UNCERTAINTY PRINCIPLES AND ASYMPTOTIC BEHAVIOR

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Abstract

Various uncertainty principles for univariate functions are studied, including classes of such principles not considered before. For many uncertainty principles for periodic functions, the lower bound on the uncertainty is not attained. By considering Riemann sums, we show that for functions whose Fourier coefficients are sampled from the Gaussian with spacing h, the uncertainty approaches the lower bound as $h \to 0$ with order $O(h^2)$, whereas earlier work had shown at best $O(h)$. We deduce that there is a sequence of trigonometric polynomials of degree k whose uncertainty approaches the lower bound with order $O(1/k^2)$ as $k \to \infty$. We also establish a general uncertainty principle for n pairs of operators on a Hilbert space, $n = 2, 3, \ldots$, which allows us to extend the above univariate uncertainty principles to such principles for functions of n variables. Furthermore we deduce an uncertainty principle for functions on the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, $n = 2, 3, \ldots$, generalizing known results for radial functions and for real-valued functions on $\mathbb{S}^2$. By considering the above work on univariate uncertainty principles, we can similarly derive, for all our multivariate uncertainty principles, sequences of functions for which the lower bound on the uncertainty is approached.

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1. Introduction

If uncertainty is inevitable, then one naturally wants to be as certain as possible. In the famous Heisenberg uncertainty principle (in the physics community, it is often referred to as the Heisenberg uncertainty relation, see [9]), the uncertainty for functions in $\mathcal{L}^2(\mathbb{R})$ is minimized by the Gaussian function $G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. This uncertainty principle, and many others, can be derived from a very general uncertainty principle, established in [4], involving a pair of quite general linear operators on a Hilbert space. Moreover [4] gives necessary and sufficient conditions on when the uncertainty is minimized. However, in many interesting cases the lower bound on the uncertainty cannot be attained and in these cases it is of importance to consider how this lower bound can be approached.

For example [4] derived, based on some function w in a large class of 2π-periodic functions, an uncertainty principle for functions in $\mathcal{L}^2[-\pi, \pi]$. For the special case $w(t) = e^{it}$, this was proved earlier in [1] where, like the Heisenberg uncertainty principle, the motivation arose in physics. Despite further study of this special case in [4], [5], [7], [10] and [11], it was not noted that the lower bound on the uncertainty is approached by the simple sequence of trigonometric polynomials $p_k(t) := (1 + \cos t)^k$, $k = 1, 2, \ldots$. (We show near the end of Section 2 that the uncertainty for p_k differs from the minimum by $\frac{1}{2(4k-1)}$.) Instead, it was shown in [10] that the lower bound is approached by functions whose Fourier coefficients are uniform samplings of the Gaussian function, i.e. of the form $g_h(t) = \sum_{j=-\infty}^{\infty} G(jh) e^{ijt}$, $-\pi \le t \le \pi$, where $h > 0$. In connection with the construction of refinable functions and wavelets with good time-frequency localization, the authors considered the order of convergence and proved that the uncertainty for g_h differs from the lower bound by $O(\sqrt{h})$ as $h \to 0$.

In [4] this was extended to the uncertainty principles based on quite general functions w and the order of approximation improved to give $O(h)$. In Section 3 we improve this further to give order $O(h^2)$. Although the technical details are rather involved, the basic idea of the proof is simple. Firstly, in Section 2, we rewrite the uncertainty principle for a function f in $\mathcal{L}^2[-\pi, \pi]$ in terms of the Fourier coefficients of f. Then in Section 3, we note that for $f = g_h$ the terms in this uncertainty principle are essentially Riemann sums of the integrals, involving G, which appear in the Heisenberg uncertainty principle. This allows us to prove that the uncertainty for g_h differs from that of G by $O(h^2)$ and the result follows. By suitably truncating the Fourier series for g_h, we also show in Section 3 that there is a sequence of trigonometric polynomials of degree k whose uncertainty differs from the lower bound by $O(1/k^2)$ as $k \to \infty$.

In Section 2 we also derive a new class of uncertainty principles for functions in weighted $\mathcal{L}^2$ spaces on $[0, \infty)$. When the weight function is constant, this uncertainty principle is equivalent to the Heisenberg uncertainty principle applied to even functions. Moreover we show that for these uncertainty principles, the uncertainty is minimized by the Gaussian function G. Similarly we also derive in Section 2 a new class of uncertainty principles for functions in weighted $\mathcal{L}^2$ spaces on $[0, \pi]$ which, when the weight function is constant, reduce to the uncertainty principle in [1] applied to even functions. In Section 3, the above results on order of convergence to the minimum uncertainty are extended to the case of weighted $\mathcal{L}^2$ spaces on $[0, \pi]$, provided that the weight function does not vanish at zero.

