SYMMETRIC AND ANTISYMMETRIC TIGHT WAVELET FRAMES

SAY SONG GOH   ZHI YUAN LIM   ZUOWEI SHEN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
10 KENT RIDGE CRESCENT, SINGAPORE 119260
REPUBLIC OF SINGAPORE

Abstract. For a given set of wavelets Ψ, we provide a general, and yet simple, method to derive a new set of wavelets Ψ' such that each wavelet in Ψ' is either symmetric or antisymmetric. The affine system generated by Ψ' is a tight frame for the space $L_2(\mathbb{R}^d)$ whenever the affine system generated by Ψ is so. Further, when Ψ is constructed via a multiresolution analysis, Ψ' can also be derived from a, but possibly different, multiresolution analysis. If moreover the multiresolution analysis for constructing Ψ is generated by a symmetric refinable function, then Ψ' is obtained from the same multiresolution analysis.

1. Introduction

Symmetric or antisymmetric compactly supported wavelets are very much desirable in various applications, since they preserve linear phase properties and also allow symmetric boundary conditions in wavelet algorithms which normally perform better. However, there does not exist any real-valued symmetric or antisymmetric compactly supported orthonormal wavelet with dyadic dilation except for the Haar wavelet. Many subsequent constructions sought to remedy this by relaxing some restrictions. Indeed, in [8], symmetry was obtained at the cost of dropping orthogonality: two compactly supported dual refinable functions were needed, only one of which could be a spline function. In [7], similar non-orthogonal dual symmetric spline wavelet bases were given, but only one of them could be compactly supported. As for examples of [14], symmetry, orthonormality and compact support were combined at the price of having multiwavelets from a vector multiresolution analysis. Subsequently, it was shown in [13] that this could be done with a spline vector multiresolution analysis. In examples of [21], symmetry, orthonormality, interpolatory property and compact support were achieved at the cost of using non-dyadic dilations.

In [26], symmetry was obtained by relaxing the non-redundancy condition, where a set of compactly supported spline tight wavelet frames was constructed from an arbitrary B-spline via the unitary extension principle. However, one of the wavelets only has a vanishing moment of order one. In [12] and [5], examples of symmetric compactly supported tight wavelet frames

2000 *Mathematics Subject Classification.* 42C40, 42C15, 42C30.

Key words and phrases. Wavelet frame, multiresolution analysis, symmetry.

with high orders of vanishing moments were obtained via the oblique extension principle. The compactly supported tight wavelet frames with high orders of vanishing moments obtained from systematic constructions based on an arbitrary B-spline in [12] were not symmetric. This was remedied through the approach of bi-frames at the cost of using two dual frame systems in [5], [11], [12] and [16]. More recently, in [18], three compactly supported symmetric or antisymmetric tight frame wavelets were constructed from an arbitrary B-spline using the oblique extension principle such that the order of vanishing moments is the same as the order of the B-spline. This construction was extended in [19] to constructions from a compactly supported symmetric refinable function with stable shifts.

We also note that [17] and [24] are other papers on constructions of compactly supported symmetric tight wavelet frames using the unitary or oblique extension principle. Both papers focused on finding and using sufficient conditions that the refinement and wavelet masks should satisfy for the construction of compactly supported symmetric tight wavelet frames. This direction led to several very interesting examples and a deep understanding of the structure of the masks of symmetric and antisymmetric wavelets. But it also revealed the difficulties of obtaining a systematic construction of compactly supported symmetric and antisymmetric tight wavelet frames in general. The approach in this paper is entirely different and overcomes the above difficulties.

Let $L_2(\mathbb{R}^d)$ be the space of all complex-valued square-integrable functions on the d-dimensional Euclidean space $\mathbb{R}^d$. We say that a countable system X in $L_2(\mathbb{R}^d)$ is a *frame* for $L_2(\mathbb{R}^d)$ if there exist constants A, $B > 0$ such that for every $f \in L_2(\mathbb{R}^d)$,

$$A\,\|f\|^2 \le \sum_{g\in X} |\langle f, g\rangle|^2 \le B\,\|f\|^2\,, \tag{1.1}$$

where we use the standard inner product $\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}\mathrm{d}x$ and norm $\|\cdot\| := \langle \cdot, \cdot\rangle^{\frac{1}{2}}$ on $L_2(\mathbb{R}^d)$. A frame is a special case of a *Bessel system*, i.e. the right inequality of (1.1) holds for every $f \in L_2(\mathbb{R}^d)$. The supremum of A and the infimum of B for (1.1) to hold are called *frame bounds*. A frame X is said to be *tight* if we may take $A = B = 1$. Such a frame is sometimes referred to as a *normalized tight frame* in the literature, see for instance [20]. A tight frame X for $L_2(\mathbb{R}^d)$ becomes an *orthonormal basis* when all the functions in X have their norm equal to 1. More information about the theory of frames can be found in the books [4], [10] and [15].

The main idea here originates from the following simple, but highly useful, observation for the case when $d = 1$, i.e. $L_2(\mathbb{R})$. Consider $\psi \in L_2(\mathbb{R})$ that is not symmetric. Assume that the affine system $X(\psi) := \{2^{k/2}\psi(2^k\cdot - j) : k, j \in \mathbb{Z}\}$ generated by ψ forms a tight frame for $L_2(\mathbb{R})$. The generator ψ is known as a wavelet. Let $\Psi' := \{\psi'_1, \psi'_2\}$, where

$$\psi'_1 := \frac{1}{2}(\psi + \psi(-\cdot)), \quad \psi'_2 := \frac{1}{2}(\psi - \psi(-\cdot)).$$

Then ψ'_1 is symmetric and ψ'_2 is antisymmetric about the origin. Further, the orders of the smoothness and vanishing moments of ψ are not reduced. It turns out that $X(\Psi') := X(\psi'_1) \cup$

$X(\psi_2')$ also forms a tight frame for $L_2(\mathbb{R})$. Therefore this method converts any nonsymmetric wavelet that generates an affine tight frame to a pair of symmetric and antisymmetric wavelets that generate an affine tight frame. The idea here can be refined to ensure that the supports of the new wavelets ψ_1' and ψ_2' are almost the same, if not identical, as that of ψ. In particular, if we begin with an orthonormal basis generated by one wavelet ψ, then the method gives a tight frame generated by two wavelets ψ_1' and ψ_2' with symmetry and of similar support as ψ. It can also be adjusted easily to suit the case when the original affine tight frame is generated by more than one wavelet. The number of new wavelets is at most twice the number of the original wavelets.

In this paper, the approach outlined above is developed under the most general setting of $L_2(\mathbb{R}^d)$. We begin in Section 2 with affine systems and show that both the frame property and frame bounds are preserved under the symmetrization process. In Section 3, we consider the case when the original wavelets are obtainable from a multiresolution analysis (MRA), i.e. the setting of framelets. We prove that for a given MRA-based tight frame system, one can always derive a symmetric and antisymmetric tight frame system that arise from a, but possibly different, MRA generated by symmetric or antisymmetric refinable functions. When the original MRA is generated by a symmetric refinable function, the symmetric and antisymmetric tight frame system is obtained from the same MRA. This enables us to convert the systematic construction of spline tight framelets of [12] to a systematic construction of symmetric and antisymmetric spline tight framelets with given orders of smoothness and vanishing moments. Further, framelets constructed via the oblique or unitary extension principle are also considered in Section 3. Finally, in Section 4, we illustrate with examples the constructions given by our method. We also discuss practical issues related to minimizing the supports of the resulting refinable functions and wavelets as well as improving their spreads in the time domain.

