Uncertainty Principles and Optimality on Circles and Spheres

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Abstract. From a general uncertainty principle we derive uncertainty principles on spheres in any dimension which extend, for real-valued functions, known uncertainty principles on spheres in two and three dimensions. For the uncertainty principle on the circle, we show that for $0 < \epsilon < 1$, there is a sequence of trigonometric polynomials of degree k whose uncertainty differs from the optimal by $O(1/k^{2(1-\epsilon)})$ as $k \to \infty$, and similar results are given for spheres in higher dimensions. An alternative uncertainty principle on the circle is also considered for which minimum uncertainty is attained, and the minimizing functions are shown to have similarities to the Gaussian (which gives minimum uncertainty in the Heisenberg uncertainty principle).

§1. Introduction

The famous Heisenberg uncertainty principle can be derived from the following general uncertainty principle which is a special case of a more general uncertainty principle derived in [3]. Let $\mathcal{H}$ be a Hilbert space. For any linear operator T with domain and range in $\mathcal{H}$, and for any nonzero v in its domain $\mathcal{D}(T)$, we define

$$\Delta_v(T) := \left(\|Tv\|^2 - \frac{|\langle Tv, v\rangle|^2}{\|v\|^2} \right)^{1/2}.$$

Let A and B be such operators which are each either symmetric or normal. Then for any nonzero v in $\mathcal{D}(AB) \cap \mathcal{D}(BA)$,

$$\frac{1}{4}|\langle [A,B]v, v\rangle|^2 \le \Delta_v(A)^2 \Delta_v(B)^2, \qquad (1)$$

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where $[A, B]$ denotes the commutator $AB - BA$. When A and B are self-adjoint, the inequality is well known, see [5].

Taking $\mathcal{H} = L^2(\mathbb{R})$, the space of complex-valued, square-integrable functions on $\mathbb{R}$, $Af(x) = xf(x)$, $Bf = if'$, leads to the Heisenberg uncertainty principle:

$$\nu(f) \geq \frac{1}{4}, \tag{2}$$

where

$$\nu(f) := \frac{\left(\|xf(x)\|^2 - \dfrac{|\langle xf(x), f(x)\rangle|^2}{\|f\|^2}\right)\left(\|f'\|^2 - \dfrac{|\langle f', f\rangle|^2}{\|f\|^2}\right)}{\|f\|^4}. \tag{3}$$

Taking $\mathcal{H} = L^2[-\pi, \pi]$, the space of 2π-periodic, complex-valued, square-integrable functions, $Ag(t) = e^{it}g(t)$, $Bg = ig'$, gives

$$\mu(g) \geq \frac{1}{4}, \tag{4}$$

where

$$\mu(g) := \frac{\left(\|g\|^2 - \dfrac{|\langle e^{it}g(t), g(t)\rangle|^2}{\|g\|^2}\right)\left(\|g'\|^2 - \dfrac{|\langle g', g\rangle|^2}{\|g\|^2}\right)}{|\langle e^{it}g(t), g(t)\rangle|^2}. \tag{5}$$

This was first established in [1] and, like the Heisenberg uncertainty principle, has an interpretation in physics. Subsequently (4) has been much studied, see [3], [4], [6], [8], [9].

In [7], an uncertainty principle was derived for functions f on the unit sphere $\mathcal{S}^2$ in $\mathbb{R}^3$:

$$\left|\int_{\mathcal{S}^2} x|f(x)|^2\, d\sigma(x)\right|^2 \leq \left(\int_{\mathcal{S}^2} |f|^2\, d\sigma - \frac{\left|\int_{\mathcal{S}^2} x|f(x)|^2\, d\sigma(x)\right|^2}{\int_{\mathcal{S}^2} |f|^2\, d\sigma}\right) \times \int_{\mathcal{S}^2} |\Omega f - a(f)f|^2\, d\sigma, \tag{6}$$