For the special case when the weight function is given by a constant multiple of $|\sin t|^\alpha$ for some $\alpha > 0$, the above uncertainty principle was considered in [14]. For this case we show at the end of Section 2 that the uncertainty for the trigonometric polynomials $p_k(t) := (1 + \cos t)^k$, $k = 1, 2, \ldots$, differs from the minimum by $\dfrac{1}{2(4k+\alpha-1)}$. Moreover in Section 3 we derive results on the order of convergence to the minimum uncertainty by employing Riemann sums as before, but in this case expressing the function in the weighted $\mathcal{L}^2$ space on $[0,\pi]$ not in its Fourier series but, following [14], as a series in terms of $P_j(\cos t)$, $j=0,1,\ldots$, where P_j, $j=0,1,\ldots$, are the ultraspherical polynomials of index $\frac{1}{2}(\alpha-1)$.

All the above uncertainty principles involve only univariate functions. In Section 4 we show that the general uncertainty principle in [4] can be extended from a pair of operators on a Hilbert space $\mathcal{H}$ to n pairs of operators on $\mathcal{H}$, $n=1,2,\ldots$. Moreover we give necessary and sufficient conditions on when this uncertainty is minimized. By choosing suitable operators we can extend the uncertainty principles derived in Section 2 to uncertainty principles for functions f of n variables. When $f(x_1,\ldots,x_n)=f_0(x_1)\cdots f_0(x_n)$, for a univariate function f_0 and $x_1,\ldots,x_n\in\mathbb{R}$, these reduce to the corresponding uncertainty principles for f_0. Thus by taking f_0 to be a function which attains or approximates the lower bound in one of the univariate uncertainty principles, the function f will attain or approximate similarly the lower bound in the corresponding multivariate uncertainty principle.

Finally, in Section 5, we consider uncertainty principles for functions on the unit n-sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, $n=2,3,\ldots$. For $n=2$, such an uncertainty principle was derived in [10], but the techniques of the proof do not extend to $n\geq 3$. In [14] an uncertainty principle was given for the general case $n\geq 2$, but only for a function f on $\mathbb{S}^n$ which is radial, i.e. for some unit vector u, $f(x)$ depends only on $x\cdot u$, $x\in\mathbb{S}^n$. In this case, putting $x\cdot u=\cos t$, $f(x)=g(t)$, $0\leq t\leq\pi$, shows that the uncertainty principle for f on $\mathbb{S}^n$ is equivalent to considering g in the uncertainty principle for a weighted $\mathcal{L}^2$ space on $[0,\pi]$ with weight function a constant multiple of $|\sin t|^{n-1}$.

From our general uncertainty principle for a suitable choice of $n+1$ pairs of operators, we deduce, at the end of Section 4, an uncertainty principle for a general function f on $\mathbb{S}^n$ for all $n=1,2,\ldots$. In the special case that f is radial, this implies the uncertainty principle in [14]. Moreover for $n=2$, our uncertainty principle implies that given in [8] when f is real-valued, though curiously these uncertainty principles may differ when f is complex-valued.

Considering, for general $n\geq 1$, the radial functions $f_k(x):=(1+x\cdot u)^k$, $x\in\mathbb{S}^n$, $k=1,2,\ldots$, for a unit vector $u\in\mathbb{S}^n$, the work at the end of Section 2 shows that the uncertainty of f_k differs from the minimum by $\dfrac{1}{2(4k+n-2)}$. Similarly the results in Section 3 on the order of convergence to the minimum uncertainty for functions in weighted $\mathcal{L}^2$ spaces on $[0,\pi]$ give rise to corresponding results for radial functions on $\mathbb{S}^n$, $n\geq 1$.

2. Uncertainty principles

We first recall a general uncertainty principle in [4]. Let $\mathcal{H}$ be a Hilbert space and A and B be any (possibly unbounded) linear operators with domain and range in $\mathcal{H}$ with adjoints A^* and B^*. Suppose that $\mathcal{D}(AB)$, $\mathcal{D}(BA)$, $\mathcal{D}(A^*)$, $\mathcal{D}(B^*)$ are the domains of AB, BA, A^*, B^* respectively. For any nonzero $v\in\mathcal{D}(AB)\cap\mathcal{D}(BA)\cap\mathcal{D}(A^*)\cap\mathcal{D}(B^*)$, we have that

$$\begin{aligned}|\langle[A,B]v,v\rangle| \;\leq\;& \left(\|Av\|^2-\frac{|\langle Av,v\rangle|^2}{\|v\|^2}\right)^{1/2}\left(\|B^*v\|^2-\frac{|\langle B^*v,v\rangle|^2}{\|v\|^2}\right)^{1/2}\\ &+\left(\|A^*v\|^2-\frac{|\langle A^*v,v\rangle|^2}{\|v\|^2}\right)^{1/2}\left(\|Bv\|^2-\frac{|\langle Bv,v\rangle|^2}{\|v\|^2}\right)^{1/2}, \qquad (2.1)\end{aligned}$$

where $[A,B] := AB - BA$ is the *commutator* of A and B. The quantity $\|Av\|^2 - \dfrac{|\langle Av, v\rangle|^2}{\|v\|^2}$ gives the variance of the operator A at the nonzero element v, and it exists when $v \in \mathcal{D}(A)$ which will be assumed throughout the paper. However, we note that in physics, there are instances when the variance fails to exist.