2. Construction

Let Ψ be a finite subset of $L_2(\mathbb{R}^d)$. We use Ψ to denote both a set and a column vector. Define the *affine system* $X(\Psi)$ generated by Ψ to be

$$X(\Psi) := \{D^k E^j \psi : k \in \mathbb{Z}, j \in \mathbb{Z}^d, \psi \in \Psi\}, \tag{2.1}$$

where $E^j : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is the *shift operator* given by

$$E^j : f \mapsto f(\cdot - j),$$

and $D : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is the *dilation operator*

$$D : f \mapsto |\det s|^{\frac{1}{2}} f(s\cdot),$$

with s a $d \times d$ invertible matrix with integer entries such that s^{-1} is contractive. An affine system that forms a frame for $L_2(\mathbb{R}^d)$ is known as a *wavelet frame*. For a wavelet frame, the functions $\psi \in \Psi$ in (2.1) are known as *mother wavelets* or simply *wavelets*.

Our objective is to obtain symmetric and antisymmetric wavelets through appropriate modifications and transformations of known wavelets. The general setup is as follows.

Construction 2.1. *Let $\Psi := [\psi_l]_{l=1}^r \subset L_2(\mathbb{R}^d)$ be a finite set of functions. Consider*

$$\Upsilon := \begin{bmatrix} \frac{1}{\sqrt{2}}\psi_l \\ \frac{1}{\sqrt{2}}\psi_l(\kappa_l - \cdot) \end{bmatrix}_{l=1}^r,$$

where $\kappa_l \in \mathbb{Z}^d$, as a $2r \times 1$ vector arranged in the order of $\frac{1}{\sqrt{2}}\psi_l$ followed by $\frac{1}{\sqrt{2}}\psi_l(\kappa_l - \cdot)$ for $l = 1, ..., r$. Define $\Psi' := U_{2r}\Upsilon$, where U_{2r} is the $2r \times 2r$ unitary matrix given by

$$U_{2r} := \begin{bmatrix} U_0 & & \\ & \ddots & \\ & & U_0 \end{bmatrix}, \quad U_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.2}$$

Then Ψ' consists of symmetric and antisymmetric functions, where a typical symmetric function $\frac{1}{2}(\psi_l + \psi_l(\kappa_l - \cdot))$ is symmetric about $\frac{\kappa_l}{2}$ and a typical antisymmetric function $\frac{1}{2}(\psi_l - \psi_l(\kappa_l - \cdot))$ is antisymmetric about $\frac{\kappa_l}{2}$.

The above is a very natural way of obtaining symmetric and antisymmetric functions from a given collection of functions. The main issue here is to show that whenever $X(\Psi)$ is a frame for $L_2(\mathbb{R}^d)$, Construction 2.1 gives a frame $X(\Psi')$ for $L_2(\mathbb{R}^d)$ with the same frame bounds. Our proof will utilize the following elementary lemma obtained from the frame condition (1.1) and a change of variables.

Lemma 2.2. *Let the ordered set $\Psi := [\psi_l]_{l=1}^r$ be a subset of $L_2(\mathbb{R}^d)$. If the affine system $X(\Psi)$ as in (2.1) is a frame for $L_2(\mathbb{R}^d)$, then the affine system $X([\psi_l(\kappa_l - \cdot)]_{l=1}^r)$, where $\kappa_l \in \mathbb{Z}^d$, is also a frame for $L_2(\mathbb{R}^d)$ with the same frame bounds.*

The next lemma will also be used. Although it is a special case of Theorem 4 in [1], we include its simple proof for completeness.

Lemma 2.3. *Let $\{g_n\}_{n\in K}$ be a frame for $L_2(\mathbb{R}^d)$. Then $\{h_n\}_{n\in K} := \mathcal{U}\{g_n\}_{n\in K}$, where $\mathcal{U}$ is a unitary matrix with finitely many nonzero entries in each row and column, is also a frame for $L_2(\mathbb{R}^d)$ with the same frame bounds as $\{g_n\}_{n\in K}$.*

Proof. The matrix $\mathcal{U}$ defines a unitary operator from $l_2(K)$, the space of all complex square-summable sequences indexed by K, onto $l_2(K)$ by

$$\mathcal{U} : \{c_k\}_{k\in K} \to \{\sum_{k\in K} u_{jk}c_k\}_{j\in K}.$$

Indeed, $\|\mathcal{U}\{c_k\}_{k\in K}\|_{l_2(K)}^2 = \|\{c_k\}_{k\in K}\|_{l_2(K)}^2$ for all finite sequences $\{c_k\}_{k\in K}$, which also holds for all sequences in $l_2(K)$ since $\mathcal{U}$ is a bounded linear operator on the densely defined subspace of finite sequences in $l_2(K)$. For $f \in L_2(\mathbb{R}^d)$, since

$$\{\langle h_j, f\rangle\}_{j\in K} = \{\langle \sum_{k\in K} u_{jk} g_k, f\rangle\}_{j\in K} = \{\sum_{k\in K} u_{jk}\langle g_k, f\rangle\}_{j\in K} = \mathcal{U}\{\langle g_k, f\rangle\}_{k\in K},$$

the result follows from the fact that $\mathcal{U}$ is a unitary operator on $l_2(K)$ and the frame condition (1.1). □

Theorem 2.4. *Let $\Psi := [\psi_l]_{l=1}^r$ such that the affine system $X(\Psi)$ as in* (2.1) *is a frame for $L_2(\mathbb{R}^d)$. Let Ψ' be constructed from Ψ as in Construction* 2.1*. Then the affine system $X(\Psi')$ is also a frame for $L_2(\mathbb{R}^d)$ with the same frame bounds as $X(\Psi)$. In particular, if $X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$, then $X(\Psi')$ is also a tight frame for $L_2(\mathbb{R}^d)$.*

Proof. Let $\tilde{\Psi} := [\psi_l(\kappa_l - \cdot)]_{l=1}^r$, $\kappa_l \in \mathbb{Z}^d$, and Υ be as in Construction 2.1. Lemma 2.2 shows that $X(\tilde{\Psi})$ is a frame for $L_2(\mathbb{R}^d)$ with the same frame bounds as $X(\Psi)$. When we combine $X(\Psi)$ with $X(\tilde{\Psi})$ under the appropriate normalization as $X(\Upsilon)$, $X(\Upsilon)$ remains a frame for $L_2(\mathbb{R}^d)$ with the same frame bounds. This is because the frame condition (1.1) implies that

$$A\,\|f\|^2 \le \sum_{g\in X(\Psi)} \left|\langle f, \frac{1}{\sqrt{2}} g\rangle\right|^2 + \sum_{g\in X(\tilde{\Psi})} \left|\langle f, \frac{1}{\sqrt{2}} g\rangle\right|^2 \le B\,\|f\|^2, \quad f \in L_2(\mathbb{R}^d),$$

where A and B are the frame bounds of $X(\Psi)$.