where σ denotes surface measure on $\mathcal{S}^2$, $\Omega f(x) := -ix \times \nabla f(x)$ and $a(f) := \left(\int_{\mathcal{S}^2}(\Omega f)\overline{f}\, d\sigma\right) / \left(\int_{\mathcal{S}^2} |f|^2\, d\sigma\right)$. This inequality is derived in a similar manner to (1) using vector operators A, B and essentially replacing $[A, B]$ by $A \times B + B \times A$. This technique does not extend to higher dimensions but in [2] different techniques were used to derive an uncertainty principle for functions on the unit sphere $\mathcal{S}^n$ in $\mathbb{R}^{n+1}$ for any $n \geq 1$.

We recall the inequality in Section 2 and observe that for real-valued functions, it implies (4) for $n = 1$ and (6) for $n = 2$.

For the Heisenberg uncertainty principle (2), equality is attained for the Gaussian $G(x) = e^{-x^2/2}$. However, for the uncertainty principle (4) for periodic functions, equality is not attained. We also see in Section 2 that there is a simple sequence of trigonometric polynomials p_k of degree k such that as $k \to \infty$, $\mu(p_k) = \frac{1}{4} + O(\frac{1}{k})$, and this result extends to spheres $\mathcal{S}^n$, $n \geq 2$. By taking $0 < \epsilon < 1$ and considering trigonometric polynomials whose coefficients are uniformly sampled from the Gaussian, we can obtain the order of $O(\frac{1}{k^{2(1-\epsilon)}})$ for $n = 1$, and similar results are gained for $n \geq 3$ by considering expansions in ultraspherical polynomials. The work presented in Section 2 is taken mostly from [2] but the exposition here is much clearer as [2] is rather long and technical, giving results in greater generality than here.

An alternative uncertainty principle on the circle is considered in Section 3 which is similar to (4) but includes factors $\cos t$ and $\sin t$ instead of e^{it}. In contrast to (4), equality is attained in this case by the function $g^a(t) := e^{a(\cos t - 1)}$, $-\pi \leq t \leq \pi$, for any real $a \neq 0$. We see that, in both time and frequency domains, g^a satisfies equations analogous to $G'(x) = -x\, G(x)$, $x \in \mathbb{R}$, and g^a converges to a scaled version of G as $a \to \infty$.

§2. Uncertainty Principles on Spheres

Take $n \geq 1$ and let $A_0, \ldots, A_n$, $B_0, \ldots, B_n$ be symmetric or normal operators with domain and range in the same Hilbert space $\mathcal{H}$. Then applying (1) to any nonzero v in $\mathcal{D}(A_j B_j) \cap \mathcal{D}(B_j A_j)$, $j = 0, 1, \ldots, n$, gives

$$\frac{1}{2} \sum_{j=0}^{n} |\langle [A_j, B_j] v, v \rangle| \leq \sum_{j=0}^{n} \Delta_v(A_j) \Delta_v(B_j),$$

and so by the Cauchy-Schwarz inequality,

$$\frac{1}{4} \left(\sum_{j=0}^{n} |\langle [A_j, B_j] v, v \rangle| \right)^2 \leq \left(\sum_{j=0}^{n} \Delta_v(A_j)^2 \right) \left(\sum_{j=0}^{n} \Delta_v(B_j)^2 \right). \tag{7}$$

Now consider $\mathcal{H} = L^2(\mathcal{S}^n)$, the space of complex-valued, square-integrable functions on $\mathcal{S}^n$. For a C^1-function f in $L^2(\mathcal{S}^n)$, let $\nabla_{\mathcal{S}^n} f = (D_0 f, \ldots, D_n f)$ denote the gradient operator restricted to $\mathcal{S}^n$, i.e. for $x \in \mathcal{S}^n$, $x \cdot \nabla_{\mathcal{S}^n} f(x) = 0$ and for $y \in \mathbb{R}^{n+1} \backslash \{0\}$, $y \cdot x = 0$, $y \cdot \nabla_{\mathcal{S}^n} f(x) = y \cdot \nabla f(x)$. For $j = 0, 1, \ldots, n$, define

$$A_j f(x) = x_j f(x), \quad B_j f(x) = i D_j f(x) - \frac{in}{2} x_j f(x), \quad x \in \mathcal{S}^n.$$