If each of A and B is either symmetric or normal, then (2.1) reduces to the inequality

$$\frac{1}{4}|\langle [A,B]v, v\rangle|^2 \le \left(\|Av\|^2 - \frac{|\langle Av, v\rangle|^2}{\|v\|^2}\right)\left(\|Bv\|^2 - \frac{|\langle Bv, v\rangle|^2}{\|v\|^2}\right), \tag{2.2}$$

which is valid for all nonzero $v \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$. The inequality (2.2) under these conditions was also noted in [16]. Since the operators A and B could be unbounded, even if they are symmetric, they need not be self-adjoint as their domains may not be the same as that of their adjoints. Of course, all self-adjoint operators are symmetric.

Here, we shall mainly be concerned with (2.2). For the special case of A and B being self-adjoint operators, the inequality (2.2) is well known and it is a generalization of the classical Heisenberg uncertainty principle (see, for instance, [3], [6]). Indeed, take $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ the space of complex-valued square-integrable functions with the inner product $\langle f, g\rangle = \int_{\mathbb{R}} f\overline{g}$, $Af(x) = xf(x)$, $Bf(x) = if'(x)$. The domain $\mathcal{D}(A)$ consists of all $f \in \mathcal{L}^2(\mathbb{R})$ for which $Af \in \mathcal{L}^2(\mathbb{R})$, and the domain $\mathcal{D}(B)$ contains all $f \in \mathcal{L}^2(\mathbb{R})$ that are absolutely continuous on every compact interval of $\mathbb{R}$ with $f' \in \mathcal{L}^2(\mathbb{R})$. We define

$$\nu_{\mathbb{R}}(f) := \frac{\left(\|xf(x)\|_2^2 - \dfrac{|\langle xf(x), f(x)\rangle|^2}{\|f\|_2^2}\right)\left(\|f'\|_2^2 - \dfrac{|\langle f', f\rangle|^2}{\|f\|_2^2}\right)}{\|f\|_2^4} \tag{2.3}$$

and (2.2) becomes

$$\frac{1}{4} \le \nu_{\mathbb{R}}(f). \tag{2.4}$$

The quantity $\nu_{\mathbb{R}}(f)$ actually involves two unrelated operators, which is also the case for other expressions of uncertainty in the paper. The domain of validity of the inequality (2.4) can be extended to all f in $\mathcal{L}^2(\mathbb{R})$ for which (2.3) is defined (see, for instance, [12]). Note that when f is real-valued, then $\langle f', f\rangle = 0$.

Our next example was shown in [4]. Take $\mathcal{H} = \mathcal{L}^2[-\pi,\pi]$ the space of 2π-periodic, complex-valued square-integrable functions with the inner product $\langle f, g\rangle_{2\pi} := \int_{-\pi}^{\pi} f\overline{g}$, $Af = wf$, $Bf = if'$, where w is an absolutely continuous 2π-periodic function with $w' \in \mathcal{L}^\infty[-\pi,\pi]\backslash\{0\}$. The operator A is a bounded normal operator, while the operator B is unbounded self-adjoint with domain consisting of all absolutely continuous functions f on $[-\pi,\pi]$ for which $f' \in \mathcal{L}^2[-\pi,\pi]$. Here we define

$$\nu_{2\pi}(f; w) := \frac{\left(\|wf\|_{2\pi}^2 - \dfrac{|\langle wf, f\rangle_{2\pi}|^2}{\|f\|_{2\pi}^2}\right)\left(\|f'\|_{2\pi}^2 - \dfrac{|\langle f', f\rangle_{2\pi}|^2}{\|f\|_{2\pi}^2}\right)}{|\langle w'f, f\rangle_{2\pi}|^2}, \tag{2.5}$$

and (2.2) becomes

$$\frac{1}{4} \le \nu_{2\pi}(f; w), \tag{2.6}$$

for all absolutely continuous f on $[-\pi,\pi]$ with $f' \in \mathcal{L}^2[-\pi,\pi]$ for which (2.5) is defined. When f is real-valued, $\langle f', f\rangle_{2\pi} = 0$. The special case of (2.6) when $w(t) = e^{it}$, i.e.

$$\frac{1}{4} \le \frac{\left(\int_{-\pi}^{\pi} |f|^2 - \frac{\left|\int_{-\pi}^{\pi} e^{it}|f(t)|^2\,dt\right|^2}{\int_{-\pi}^{\pi}|f|^2}\right)\left(\int_{-\pi}^{\pi} |f'|^2 - \frac{\left|\int_{-\pi}^{\pi} f'(t)\overline{f(t)}\,dt\right|^2}{\int_{-\pi}^{\pi}|f|^2}\right)}{\left|\int_{-\pi}^{\pi} e^{it}|f(t)|^2\,dt\right|^2}, \tag{2.7}$$

was first established in [1] and was subsequently studied in detail in [4], [5], [7], [10] and [11]. Although the original form of this inequality in [1] was more compact, we need it as in (2.7) for subsequent discussions.

