v_{k+1,j}\|^2}\right) \leq B, \quad j \in \mathbb{E}_k, \tag{3.6}$$

then the collection $\{\phi_0\} \cup \bigcup_{k\in\mathbb{N}}\{\psi_k^\ell : \ell \in \mathbb{E}_k\}$ forms a Riesz basis of $L^2_{2\pi}$.

Remark 3.1 For $k \in \mathbb{N}$, if we define

$$u_{k,j} := \sum_{q\in\mathbb{Z}} \widehat{\psi}_k(j + 2^k q)e^{i(j+2^k q)\cdot}, \quad j \in \mathbb{E}_k,$$

then by standard arguments in [4],

$$\|u_{k,j}\|^2 = |\widehat{g}_{k+1}(j)|^2\|v_{k+1,j}\|^2 + |\widehat{g}_{k+1}(j + 2^k)|^2\|v_{k+1,j+2^k}\|^2, \quad j \in \mathbb{E}_k.$$

Consequently, the condition (3.6) is equivalent to the condition that for every $k \in \mathbb{N}$,

$$\frac{A}{2^k} \leq \|u_{k,j}\|^2 \leq \frac{B}{2^k}, \quad j \in \mathbb{E}_k,$$

which is the condition for uniform stability of the sequence $\{\psi_k\}_{k\in\mathbb{N}}$. Thus a sequence of uniformly stable wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ will lead to a Riesz basis of the entire space $L^2_{2\pi}$.

As to how uniformly stable wavelets can be constructed, the following theorem utilizes Theorem 3.1 to provide suitable sequences $\widehat{g}_{k+1}$, $k \in \mathbb{N}$. In fact, the sequences $\widehat{g}_{k+1}$ identified yield wavelets that are also uniformly local and uniformly regular.

Theorem 3.2 *Let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence of uniformly stable, uniformly (α,β)-regular and uniformly local scaling functions in $L^2_{2\pi}$, where $\alpha > \frac{1}{2}$ and $\beta \geq \frac{1}{2}$, with $h_{k+1} \in \mathcal{S}_{k+1}$, $k \in \mathbb{N}$, as in (1.1). Suppose that there exists a positive integer N such that for sufficiently large values of k, $\operatorname{supp}(\phi_k) \subseteq [0, \frac{2\pi N}{2^k}]$ and $h_{k+1}(\ell) = 0$ for $\ell \in \mathbb{E}_{k+1} \setminus \{0, 1, \ldots, N\}$. For $k \in \mathbb{N}$, define $\widehat{g}_{k+1} \in \mathcal{S}_{k+1}$ by*

$$\widehat{g}_{k+1}(j) := 2^{k+1}\|v_{k+1,j+2^k}\|^2 e^{-\frac{i2\pi j}{2^{k+1}}}\overline{\widehat{h}_{k+1}(j+2^k)}, \quad j \in \mathbb{E}_{k+1}. \tag{3.7}$$

Then the corresponding sequence of wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ given by (1.3) is also uniformly stable, uniformly (α,β)-regular and uniformly local. Furthermore, the collection $\{\phi_0\} \cup \bigcup_{k\in\mathbb{N}}\{\psi_k^\ell : \ell \in \mathbb{E}_k\}$ forms a Riesz basis of $L^2_{2\pi}$.

Proof. Using standard arguments in [4], we have

$$\|v_{k,j}\|^2 = |\widehat{h}_{k+1}(j)|^2\|v_{k+1,j}\|^2 + |\widehat{h}_{k+1}(j+2^k)|^2\|v_{k+1,j+2^k}\|^2, \quad j \in \mathbb{E}_k. \tag{3.8}$$

Since $\{\phi_k\}_{k\in\mathbb{N}}$ is uniformly stable, it follows from Proposition 3.1 that there exist positive constants A and B such that for every $k \in \mathbb{N}$,

$$\frac{A}{2^k} \leq \|v_{k,j}\|^2 \leq \frac{B}{2^k}, \quad j \in \mathbb{E}_k. \tag{3.9}$$

