Wavelets, multiwavelets and wavelet frames for periodic functions

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Abstract. Various results on constructing wavelets, multiwavelets and wavelet frames for periodic functions are reviewed. The orthonormal and Riesz bases as well as frames are constructed from sequences of subspaces called multiresolution analyses. These studies employ general frequency-based approaches facilitated by functions known as orthogonal splines and polyphase splines. While the focus is on the intrinsic nature of the periodic setup, the exposition highlights the main ideas developed in the evolution of wavelet theory, from wavelets to multiwavelets and then wavelet frames.

1 Introduction

Wavelet analysis studies and applies functions of the form

$$\psi_{b,a}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R},\ a \neq 0,$$

which are obtained by *dilating* and *translating* a single function ψ in $L^2(\mathbb{R})$, the space of all square-integrable complex-valued functions over the real line $\mathbb{R}$. The function ψ is called a *mother wavelet* or simply a *wavelet*.

In the eighties, Goupillard, Grossmann and Morlet [22] introduced the notion of wavelets, and applied them with much success to the analysis of seismic data. Major breakthroughs of the subject came with the work of Mallat [25] and Daubechies [10] on constructing suitable mother wavelets ψ to represent functions in $L^2(\mathbb{R})$ as orthogonal series expansions in terms of the collection $\{\psi_{2^{-k}\ell,\,2^{-k}} : k, \ell \in \mathbb{Z}\}$. In particular, Mallat introduced the general approach of multiresolution analysis in obtaining orthonormal wavelets and Daubechies constructed new compactly supported orthonormal wavelets with desirable properties. With all these developments, wavelets attracted the attention of many scientists and engineers. Since then, there has been active research in both theoretical and practical aspects of the subject. Examples of applications of wavelets include image and video compression, image restoration, signal analysis, transient detection, and missing data recovery. Many books on wavelets have also been written, such as [3, 5, 11, 26, 32].

On the theoretical front, apart from the orthonormal wavelet bases constructed by Daubechies, the framework of multiresolution analysis Mallat introduced could be readily adapted for more general constructions. This gave rise to the rapid development of Riesz wavelet bases, orthonormal and Riesz multiwavelet bases, and wavelet frames. Among others, pioneering papers on the topics of wavelet bases, multiwavelet bases and wavelet frames for $L^2(\mathbb{R})$ include $[1, 7, 9, 20, 21, 31]$.

Inspired by the impact of wavelet analysis on $L^2(\mathbb{R})$ and motivated by the fact that many signals in practice are periodic, a natural topic of investigation would be the development of a parallel theory for $L^2[0, 2\pi]$, the space of all 2π-periodic square-integrable complex-valued functions over $\mathbb{R}$. Here we focus on this line of research. Fixing notations, the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ of $L^2[0, 2\pi]$ are given by $\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx$, where $f, g \in L^2[0, 2\pi]$, and $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. The Fourier series of a function $f \in L^2[0, 2\pi]$ is written as $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\cdot}$, where $\widehat{f}(n) := \langle f, e^{in\cdot} \rangle$, $n \in \mathbb{Z}$, are its Fourier coefficients.

For $k \geq 0$, we set $\mathcal{R}_k := \{0, 1, \ldots, 2^k - 1\}$ and let $\mathcal{S}(2^k)$ be the space of all complex 2^k-periodic sequences. The finite Fourier transform of $c_k \in \mathcal{S}(2^k)$ is defined by

$$\widehat{c_k}(j) := \sum_{\ell\in\mathcal{R}_k} c_k(\ell)e^{-\frac{2\pi i j\ell}{2^k}}, \quad j \in \mathcal{R}_k,$$

which maps $\mathcal{S}(2^k)$ bijectively onto $\mathcal{S}(2^k)$. Now, consider the $\frac{2\pi}{2^k}$-shift operator $T_k : L^2[0, 2\pi] \longrightarrow L^2[0, 2\pi]$ defined by

$$T_k f := f(\cdot - \tfrac{2\pi}{2^k}), \quad f \in L^2[0, 2\pi].$$

For any $f \in L^2[0, 2\pi]$, due to its periodicity, $T_k^{2^k} f = f$ and so the collection of all shifts of f is only the finite set $\{T_k^\ell f : \ell \in \mathcal{R}_k\}$. This is a major difference between wavelet analysis on $L^2[0, 2\pi]$ and that on $L^2(\mathbb{R})$. Another difference is that the periodic case is nonstationary in nature, in the sense that for different levels k, different wavelet functions are utilized, instead of dilating a single mother wavelet.

For ϕ_0, ψ_k, $k \geq 0$, in $L^2[0, 2\pi]$, we say that the collection

$$\Psi := \{\phi_0\} \cup \{T_k^\ell \psi_k : k \geq 0,\ \ell \in \mathcal{R}_k\} \tag{1}$$

is a *Riesz wavelet basis* of $L^2[0, 2\pi]$ if the linear span of the functions in Ψ is dense in $L^2[0, 2\pi]$ and if there exist positive constants A and B such that

$$A\left(|c_0|^2 + \sum_{k=0}^{\infty}\sum_{\ell\in\mathcal{R}_k} |d_k(\ell)|^2\right) \leq \left\| c_0\phi_0 + \sum_{k=0}^{\infty}\sum_{\ell\in\mathcal{R}_k} d_k(\ell)T_k^\ell\psi_k \right\|^2 \leq B\left(|c_0|^2 + \sum_{k=0}^{\infty}\sum_{\ell\in\mathcal{R}_k} |d_k(\ell)|^2\right)$$

for all constants c_0, $d_k(\ell)$, $k \geq 0$, $\ell \in \mathcal{R}_k$, satisfying $|c_0|^2 + \sum_{k=0}^{\infty} \sum_{\ell \in \mathcal{R}_k} |d_k(\ell)|^2 < \infty$. The functions ψ_k, $k \geq 0$, are called *periodic mother wavelets* or simply *wavelets*. The constants A and B are known as *Riesz bounds* of the Riesz basis Ψ. If all the functions in such a Riesz basis Ψ have unit norm and are pairwise orthogonal, then Ψ is an *orthonormal wavelet basis* of $L^2[0, 2\pi]$.