We now write f and w in their Fourier series

$$f(t) = \sum_{j=-\infty}^{\infty} a_j e^{ijt}, \quad w(t) = \sum_{j=-\infty}^{\infty} w_j e^{ijt}, \quad -\pi \le t \le \pi, \tag{2.8}$$

where we assume that $\sum_{j=-\infty}^{\infty} |j|\,|w_j| < \infty$. Then setting $W_j = \sum_{k=-\infty}^{\infty} w_k\overline{w_{k-j}}$, $j \in \mathbb{Z}$, a routine calculation shows that we can rewrite (2.5) as

$$\nu_{2\pi}(f;w) = \frac{\left(\sum_{\ell=-\infty}^{\infty} W_\ell \sum_{j=-\infty}^{\infty} a_j\overline{a_{j+\ell}} - \frac{\left|\sum_{\ell=-\infty}^{\infty} w_\ell \sum_{j=-\infty}^{\infty} a_j\overline{a_{j+\ell}}\right|^2}{\sum_{j=-\infty}^{\infty}|a_j|^2}\right)\left(\sum_{j=-\infty}^{\infty} j^2|a_j|^2 - \frac{\left(\sum_{j=-\infty}^{\infty} j|a_j|^2\right)^2}{\sum_{j=-\infty}^{\infty}|a_j|^2}\right)}{\left|\sum_{\ell=-\infty}^{\infty} \ell w_\ell \sum_{j=-\infty}^{\infty} a_j\overline{a_{j+\ell}}\right|^2}. \tag{2.9}$$

We now consider two special cases of (2.9). Let $\mathcal{D}$ denote all $c = (c_j)$ in $\ell^2(\mathbb{Z})$, the space of complex square-summable sequences, with $\sum_{j=-\infty}^{\infty} j^2|c_j|^2 < \infty$ and $\sum_{j=-\infty}^{\infty} c_j\overline{c_{j+1}} \neq 0$, and for $c \in \mathcal{D}$ write

$$\Lambda_1(c) := \sum_{j=-\infty}^{\infty} j^2|c_j|^2 - \frac{\left(\sum_{j=-\infty}^{\infty} j|c_j|^2\right)^2}{\sum_{j=-\infty}^{\infty}|c_j|^2}, \qquad \Lambda_2(c) := \sum_{j=-\infty}^{\infty} |c_j|^2 - \frac{\left|\sum_{j=-\infty}^{\infty} c_j\overline{c_{j+1}}\right|^2}{\sum_{j=-\infty}^{\infty}|c_j|^2},$$

$$\Lambda_3(c) := \sum_{j=-\infty}^{\infty} |c_{j+1}-c_j|^2 - \frac{\left|\displaystyle\sum_{j=-\infty}^{\infty} (c_{j+1}-c_j)\overline{c_j}\right|^2}{\displaystyle\sum_{j=-\infty}^{\infty} |c_j|^2}, \qquad \Lambda_4(c) := \left|\sum_{j=-\infty}^{\infty} c_j\overline{c_{j+1}}\right|^2.$$

Then putting $w(t) = e^{it}$, $w(t) = e^{it} - 1$, respectively, in (2.9) gives

$$\frac{1}{4} \le \nu_{\ell^2}(c), \qquad \frac{1}{4} \le \mu_{\ell^2}(c), \tag{2.10}$$

for all $c \in \mathcal{D}$, where

$$\nu_{\ell^2}(c) := \frac{\Lambda_1(c)\Lambda_2(c)}{\Lambda_4(c)}, \qquad \mu_{\ell^2}(c) := \frac{\Lambda_1(c)\Lambda_3(c)}{\Lambda_4(c)}. \tag{2.11}$$

Note that the first inequality in (2.10) is the well-known inequality (2.7), and the second inequality in (2.10) is a discrete analogue of (2.3) apart from a slight difference in the term in the denominator.

For our next example we consider the space $\mathcal{L}^2_w[-\pi,\pi]$ of complex-valued functions on $[-\pi,\pi]$ with the weighted inner product $\langle f,g\rangle_w = \int_{-\pi}^{\pi} f\overline{g}w$. Here $w \in \mathcal{C}[-\pi,\pi]$ is a non-negative, even function with at most a finite number of zeros such that the function $\frac{t(\pi-t)w'(t)}{w(t)}$, $0<t<\pi$, lies in $\mathcal{L}^\infty[0,\pi]$. Define $Af(t) = e^{it}f(t)$ and $Bf(t) = if'(t) + \frac{i}{2}\frac{w'(t)}{w(t)}(f(t)-f(-t))$. The operator A is bounded and normal. We take the domain of B to be $\mathcal{D}(B) = \{f \in \mathcal{C}^1[-\pi,\pi] : f(-\pi) = f(\pi)\}$. Since $\mathcal{D}(B)$ contains all trigonometric polynomials, $\mathcal{D}(B)$ is dense in $\mathcal{L}^2_w[-\pi,\pi]$. Furthermore, it is straightforward to show that B is symmetric. Thus (2.2) holds. Note that $[A,B]f(t) = e^{it}f(t) + \frac{w'(t)}{w(t)}f(-t)\sin t$.