We order the functions in $X(\Upsilon)$ such that the $2r$ wavelets $\psi_1, \psi_1(\kappa_1 - \cdot), ..., \psi_r, \psi_r(\kappa_r - \cdot)$ are always grouped together under the various applications of the dilation operator D and the shift operator E. By selecting the same ordering for the functions in $X(\Psi')$, it follows that $X(\Psi') = \mathcal{U}X(\Upsilon)$, where $\mathcal{U}$ is the block diagonal matrix of bi-infinite order with the matrix U_{2r} as the diagonal blocks. Then we apply Lemma 2.3 to $X(\Upsilon)$ to conclude that $X(\Psi')$ is a frame with the same frame bounds as $X(\Upsilon)$. □

3. Framelets

Let $\Phi \subset L_2(\mathbb{R}^d)$ be a finite set and let $S(\Phi)$ be the closed shift-invariant linear subspace generated by Φ, i.e. $S(\Phi) = \overline{\text{span}\,\{E^j\phi : \phi \in \Phi, j \in \mathbb{Z}^d\}}$. We shall consider wavelets that are derived from a *multiresolution analysis* (MRA) of $L_2(\mathbb{R}^d)$, i.e. *framelets*. Following [2], an MRA generated by a finite ordered set $\Phi \subset L_2(\mathbb{R}^d)$ is a sequence of subspaces $\{S_k(\Phi)\}$ with $S_k(\Phi) := \{D^k f : f \in S(\Phi)\}$ such that (i) $S_k(\Phi) \subset S_{k+1}(\Phi)$, (ii) $\bigcup_{k\in\mathbb{Z}} S_k(\Phi)$ is dense in $L_2(\mathbb{R}^d)$ and (iii) $\bigcap_{k\in\mathbb{Z}} S_k(\Phi) = \{0\}$. Fixing notations, the Fourier transform $\hat{f}$ of a function f in $L_1(\mathbb{R}^d)$, the space of all complex-valued integrable functions on $\mathbb{R}^d$, is defined as $\hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-\mathrm{i}\omega\cdot x}\mathrm{d}x$, and is extended in the standard manner to a unitary operator on $L_2(\mathbb{R}^d)$. Condition (i) requires the vector Φ to be *refinable*, i.e.

$$\widehat{\Phi}(s^*\cdot) = \tau_\Phi \widehat{\Phi}, \tag{3.1}$$

where τ_Φ is a $2\pi\mathbb{Z}^d$-periodic matrix-valued measurable function known as the *refinement mask*. The vector Φ is known as a *refinable vector* and (3.1) is the *refinement equation*. For Φ satisfying (i), Condition (ii) always holds if Φ is also compactly supported (see [22]). Condition (iii) is automatically satisfied since Φ is a finite subset of $L_2(\mathbb{R}^d)$ (see Corollary 4.14 of [2] and Theorem 2.2 and Remark 2.6 of [23]).

Suppose that $\{S_k(\Phi)\}$ is an MRA of $L_2(\mathbb{R}^d)$ generated by a finite refinable vector Φ. Let Ψ be a finite ordered subset of $S_1(\Phi)$. Then there exists a $2\pi\mathbb{Z}^d$-periodic matrix-valued measurable function τ_Ψ known as the *wavelet mask* (see [12]) such that

$$\widehat{\Psi}(s^*\cdot) = \tau_\Psi \widehat{\Phi}. \tag{3.2}$$

Equation (3.2) defines a vector of wavelets and is called the *wavelet equation*. We define the *combined MRA mask* to be the $(\Phi \cup \Psi) \times \Phi$-indexed matrix

$$\tau := \begin{bmatrix} \tau_\Phi \\ \tau_\Psi \end{bmatrix}, \tag{3.3}$$

and in the event of Φ being a singleton set, i.e. $\Phi := \{\phi\}$, we denote $\tau_\phi := \tau_\Phi$.

Under the assumption that the entries of τ lie in $L_\infty(\mathbb{T}^d)$, the space of all essentially bounded complex-valued functions on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$, we define the Fourier coefficients of the masks τ_Φ and τ_Ψ, which we shall term simply as *lowpass filter* H and *highpass filter* G, by

$$\tau_\Phi(\omega) = \sum_{n\in\mathbb{Z}^d} H(n)\mathrm{e}^{-\mathrm{i}n\cdot\omega}, \quad \tau_\Psi(\omega) = \sum_{n\in\mathbb{Z}^d} G(n)\mathrm{e}^{-\mathrm{i}n\cdot\omega}.$$

We shall use the notations τ_ϕ, τ_ψ, h and g in place of τ_Φ, τ_Ψ, H and G respectively when $H(n)$ and $G(n)$, $n \in \mathbb{Z}^d$, are scalars. The refinement and wavelet equations (3.1) and (3.2) are equivalent to

$$\Phi = |\det s| \sum_{n\in\mathbb{Z}^d} H(n)\Phi(s\cdot -n), \quad \Psi = |\det s| \sum_{n\in\mathbb{Z}^d} G(n)\Phi(s\cdot -n). \tag{3.4}$$

A straightforward calculation gives explicit expressions of the lowpass and highpass filters for refinable functions and wavelets under certain affine transformations. We record them in the following proposition, which will be used in our subsequent construction of symmetric and antisymmetric framelets.

Proposition 3.1. *Let* $\Phi := [\phi_l]_{l=1}^q$ *and* $\Psi := [\psi_l]_{l=1}^r$ *satisfy the refinement and wavelet equations in* (3.4) *respectively with matrix filters* $H := [H_{l,m}]_{l,m=1}^q$ *and* $G := [G_{l,m}]_{l=1,m=1}^{r,q}$. *Let* $\tilde{\Phi} := [\phi_l(\lambda\cdot +\eta_l)]_{l=1}^q$ *and* $\tilde{\Psi} := [\psi_l(\lambda\cdot +\kappa_l)]_{l=1}^r$, *where* $\lambda \in \{\pm 1\}$ *and* $\eta_l, \kappa_l \in \mathbb{Z}^d$. *Then*

$$\tilde{\Phi} = |\det s| \sum_{n\in\mathbb{Z}^d} \tilde{H}(n)\tilde{\Phi}(s\cdot -n), \quad \tilde{\Psi} = |\det s| \sum_{n\in\mathbb{Z}^d} \tilde{G}(n)\tilde{\Phi}(s\cdot -n),$$

where $\tilde{H}(n) := [H_{l,m}(s\eta_l - \eta_m + \lambda n)]_{l,m=1}^{q}$ *and* $\tilde{G}(n) := [G_{l,m}(s\kappa_l - \eta_m + \lambda n)]_{l=1,m=1}^{r,q}$ *for* $n \in \mathbb{Z}^d$. *Further,*

$$\tau_{\tilde{\Phi}}(\omega) = \operatorname{diag}\left[\mathrm{e}^{\mathrm{i}s\eta_l\cdot\lambda\omega}\right]_{l=1}^{q} \tau_{\Phi}(\lambda\omega) \operatorname{diag}\left[\mathrm{e}^{-\mathrm{i}\eta_m\cdot\lambda\omega}\right]_{m=1}^{q},$$
$$\tau_{\tilde{\Psi}}(\omega) = \operatorname{diag}\left[\mathrm{e}^{\mathrm{i}s\kappa_l\cdot\lambda\omega}\right]_{l=1}^{r} \tau_{\Psi}(\lambda\omega) \operatorname{diag}\left[\mathrm{e}^{-\mathrm{i}\eta_m\cdot\lambda\omega}\right]_{m=1}^{q}.$$

Now, consider the affine system $X(\Psi)$ in (2.1) generated by Ψ. Theorem 2.4 shows that if $X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$, then $X(\Psi')$ is also a tight frame for $L_2(\mathbb{R}^d)$, where Ψ' is constructed from Ψ as in Construction 2.1. Given, in addition, that $X(\Psi)$ is derived from an MRA, we are interested to know whether $X(\Psi')$ comes from an MRA, and further, the same MRA or a different MRA. In this connection, we need the following lemma.