The notions of Riesz and orthonormal wavelet bases of $L^2[0, 2\pi]$ can be extended to the more general setting of multiwavelets. More precisely, for ϕ_0^m, ψ_k^m, $k \geq 0$, $m = 1, 2, \dots, r$, in $L^2[0, 2\pi]$, we consider the collection

$$\Psi := \{\phi_0^m : m = 1, 2, \dots, r\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \dots, r,\ \ell \in \mathcal{R}_k\} \qquad (2)$$

which has multiple mother wavelets, sometimes called *multiwavelets*, for each $k \geq 0$, in contrast to the collection in (1). Similar to the definitions of Riesz and orthonormal wavelet bases, we define the notions of *Riesz multiwavelet basis* and *orthonormal multiwavelet basis* for the collection Ψ in (2).

If the collection Ψ in (1) or (2) forms a Riesz or orthonormal basis of $L^2[0, 2\pi]$, then it contains no redundant element in representing functions in $L^2[0, 2\pi]$. In other words, when any element in Ψ is removed, the remaining elements in Ψ are no longer sufficient to represent all the functions in $L^2[0, 2\pi]$. Sometimes in practical applications, it is useful to have redundancy in the spanning set of $L^2[0, 2\pi]$ that provides both additional information and sparse representations for appropriate functions. This redundancy is supplied by the notion of frames.

For ϕ_0^m, $m = 1, 2, \dots, r$, ψ_k^m, $k \geq 0$, $m = 1, 2, \dots, \rho_k$, $\rho_k \geq r$, in $L^2[0, 2\pi]$, we say that the collection

$$\Psi := \{\phi_0^m : m = 1, 2, \dots, r\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \dots, \rho_k,\ \ell \in \mathcal{R}_k\} \qquad (3)$$

is a *wavelet frame* for $L^2[0, 2\pi]$ if there exist positive constants A and B such that

$$A \|f\|^2 \leq \sum_{m=1}^{r} |\langle f, \phi_0^m \rangle|^2 + \sum_{k=0}^{\infty} \sum_{m=1}^{\rho_k} \sum_{\ell \in \mathcal{R}_k} |\langle f, T_k^\ell \psi_k^m \rangle|^2 \leq B \|f\|^2$$

for all $f \in L^2[0, 2\pi]$. The functions ψ_k^m, $k \geq 0$, $m = 1, 2, \dots, \rho_k$, are also referred to as wavelets (or multiwavelets). The constants A and B are called *bounds* of the frame. If it is possible to have $A = B = 1$, then Ψ is a *normalized tight wavelet frame* or simply *tight wavelet frame*. In contrast to the collection in (2), for each $k \geq 0$, the number of functions ψ_k^m in (3) is ρ_k, which need not be r. Therefore a wavelet frame can be viewed as a generalization of a Riesz multiwavelet basis. By standard frame theory (see for instance [3]), if a frame Ψ of the form (3) ceases to be a frame when any one of its elements is removed, then it must be a Riesz basis of $L^2[0, 2\pi]$.

This paper provides a survey of various results on constructing wavelets, multiwavelets and wavelet frames for periodic functions. While the focus is on the intrinsic nature of the periodic setup for $L^2[0, 2\pi]$, the presentation will also provide a snapshot of the evolution of ideas involved in the development of wavelet theory during the past twenty years. The paper will highlight fundamental results from the papers [15, 17–19, 23]. Although the theories in some of these papers were developed for a more general multidimensional setup on $L^2([0, 2\pi]^s)$, only the one-dimensional case will be considered here to better present the ideas. Additional papers related to these ideas are [4, 13, 14, 16, 24], and examples of other papers on periodic wavelets include [2, 8, 27–30].

The paper is organized as follows. Wavelet bases of the form (1) are often constructed from a nested sequence of shift-invariant subspaces known as a multiresolution analysis. Each of these subspaces in a multiresolution analysis is generated by a scaling function. A general frequency-based approach for constructing wavelet bases from multiresolution analyses was developed in [23], and this will be described in Section 2. Our starting point in constructing multiwavelet bases of the form (2) is again a multiresolution analysis, but the multiresolution analysis involves multiscaling functions which are reflected in the multiplicity r. A general approach for this, as developed in [15], will be presented in Section 3. As for wavelet frames of the form (3), they can be constructed by either considering orthogonal complements of subspaces in a multiresolution analysis or employing the unitary and oblique extension principles for periodic functions. The methods for the former from [17] and the latter from [18] will be presented in Sections 4 and 5 respectively.

2 Multiresolution analyses and wavelet bases

The approach for constructing wavelet bases of $L^2[0, 2\pi]$ in [23] begins with a multiresolution analysis. We say that a sequence of subspaces $\{V_k\}_{k\geq 0}$ of $L^2[0, 2\pi]$ is a *multiresolution analysis* (MRA) of $L^2[0, 2\pi]$ if it satisfies the following three conditions:

- MRA1 For every $k \geq 0$, there exists a function ϕ_k in V_k such that the collection $\{T_k^\ell \phi_k : \ell \in \mathcal{R}_k\}$ forms a basis of V_k.
- MRA2 For every $k \geq 0$, $V_k \subseteq V_{k+1}$.
- MRA3 $\overline{\bigcup_{k\geq 0} V_k} = L^2[0, 2\pi]$.

The functions ϕ_k, $k \geq 0$, are known as *scaling functions* or *refinable functions*, and they are said to generate the MRA $\{V_k\}_{k\geq 0}$.