For all even functions $f \in \mathcal{D}(B)$ such that $\langle [A,B]f,f\rangle_w \ne 0$, (2.2) becomes

$$\frac{1}{4} \le \mu_\pi(f;w), \tag{2.12}$$

where

$$\mu_\pi(f;w) := \frac{\displaystyle\int_0^\pi |f'(t)|^2 w(t)\,dt \left(\int_0^\pi |f(t)|^2 w(t)\,dt - \frac{\left(\displaystyle\int_0^\pi \cos t\,|f(t)|^2 w(t)\,dt\right)^2}{\displaystyle\int_0^\pi |f(t)|^2 w(t)\,dt}\right)}{\left(\displaystyle\int_0^\pi |f(t)|^2\left(\cos t\, w(t) + \sin t\, w'(t)\right)\,dt\right)^2}. \tag{2.13}$$

To our knowledge, (2.12) has not been given before in this generality. However it was established in [14], by different methods, the special case of (2.12) when for $\alpha \ge 0$, $w = w_\alpha$ which is given by

$$w_\alpha(t) = c_\alpha |\sin t|^\alpha, \quad -\pi \le t \le \pi, \tag{2.14}$$

where c_α is chosen so that $\int_0^\pi w_\alpha(t)\,dt = 1$. We now take $\alpha \ge 0$ and rewrite (2.13) in terms of sequences in a similar manner to (2.5) being written as (2.9). Let P_j, $j = 0,1,\ldots,$ be the ultraspherical polynomials of index $\frac{1}{2}(\alpha-1)$, normalized by $P_j(1) = 1$, $j = 0,1,\ldots$. (For information on ultraspherical polynomials, see [17].) Define

$$h_0 := 1, \quad h_j := \frac{(2j+\alpha)\alpha(\alpha+1)\cdots(\alpha+j-1)}{\alpha\, j!}, \quad j = 1, 2, \dots , \tag{2.15}$$

$$\lambda_0 := 1, \quad \lambda_j := \frac{j+\alpha}{2j+\alpha}, \quad j = 1, 2, \dots . \tag{2.16}$$

For later use we note that for $j = 0, 1, \dots,$

$$\lambda_j h_j = \frac{(\alpha+1)\cdots(\alpha+j)}{j!} = (1-\lambda_{j+1})h_{j+1}. \tag{2.17}$$

Then proceeding as in [14], it follows from (2.13) that

$$\mu_\pi(f; w_\alpha) = \frac{\sum_{j=0}^{\infty} h_j j(j+\alpha)|a_j|^2}{(\alpha+1)^2 \sum_{j=0}^{\infty} h_j |a_j|^2} \left[\left(\frac{\sum_{j=0}^{\infty} h_j |a_j|^2}{\sum_{j=0}^{\infty} 2\lambda_j h_j \operatorname{Re}\{a_j \overline{a_{j+1}}\}} \right)^2 - 1 \right]. \tag{2.18}$$

For our final example of this section, we give a generalization of (2.4) for even functions f. Let $w \in \mathcal{C}(\mathbb{R})$ be a non-negative, even function with at most a finite number of zeros for which $\frac{xw'(x)}{w(x)}$ is bounded on compact subsets of $[0, \infty)$, and consider the space $\mathcal{L}^2_w(\mathbb{R})$ of complex-valued functions on $\mathbb{R}$ with the weighted inner product $\langle f, g\rangle_w = \int_{\mathbb{R}} f\overline{g}w$. Define $Af(x) = xf(x)$ and $Bf(x) = if'(x) + \frac{i}{2}\frac{w'(x)}{w(x)}(f(x) - f(-x))$. The operator A is self-adjoint with domain consisting of all $f \in \mathcal{L}^2_w(\mathbb{R})$ for which $Af \in \mathcal{L}^2_w(\mathbb{R})$. We set the domain of B to be $\mathcal{D}(B) = \{f \in \mathcal{L}^2_w(\mathbb{R}) : f \in \mathcal{C}^1(\mathbb{R}), Bf \in \mathcal{L}^2_w(\mathbb{R})\}$. This set includes all $\mathcal{C}^\infty$ functions of compact support. As in the previous example, $\mathcal{D}(B)$ is dense in $\mathcal{L}^2_w(\mathbb{R})$ and B is symmetric. Here $[A, B]f(x) = -if(x) - i\frac{w'(x)}{w(x)}xf(-x)$.

When $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ is even with $\langle [A, B]f, f\rangle_w \neq 0$, (2.2) then becomes

$$\frac{1}{4} \le \mu_{\mathbb{R}}(f; w), \tag{2.19}$$

where

$$\mu_{\mathbb{R}}(f; w) := \frac{\int_0^\infty x^2|f(x)|^2 w(x)\,dx \int_0^\infty |f'(x)|^2 w(x)\,dx}{\left(\int_0^\infty |f(x)|^2 \left(w(x) + xw'(x)\right)\,dx\right)^2}. \tag{2.20}$$

For the special case when for $\alpha \ge 0$, $w = \omega_\alpha$ which is given by

$$\omega_\alpha(x) = |x|^\alpha, \quad x \in \mathbb{R}, \tag{2.21}$$

(2.19) essentially reduces to a particular case of an uncertainty principle considered in [15].