Lemma 3.2. *Suppose that* $\{S_k(\Phi)\}$ *is an MRA of* $L_2(\mathbb{R}^d)$, *where* $\Phi := [\phi_l]_{l=1}^{q}$. *Let* $\tilde{\Phi} := [\phi_l(\eta_l - \cdot)]_{l=1}^{q}$, *where* $\eta_l \in \mathbb{Z}^d$. *Then* $\{S_k(\Phi \cup \tilde{\Phi})\}$ *is an MRA of* $L_2(\mathbb{R}^d)$.

Proof. Proposition 3.1 shows that $\tilde{\Phi}$ is a refinable vector-valued function. By (3.1) for both Φ and $\tilde{\Phi}$, $\Phi \cup \tilde{\Phi}$ is also refinable. The density of $\bigcup_{k\in\mathbb{Z}} S_k(\Phi)$ in $L_2(\mathbb{R}^d)$ implies the density of $\bigcup_{k\in\mathbb{Z}} S_k(\Phi \cup \tilde{\Phi})$. Therefore $\{S_k(\Phi \cup \tilde{\Phi})\}$ is an MRA of $L_2(\mathbb{R}^d)$. ☐

We shall build upon Construction 2.1 in the following way. Given that $\Psi := [\psi_l]_{l=1}^{r}$ is a vector-valued function satisfying the wavelet equation (3.2) of the MRA $\{S_k(\Phi)\}$ of $L_2(\mathbb{R}^d)$, let $\tilde{\Phi} := [\phi_l(\eta_l - \cdot)]_{l=1}^{q}$ and $\tilde{\Psi} := [\psi_l(\kappa_l - \cdot)]_{l=1}^{r}$, for some $\eta_l, \kappa_l \in \mathbb{Z}^d$. Then we define

$$\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_l \\ \phi_l(\eta_l - \cdot)\end{bmatrix}_{l=1}^{q}, \quad \Upsilon := \frac{1}{\sqrt{2}}\begin{bmatrix}\psi_l \\ \psi_l(\kappa_l - \cdot)\end{bmatrix}_{l=1}^{r}, \quad \Phi' := U_{2q}\Xi, \quad \Psi' := U_{2r}\Upsilon, \tag{3.5}$$

where U_{2q} and U_{2r} are $2q \times 2q$ and $2r \times 2r$ block diagonal matrices respectively with the matrix U_0 in (2.2) as their blocks.

Theorem 3.3. *Let* $\Psi := [\psi_l]_{l=1}^{r}$ *be a finite set of tight framelets obtained from the MRA* $\{S_k(\Phi)\}$ *of* $L_2(\mathbb{R}^d)$ *generated by* $\Phi := [\phi_l]_{l=1}^{q}$. *Define* Φ' *and* Ψ' *as in* (3.5). *Then* Ψ' *is a finite set of symmetric or antisymmetric tight framelets obtained from the MRA generated by* Φ'.

Proof. Let $\tilde{\Phi} := [\phi_l(\eta_l - \cdot)]_{l=1}^{q}$ and $\tilde{\Psi} := [\psi_l(\kappa_l - \cdot)]_{l=1}^{r}$. From Lemma 3.2, we know that $\{S_k(\Xi)\}$ is an MRA of $L_2(\mathbb{R}^d)$. By Proposition 3.1,

$$\widehat{\tilde{\Phi}}(s^*\cdot) = \tau_{\tilde{\Phi}}\widehat{\tilde{\Phi}}, \quad \widehat{\tilde{\Psi}}(s^*\cdot) = \tau_{\tilde{\Psi}}\widehat{\tilde{\Phi}}.$$

Combining with (3.1) and (3.2), we obtain

$$\begin{bmatrix}\widehat{\Phi}(s^*\cdot) \\ \widehat{\tilde{\Phi}}(s^*\cdot)\end{bmatrix} = \begin{bmatrix}\tau_\Phi & 0 \\ 0 & \tau_{\tilde{\Phi}}\end{bmatrix}\begin{bmatrix}\widehat{\Phi} \\ \widehat{\tilde{\Phi}}\end{bmatrix}, \quad \begin{bmatrix}\widehat{\Psi}(s^*\cdot) \\ \widehat{\tilde{\Psi}}(s^*\cdot)\end{bmatrix} = \begin{bmatrix}\tau_\Psi & 0 \\ 0 & \tau_{\tilde{\Psi}}\end{bmatrix}\begin{bmatrix}\widehat{\Phi} \\ \widehat{\tilde{\Phi}}\end{bmatrix}. \tag{3.6}$$

Rearranging the rows of the vectors in (3.6) based on the ordering in Ξ and Υ gives

$$\widehat{\Xi}(s^*\cdot) = \tau_\Xi\widehat{\Xi}, \quad \widehat{\Upsilon}(s^*\cdot) = \tau_\Upsilon\widehat{\Xi},$$

where τ_Ξ and τ_Υ are the refinement and wavelet masks of Ξ and Υ respectively. By Theorem 2.4, $X(\Psi')$ is a tight frame for $L_2(\mathbb{R}^d)$. Note that Φ' generates the same MRA as Ξ with refinement mask $\tau_{\Phi'} := U_{2q}\tau_\Xi U_{2q}^*$ because Φ' is obtained from a unitary transformation of Ξ. Similarly, the wavelet mask of Ψ' is $\tau_{\Psi'} := U_{2r}\tau_\Upsilon U_{2q}^*$ with the tight frame $X(\Psi')$ arising from the MRA $\{S_k(\Phi')\}$. □

In practice, fast wavelet decomposition and reconstruction algorithms are needed. These algorithms exist for tight framelets derived from the *oblique extension principle* (OEP) (see [26], [5] and [12]). In [12], tight framelets were constructed from an MRA generated by a refinable B-spline with the desired approximation order using the OEP. However, the framelets are not symmetric even though B-splines are symmetric. Next, we shall prove that when the refinable function in the OEP is symmetric, Construction 2.1 gives symmetric and antisymmetric tight framelets arising from the same MRA, and the corresponding new fundamental function in the OEP can also be found. Knowing the fundamental function is important in applying the fast decomposition and reconstruction algorithms (see [12]) for tight framelets derived from the OEP.

Before we state the OEP, recall that the *spectrum* of a shift-invariant space $S(\Phi)$ is defined (up to measure zero sets) as

$$\sigma(S(\Phi)) := \{\omega \in \mathbb{T}^d : \sum_{j\in 2\pi\mathbb{Z}^d} |\hat{\phi}(\omega + j)|^2 > 0 \text{ for some } \phi \in \Phi\},$$

where $\sum_{j\in 2\pi\mathbb{Z}^d} |\hat{\phi}(\omega + j)|^2$ is well defined for almost every $\omega \in \mathbb{T}^d$ since $\phi \in L_2(\mathbb{R}^d)$. The spectrum of $S(\Phi)$ only depends on the space and is independent of the choice of generators of the space (see [3] and [25]). In all our discussion that follows, we shall assume that every $\phi \in \Phi$ satisfies

$$\sigma(S(\phi)) = \sigma(S(\phi(\eta - \cdot))) \tag{3.7}$$

for some $\eta \in \mathbb{Z}^d$. Equation (3.7) holds when all the functions $\phi \in \Phi$ are compactly supported (since $\sigma(S(\phi)) = \mathbb{T}^d$) or satisfy $\left|\hat{\phi}(\omega)\right|^2 = \left|\hat{\phi}(-\omega)\right|^2$ a.e. on $\mathbb{R}^d$, which is valid for real-valued or symmetric ϕ.