It can be checked that A_j and B_j are symmetric. Moreover some calculation shows that

$$\sum_{j=0}^{n} |\langle [A_j, B_j] f, f\rangle| = n\|f\|^2, \tag{8}$$

$$\sum_{j=0}^{n} \Delta_f(A_j)^2 = \|f\|^2 - \frac{|\langle xf(x), f(x)\rangle|^2}{\|f\|^2}, \tag{9}$$

$$\sum_{j=0}^{n} \Delta_f(B_j)^2 = \|\nabla_{\mathcal{S}^n} f\|^2 + \frac{n^2}{4}\|f\|^2 - \frac{|\langle \nabla_{\mathcal{S}^n} f, f\rangle|^2 - \frac{n^2}{4}|\langle xf(x), f(x)\rangle|^2}{\|f\|^2}. \tag{10}$$

Substituting (8)–(10) into (7) gives, after some rearrangement,

$$\frac{n^2}{4}\frac{|\langle xf(x), f(x)\rangle|^4}{\|f\|^4} \le \left(\|f\|^2 - \frac{|\langle xf(x), f(x)\rangle|^2}{\|f\|^2}\right) \times \left(\|\nabla_{\mathcal{S}^n} f\|^2 - \frac{|\langle \nabla_{\mathcal{S}^n} f, f\rangle|^2}{\|f\|^2}\right). \tag{11}$$

Some further calculation shows that (11) implies the weaker inequality

$$\frac{n^2}{4}|\langle xf(x), f(x)\rangle|^2 \le \left(\|f\|^2 - \frac{|\langle xf(x), f(x)\rangle|^2}{\|f\|^2}\right)\|\nabla_{\mathcal{S}^n} f\|^2. \tag{12}$$

We first illustrate the uncertainty principle (12) with the case $n = 1$. Here we write $f(\cos t, \sin t) = g(t)$, $-\pi \le t \le \pi$. Then $\nabla_{\mathcal{S}^1} f(\cos t, \sin t) = (-\sin t, \cos t)g'(t)$ and (12) becomes

$$\frac{1}{4}|\langle e^{it}g(t), g(t)\rangle|^2 \le \left(\|g\|^2 - \frac{|\langle e^{it}g(t), g(t)\rangle|^2}{\|g\|^2}\right)\|g'\|^2, \tag{13}$$

where the inner product is that of $L^2[-\pi, \pi]$. If g is real-valued, $\langle g', g\rangle = 0$ and so (13) coincides with (4).

We also find that for $n = 2$ the inequality (12) for real-valued functions coincides with the inequality (6) derived in [7]. To see this we note that for $x \in \mathcal{S}^2$, $|x \times \nabla f(x)| = |x \times \nabla_{\mathcal{S}^2} f(x)|$ and since $x \cdot \nabla_{\mathcal{S}^2} f(x) = 0$, $|\Omega f(x)| = |x|\,|\nabla_{\mathcal{S}^2} f(x)| = |\nabla_{\mathcal{S}^2} f(x)|$. Also for real-valued f, some calculation shows that $\int_{\mathcal{S}^2} (\Omega f) f \, d\sigma = 0$ and therefore $a(f) = 0$. Thus

$$\int_{\mathcal{S}^2} |\Omega f - a(f) f|^2 \, d\sigma = \int_{\mathcal{S}^2} |\nabla_{\mathcal{S}^2} f|^2 \, d\sigma$$

and so (12) and (6) coincide.