In this paper, we are concerned with when the lower bound in (2.2) can be attained or approached. This issue will be addressed for all the above examples. A characterization of equality in a more general setting is given in Theorem 7 of [4] which implies the following for symmetric operators.

Theorem 2.1. *Let A and B be any symmetric operators with domain and range in the same Hilbert space $\mathcal{H}$, and $v \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be nonzero. Either v is an eigenvector of A or B and equality holds in* (2.2) *or equality holds in* (2.2) *if and only there exists a nonzero constant $\mu \in \mathbb{R}$ such that v is an eigenvector of the operator $A - i\mu B$.*

Note that the upper bound of (2.2) equals zero if and only if v is an eigenvector of A or B. Thus if $\langle [A, B]v, v\rangle \neq 0$, then equality in (2.2) is equivalent to v being an eigenvector of $A - i\mu B$ for some nonzero $\mu \in \mathbb{R}$.

We begin with the inequality (2.19) for even $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ such that $\langle [A, B]f, f\rangle_w \neq 0$. In this case, equality holds if and only if f is an eigenvector of the operator $A - i\mu B$ for some nonzero $\mu \in \mathbb{R}$. Since f is even, this condition amounts to the differential equation $xf(x) + \mu f'(x) = 0$, $x \in \mathbb{R}$. For $\mu > 0$, the solution of this differential equation is of the form

$$f(x) = c\, G(x/\sqrt{\mu}), \quad x \in \mathbb{R}, \tag{2.22}$$

where $c \in \mathbb{C}\backslash\{0\}$ and G is the Gaussian function given by

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}. \tag{2.23}$$

Hence, if w is a weight function for which G lies in $\mathcal{D}(AB) \cap \mathcal{D}(BA)$ (for example, $w = \omega_\alpha$, $\alpha \geq 0$, of (2.21)), then equality holds in (2.19) if f is of the form (2.22). For the special case $w = 1$, this shows that equality is attained in (2.4) for $f = c\, G(\cdot/\sqrt{\mu})$, where $c \in \mathbb{C}\backslash\{0\}$ and $\mu > 0$, as is well known.

We next address the issue of equality in (2.12). Since the operator A in this case is normal, we have to apply Theorem 7 of [4] instead of Theorem 2.1 above. Following the same arguments as Proposition 5 of [4], we see that it is not possible for equality to hold in (2.12) for an even function $f \in \mathcal{D}(B)$ that satisfies $\langle [A, B]f, f\rangle_w \neq 0$. Thus we turn to the possibility of asymptotic equality. At this moment, we consider only the case $w = w_\alpha$, $\alpha \geq 0$, of (2.14), and postpone the study of other situations to Section 3.

Theorem 2.2. *For $\alpha \geq 0$, $p_k(t) := (1 + \cos t)^k$, $k = 1, 2, \ldots,$*

$$\mu_\pi(p_k; w_\alpha) = \frac{1}{4} + \frac{1}{2(4k + \alpha - 1)}.$$

Proof: Here,

$$2^{-2k} \int_0^\pi p_k(t)^2 \sin^\alpha t\, dt = \int_0^\pi \cos^{4k}\left(\frac{t}{2}\right) \left(2 \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right)\right)^\alpha dt = 2^{\alpha+1} I_{4k,\alpha}, \tag{2.24}$$

where for $m \geq 0$,

$$I_{m,\alpha} := \int_0^{\pi/2} \cos^{m+\alpha} u\, \sin^\alpha u\, du.$$

By integration by parts, for $m \geq 2$,

$$I_{m,\alpha} = \frac{m + \alpha - 1}{\alpha + 1} I_{m-4,\alpha+2} = \frac{m + \alpha - 1}{\alpha + 1} (I_{m-2,\alpha} - I_{m,\alpha}) \tag{2.25}$$

and so

$$I_{m,\alpha} = \frac{m + \alpha - 1}{m + 2\alpha} I_{m-2,\alpha}. \tag{2.26}$$

Also

$$2^{-2k} \int_0^\pi p_k'(t)^2 \sin^\alpha t\, dt = k^2 2^{\alpha+1} I_{4k-4,\alpha+2} = \frac{k^2 2^{\alpha+1}(\alpha + 1)}{4k + \alpha - 1} I_{4k,\alpha}, \tag{2.27}$$

by (2.25). Moreover

$$2^{-2k}\int_0^\pi \cos t\, p_k(t)^2 \sin^\alpha t\, dt = 2^{\alpha+2} I_{4k+2,\alpha} - 2^{\alpha+1} I_{4k,\alpha} = \frac{2^{\alpha+2}k}{2k+1+\alpha} I_{4k,\alpha}, \tag{2.28}$$

by (2.26). Thus by (2.13), (2.24), (2.27) and (2.28),

$$\mu_\pi(p_k; w_\alpha) = \frac{1}{(\alpha+1)^2}\frac{k^2(\alpha+1)}{4k+\alpha-1}\left[\left(\frac{2k+1+\alpha}{2k}\right)^2 - 1\right] = \frac{1}{4}\frac{4k+\alpha+1}{4k+\alpha-1}.$$

■

Putting $\alpha = 0$ in Theorem 2.2, i.e. $w = 1$, gives asymptotic equality in (2.7), i.e.

$$\nu_{2\pi}(p_k; e^{i\cdot}) = \frac{1}{4} + \frac{1}{2(4k-1)}.$$

The trigonometric polynomials p_k, $k = 1, 2, \ldots$, form a new and simple sequence of functions in $\mathcal{L}^2[-\pi, \pi]$ that attains asymptotic equality for the well-studied inequality (2.7).