The following theorem is known as the *oblique extension principle* (OEP). It is stated in the setting of Φ being a singleton set $\{\phi\}$.

Theorem 3.4. [12] (Oblique Extension Principle) *Let $\{S_k(\phi)\}$ be an MRA of $L_2(\mathbb{R}^d)$ with combined mask τ defined as in* (3.3) *having entries in $L_\infty(\mathbb{T}^d)$ and such that $E(\phi)$ is a Bessel system. Suppose that $\lim_{\omega\to 0}\hat{\phi}(\omega) = 1$ and there exists a $2\pi\mathbb{Z}^d$-periodic nonnegative essentially bounded function Θ, which is continuous at the origin, with $\Theta(0) = 1$ and satisfies*

$$\overline{\tau_\phi(\omega)}\Theta(s^*\omega)\tau_\phi(\omega + \nu) + \tau_\Psi(\omega)^*\tau_\Psi(\omega + \nu) = \delta_\nu\Theta(\omega), \tag{3.8}$$

whenever $\omega \in \sigma(S(\phi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ is such that $\omega + \nu \in \sigma(S(\phi))$. Then the affine system $X(\Psi)$ as in (2.1) *defined by τ is a tight frame for $L_2(\mathbb{R}^d)$.*

The function Θ in Theorem 3.4 is known as the *fundamental function*. The OEP was also proved independently in [5].

We shall now show that if $X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$ derived from an MRA generated by a symmetric refinable function using the OEP, then for Ψ' constructed from Ψ as in Construction 2.1, $X(\Psi')$ is also a tight frame for $L_2(\mathbb{R}^d)$ derived from the same MRA using the OEP. In view of various available examples in the literature (see also Section 4), instead of the more general case as discussed in Theorem 3.3, here we only deal with the situation in which the MRA is generated by a single symmetric refinable function.

Theorem 3.5. *Let* $\Psi := [\psi_l]_{l=1}^r$ *such that* $X(\Psi)$ *as in* (2.1) *is a tight frame for* $L_2(\mathbb{R}^d)$ *derived from the OEP with* $\{S_k(\phi)\}$ *as the underlying MRA of* $L_2(\mathbb{R}^d)$, ϕ *being symmetric about* $\frac{\eta}{2}$, *where* $\eta \in \mathbb{Z}^d$, Θ *as the fundamental function, and* $\tau := \begin{bmatrix}\tau_\phi \\ \tau_\Psi\end{bmatrix}$ *as the combined MRA mask. Let the set of symmetric and antisymmetric wavelets* Ψ' *be constructed from* Ψ *as in Construction* 2.1. *Then* $X(\Psi')$ *is a tight frame for* $L_2(\mathbb{R}^d)$ *derived from the same MRA* $\{S_k(\phi)\}$ *using the OEP with the fundamental function* $\Theta' := \frac{1}{2}[\Theta + \Theta(-\cdot)]$ *and the combined MRA mask* $\tau' := \begin{bmatrix}\tau_\phi \\ \tau_{\Psi'}\end{bmatrix}$, *where* $\tau_{\Psi'}$ *is the* $2r \times 1$ *vector given by*

$$\tau_{\Psi'}(\omega) := \frac{1}{2}\begin{bmatrix}\tau_{\psi_l}(\omega) + \mathrm{e}^{-\mathrm{i}(s\kappa_l - \eta)\cdot\omega}\tau_{\psi_l}(-\omega) \\ \tau_{\psi_l}(\omega) - \mathrm{e}^{-\mathrm{i}(s\kappa_l - \eta)\cdot\omega}\tau_{\psi_l}(-\omega)\end{bmatrix}_{l=1}^r, \tag{3.9}$$

$\kappa_l \in \mathbb{Z}^d$.

Proof. We first apply Proposition 3.1 to see that for $\tilde{\phi} := \phi(\eta - \cdot)$ and $\tilde{\Psi} := [\psi_l(\kappa_l - \cdot)]_{l=1}^r$,

$$\overline{\tau_{\tilde{\phi}}(\omega)}\tau_{\tilde{\phi}}(\omega+\nu) = \mathrm{e}^{\mathrm{i}\eta\cdot\nu}\overline{\tau_\phi(-\omega)}\tau_\phi(-\omega-\nu), \;\; \tau_{\tilde{\Psi}}(\omega)^*\tau_{\tilde{\Psi}}(\omega+\nu) = \mathrm{e}^{\mathrm{i}\eta\cdot\nu}\tau_\Psi(-\omega)^*\tau_\Psi(-\omega-\nu), \tag{3.10}$$

where $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$, since $\mathrm{e}^{-\mathrm{i}s\eta\cdot\nu} = \mathrm{e}^{-\mathrm{i}s\kappa_l\cdot\nu} = 1$. By the symmetry of ϕ, $\omega \in \sigma(S(\phi))$ if and only if $-\omega \in \sigma(S(\phi))$ for almost every $\omega \in \mathbb{T}^d$. Thus

$$\overline{\tau_{\tilde{\phi}}(\omega)}\Theta(-s^*\omega)\tau_{\tilde{\phi}}(\omega+\nu) + \tau_{\tilde{\Psi}}(\omega)^*\tau_{\tilde{\Psi}}(\omega+\nu) = \delta_\nu\Theta(-\omega) \tag{3.11}$$

holds for $\omega \in \sigma(s(\phi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ such that $\omega+\nu \in \sigma(S(\phi))$ as we may replace ω by $-\omega$ and ν by $-\nu$ in (3.8). Let Υ be as in Construction 2.1. Since $\tilde{\phi} = \phi$, adding (3.8) and (3.11) leads to

$$\overline{\tau_\phi(\omega)}\Theta'(s^*\omega)\tau_\phi(\omega+\nu) + \tau_\Upsilon(\omega)^*\tau_\Upsilon(\omega+\nu) = \delta_\nu\Theta'(\omega), \tag{3.12}$$

whenever $\omega \in \sigma(S(\phi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ is such that $\omega+\nu \in \sigma(S(\phi))$.

Next, as $\Psi' := U_{2r}\Upsilon$, where U_{2r} is the constant unitary matrix in (2.2), it follows that the final wavelet mask is given by $\tau_{\Psi'} := U_{2r}\tau_\Upsilon$. Let $\omega \in \sigma(S(\phi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ such that $\omega+\nu \in \sigma(S(\phi))$. Then $\tau_{\Psi'}(\omega)^*\tau_{\Psi'}(\omega+\nu) = \tau_\Upsilon(\omega)^*\tau_\Upsilon(\omega+\nu)$ and so (3.12) yields

$$\overline{\tau_\phi(\omega)}\Theta'(s^*\omega)\tau_\phi(\omega+\nu) + \tau_{\Psi'}(\omega)^*\tau_{\Psi'}(\omega+\nu) = \delta_\nu\Theta'(\omega).$$

Hence by Theorem 3.4, $X(\Psi')$ is a tight frame for $L_2(\mathbb{R}^d)$ derived from the MRA $\{S_k(\phi)\}$ using the OEP with the fundamental function Θ'. □

Let us highlight an application of Theorem 3.5 which gives a systematic approach to constructing symmetric and antisymmetric framelets, with given approximation order, for the univariate case with dilation factor 2. In Section 3.2 of [12], starting from a B-spline ϕ of order m (which is symmetric), tight frame systems were constructed by choosing appropriate trigonometric polynomials Θ to be the fundamental function in the OEP, according to m and the approximation order of the system required. The approximation order is closely related to the order of vanishing moments of the framelets, which in turn depends on ϕ and Θ (see Theorems 2.8 and 2.11 of [12]). One choice of the fundamental function Θ gave a total of three mother wavelets, while another choice produced two. None of the wavelets was symmetric, though both fundamental functions were symmetric. Applying Theorem 3.5 to these two sets of wavelets, we see that Construction 2.1 gives three symmetric and three antisymmetric wavelets for the first set, and two symmetric and two antisymmetric wavelets for the second. In both instances, since ϕ and Θ are unchanged, the approximation order of the resulting tight frame system remains the same.