The subspaces in a multiresolution analysis can be expressed in terms of the orthogonal splines introduced in [23]. For $k \geq 0$ and $\phi_k \in L^2[0, 2\pi]$, we define *orthogonal splines* $v_{k,j}$, $j \in \mathcal{R}_k$, associated with ϕ_k by

$$v_{k,j} := \sum_{p\in\mathbb{Z}} \widehat{\phi}_k(j + 2^k p) e^{i(j+2^k p)\cdot}, \quad j \in \mathcal{R}_k. \tag{4}$$

For different values of $j \in \mathcal{R}_k$, the functions $v_{k,j}$ involve different Fourier coefficients of ϕ_k, and hence they are pairwise orthogonal.

Now, since $\phi_k = \sum_{j\in\mathcal{R}_k} v_{k,j}$, it follows that $T_k^\ell \phi_k = \sum_{j\in\mathcal{R}_k} e^{-\frac{2\pi i j\ell}{2^k}} v_{k,j}$ for all $\ell \in \mathcal{R}_k$. This gives the matrix equation

$$\left(\phi_k, T_k\phi_k, \ldots, T_k^{2^k-1}\phi_k\right)^T = \left(e^{-\frac{2\pi i j\ell}{2^k}}\right)_{\ell,j=0}^{2^k-1} \left(v_{k,0}, v_{k,1} \ldots, v_{k,2^k-1}\right)^T, \qquad (5)$$

where $\frac{1}{\sqrt{2^k}}\left(e^{-\frac{2\pi i j\ell}{2^k}}\right)_{\ell,j=0}^{2^k-1}$ is the $2^k \times 2^k$ Fourier matrix which is unitary. In other words, (5) provides a change of basis of V_k from the collection $\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}$ to $\{v_{k,j} : j \in \mathcal{R}_k\}$, that is, V_k can be written as

$$V_k = \langle\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}\rangle = \langle\{v_{k,j} : j \in \mathcal{R}_k\}\rangle. \qquad (6)$$

Consequently, the orthogonal splines can be used to characterize various basis properties of V_k, where Riesz and orthonormal bases of the finite-dimensional subspace V_k are defined similarly as those for $L^2[0, 2\pi]$.

Theorem 1. **([19, 23])** *For $k \geq 0$ and $\phi_k \in L^2[0, 2\pi]$, let V_k be as defined in* (6) *and $v_{k,j}$, $j \in \mathcal{R}_k$, as defined in* (4)*.*

(a) *The collection $\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}$ is a basis of V_k if and only if*

$$v_{k,j} \neq 0, \quad j \in \mathcal{R}_k.$$

(b) *The collection $\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}$ is a Riesz basis of V_k with Riesz bounds A and B if and only if*

$$\frac{A}{2^k} \leq \|v_{k,j}\|^2 \leq \frac{B}{2^k}, \quad j \in \mathcal{R}_k. \qquad (7)$$

(c) *The collection $\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}$ is an orthonormal basis of V_k if and only if*

$$\|v_{k,j}\|^2 = \frac{1}{2^k}, \quad j \in \mathcal{R}_k.$$

The other two conditions of a multiresolution analysis, MRA2 and MRA3, are characterized in the following results from [23]. As observed in [17], these two results only require $\{T_k^\ell\phi_k : \ell \in \mathcal{R}_k\}$ to be a spanning set (and not a basis) of V_k in (6). This will enable them to remain applicable for the more general setting of frames in Section 4.

Proposition 1. **([23])** *For $k \geq 0$ and $\phi_k \in L^2[0, 2\pi]$, let V_k be as defined in* (6) *and $v_{k,j}$, $j \in \mathcal{R}_k$, as defined in* (4)*. Then the following are equivalent for every $k \geq 0$.*

(i) $V_k \subseteq V_{k+1}$.

(ii) *There exists* $h_{k+1} \in \mathcal{S}(2^{k+1})$ *such that*

$$\phi_k = \sum_{\ell \in \mathcal{R}_{k+1}} h_{k+1}(\ell) T_{k+1}^{\ell} \phi_{k+1}. \tag{8}$$

(iii) *There exists* $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ *such that*

$$\widehat{\phi}_k(n) = \widehat{h}_{k+1}(n) \widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}. \tag{9}$$

(iv) *There exists* $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ *such that*

$$v_{k,j} = \widehat{h}_{k+1}(j) v_{k+1,j} + \widehat{h}_{k+1}(j + 2^k) v_{k+1,j+2^k}, \quad j \in \mathcal{R}_k. \tag{10}$$

The equation (8) is known as the *periodic refinement equation* and the periodic sequence h_{k+1} *refinement mask*. The sequence $\widehat{h}_{k+1}$ in (9) and (10) is the finite Fourier transform of h_{k+1}.

The next theorem characterizes the third condition of an MRA and it provides an easy-to-check criterion for this condition.

Theorem 2. **([23])** *Suppose that* $\{V_k\}_{k \geq 0}$ *is a sequence of subspaces of* $L^2[0, 2\pi]$ *of the form* (6) *for which* $V_k \subseteq V_{k+1}$ *for every* $k \geq 0$. *Then* $\overline{\bigcup_{k \geq 0} V_k} = L^2[0, 2\pi]$ *if and only if the set* $\{n \in \mathbb{Z} : \widehat{\phi}_k(n) = 0 \text{ for all } k \geq 0\}$ *is empty.*

Starting from an MRA $\{V_k\}_{k \geq 0}$, the approach to construct wavelet bases of $L^2[0, 2\pi]$ is as follows. For each $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}, that is, $V_{k+1} = V_k \oplus^{\perp} W_k$, where the notation $\oplus^{\perp}$ denotes orthogonal direct sum. Then the space $L^2[0, 2\pi]$ can be decomposed into the orthogonal direct sum

$$L^2[0, 2\pi] = V_0 \oplus^{\perp} W_0 \oplus^{\perp} W_1 \oplus^{\perp} \cdots.$$