3. Riemann sums

For our study of asymptotic behavior of uncertainty products, we need some preliminary results on approximating integrals by Riemann sums. For $\alpha > 0$ we denote by $\mathcal{S}(\alpha)$ the set of all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $|f(x)| \le C(1+|x|)^{-\alpha}$, $x \in \mathbb{R}$, for some constant C.

Lemma 3.1. *Take f in $\mathcal{S}(\alpha_0)$ with f'' in $\mathcal{S}(\alpha_2)$ for some $\alpha_0, \alpha_2 > 1$. Then for $h > 0$,*

$$h\sum_{j=-\infty}^{\infty} f(jh) - \int_{-\infty}^{\infty} f = O(h^2) \tag{3.1}$$

as $h \to 0$. Moreover if $N_h^{-1} = O\left(h^{1+\frac{2}{\alpha_0-1}}\right)$, then as $h \to 0$,

$$h\sum_{j=-N_h}^{N_h} f(jh) - \int_{-\infty}^{\infty} f = O(h^2). \tag{3.2}$$

Proof: Take $a \in \mathbb{R}$, $h > 0$. Then for each x in $[a, a+h]$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(\eta)(x-a)^2$$

for some η in (a, x). In particular,

$$\frac{1}{h}\left(f(a+h) - f(a)\right) = f'(a) + \frac{1}{2}hf''(\tau)$$

for some τ in $(a, a+h)$. So letting

$$\ell(x) = f(a) + \frac{1}{h}\left(f(a+h) - f(a)\right)(x-a), \quad x \in \mathbb{R},$$

we have for $a \le x \le a+h$,

$$|f(x) - \ell(x)| \le \frac{1}{2}h|f''(\tau)|\,|x-a| + \frac{1}{2}|f''(\eta)|\,|x-a|^2 \le \frac{1}{2}h^2\left(|f''(\tau)| + |f''(\eta)|\right).$$

Since

$$\int_a^{a+h} \ell = \frac{1}{2}h\left(f(a) + f(a+h)\right),$$

we have for $N \geq 1$,

$$h \sum_{j=-N+1}^{N-1} f(jh) + \frac{1}{2} h\left(f(-Nh) + f(Nh)\right) - \int_{-Nh}^{Nh} f = O(h^2) \int_0^{Nh} (1+x)^{-\alpha_2}\, dx = O(h^2)$$

as $h \to 0$. Letting $N \to \infty$ gives (3.1).

Now (3.2) holds provided

$$\frac{1}{2} h\left(f(-N_h h) + f(N_h h)\right) + \int_{N_h h}^{\infty} |f| + \int_{-\infty}^{-N_h h} |f| = O(h^2)$$

as $h \to 0$. Note that the left-hand side is of order $h(N_h h)^{-\alpha_0} + (N_h h)^{-\alpha_0+1}$, which is $O(h^2)$ if $N_h^{-1} = O\left(h^{1+\frac{2}{\alpha_0 - 1}}\right)$. ■

By applying Lemma 3.1 to the functions f^2, $xf(x)^2$ and $x^2 f(x)^2$, we can deduce the following.

Lemma 3.2. *Take f in $\mathcal{S}(\alpha_0)$ with $f^{(\ell)}$ in $\mathcal{S}(\alpha_\ell)$, $\ell = 1, 2$, where $\alpha_0, \alpha_1, \alpha_2 > \frac{3}{2}$. Then for $h > 0$, $a_j = f(jh)$, $j \in \mathbb{Z}$,*

$$h \sum_{j=-\infty}^{\infty} a_j^2 - \int_{-\infty}^{\infty} f^2 = O(h^2), \tag{3.3}$$

$$h^2 \sum_{j=-\infty}^{\infty} j a_j^2 - \int_{-\infty}^{\infty} x f(x)^2\, dx = O(h^2), \tag{3.4}$$

$$h^3 \sum_{j=-\infty}^{\infty} j^2 a_j^2 - \int_{-\infty}^{\infty} x^2 f(x)^2\, dx = O(h^2), \tag{3.5}$$

as $h \to 0$. Moreover if $N_h^{-1} = O\left(h^{1+\frac{2}{2\alpha_0 - 3}}\right)$, then for $h > 0$, (3.3)–(3.5) hold as $h \to 0$ with

$$a_j = \begin{cases} f(jh), & -N_h \leq j \leq N_h, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Lemma 3.3. *Take f in $\mathcal{S}(\alpha_0)$ with $f^{(\ell)}$ in $\mathcal{S}(\alpha_\ell)$, $\ell = 1, 2, 3$, where $\alpha_0 > 0$ and $\alpha_1, \alpha_2, \alpha_3 > \frac{1}{2}$. Then for $h > 0$, $a_j = f(jh)$, $j \in \mathbb{Z}$,*

$$h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{n^2} (a_{j+n} - a_j)^2 - \int_{-\infty}^{\infty} f'^2 = O(n^2 h^2), \tag{3.7}$$

as $h \to 0$, uniformly for integers $n \geq 1$. Moreover if $N_h^{-1} = O(h^{1+\beta})$, $\beta \geq \frac{3}{\alpha_0}, \frac{2}{2\alpha_1 - 1}$, then for $h > 0$, (3.7) holds as $h \to 0$ with (a_j) given by (3.6).