In [18], three symmetric and antisymmetric framelets were constructed directly from the B-spline of order m. This method was extended to constructions based on a compactly supported symmetric refinable function with stable shifts in [19]. Our construction does not require the stability assumption of the refinable function and reduces the construction of symmetric tight framelets to the construction of tight framelets, which is easier. It combines the procedure in [12] with Construction 2.1 to give a systematic procedure for obtaining symmetric and antisymmetric framelets with at least the same vanishing moments, smoothness and approximation orders as the original wavelets. While the construction in [18] resulted in framelets with the highest possible order of vanishing moments, the flexibility of our construction allows us to tailor the approximation order of our framelet system and the order of vanishing moments of the framelets according to the needs of our application.

Let us now return to the general setting of $L_2(\mathbb{R}^d)$ and arbitrary dilation matrix s. We have shown that when $\phi \in L_2(\mathbb{R}^d)$ is symmetric, the new set of symmetric and antisymmetric framelets is obtained from the same MRA generated by ϕ. However, in many cases, the scaling function ϕ such as one of the Daubechies scaling functions or a pseudo-spline (see [12]) is not symmetric, and the corresponding wavelets are obtainable from the *unitary extension principle* (UEP), i.e. the OEP with fundamental function $\Theta = 1$. We shall see that in these instances, notwithstanding that the scaling function ϕ is not symmetric, it is still possible to construct a symmetric and antisymmetric tight frame system from the UEP. However, the set of framelets comes from an MRA generated by two functions, which is different from the original MRA $\{S_k(\phi)\}$, and the proof requires the following vector version of the UEP (see [26]).

Theorem 3.6. [26] (Unitary Extension Principle). *Let $\{S_k(\Phi)\}$ be an MRA of $L_2(\mathbb{R}^d)$ with combined mask τ defined as in (3.3) having entries in $L_\infty(\mathbb{T}^d)$ and such that $E(\Phi)$ is a Bessel system. Suppose that $\lim_{\omega\to 0}(\widehat{\Phi}^*\widehat{\Phi})(\omega) = 1$ and*

$$\tau(\omega)^*\tau(\omega+\nu) = \delta_\nu I, \tag{3.13}$$

whenever $\omega \in \sigma(S(\Phi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ is such that $\omega + \nu \in \sigma(S(\Phi))$. Then the affine system $X(\Psi)$ defined by τ is a tight frame for $L_2(\mathbb{R}^d)$.

Our next result is analogous to Theorem 3.5 for the UEP setting, except that the refinable function ϕ may not be symmetric but satisfies (3.7). Again, based on examples of interest (see Section 4), we focus on the case when the original MRA is generated by a single refinable function.

Theorem 3.7. *Let $\Psi := [\psi_l]_{l=1}^r$ such that $X(\Psi)$ as in (2.1) is a tight frame for $L_2(\mathbb{R}^d)$ derived from the UEP with $\{S_k(\phi)\}$ as the underlying MRA of $L_2(\mathbb{R}^d)$ under the condition that ϕ satisfies (3.7) and $\tau := \begin{bmatrix}\tau_\phi \\ \tau_\Psi\end{bmatrix}$ as the combined MRA mask. Let $\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix}\phi \\ \phi(\eta-\cdot)\end{bmatrix}$, where $\eta \in \mathbb{Z}^d$. Suppose that $\Phi' := U_0\Xi$, where U_0 is the unitary matrix in (2.2), and the set of symmetric and antisymmetric wavelets Ψ' is constructed from Ψ as in Construction 2.1. Then $X(\Psi')$ is a tight frame for $L_2(\mathbb{R}^d)$ derived from the MRA $\{S_k(\Phi')\}$ using the UEP with the combined MRA mask $\tau' := \begin{bmatrix}\tau_{\Phi'} \\ \tau_{\Psi'}\end{bmatrix}$, where $\tau_{\Phi'}$ and $\tau_{\Psi'}$ are the 2×2 and $2r \times 2$ matrices given by*

$$\tau_{\Phi'}(\omega) := \frac{1}{2}\begin{bmatrix}\tau_\phi(\omega) + \mathrm{e}^{-\mathrm{i}(s\eta-\eta)\cdot\omega}\tau_\phi(-\omega) & \tau_\phi(\omega) - \mathrm{e}^{-\mathrm{i}(s\eta-\eta)\cdot\omega}\tau_\phi(-\omega) \\ \tau_\phi(\omega) - \mathrm{e}^{-\mathrm{i}(s\eta-\eta)\cdot\omega}\tau_\phi(-\omega) & \tau_\phi(\omega) + \mathrm{e}^{-\mathrm{i}(s\eta-\eta)\cdot\omega}\tau_\phi(-\omega)\end{bmatrix}, \tag{3.14}$$

$$\tau_{\Psi'}(\omega) := \frac{1}{2}\begin{bmatrix}\tau_{\psi_l}(\omega) + \mathrm{e}^{-\mathrm{i}(s\kappa_l-\eta)\cdot\omega}\tau_{\psi_l}(-\omega) & \tau_{\psi_l}(\omega) - \mathrm{e}^{-\mathrm{i}(s\kappa_l-\eta)\cdot\omega}\tau_{\psi_l}(-\omega) \\ \tau_{\psi_l}(\omega) - \mathrm{e}^{-\mathrm{i}(s\kappa_l-\eta)\cdot\omega}\tau_{\psi_l}(-\omega) & \tau_{\psi_l}(\omega) + \mathrm{e}^{-\mathrm{i}(s\kappa_l-\eta)\cdot\omega}\tau_{\psi_l}(-\omega)\end{bmatrix}_{l=1}^r, \tag{3.15}$$

$\kappa_l \in \mathbb{Z}^d$, respectively.

Proof. By Lemma 3.2, $\{S_k(\Xi)\}$ is an MRA of $L_2(\mathbb{R}^d)$. Further, $E(\Xi)$ is also a Bessel system. Let Υ be as in Construction 2.1. The combined MRA mask $\begin{bmatrix}\tau_\Xi \\ \tau_\Upsilon\end{bmatrix}$ has entries in $L_\infty(\mathbb{T}^d)$ and τ_Ξ is a 2×2 diagonal matrix while τ_Υ is a vector of r 2×2 diagonal matrices. In addition, $\lim_{\omega\to 0}(\hat{\Xi}^*\hat{\Xi})(\omega) = 1$. We shall show that

$$\tau_\Xi(\omega)^*\tau_\Xi(\omega+\nu) + \tau_\Upsilon(\omega)^*\tau_\Upsilon(\omega+\nu) = \delta_\nu I, \tag{3.16}$$

whenever $\omega \in \sigma(S(\Xi))$ and $\nu \in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ is such that $\omega+\nu \in \sigma(S(\Xi))$. We note from (3.7) that $\sigma(S(\phi)) = \sigma(S(\tilde{\phi}))$, where $\tilde{\phi} := \phi(\eta-\cdot)$, and hence $\sigma(S(\Xi)) = \sigma(S(\phi)) \cup \sigma(S(\tilde{\phi})) = \sigma(S(\phi))$.