Thus by setting σ_{k+1}, $k \in \mathbb{N}$, in Theorem 3.1 to be

$$\sigma_{k+1}(j) := 2^{k+1}\|v_{k+1,j+2^k}\|^2\|v_{k+1,j}\|^2 e^{-\frac{i2\pi j}{2^{k+1}}}, \quad j \in \mathbb{E}_{k+1},$$

we see that for every $k \in \mathbb{N}$,

$$A^3 \leq 2^k|\sigma_{k+1}(j)|^2\left(\frac{|\widehat{h}_{k+1}(j)|^2}{\|v_{k+1,j+2^k}\|^2} + \frac{|\widehat{h}_{k+1}(j+2^k)|^2}{\|v_{k+1,j}\|^2}\right) \leq B^3, \quad j \in \mathbb{E}_k.$$

This shows that the sequence of wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ is uniformly stable.

Now, (3.8) and (3.9) also imply that for every $k \in \mathbb{N}$,

$$|\widehat{h}_{k+1}(j)|^2 + |\widehat{h}_{k+1}(j+2^k)|^2 \leq \frac{2B}{A}, \quad j \in \mathbb{E}_k.$$

Consequently, we have

$$|\widehat{h}_{k+1}(j)|^2 \leq \frac{2B}{A}, \quad j \in \mathbb{E}_{k+1},$$

and it follows from (3.7) that

$$|\widehat{g}_{k+1}(j)|^2 \leq \frac{2B^3}{A}, \quad j \in \mathbb{E}_{k+1}.$$

Since $\{\phi_k\}_{k\in\mathbb{N}}$ is uniformly (α,β)-regular, this inequality together with (1.4) show that $\{\psi_k\}_{k\in\mathbb{N}}$ is also uniformly (α,β)-regular.

It remains to show that $\{\psi_k\}_{k\in\mathbb{N}}$ is uniformly local. To this end, define $\widehat{\lambda}_{k+1} \in \mathcal{S}_{k+1}$ by $\widehat{\lambda}_{k+1}(j) := 2^{k+1}\|v_{k+1,j}\|^2$ for $j \in \mathbb{E}_{k+1}$. Observe that

$$\sum_{\ell\in\mathbb{E}_{k+1}} \langle \phi_{k+1}, \phi_{k+1}^{\ell}\rangle e^{-\frac{i2\pi j\ell}{2^{k+1}}} = \widehat{\lambda}_{k+1}(j), \quad j \in \mathbb{E}_{k+1}.$$

This yields

$$\lambda_{k+1}(\ell) = \langle \phi_{k+1}, \phi_{k+1}^{\ell}\rangle, \quad \ell \in \mathbb{E}_{k+1}.$$

Using (3.7), we obtain

$$\begin{aligned} g_{k+1}(\ell) &= (-1)^{1-\ell} \sum_{\nu\in\mathbb{E}_{k+1}} \overline{h_{k+1}(\nu)\lambda_{k+1}(1-\nu-\ell)} \\ &= (-1)^{1-\ell} \sum_{\nu=0}^{N} \overline{h_{k+1}(\nu)\langle \phi_{k+1}, \phi_{k+1}^{1-\nu-\ell}\rangle}, \quad \ell \in \mathbb{E}_{k+1}. \end{aligned}$$

Since $\operatorname{supp}(\phi_{k+1}) \subseteq [0, \frac{2\pi N}{2^{k+1}}]$ for sufficiently large values of k, we conclude that $g_{k+1}(\ell) = 0$ for $\ell \in \mathbb{E}_{k+1} \setminus \big(\{0,1,\ldots,N\} \cup \{2^{k+1}-(2N-2),\ldots,2^{k+1}-1\}\big)$. Hence, it follows from (1.3) that $\operatorname{supp}(\psi_k) \subseteq [-\frac{2\pi(N-1)}{2^k}, \frac{2\pi N}{2^k}]$ for sufficiently large values of k, which means that $\{\psi_k\}_{k\in\mathbb{N}}$ is uniformly local. ■

Let us combine all our results together to analyze the angle-frequency localization of scaling functions and wavelets that are uniformly stable, uniformly (α,β)-regular and uniformly local. In view of the bounds in (2.3), to have good frequency localization, we shall assume that $\beta = \alpha > \frac{3}{2}$.