Next we consider (12) for the case when f is radial i.e. for some $u \in \mathcal{S}^n$, $f(x)$ depends only on $x \cdot u$, $x \in \mathcal{S}^n$. Writing $x \cdot u = \cos t$, $0 \le t \le \pi$, we put $f(x) = g(t)$, $0 \le t \le \pi$. Then (12) becomes

$$\frac{n^2}{4}\left(\int_0^\pi \cos t\,|g(t)|^2 w_n(t)\,dt\right)^2 \le \left(\int_0^\pi |g|^2 w_n - \frac{\left(\int_0^\pi \cos t\,|g(t)|^2 w_n(t)\,dt\right)^2}{\int_0^\pi |g|^2 w_n}\right)\int_0^\pi |g'|^2 w_n, \tag{14}$$

where $w_n(t) = (\sin t)^{n-1}$, $0 \le t \le \pi$. We note that if g in (4) is real and even, then (4) reduces to (14).

Inequality (14) was given in [10] for the more general case when n is replaced by any real number $\alpha \ge 1$. In [2] it was extended further by replacing w_n by a much more general weight function w and the left hand side of (14) by

$$\frac{1}{4}\left(\int_0^\pi |g(t)|^2\left(\cos t\, w(t) + \sin t\, w'(t)\right)\,dt\right)^2.$$

It can be shown, see [2], that equality cannot be attained in (14). Rewriting (14) in the form

$$\mu_n(g) \ge \frac{1}{4},$$

we shall consider sequences (g_k) for which $\lim_{k\to\infty} \mu_n(g_k) = \frac{1}{4}$. This then gives sequences of functions for which equality is approached in (12) and hence also in (11).

Indeed we may simply put

$$p_k(t) := (1+\cos t)^k, \quad k = 1, 2, \ldots.$$

Then a routine calculation gives for $n \ge 1$,

$$\mu_n(p_k) = \frac{1}{4} + \frac{1}{2(4k+n-2)}. \tag{15}$$

So equality is approached in (12) for the corresponding sequence (f_k) of radial functions, where for any u in $\mathcal{S}^n$,

$$f_k(x) := (1 + x \cdot u)^k, \quad k = 1, 2, \ldots.$$

It is natural to ask whether we can improve the order of convergence of (15). For simplicity we first consider the case $n = 1$. For g as in (4), we write it in its Fourier series $g(t) = \sum_{j=-\infty}^{\infty} c_j e^{ijt}$, $-\pi \le t \le \pi$. Then a simple calculation from (5) shows that

$$\mu(g) = \frac{\left(\sum |c_j|^2 - \dfrac{\left|\sum c_j \overline{c_{j+1}}\right|^2}{\sum |c_j|^2}\right)\left(\sum j^2 |c_j|^2 - \dfrac{\left(\sum j|c_j|^2\right)^2}{\sum |c_j|^2}\right)}{\left|\sum c_j \overline{c_{j+1}}\right|^2}, \tag{16}$$

where all summations are taken over $j \in \mathbb{Z}$.

Now for $h > 0$ and f a real-valued function in $C^3(\mathbb{R})$, let

$$f_h(t) := \sum_{j=-\infty}^{\infty} c_j e^{ijt}, \qquad c_j = f(jh), \;\; j \in \mathbb{Z}.$$

Then if $f^{(\ell)}$ decays fast enough, $0 \le \ell \le 3$, a consideration of Riemann sums shows, see [2], that as $h \to 0$,

$$h^{\ell+1} \sum_{j=-\infty}^{\infty} j^\ell c_j^2 = \int_{-\infty}^{\infty} x^\ell f(x)^2 \, dx + O(h^2), \quad \ell = 0, 1, 2.$$

Moreover

$$\sum_{j=-\infty}^{\infty} c_j c_{j+1} = \sum_{j=-\infty}^{\infty} c_j^2 - \frac{1}{2} \sum_{j=-\infty}^{\infty} (c_{j+1} - c_j)^2,$$

and as $h \to 0$,

$$h^{-1} \sum_{j=-\infty}^{\infty} (c_{j+1} - c_j)^2 = \int_{-\infty}^{\infty} f'^2 + O(h^2).$$