The $(1,1)$-entry of (3.16) is exactly (3.13). By the structure of the 2×2 diagonal matrices in τ_Ξ and τ_Υ, we see that the $(1,2)$- and $(2,1)$-entries of (3.16) are both zero. It remains to prove the equality of the $(2,2)$-entry on both sides of (3.16), i.e.

$$\overline{\tau_{\tilde{\phi}}(\omega)}\tau_{\tilde{\phi}}(\omega+\nu)+\tau_{\tilde{\Psi}}(\omega)^*\tau_{\tilde{\Psi}}(\omega+\nu)=\delta_\nu, \tag{3.17}$$

where $\tilde{\Psi} := [\psi_l(\kappa_l-\cdot)]_{l=1}^r$. As in the proof of Theorem 3.5, we use Proposition 3.1 to obtain (3.10). Since $\omega\in\sigma(S(\phi))$ if and only if $-\omega\in\sigma(S(\tilde{\phi}))$ for almost every $\omega\in\mathbb{T}^d$, it follows from (3.7) that $\omega\in\sigma(S(\phi))$ if and only if $-\omega\in\sigma(S(\phi))$ for almost every $\omega\in\mathbb{T}^d$. Thus in view of (3.10), (3.17) holds for $\omega\in\sigma(S(\phi))$ and $\nu\in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ such that $\omega+\nu\in\sigma(S(\phi))$, because we can replace ω by $-\omega$ and ν by $-\nu$ in (3.13).

Now, let $\Psi' := U_{2r}\Upsilon$, where U_{2r} is the constant unitary matrix in (2.2). We first observe from the refinement equation (3.1) that the vector Φ' is refinable with refinement mask $\tau_{\Phi'} := U_0\tau_\Xi U_0^*$, generating the same MRA as Ξ. Using the wavelet equation (3.2), the final wavelet mask is given by $\tau_{\Psi'} := U_{2r}\tau_\Upsilon U_0^*$. Clearly, the entries of the combined MRA mask $\tau' := \begin{bmatrix}\tau_{\Phi'}\\ \tau_{\Psi'}\end{bmatrix}$ lie in $L_\infty(\mathbb{T}^d)$. Also, we have $\lim_{\omega\to 0}(\widehat{\Phi}'^*\widehat{\Phi}')(\omega)=1$. Let $\omega\in\sigma(S(\Xi))$ and $\nu\in 2\pi((s^*)^{-1}\mathbb{Z}^d/\mathbb{Z}^d)$ such that $\omega+\nu\in\sigma(S(\Xi))$. Then $\tau_{\Phi'}(\omega)^*\tau_{\Phi'}(\omega+\nu)=U_0\tau_\Xi(\omega)^*\tau_\Xi(\omega+\nu)U_0^*$ and $\tau_{\Psi'}(\omega)^*\tau_{\Psi'}(\omega+\nu)=U_0\tau_\Upsilon(\omega)^*\tau_\Upsilon(\omega+\nu)U_0^*$. This enables us to conclude from (3.16) that (3.13) holds for τ', i.e.

$$\tau_{\Phi'}(\omega)^*\tau_{\Phi'}(\omega+\nu)+\tau_{\Psi'}(\omega)^*\tau_{\Psi'}(\omega+\nu)=\delta_\nu I.$$

Applying Theorem 3.6 to τ' gives the result. $\square$

4. Examples

We shall now illustrate the results in Section 3 with concrete examples for the univariate case with dilation factor 2. We begin with a discussion on practical issues related to the flexibility we have in the construction of symmetric and antisymmetric wavelets. When we utilize Construction 2.1 to construct our wavelets, we need to consider the positions of reflection of the original wavelets. Since we have the freedom of reflecting the wavelets about any half-integer point, we may choose to reflect them about half-integer points around the midpoints of their individual supports. This minimizes the supports of the resulting wavelets, in the sense that they are almost the same as the supports of the original wavelets. However, this may not always be ideal since we may obtain more than one peak or have more oscillations when we essentially take the sum and difference, using the matrix U_0 in (2.2), of the original wavelets and their reflections. Therefore it could be more desirable to reflect about the positions where their peaks occurred so that the resulting wavelets will have better spreads in the time domain. It should also be mentioned that in some cases, other positions may be even more appropriate, depending on the graphs of the original wavelets. For situations when the original refinable functions are not symmetric, similar considerations in choosing the positions of reflection apply.

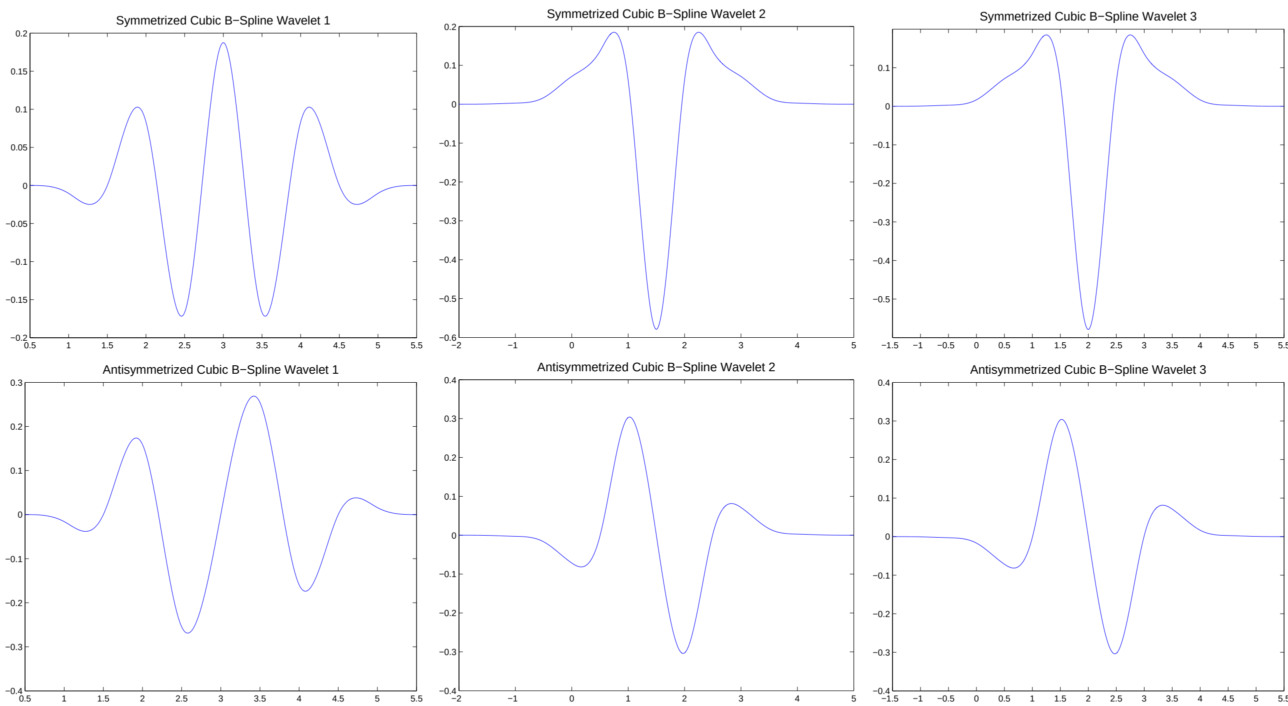

FIGURE 1. Symmetric and antisymmetric wavelets obtained in Example 4.1 from a systematic construction based on the cubic B-spline.