Theorem 3.3 *Let* $\{\phi_k\}_{k\in\mathbb{N}}$ *and* $\{\psi_k\}_{k\in\mathbb{N}}$ *be as in Theorem* 3.2 *with* $\beta = \alpha > \frac{3}{2}$. *Then as* $k \to \infty$,

$$\triangle_m(\phi_k)\triangle_\theta(\phi_k) = O(1), \qquad \triangle_m(\psi_k)\triangle_\theta(\psi_k) = O(1). \tag{3.10}$$

Proof. The result follows from Theorems 3.2, 2.1 and 2.2. ∎

Remark 3.2 As mentioned in Section 1, the growth order of $O(1)$ of the uncertainty products in (3.10) for both scaling functions and wavelets is the best possible. In fact, the uncertainty principle (1.8) shows that for scaling functions and wavelets which are uniformly stable, uniformly (α, β)-regular, where $\beta = \alpha > \frac{3}{2}$, and uniformly local, the growth order of $\triangle_m(\phi_k)$ and $\triangle_m(\psi_k)$ is exactly $O(2^k)$ as $k \to \infty$, and that of $\triangle_\theta(\phi_k)$ and $\triangle_\theta(\psi_k)$ is exactly $O(\frac{1}{2^k})$.

4 Scaling functions constructed from periodic weighted convolutions

In the previous section, we have seen that if a sequence of functions $\{\phi_k\}_{k\in\mathbb{N}}$ in $L^2_{2\pi}$ satisfies the hypothesis of Proposition 3.2, then it is a sequence of uniformly stable scaling functions. With the additional assertions in Theorem 3.2, $\{\phi_k\}_{k\in\mathbb{N}}$ yields a sequence of wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ that generates a Riesz basis of $L^2_{2\pi}$, and the uncertainty products of both $\{\phi_k\}_{k\in\mathbb{N}}$ and $\{\psi_k\}_{k\in\mathbb{N}}$ attain the optimal rate of growth of $O(1)$ as $k \to \infty$. Thus it is of interest to construct sequences of functions $\{\phi_k\}_{k\in\mathbb{N}}$ that satisfy the conditions of Proposition 3.2 and Theorem 3.2. For this purpose, we shall introduce a periodic weighted convolution operator.

Given $m \in \mathbb{Z}$, we define the 2π*-periodic* m*-weighted convolution operator* $\otimes_m$ on $L^2_{2\pi}$ by

$$(f \otimes_m g)(x) := \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)e^{imy}\,dy, \quad x \in \mathbb{R},$$

where $f, g \in L^2_{2\pi}$. Note that $f \otimes_m g \in L^2_{2\pi}$ and

$$(\widehat{f \otimes_m g})(n) = \widehat{f}(n)\widehat{g}(n-m), \quad n \in \mathbb{Z}. \tag{4.1}$$

A discrete analogue of the operator $\otimes_m$ is as follows. For $k \in \mathbb{N}$, we define the *discrete 2^{k+1}-periodic m-weighted convolution operator* $\otimes_m^D$ on $\mathcal{S}_{k+1}$ by

$$(h_{k+1} \otimes_m^D g_{k+1})(\ell) := \sum_{\nu \in \mathbb{E}_{k+1}} h_{k+1}(\ell - \nu) g_{k+1}(\nu) e^{\frac{i2\pi m\nu}{2^{k+1}}}, \quad \ell \in \mathbb{E}_{k+1},$$

where $h_{k+1}, g_{k+1} \in \mathcal{S}_{k+1}$. Then $h_{k+1} \otimes_m^D g_{k+1} \in \mathcal{S}_{k+1}$ and

$$\widehat{h_{k+1} \otimes_m^D g_{k+1}}(j) = \widehat{h}_{k+1}(j)\widehat{g}_{k+1}(j-m), \quad j \in \mathbb{E}_{k+1}. \tag{4.2}$$