To obtain a Riesz (or orthonormal) wavelet basis of $L^2[0, 2\pi]$, the strategy is to find, for every $k \geq 0$, a wavelet ψ_k in V_{k+1} such that the collection $\{T_k^{\ell} \psi_k : \ell \in \mathcal{R}_k\}$ forms a Riesz (or orthonormal) basis of the subspace W_k. For such a function ψ_k, we define orthogonal splines associated with ψ_k by

$$u_{k,j} := \sum_{p \in \mathbb{Z}} \widehat{\psi}_k(j + 2^k p) e^{i(j+2^k p)\cdot}, \quad j \in \mathcal{R}_k. \tag{11}$$

Then Theorem 1 is employed to obtain equivalent conditions for Riesz and orthonormal bases of the subspace W_k which can be written as

$$W_k = \langle \{T_k^{\ell} \psi_k : \ell \in \mathcal{R}_k\} \rangle = \langle \{u_{k,j} : j \in \mathcal{R}_k\} \rangle.$$

In addition, analogous to Proposition 1, for $k \geq 0$, $W_k \subseteq V_{k+1}$ if and only if there exists $g_{k+1} \in \mathcal{S}(2^{k+1})$ such that

$$\psi_k = \sum_{\ell \in \mathcal{R}_{k+1}} g_{k+1}(\ell) T_{k+1}^{\ell} \phi_{k+1}. \tag{12}$$

The equation (12) is known as the *periodic wavelet equation* and the periodic sequence g_{k+1} *wavelet mask*. Furthermore, (12) is the same as having $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})$ such that

$$\widehat{\psi}_k(n) = \widehat{g}_{k+1}(n) \widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}, \tag{13}$$

or equivalently,

$$u_{k,j} = \widehat{g}_{k+1}(j) v_{k+1,j} + \widehat{g}_{k+1}(j + 2^k) v_{k+1,j+2^k}, \quad j \in \mathcal{R}_k. \tag{14}$$

Consequently, the wavelet construction problem reduces to finding an appropriate wavelet mask g_{k+1} from a given refinement mask h_{k+1}. This always has an explicit solution by employing a frequency-based approach to construct $\widehat{g}_{k+1}$ from $\widehat{h}_{k+1}$.

Theorem 3. ([19, 23]) *Suppose that $\{V_k\}_{k \geq 0}$ is an MRA of $L^2[0, 2\pi]$ generated by scaling functions ϕ_k, $k \geq 0$, whose corresponding orthogonal splines $v_{k,j}$, $j \in \mathcal{R}_k$, as defined in (4), satisfy (7) with common Riesz bounds A and B. For $k \geq 0$, define $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})$ by*

$$\widehat{g}_{k+1}(j) := 2^{k+1} \|v_{k+1,j+2^k}\|^2 e^{-\frac{2\pi i j}{2^{k+1}}} \overline{\widehat{h}_{k+1}(j + 2^k)}, \quad j \in \mathcal{R}_{k+1}. \tag{15}$$

Then for the wavelet functions ψ_k, $k \geq 0$, given by (13), the collection Ψ in (1) forms a Riesz basis of $L^2[0, 2\pi]$.

By the derivations in [23], if $\{T_k^{\ell} \phi_k : \ell \in \mathcal{R}_k\}$ is orthonormal for every $k \geq 0$, then Ψ in (1) obtained from Theorem 3 is an orthonormal basis of $L^2[0, 2\pi]$.

Example 1. Take a fixed positive integer ν. For $k \geq 0$, let $\phi_k \in L^2[0, 2\pi]$ be the periodic B-spline of order ν defined by

$$\widehat{\phi}_k(n) := \begin{cases} \frac{1}{\sqrt{2^k}} \left(\frac{\sin \frac{\pi n}{2^k}}{\frac{\pi n}{2^k}} \right)^{\nu} e^{-\frac{\nu \pi i n}{2^k}}, & \text{if } n \in \mathbb{Z} \backslash \{0\}, \\ \frac{1}{\sqrt{2^k}}, & \text{if } n = 0. \end{cases}$$

Then ϕ_k, $k \geq 0$, satisfy (9) with $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$, $k \geq 0$, given by

$$\widehat{h}_{k+1}(j) = \sqrt{2} \left(\cos \frac{\pi j}{2^{k+1}} \right)^{\nu} e^{-\frac{\nu \pi i j}{2^{k+1}}}, \quad j \in \mathcal{R}_{k+1}.$$

As shown in [19], it follows from Proposition 1 and Theorems 1–3 that ϕ_k, $k \geq 0$, generate an MRA $\{V_k\}_{k \geq 0}$ of $L^2[0, 2\pi]$, and the resulting wavelets ψ_k,

$k \geq 0$, from (15) and (13), produce a Riesz wavelet basis Ψ, as defined in (1), of $L^2[0, 2\pi]$. In this case, the functions $v_{k,j}$, $j \in \mathcal{R}_k$, in (4) are closely related to the orthogonal periodic polynomial splines of order ν studied in [24], which motivated the notion of orthogonal splines in [23].

Example 2. For $k \geq 0$, define $\phi_k \in L^2[0, 2\pi]$ by

$$\widehat{\phi}_k(n) := \begin{cases} \frac{1}{\sqrt{2^k}}, & \text{if } n = -2^{k-1}+1, \dots, 2^{k-1}-1, \\ \frac{1}{\sqrt{2^{k+1}}}, & \text{if } n = \pm 2^{k-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the trigonometric polynomials ϕ_k, $k \geq 0$, satisfy Proposition 1 and Theorems 1 and 2, and therefore generate an MRA $\{V_k\}_{k \geq 0}$ of $L^2[0, 2\pi]$, where $\{T_k^\ell \phi_k : \ell \in \mathcal{R}_k\}$ is orthonormal for every $k \geq 0$. For $k \geq 0$, the sequence $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ in (9) is given by

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2}, & \text{if } j = -2^{k-1}+1, \dots, 2^{k-1}-1, \\ 1, & \text{if } j = \pm 2^{k-1}, \\ 0, & \text{if } j \in \{-2^k, \dots, 2^k - 1\} \backslash \{-2^{k-1}, \dots, 2^{k-1}\}. \end{cases}$$

The corresponding $\widehat{g}_{k+1}$ in (15) gives the trigonometric polynomial wavelet ψ_k from (13). The collection Ψ as defined in (1) forms an orthonormal basis of $L^2[0, 2\pi]$.