Substituting these into (16) and recalling (3) shows that as $h \to 0$,

$$\mu(f_h) = \nu(f) + O(h^2).$$

Since the Gaussian G satisfies the decay conditions, then as $h \to 0$,

$$\mu(G_h) = \nu(G) + O(h^2) = \frac{1}{4} + O(h^2).$$

A modification of the above argument shows that if

$$g_h(t) := \sum_{j=-N_h}^{N_h} G(jh)e^{ijt}, \quad -\pi \le t \le \pi, \tag{17}$$

where $N_h^{-1} = O(h^{1+\beta})$ as $h \to 0$ for some $\beta > 0$, then as $h \to 0$,

$$\mu(g_h) = \frac{1}{4} + O(h^2). \tag{18}$$

So taking $c > 0$, $0 < \epsilon < 1$, and defining for $k = 1, 2, \ldots$,

$$p_k(t) := \sum_{j=-k}^{k} G\left(\frac{cj}{k^{1-\epsilon}}\right) e^{ijt}, \quad -\pi \le t \le \pi,$$

(18) gives as $k \to \infty$,

$$\mu(p_k) = \frac{1}{4} + O\left(\frac{1}{k^{2(1-\epsilon)}}\right).$$

To finish this section we consider (14) for $n \ge 2$. In this case we replace (17) by

$$g_{h,n}(t) := \sum_{j=0}^{N_h} h_j G\left(\left(j + \frac{n-1}{2}\right) h\right) P_j(\cos t), \quad 0 \le t \le \pi,$$

where P_j, $j = 0, 1, \ldots$, are the ultraspherical polynomials of index $\frac{1}{2}(n-2)$ normalized by $P_j(1) = 1$, $j = 0, 1, \ldots$, and

$$h_j = \frac{2j + n - 1}{j + n - 1} \binom{j + n - 1}{j}, \quad j = 0, 1, \ldots.$$

Then it is shown in [2] that a similar argument to before gives as $h \to 0$,

$$\mu_n(g_{h,n}) = \frac{1}{4} + O(h^{\min\{n-1,\, 2\}}).$$

So for $n \ge 3$, taking $0 < \epsilon < 1$, we can construct as before a sequence $p_{n,k}$ of trigonometric polynomials of degree k such that as $k \to \infty$,

$$\mu_n(p_{n,k}) = \frac{1}{4} + O\left(\frac{1}{k^{2(1-\epsilon)}}\right).$$

Curiously, for $n = 2$ this approach does not improve on the order of convergence already obtained in (15).

§3. Another Uncertainty Principle on the Circle

We now return to the general uncertainty principle (1) and put $\mathcal{H} = L^2[-\pi,\pi]$, $Bg = ig'$, as before, but now define $Ag(t) = \sin t\, g(t)$. Then (1) gives

$$\frac{1}{4}\langle \cos t\, g(t), g(t)\rangle^2 \le \left(\|\sin t\, g(t)\|^2 - \frac{\langle \sin t\, g(t), g(t)\rangle^2}{\|g\|^2}\right)\left(\|g'\|^2 - \frac{|\langle g', g\rangle|^2}{\|g\|^2}\right). \tag{19}$$

Characterization of equality in (1) is given in [3], which implies that equality holds for v if $(A - i\frac{1}{a}B)v = 0$ for a nonzero real number a. Thus equality holds in (19) if $a\sin t\, g(t) + g'(t) = 0$, $-\pi \le t \le \pi$, which is true when g is the function

$$g^a(t) := e^{-a}e^{a\cos t}, \quad -\pi \le t \le \pi, \tag{20}$$

where we have chosen the normalization $g^a(0) = 1$.