Example 4.1. This example, illustrated in Figure 1, is based on the systematic construction in Example 3.7 of [12]. The original wavelets are obtained from an MRA generated by a symmetric refinable function using the OEP, and we apply Theorem 3.5. Here the lowpass filter h is that of the cubic B-spline ϕ supported on $[0,4]$, and there are three wavelets ψ_1, ψ_2 and ψ_3 in the construction with filters g_1, g_2 and g_3 respectively. The approximation order of the framelet system generated by ψ_1, ψ_2 and ψ_3 is 4. We define $\Psi := \frac{1}{\sqrt{2}}\left[\psi_1, \psi_1(6-\cdot), \psi_2, \psi_2(3-\cdot), \psi_3, \psi_3(4-\cdot)\right]^T$ and $\Psi' := U_6\Psi$, where U_6 is the 6×6 block diagonal matrix with the matrix U_0 defined in (2.2) as its blocks. For ψ_1, we reflect at the midpoint of its support as this happens to reduce the oscillations in the resulting antisymmetric wavelet. As for ψ_2 and ψ_3, we choose to reflect at the nearest half-integers where their peaks occur. It follows from (3.9) that the matrix filter of Ψ' is given by $G' := [G'_l]_{l=1}^3$, where $G'_l(n) := \frac{1}{2}\begin{bmatrix} g_l(n) + g_l(\mu_l - n) \\ g_l(n) - g_l(\mu_l - n) \end{bmatrix}$ for $l = 1,2,3$ with $\mu_1 = 8$, $\mu_2 = 2$ and $\mu_3 = 4$.

Example 4.2. Consider the Daubechies-4 refinable function ϕ with filter h supported on $\{1,\ldots,4\}$ and the corresponding wavelet ψ with filter g given by $g(n) := (-1)^{3-n}h(3-n)$ (see [9] and [10]). As ϕ is not symmetric, we apply Theorem 3.7. Let $\Phi := \frac{1}{\sqrt{2}}\left[\phi, \phi(4-\cdot)\right]^T$, $\Psi := \frac{1}{\sqrt{2}}\left[\psi, \psi(3-\cdot)\right]^T$, $\Phi' := U_0\Phi$ and $\Psi' := U_0\Psi$, where U_0 is as defined in (2.2). Using (3.15), the matrix filter G' of Ψ' can be expressed as $G'(n) = (-1)^{3-n}H'(3-n)$, where the matrix filter

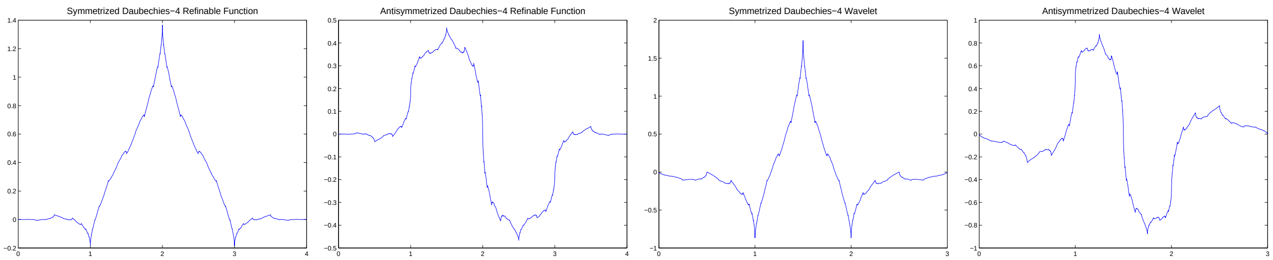

FIGURE 2. Symmetric and antisymmetric refinable functions and wavelets obtained in Example 4.2 from the Daubechies-4 refinable function and wavelet.

H' of Φ' is given by $H'(n) := \frac{1}{2}\begin{bmatrix} h(n)+h(4-n) & h(n)-h(4-n) \\ h(n)-h(4-n) & h(n)+h(4-n) \end{bmatrix}$ from (3.14). The graphs of the resulting refinable functions and wavelets are shown in Figure 2. Both the original refinable function and wavelet are reflected around their peaks. The supports of the resulting wavelets are the same as that of the original, since the reflection point occurs at the midpoint.

References

[1] A. Aldroubi, Portraits of frames, Proc. Amer. Math. Soc. 123 (1995) 1661–1668.

[2] C. de Boor, R. DeVore, A. Ron, On the construction of multivariate (pre)wavelets, Constr. Approx. 9 (1993) 123–126.

[3] C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78.

[4] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[5] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.

[6] C.K. Chui, X. Shi, J. Stöckler, Affine frames, quasi-affine frames, and their duals, Adv. Comput. Math. 8 (1998) 1–17.

[7] C.K. Chui, J. Wang, On compactly supported spline wavelets and a duality principle, Trans. Amer. Math. Soc. 330 (1992) 903–915.

[8] A. Cohen, I. Daubechies, J.-C. Feauveau, Biorthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math. 45 (1992) 485–560.

[9] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988) 909–996.

[10] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[11] I. Daubechies, B. Han, Pairs of dual wavelet frames from any two refinable functions, Constr. Approx. 20 (2004) 325–352.

[12] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46.

[13] G.C. Donovan, J.S. Geronimo, D.P. Hardin, Intertwining multiresolution analyses and the construction of piecewise-polynomial wavelets, SIAM J. Math. Anal. 27 (1996) 1791–1815.

[14] J.S. Geronimo, D.P. Hardin, P.R. Massopust, Fractal functions and wavelet expansions based on several scaling functions, J. Approx. Theory 78 (1994) 373–401.

[15] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.

[16] B. Han, Q. Mo, Multiwavelet frames from refinable function vectors, Adv. Comput. Math. 18 (2003) 211–245.

[17] B. Han, Q. Mo, Splitting a matrix of Laurent polynomials with symmetry and its application to symmetric framelet filter banks, SIAM J. Matrix Anal. Appl. 26 (2004) 97–124.

[18] B. Han, Q. Mo, Tight wavelet frames generated by three symmetric B-spline functions with high vanishing moments, Proc. Amer. Math. Soc. 132 (2004) 77–86.

[19] B. Han, Q. Mo, Symmetric MRA tight wavelet frames with three generators and high vanishing moments, Appl. Comput. Harmon. Anal. 18 (2005) 67–93.

[20] D. Han, D.R. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc. 147 (2000).

[21] H. Ji, Z. Shen, Compactly supported (bi)orthogonal wavelets generated by interpolatory refinable functions, Adv. Comput. Math. 11 (1999) 81–104.

[22] R.Q. Jia, C.A. Micchelli. Using the refinement equations for the construction of pre-wavelets II: powers of two, in: P.J. Laurent, A. Le Méhauté, L.L. Schumaker (Eds.), Curves and Surfaces, Academic Press, Boston, 1991, pp. 209–246.

[23] R.Q. Jia, Z. Shen, Multiresolution and wavelets, Proc. Edinburgh Math. Soc. 37 (1994) 271–300.

[24] A. Petukhov, Symmetric framelets, Constr. Approx. 19 (2003) 309–328.

[25] A. Ron, Z. Shen, Frames and stable bases for shift invariant subspaces of $L_2(\mathbb{R}^d)$, Can. J. Math. 47 (1995) 1051–1094.

[26] A. Ron, Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, J. Funct. Anal. 148 (1997) 408–447.