3 Multiscaling functions and multiwavelets

To provide greater flexibility in constructing wavelets with desired properties, multiwavelets for $L^2[0, 2\pi]$ were studied in [15]. The construction of multiwavelets involves multiresolution analyses of multiplicity r. A sequence of subspaces $\{V_k\}_{k \geq 0}$ of $L^2[0, 2\pi]$ is a *multiresolution analysis of multiplicity* r of $L^2[0, 2\pi]$ if it satisfies the condition

MRA1 For every $k \geq 0$, there exist functions $\phi_k^1, \dots, \phi_k^r$ in V_k such that the collection $\{T_k^\ell \phi_k^m : m = 1, 2, \dots, r,\ \ell \in \mathcal{R}_k\}$ forms a basis of V_k;

together with the conditions MRA2 and MRA3 in Section 2. In contrast to the definition of an MRA in Section 2, here for each $k \geq 0$, there are altogether r scaling functions, known as *multiscaling functions*, which can be written as a vector-valued function

$$\phi_k := (\phi_k^1, \dots, \phi_k^r)^T. \tag{16}$$

This is why the setup for multiscaling functions and multiwavelets is often referred to as the vector case, while the original formulation corresponding to $r = 1$ is called the scalar case.

Analogous to the orthogonal splines described in Section 2, the notion of polyphase splines was introduced in [15] for the vector case. For $k \geq 0$ and $\phi_k^1, \dots, \phi_k^r \in L^2[0, 2\pi]$, the *polyphase splines* $v_{k,j}^1, \dots, v_{k,j}^r$, $j \in \mathcal{R}_k$, associated with $\phi_k^1, \dots, \phi_k^r$ are defined by

$$v_{k,j} := \sum_{p \in \mathbb{Z}} \widehat{\phi}_k(j + 2^k p) e^{i(j+2^k p)\cdot}, \quad j \in \mathcal{R}_k, \tag{17}$$

where

$$\widehat{\phi}_k := (\widehat{\phi}_k^1, \dots, \widehat{\phi}_k^r)^T, \qquad v_{k,j} := (v_{k,j}^1, \dots, v_{k,j}^r)^T, \quad j \in \mathcal{R}_k. \tag{18}$$

For a given $j \in \mathcal{R}_k$, we have r polyphase splines $v_{k,j}^1, \dots, v_{k,j}^r$, instead of a single orthogonal spline in the scalar case. This makes the analysis for the vector case more complicated. Fortunately, the problem is still tractable by considering the Gram matrix of the functions $v_{k,j}^1, \dots, v_{k,j}^r$ given by

$$M_k(j) := \left(\langle v_{k,j}^m, v_{k,j}^\mu \rangle \right)_{m,\mu=1}^r, \quad j \in \mathcal{R}_k. \tag{19}$$

Theorem 4. **([15])** *For $k \geq 0$ and $\phi_k^1, \dots, \phi_k^r \in L^2[0, 2\pi]$, consider*

$$V_k := \langle \{ T_k^\ell \phi_k^m : m = 1, 2, \dots, r, \ \ell \in \mathcal{R}_k \} \rangle, \tag{20}$$

and let $v_{k,j}^m$, $m = 1, 2, \dots, r$ and $j \in \mathcal{R}_k$, be as defined in (17).

(a) *The collection $\{T_k^\ell \phi_k^m : m = 1, 2, \dots, r, \ \ell \in \mathcal{R}_k\}$ is a basis of V_k if and only if*

$$\det(M_k(j)) > 0, \quad j \in \mathcal{R}_k.$$

(b) *The collection $\{T_k^\ell \phi_k^m : m = 1, 2, \dots, r, \ \ell \in \mathcal{R}_k\}$ is an orthonormal basis of V_k if and only if*

$$M_k(j) = \frac{1}{2^k} I, \quad j \in \mathcal{R}_k.$$

As shown in [15], the conditions MRA2 and MRA3 for the vector case can be characterized in similar ways as the scalar case. Indeed, for subspaces V_k, $k \geq 0$, of the form (20), Proposition 1 and Theorem 2 also hold for the vector-valued functions ϕ_k, $\widehat{\phi}_k$ and $v_{k,j}$ as defined in (16) and (18). Here, $h_{k+1}, \widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r \times r}$, the space of all 2^k-periodic sequences of $r \times r$ matrices, and $\widehat{h}_{k+1}$ is the matrix whose entries are the finite Fourier transform of the entries of h_{k+1}.

With an MRA $\{V_k\}_{k \geq 0}$ of multiplicity r as the starting point, the strategy of constructing multiwavelet bases of $L^2[0, 2\pi]$ is analogous to that for the scalar case. For $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}. Then $\dim W_k = \dim V_{k+1} - \dim V_k = r2^{k+1} - r2^k = r2^k$, and so we need to find r wavelets $\psi_k^1, \dots, \psi_k^r \in W_k \subseteq V_{k+1}$ whose $\frac{2\pi}{2^k}$-shifts form a basis of W_k. Once again, we employ polyphase splines. As in (17), we define polyphase splines $u^1_{k,j}, \ldots, u^r_{k,j}$, $j \in \mathcal{R}_k$, associated with $\psi^1_k, \ldots, \psi^r_k$ by (11), where

$$\widehat{\psi}_k := (\widehat{\psi}^1_k, \ldots, \widehat{\psi}^r_k)^T, \qquad u_{k,j} := (u^1_{k,j}, \ldots, u^r_{k,j})^T, \quad j \in \mathcal{R}_k. \tag{21}$$

Then for the subspaces

$$W_k = \langle \{T^\ell_k \psi^m_k : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\} \rangle \tag{22}$$

and V_k in (20) where $k \geq 0$, (12)–(14) hold for $\psi_k := (\psi^1_k, \ldots, \psi^r_k)^T$, $\widehat{\psi}_k$ and $u_{k,j}$ in (21), and some $g_{k+1}, \widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$. As a result, the problem of constructing multiwavelets can be formulated into obtaining an appropriate matrix $\widehat{g}_{k+1}$ from a given matrix $\widehat{h}_{k+1}$.