Thus (19) gives an uncertainty principle on the circle for which equality is attained by g^a for any real $a \neq 0$. Recall that for the Heisenberg uncertainty principle, equality is attained by the Gaussian G or more generally by $G^a(x) := e^{-ax^2/2}$, $x \in \mathbb{R}$, for any real $a \neq 0$. In a sense the function g^a in (20) can be regarded as a periodic analog of the scaled Gaussian G^a. For the rest of this paper we look more closely at the connection between g^a and G^a.

In the following result we consider convergence as $a \to \infty$ of g^a to G^a on $[-\pi,\pi]$, where G^a here denotes its restriction to $[-\pi,\pi]$.

Theorem 1. *For* $1 \le p \le \infty$,

$$\lim_{a\to\infty} a^{\frac{1}{2p}}\|g^a - G^a\|_{L^p[-\pi,\pi]} = 0. \tag{21}$$

Proof: First we consider $p = \infty$. Take $\epsilon > 0$ and choose $c > 0$ with $e^{-\frac{1}{2}c^2} < \frac{1}{2}\epsilon$. For $|x| \le c$, as $a \to \infty$,

$$g^a\left(\frac{x}{\sqrt{a}}\right) = e^{-a(1-\cos\frac{x}{\sqrt{a}})} = e^{-a(\frac{1}{2}\frac{x^2}{a} + O(\frac{1}{a^2}))} = e^{-\frac{1}{2}x^2}e^{O(\frac{1}{a})}$$

and so

$$\left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| = e^{-\frac{1}{2}x^2}\left|e^{O(\frac{1}{a})} - 1\right| = O\left(\frac{1}{a}\right). \tag{22}$$

Thus we may choose $A > 0$ so that for $a > A$,

$$\left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| < \frac{1}{2}\epsilon, \quad |x| \le c.$$

Take $a > A$. Then for $c \le |x| \le \sqrt{a}\pi$,

$$g^a\left(\frac{x}{\sqrt{a}}\right) \le g^a\left(\frac{c}{\sqrt{a}}\right) < e^{-\frac{1}{2}c^2} + \frac{1}{2}\epsilon < \epsilon,$$

and so

$$\left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| < \max\left\{g^a\left(\frac{x}{\sqrt{a}}\right), e^{-\frac{1}{2}x^2}\right\} < \epsilon.$$

Therefore for all x with $|x| \le \sqrt{a}\pi$,

$$\left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| < \epsilon,$$

i.e. for all t with $|t| \le \pi$,

$$\left|g^a(t) - e^{-\frac{1}{2}at^2}\right| < \epsilon.$$

So $\lim_{a\to\infty} \|g^a - G^a\|_{L^\infty[-\pi,\pi]} = 0$.

Next we consider $p = 1$. Now for $0 \le t \le \frac{\pi}{2}$, $\sin t \ge \frac{2t}{\pi}$ and so for $0 \le x \le \sqrt{a}\pi$,

$$g^a\left(\frac{x}{\sqrt{a}}\right) = e^{-2a\sin^2\frac{x}{2\sqrt{a}}} \le e^{-\frac{2}{\pi^2}x^2}. \tag{23}$$

Take $\epsilon > 0$ and choose $c > 0$ such that

$$\int_c^\infty e^{-\frac{2}{\pi^2}x^2}\,dx < \epsilon. \tag{24}$$

By (22) we may choose $A > 0$ so that for $a > A$,

$$\int_{-c}^{c} \left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| dx < \epsilon.$$

Also for $\sqrt{a}\pi > c$,

$$\begin{aligned}\int_c^{\sqrt{a}\pi} \left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| dx &\le \int_c^{\sqrt{a}\pi} g^a\left(\frac{x}{\sqrt{a}}\right) dx \\ &\quad + \int_c^{\sqrt{a}\pi} e^{-\frac{1}{2}x^2}\,dx < 2\epsilon,\end{aligned}$$

by (23) and (24) and so $\int_{-\sqrt{a}\pi}^{\sqrt{a}\pi} \left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| dx < 5\epsilon$. Thus

$$\lim_{a\to\infty} \int_{-\sqrt{a}\pi}^{\sqrt{a}\pi} \left|g^a\left(\frac{x}{\sqrt{a}}\right) - e^{-\frac{1}{2}x^2}\right| dx = 0,$$

i.e. $\lim_{a\to\infty}\sqrt{a}\,\|g^a - G^a\|_{L^1[-\pi,\pi]} = 0$.