Theorem 4.1 *For $\nu = 1, 2, \ldots, s$, suppose that $\{\phi_{k,\nu}\}_{k\in\mathbb{N}}$ is a sequence of uniformly (α_ν, β_ν)-regular functions in $L^2_{2\pi}$, where $\alpha_\nu > \frac{1}{2}$ and $\beta_\nu \geq \frac{1}{2}$, satisfying*

$$\phi_{k,\nu} = \sum_{\ell \in \mathbb{E}_{k+1}} h_{k+1,\nu}(\ell)\phi^{\ell}_{k+1,\nu}, \tag{4.3}$$

with $h_{k+1,\nu} \in \mathcal{S}_{k+1}$, $k \in \mathbb{N}$, and there exists a positive integer N_ν such that for sufficiently large values of k, $\mathrm{supp}(\phi_{k,\nu}) \subseteq [0, \frac{2\pi N_\nu}{2^k}]$ and $h_{k+1,\nu}(\ell) = 0$ for $\ell \in \mathbb{E}_{k+1} \setminus \{0, 1, \ldots, N_\nu\}$. In addition, let there be positive constants $A_1, \ldots, A_s$, $B_1, \ldots, B_s$, integers $m_2, \ldots, m_s$, and a nonnegative integer k_0 such that for every $k \geq k_0$ and $\nu = 1, 2, \ldots, s$,

$$|\widehat{\phi}_{k,\nu}(n - m_\nu)|^2 \geq \frac{A_\nu}{2^k}, \quad |n| \in \mathbb{E}_{k-1} \cup \{2^{k-1}\}, \tag{4.4}$$

and

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_{k,\nu}(j + 2^k q)|^2 \leq \frac{B_\nu}{2^k}, \quad j \in \mathbb{E}_k, \tag{4.5}$$

where $m_1 := 0$. For $k \in \mathbb{N}$, define

$$\phi_k := (\sqrt{2^{k+k_0}})^{s-1} \phi_{k+k_0,1} \otimes_{m_2} \phi_{k+k_0,2} \otimes_{m_3} \cdots \otimes_{m_s} \phi_{k+k_0,s}. \tag{4.6}$$

Then $\{\phi_k\}_{k\in\mathbb{N}}$ *generates a periodic multiresolution and is uniformly local, uniformly stable and uniformly* (α,β)*-regular with* $\alpha := \alpha_1 + \cdots + \alpha_s$, $\beta := \beta_1 + \cdots + \beta_s$. *Moreover, the sequences* $h_{k+1} \in \mathcal{S}_{k+1}$, $k \in \mathbb{N}$, *in* (1.1) *are given by*

$$h_{k+1}(\ell) := \frac{1}{(\sqrt{2})^{s-1}}(h_{k+k_0+1,1} \otimes^D_{m_2} h_{k+k_0+1,2} \otimes^D_{m_3} \cdots \otimes^D_{m_s} h_{k+k_0+1,s})(\ell), \quad \ell \in \mathbb{E}_{k+1}, \tag{4.7}$$

and for sufficiently large values of k, $\operatorname{supp}(\phi_k) \subseteq [0, \frac{2\pi N}{2^k}]$ *and* $h_{k+1}(\ell) = 0$ *for* $\ell \in \mathbb{E}_{k+1} \setminus \{0, 1, \ldots, N\}$, *where* $N := N_1 + \cdots + N_s$.

Proof. First, we observe from (4.6) and (4.1) that for $k \in \mathbb{N}$, (1.2) holds with

$$\widehat{h}_{k+1}(n) := \frac{1}{(\sqrt{2})^{s-1}}\widehat{h}_{k+k_0+1,1}(n)\widehat{h}_{k+k_0+1,2}(n-m_2)\cdots\widehat{h}_{k+k_0+1,s}(n-m_s).$$