Theorem 5. ([15]) *Let $\{V_k\}_{k\geq 0}$ be an MRA of multiplicity r of $L^2[0, 2\pi]$ generated by multiscaling functions $\phi^1_k, \ldots, \phi^r_k \in L^2[0, 2\pi]$, $k \geq 0$, satisfying* (9) *for $\widehat{\phi}_k$ as defined in* (18) *and some $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$, with Gram matrices $M_k(j)$, $j \in \mathcal{R}_k$, of the corresponding polyphase splines given by* (19)*. Then for every $k \geq 0$, there exists $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$ such that*

$$\det\left(\widehat{g}_{k+1}(j) M_{k+1}(j) \widehat{g}_{k+1}(j)^* + \widehat{g}_{k+1}(j + 2^k) M_{k+1}(j + 2^k) \widehat{g}_{k+1}(j + 2^k)^*\right) > 0 \tag{23}$$

and

$$\widehat{g}_{k+1}(j) M_{k+1}(j) \widehat{h}_{k+1}(j)^* + \widehat{g}_{k+1}(j + 2^k) M_{k+1}(j + 2^k) \widehat{h}_{k+1}(j + 2^k)^* = 0 \tag{24}$$

for all $j \in \mathcal{R}_k$. If, in addition,

$$\begin{aligned} \frac{A}{2^k} I &\leq \widehat{g}_{k+1}(j) M_{k+1}(j) \widehat{g}_{k+1}(j)^* + \widehat{g}_{k+1}(j + 2^k) M_{k+1}(j + 2^k) \widehat{g}_{k+1}(j + 2^k)^* \\ &\leq \frac{B}{2^k} I, \quad k \geq 0,\ j \in \mathcal{R}_k, \end{aligned} \tag{25}$$

for some positive constants A and B, then the collection Ψ in (2)*, where the functions $\psi^1_k, \ldots, \psi^r_k$, $k \geq 0$, are defined via* (21) *and* (13)*, forms a Riesz basis of $L^2[0, 2\pi]$.*

The proof of Theorem 5 given in [15] is constructive. For every $k \geq 0$, starting from a given $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$, an algorithmic approach based on matrix extension enables $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$ satisfying (23) and (24) to be constructed. Having (23) holds for all $j \in \mathcal{R}_k$ is equivalent to $\{T^\ell_k \psi^m_k : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ being linearly independent, and equipping with (24) for all $j \in \mathcal{R}_k$ amounts to $\{T_k^\ell \psi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ lying inside W_k, the orthogonal complement of V_k in V_{k+1}. As $\widehat{g}_{k+1}(j)$, $M_{k+1}(j)$ and I are $r \times r$ complex matrices, the inequalities in (25) mean that for every nonzero column vector $z \in \mathbb{C}^r$,

$$\frac{A}{2^k} z^* z \le z^* \widehat{g}_{k+1}(j) M_{k+1}(j) \widehat{g}_{k+1}(j)^* z + z^* \widehat{g}_{k+1}(j + 2^k) M_{k+1}(j + 2^k) \widehat{g}_{k+1}(j + 2^k)^* z \le \frac{B}{2^k} z^* z.$$

If the multiscaling functions $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ are orthonormal, then by employing unitary matrices in the matrix extension procedure, one can derive $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{r \times r}$, $k \ge 0$, for which the resulting Ψ in (2) forms an orthonormal multiwavelet basis of $L^2[0, 2\pi]$. An algorithm for this was also derived in [15].

4 Shift-invariant subspaces and wavelet frames

In the definition of an MRA $\{V_k\}_{k \ge 0}$ with multiplicity r, the condition MRA1 assumes that the collection $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ forms a basis of V_k. Now we relax this condition by no longer requiring $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ to be linearly independent. Here, we assume the weaker condition of

MRA1 For every $k \ge 0$, there exist functions $\phi_k^1, \ldots, \phi_k^r$ in V_k such that the collection $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ spans V_k;

while keeping the conditions MRA2 and MRA3. Henceforward we shall use this definition of an MRA with multiplicity r and describe how wavelet frames for $L^2[0, 2\pi]$ were readily constructed in [17].

The above weaker condition of MRA1 entails that for every $k \ge 0$, V_k is a *finitely generated shift-invariant subspace* of the form (20), that is, for any f in V_k, its $\frac{2\pi}{2^k}$-shift $T_k f$ also lies in V_k. Being a finite-dimensional subspace of $L^2[0, 2\pi]$, standard frame theory shows that the spanning set of V_k forms a frame for V_k. For the construction of wavelet frames, it is important to identify the bounds of this frame. In this connection, the eigenvalues of the Gram matrices of the polyphase splines play a crucial, and also natural, role.

Theorem 6. **([17])** *For $k \ge 0$ and $\phi_k^1, \ldots, \phi_k^r \in L^2[0, 2\pi]$, let V_k be as defined in* (20)*, and let $M_k(j)$, $j \in \mathcal{R}_k$, be the Gram matrices of the corresponding polyphase splines given by* (19)*. Then $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$ is a frame for V_k with bounds A and B if and only if for each $j \in \mathcal{R}_k$,*

$$\frac{A}{2^k} \le \lambda_j \le \frac{B}{2^k} \tag{26}$$

for every nonzero eigenvalue λ_j of $M_k(j)$.