Finally for any p, $1 < p < \infty$,

$$\|g^a - G^a\|_{L^p[-\pi,\pi]} \le \|g^a - G^a\|_{L^\infty[-\pi,\pi]}^{1-\frac{1}{p}} \|g^a - G^a\|_{L^1[-\pi,\pi]}^{\frac{1}{p}}$$

and hence $\lim_{a\to\infty} a^{\frac{1}{2p}}\|g^a - G^a\|_{L^p[-\pi,\pi]} = 0$. □

Next we consider convergence in the frequency domain. Let

$$g^a(t) = \sum_{j=-\infty}^{\infty} c_j e^{ijt}, \quad -\pi \le t \le \pi. \tag{25}$$

Theorem 2. *As $a \to \infty$,*

$$\sqrt{a}\left(2\pi c_j - \widehat{G}^a(j)\right) \to 0$$

uniformly over j in $\mathbb{Z}$.

Proof: From (25),

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} g^a(t)\cos jt\,dt, \quad j \in \mathbb{Z}.$$

Then by (21) for $p = 1$,

$$\lim_{a\to\infty}\sqrt{a}\left(2\pi c_j - \int_{-\pi}^{\pi} G^a(t)\cos jt\,dt\right) = 0 \tag{26}$$

uniformly over j in $\mathbb{Z}$. Now

$$\sqrt{a}\int_{-\pi}^{\pi} G^a(t)\cos jt\,dt = \int_{-\sqrt{a}\pi}^{\sqrt{a}\pi} e^{-\frac{1}{2}x^2}\cos\frac{jx}{\sqrt{a}}\,dx, \quad j \in \mathbb{Z},$$

and for any $y \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\cos(yx)\,dx = \sqrt{2\pi}e^{-\frac{1}{2}y^2}.$$

Thus

$$\lim_{a\to\infty}\left(\sqrt{a}\int_{-\pi}^{\pi} G^a(t)\cos jt\,dt - \sqrt{2\pi}e^{-\frac{j^2}{2a}}\right) = \lim_{a\to\infty}\int_{|x|\ge\sqrt{a}\pi} e^{-\frac{1}{2}x^2}\cos\frac{jx}{\sqrt{a}}\,dx = 0$$

uniformly over j in $\mathbb{Z}$. So by (26),

$$\lim_{a\to\infty}\left(2\pi\sqrt{a}\,c_j-\sqrt{2\pi}e^{-\frac{j^2}{2a}}\right)=0$$

uniformly over j in $\mathbb{Z}$. Since $\widehat{G}^a(u)=\sqrt{\frac{2\pi}{a}}e^{-\frac{u^2}{2a}}$, $u\in\mathbb{R}$, this gives the result. □

Finally we note that for any a, G^a satisfies the equation

$$(G^a)'(x)=-ax\,G^a(x),\quad x\in\mathbb{R}.$$

Similarly for any a, g^a satisfies the analogous equation

$$(g^a)'(t)=-a\sin t\,g^a(t),\quad -\pi\le t\le\pi. \tag{27}$$

The analogy is even clearer in the frequency domain. From (25) and (27) we see that for any $a\neq 0$,

$$\frac{c_{j+1}-c_{j-1}}{2}=-\frac{j}{a}c_j,\quad j\in\mathbb{Z},$$

which is the discrete analog of the equation

$$(\widehat{G}^a)'(u)=-\frac{u}{a}\widehat{G}^a(u),\quad u\in\mathbb{R}.$$

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