Then it follows from (4.2) that h_{k+1} is of the form (4.7). Using (4.4), for $k \in \mathbb{N}$, we have

$$\begin{aligned} |\widehat{\phi}_k(n)|^2 &= |(\sqrt{2^{k+k_0}})^{s-1}\widehat{\phi}_{k+k_0,1}(n)\widehat{\phi}_{k+k_0,2}(n-m_2)\cdots\widehat{\phi}_{k+k_0,s}(n-m_s)|^2 \\ &\geq \frac{A_1\cdots A_s}{2^{k+k_0}}, \quad |n| \in \mathbb{E}_{k-1} \cup \{2^{k-1}\}. \end{aligned}$$

Furthermore, by writing all integers q into the form $q = r + 2^{k_0}p$ where $r \in \mathbb{E}_{k_0}$ and $p \in \mathbb{Z}$, we observe from (4.5) that

$$\sum_{q\in\mathbb{Z}} |\widehat{\phi}_k(j + 2^k q)|^2 \leq \frac{B_1\cdots B_s}{2^k}, \quad j \in \mathbb{E}_k.$$

Thus Proposition 3.2 implies that $\{\phi_k\}_{k\in\mathbb{N}}$ is a sequence of uniformly stable scaling functions.

Since $\operatorname{supp}(\phi_{k,\nu}) \subseteq [0, \frac{2\pi N_\nu}{2^k}]$ for $\nu = 1, 2, \ldots, s$ and sufficiently large values of k, using mathematical induction, we see from (4.6) that for large enough k,

$$\operatorname{supp}(\phi_k) \subseteq [0, \frac{2\pi(N_1 + \cdots + N_s)}{2^{k+k_0}}].$$

Similarly, based on (4.7), we obtain $h_{k+1}(\ell) = 0$ for $\ell \in \mathbb{E}_{k+1} \backslash \{0, 1, \ldots, N_1 + \cdots + N_s\}$.

As for the uniform regularity of $\{\phi_k\}_{k\in\mathbb{N}}$, we note that for each $\nu = 1, 2, \dots, s$, there exist a positive constant C_ν and a nonnegative integer L_ν such that for $k \in \mathbb{N}$ and $|n| > 2^k L_\nu$, (2.2) holds with the constants α_ν and β_ν. Set $L'_\nu := \max\{2|m_\nu|, L_\nu\}$. Then for $k \in \mathbb{N}$ and $|n| > 2^k L'_\nu$,

$$|\widehat{\phi}_{k,\nu}(n - m_\nu)| \le 2C_\nu \frac{2^{k(\beta_\nu - \frac{1}{2})}}{|n|^{\alpha_\nu}}.$$

Thus for $k \in \mathbb{N}$ and $|n| > 2^k \max\{L'_1, \dots, L'_s\}$, we have

$$\begin{aligned} |\widehat{\phi}_k(n)| &= (\sqrt{2^{k+k_0}})^{s-1} |\widehat{\phi}_{k+k_0,1}(n)| |\widehat{\phi}_{k+k_0,2}(n - m_2)| \cdots |\widehat{\phi}_{k+k_0,s}(n - m_s)| \\ &\le C \frac{2^{k(\beta_1 + \cdots + \beta_s - \frac{1}{2})}}{|n|^{\alpha_1 + \cdots + \alpha_s}} \end{aligned}$$

for some $C > 0$. This completes the proof of the theorem. ■

As an illustration of what we have discussed in this section, we shall construct periodic scaling functions which are periodic spline functions.

Example 4.1 (Generalized periodic splines)
For every $\nu = 1, 2, \dots, s$, let $\{\phi_{k,\nu}\}_{k\in\mathbb{N}}$ be the sequence of periodic Haar functions $\{\phi_{k,0}\}_{k\in\mathbb{N}}$ given by

$$\phi_{k,0}(x) := \begin{cases} \sqrt{2^k}, & \text{if } x \in [0, \frac{2\pi}{2^k}], \\ 0, & \text{if } x \in (\frac{2\pi}{2^k}, 2\pi). \end{cases}$$