For the scalar case of $r = 1$, Theorem 6 shows that $\{T_k^\ell \phi_k : \ell \in \mathcal{R}_k\}$ is a frame for V_k in (6) with bounds A and B if and only if for every $j \in \mathcal{R}_k$ with $v_{k,j} \neq 0$,

$$\frac{A}{2^k} \leq \|v_{k,j}\|^2 \leq \frac{B}{2^k}.$$

This complements the series of results in Theorem 1. Here, $\{T_k^\ell \phi_k : \ell \in \mathcal{R}_k\}$ need not be linearly independent, and the lack of a basis is reflected in some of the orthogonal splines $v_{k,j}$ being the zero function.

Returning to the vector case, as the nonzero eigenvalues of the Gram matrices of polyphase splines provide explicit inequalities involving the bounds of the frame $\{T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{R}_k\}$, it suggests the possibility of using this information to find an alternative frame for V_k that is a normalized tight frame. Based on Theorem 6, if the new frame is a normalized tight frame, then the corresponding inequalities in (26) would collapse into an equation with $A = B = 1$. This was realized in a constructive proof in [17], where the new frame also possesses certain desirable orthogonality.

Theorem 7. **([17])** *For a shift-invariant subspace V_k of the form* (20)*, there exist functions $\xi_k^1, \ldots, \xi_k^r$ in V_k such that $\{T_k^\ell \xi_k^m : m = 1, 2, \ldots, r, \ell \in \mathcal{R}_k\}$ forms a normalized tight frame for V_k, with $\langle T_k^\ell \xi_k^m, T_k^\zeta \xi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, r$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{R}_k$. In particular, for every $j \in \mathcal{R}_k$, the Gram matrix $N_k(j) := (\langle w_{k,j}^m, w_{k,j}^\mu \rangle)_{m,\mu=1}^r$, where $w_{k,j}^1, \ldots, w_{k,j}^r$ are the resulting polyphase splines defined analogously to* (17)*, is a diagonal matrix with diagonal entries* 0 *or* $1/2^k$.

Equipped with Theorem 7, we are ready for our construction of normalized tight wavelet frames for $L^2[0, 2\pi]$. We begin with an MRA $\{V_k\}_{k\geq 0}$ of multiplicity r under the weaker MRA1 condition stated above. As in Section 3, for $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}, and the wavelet construction problem is equivalent to finding an appropriate $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{2r\times r}$ from a given $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$. We highlight that $\widehat{g}_{k+1}$ is a $2r \times r$ matrix here, in contrast to being a $r \times r$ matrix in Section 3. This translates to having possibly $2r$ wavelet functions $\psi_k^1, \ldots, \psi_k^{2r}$. The need for additional wavelet functions arises as the spanning set of V_k need not be linearly independent.

Theorem 8. **([17])** *Based on an MRA $\{V_k\}_{k\geq 0}$ of multiplicity r of $L^2[0, 2\pi]$, for every $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}. Then there exist $\psi_k^1, \ldots, \psi_k^{2r} \in L^2[0, 2\pi]$ such that $\{T_k^\ell \psi_k^m : m = 1, 2, \ldots, 2r, \ell \in \mathcal{R}_k\}$ forms a normalized tight frame for W_k, with $\langle T_k^\ell \psi_k^m, T_k^\zeta \psi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, 2r$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{R}_k$. In addition, the collection Ψ, as defined in* (3) *with $\rho_k = 2r$, is a normalized tight wavelet frame for $L^2[0, 2\pi]$.*

The proof of Theorem 8 is again constructive. The main steps are as follows. For every $k \geq 0$, Theorem 7 is first employed to convert the generators of

V_k into a normalized tight frame for V_k with the desired orthogonality. Then matrix extension techniques are applied to obtain a $\widehat{g}_{k+1}$ satisfying the required properties. Some of the resulting $2r$ wavelet functions could be the zero function. The minimum number of nontrivial wavelet functions are given by

$$\eta_k := \max_{j\in\mathcal{R}_k} \left\{\operatorname{rank}\left(M_{k+1}(j)\right) + \operatorname{rank}\left(M_{k+1}(j+2^k)\right) - \operatorname{rank}\left(M_k(j)\right)\right\}.$$

The Gram matrices $M_k(j)$ of polyphase splines again supply information to better understand the analysis. For details, see [17].

5 Extension principles for tight wavelet frames

The unitary extension principle obtained in [31] provides an elegant and flexible method for constructing tight wavelet frames for the space $L^2(\mathbb{R})$. It led to the oblique extension principle in [12] and [6] independently, which gives even greater flexibility in obtaining tight wavelet frames for $L^2(\mathbb{R})$. In [18], the unitary and oblique extension principles were developed for the space $L^2[0, 2\pi]$. The following theorem is the *unitary extension principle* for $L^2[0, 2\pi]$.

Theorem 9. ([18]) *Suppose that* $\phi_k^1, \ldots, \phi_k^r \in L^2[0, 2\pi]$, $k \geq 0$, *satisfy* (9) *for* $\widehat{\phi}_k$ *as defined in* (18) *and some* $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$, *and*

$$\lim_{k\to\infty} 2^k \sum_{m=1}^{r} |\widehat{\phi}_k^m(n)|^2 = 1, \quad n \in \mathbb{Z}. \tag{27}$$

For every $k \geq 0$ *and some positive integer* $\rho_k \geq r$, *let* $\psi_k^1, \ldots, \psi_k^{\rho_k} \in L^2[0, 2\pi]$ *be defined by* (13), *where* $\widehat{\psi}_k := (\widehat{\psi}_k^1, \ldots, \widehat{\psi}_k^{\rho_k})^T$ *and* $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{\rho_k\times r}$. *Suppose that for each* $j \in \mathcal{R}_k$, *the* $(r + \rho_k) \times 2r$ *matrix*

$$P_k(j) := \begin{pmatrix} \widehat{h}_{k+1}(j) & \widehat{h}_{k+1}(j+2^k) \\ \widehat{g}_{k+1}(j) & \widehat{g}_{k+1}(j+2^k) \end{pmatrix} \tag{28}$$

satisfies

$$P_k(j)^* P_k(j) = 2I. \tag{29}$$

Then the collection Ψ *as defined in* (3) *forms a normalized tight wavelet frame for* $L^2[0, 2\pi]$.

The condition (29) means that the columns of each of the matrices $\frac{1}{\sqrt{2}}P_k(j)$, $j \in \mathcal{R}_k$, are orthonormal. This condition is in general easy to check. Based on Theorem 9, for given matrices $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})^{r\times r}$, the problem of constructing normalized tight wavelet frames amounts to finding matrices $\widehat{g}_{k+1} \in \mathcal{S}(2^{k+1})^{\rho_k\times r}$

that satisfy (29) for all $j \in \mathcal{R}_k$. The number ρ_k of wavelet functions constructed can be arbitrarily chosen, which presents more flexibility in designing the wavelet frame.

A natural question one would ask is how this setup compares with that in Section 4. Defining V_k, $k \geq 0$, by (20), the sequence $\{V_k\}_{k\geq 0}$ forms an MRA of multiplicity r of $L^2[0, 2\pi]$. If W_k, $k \geq 0$, is given by (22) with ρ_k in place of r, then the construction in Theorem 9 leads to $V_{k+1} = V_k + W_k$, $k \geq 0$. In other words, we no longer fix V_{k+1} as the orthogonal direct sum of V_k and W_k, and therefore gain greater flexibility. This sum need not even be a direct sum and $\dim(V_k \cap W_k)$ could be nonzero.

For the scalar case of $r = 1$, let $\widehat{g}_{k+1} := (\widehat{g}^1_{k+1}, \ldots, \widehat{g}^{\rho_k}_{k+1})^T \in \mathcal{S}(2^{k+1})^{\rho_k \times 1}$. Then the condition of satisfying (29) for every $j \in \mathcal{R}_k$ amounts to

$$|\widehat{h}_{k+1}(j)|^2 + \sum_{m=1}^{\rho_k} |\widehat{g}^m_{k+1}(j)|^2 = 2, \quad j \in \mathcal{R}_{k+1}, \tag{30}$$

and

$$\widehat{h}_{k+1}(j)\overline{\widehat{h}_{k+1}(j + 2^k)} + \sum_{m=1}^{\rho_k} \widehat{g}^m_{k+1}(j)\overline{\widehat{g}^m_{k+1}(j + 2^k)} = 0, \quad j \in \mathcal{R}_k. \tag{31}$$

By judicious choices of $\widehat{g}^1_{k+1}, \ldots, \widehat{g}^{\rho_k}_{k+1}$, one can see that (30) and (31) are not difficult to be satisfied.

Example 3. Let $\{L_k\}_{k\geq 0}$ be a strictly increasing sequence of nonnegative integers satisfying $L_k \leq 2^{k-1}$. For $k \geq 0$, define $\phi_k \in L^2[0, 2\pi]$ by

$$\widehat{\phi}_k(n) := \begin{cases} \frac{1}{\sqrt{2^k}}, & \text{if } n = -L_k, \ldots, L_k, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing to the functions in Example 2, the trigonometric polynomials ϕ_k, $k \geq 0$, could have fewer nonzero Fourier coefficients, and hence narrower supports in the frequency domain. Observe that (27) clearly holds and so Theorem 9 applies. For $k \geq 0$, the sequence $\widehat{h}_{k+1}$ in (9) need not be unique and one possibility is given by

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2}, & \text{if } j = -L_k, \ldots, L_k, \\ 0, & \text{if } j \in \{-2^k, \ldots, 2^k - 1\} \backslash \{-L_k, \ldots, L_k\}. \end{cases}$$

By constructing $\widehat{g}_{k+1} = (\widehat{g}^1_{k+1}, \ldots, \widehat{g}^{\rho_k}_{k+1})^T \in \mathcal{S}(2^{k+1})^{\rho_k \times 1}$ that satisfies (30) and (31), we obtain narrow-band trigonometric polynomial wavelets $\psi^1_k, \ldots, \psi^{\rho_k}_k$. The collection $\{\phi_0\} \cup \{T^\ell_k \psi^m_k : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ then forms a normalized tight wavelet frame for $L^2[0, 2\pi]$.

In addition to Theorem 9, the *oblique extension principle* for $L^2[0, 2\pi]$ in [18] provides a more general sufficient condition for the collection Ψ in (3) to be a normalized tight frame for $L^2[0, 2\pi]$. Indeed, Theorem 9 still holds if we replace $P_k(j)$ in (28) by

$$P_k(j) := \begin{pmatrix} \widehat{\Theta}_k(j)\widehat{h}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1} & \widehat{\Theta}_k(j)\widehat{h}_{k+1}(j+2^k)\widehat{\Theta}_{k+1}(j+2^k)^{-1} \\ \widehat{g}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1} & \widehat{g}_{k+1}(j+2^k)\widehat{\Theta}_{k+1}(j+2^k)^{-1} \end{pmatrix},$$

where $\widehat{\Theta}_k \in \mathcal{S}(2^k)^{r\times r}$, $k \geq 0$, satisfy the conditions: $\widehat{\Theta}_k(j)$ is invertible for every $k \geq 0$ and $j \in \mathcal{R}_k$; $\lim_{k\to\infty} \widehat{\Theta}_k(n)^*\widehat{\Theta}_k(n) = I$ for all $n \in \mathbb{Z}$; and $\widehat{\Theta}_0(0) = I$.

It is interesting to remark that the oblique extension principle was obtained in [18] as a consequence of Theorem 9. On the other hand, if the matrices $\widehat{\Theta}_k$, $k \geq 0$, as above are taken to be the $r \times r$ identity matrix, then the result reduces to exactly Theorem 9. While the two results are intricately related, the oblique extension principle is a useful reformulation of Theorem 9 as it provides greater flexibility in constructing tight wavelet frames with desired properties.

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