Then $\{\phi^\ell_{k,0} : \ell \in \mathbb{E}_k\}$ is an orthonormal set and

$$\widehat{\phi}_{k,0}(n) = \frac{1}{\sqrt{2^k}} \frac{\sin \frac{\pi n}{2^k}}{\frac{\pi n}{2^k}} e^{-\frac{i\pi n}{2^k}}, \quad n \in \mathbb{Z}.$$

We shall verify that $\{\phi_{k,0}\}_{k\in\mathbb{N}}$ satisfies the hypothesis of Theorem 4.1. First, $\{\phi_{k,0}\}_{k\in\mathbb{N}}$ is uniformly (α, β)-regular with $\alpha = \beta = 1$ since

$$|\widehat{\phi}_{k,0}(n)| = \frac{1}{\sqrt{2^k}} \left| \frac{\sin \frac{\pi n}{2^k}}{\frac{\pi n}{2^k}} \right| \le \frac{1}{\pi} \frac{2^{k(1-\frac{1}{2})}}{|n|}, \quad n \ne 0.$$

Next, we observe that $\{\phi_{k,0}\}_{k\in\mathbb{N}}$ satisfies (4.3) with $\text{supp}(\phi_{k,0}) = [0, \frac{2\pi}{2^k}]$, $h_{k+1,0}(0) = h_{k+1,0}(1) = \frac{1}{\sqrt{2}}$ and $h_{k+1,0}(\ell) = 0$ for $\ell \in \mathbb{E}_{k+1} \setminus \{0, 1\}$. Since $\{\phi^\ell_{k,0} : \ell \in \mathbb{E}_k\}$ is an orthonormal set, as a consequence of Proposition 3.1, equality holds in (4.5) with $B_0 = 1$.

It remains to check the condition (4.4). Since the function $\frac{x}{\sin x}$ is bounded on $[-\frac{3\pi}{4}, \frac{3\pi}{4}]$, there exists a positive constant A_0 such that whenever $|\alpha| \le \frac{\pi}{4}$,

$$\left|\frac{\sin(x-\alpha)}{x-\alpha}\right|^2 \ge A_0, \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Thus for a fixed $m \in \mathbb{Z}$, if $k_0 \ge 2 + \log_2(|m|+1)$, then for every $k \ge k_0$, $\left|\frac{\pi m}{2^k}\right| \le \frac{\pi}{4}$ and (4.4) holds.

Now, for $k \in \mathbb{N}$, define

$$\phi_k := (\sqrt{2^{k+k_0}})^{s-1} \phi_{k+k_0,0} \otimes_{m_2} \phi_{k+k_0,0} \otimes_{m_3} \cdots \otimes_{m_s} \phi_{k+k_0,0},$$

where $s \ge 2$ and $k_0 \ge \max\{2+\log_2(|m_2|+1), \ldots, 2+\log_2(|m_s|+1)\}$. By Theorem 4.1, $\{\phi_k\}_{k\in\mathbb{N}}$ generates a multiresolution of $L^2_{2\pi}$, and is uniformly stable, uniformly local, and uniformly (α,β)-regular with $\alpha = \beta = s$. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of wavelets. Then by Theorems 3.2 and 3.3, the collection $\{\phi_0\} \cup \bigcup_{k\in\mathbb{N}} \{\psi_k^\ell : \ell \in \mathbb{E}_k\}$ forms a Riesz basis of $L^2_{2\pi}$, and as $k \to \infty$,

$$\triangle_m(\phi_k)\triangle_\theta(\phi_k) = O(1), \qquad \triangle_m(\psi_k)\triangle_\theta(\psi_k) = O(1).$$

Remark 4.1 In Example 4.1, if $m_\nu := 0$ for $\nu = 2, 3, \ldots, s$, then $\{\phi_k\}_{k\in\mathbb{N}}$ is the sequence of periodic B-splines studied in [4]. If $m_\nu := \nu - 1$ for $\nu = 2, 3, \ldots, s$, then $\{\phi_k\}_{k\in\mathbb{N}}$ is the sequence of trigonometric B-splines discussed in [3]. Thus the uncertainty products of the scaling functions and wavelets associated with the periodic B-splines and trigonometric B-splines are all uniformly bounded from above by a constant.

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