Proof: From Lemma 3.1 we have, as $h \to 0$,

$$h \sum_{j=-\infty}^{\infty} f'\left(\left(j + \frac{1}{2}\right) h\right)^2 - \int_{-\infty}^{\infty} f'^2 = O(h^2).$$

Now for $a \in \mathbb{R}$, $h > 0$, by Taylor's theorem, there are η, τ in $\left(a - \frac{1}{2}h, a + \frac{1}{2}h\right)$ with

$$\frac{1}{h}\left(f\left(a+\frac{h}{2}\right) - f\left(a-\frac{h}{2}\right)\right) = f'(a) + \frac{1}{48}h^2\left(f^{(3)}(\eta) + f^{(3)}(\tau)\right).$$

Thus

$$h\sum_{j=-\infty}^{\infty}\left(\frac{f((j+1)h) - f(jh)}{h}\right)^2 - h\sum_{j=-\infty}^{\infty} f'\left(\left(j+\frac{1}{2}\right)h\right)^2 = O(h^2)\int_0^\infty (1+x)^{-\alpha_1-\alpha_3}\,dx = O(h^2),$$

as $h \to 0$, and so

$$h^{-1}\sum_{j=-\infty}^{\infty}(f((j+1)h) - f(jh))^2 - \int_{-\infty}^{\infty} f'^2 = O(h^2).$$

By applying the above proof to $f(\cdot + \alpha h)$, we can see that

$$h^{-1}\sum_{j=-\infty}^{\infty}(f((j+1+\alpha)h) - f((j+\alpha)h))^2 - \int_{-\infty}^{\infty} f'^2 = O(h^2) \tag{3.8}$$

uniformly over α in $[0,1]$.

Now take any integer $n \geq 1$. Then for $\ell = 0, 1, \ldots, n-1$, replacing h by nh and α by $\frac{\ell}{n}$ in (3.8) gives

$$(nh)^{-1}\sum_{j=-\infty}^{\infty}(f((jn+n+\ell)h) - f((jn+\ell)h))^2 - \int_{-\infty}^{\infty} f'^2 = O(n^2h^2).$$

Summing over $\ell = 0, 1, \ldots, n-1$ gives

$$(nh)^{-1}\sum_{j=-\infty}^{\infty}(f((j+n)h) - f(jh))^2 - n\int_{-\infty}^{\infty} f'^2 = O(n^3h^2),$$

which gives (3.7) for $a_j = f(jh)$, $j \in \mathbb{Z}$.

Next take any integer $N_h \geq 1$ and let (a_j) be given by (3.6). Then for $n \geq 1$,

$$\begin{aligned}
&\left|\sum_{j=-\infty}^{\infty}(f((j+n)h) - f(jh))^2 - \sum_{j=-\infty}^{\infty}(a_{j+n} - a_j)^2\right| \\
\leq\ & \sum_{j \geq N_h+1}(f((j+n)h) - f(jh))^2 + \sum_{j+n \leq -N_h-1}(f((j+n)h) - f(jh))^2 + O(n)\sup_{|j| \geq N_h+1}|f(jh)| \\
=\ & O(n^2h)\int_{N_hh}^{\infty}(1+x)^{-2\alpha_1}\,dx + O(n)(N_hh)^{-\alpha_0} = O(n^2h^3),
\end{aligned}$$

by the condition on N_h, and (3.7) follows for (a_j) given by (3.6). ■

We can now relate, in the next two theorems, the uncertainty products $\nu_{2\pi}(f;w)$ in (2.9) and $\mu_\pi(f;w)$ in (2.13) with $\nu_{\mathbb{R}}(f)$ in (2.3).

Theorem 3.1. *Take f in $\mathcal{S}(\alpha_0)$ with $f^{(\ell)}$ in $\mathcal{S}(\alpha_\ell)$, $\ell = 1, 2, 3$, where $\alpha_0, \alpha_1, \alpha_2 > \frac{3}{2}$ and $\alpha_3 > \frac{1}{2}$. For $h > 0$, define for $-\pi \leq t \leq \pi$,*

$$f_h(t) = \sum_{j=-\infty}^{\infty} f(jh)e^{ijt}, \qquad \widetilde{f}_h(t) = \sum_{j=-N_h}^{N_h} f(jh)e^{ijt}, \tag{3.9}$$

where $N_h^{-1} = O(h^{1+\beta})$ *as* $h \to 0$, *for some* $\beta \geq \dfrac{2}{2\alpha_0 - 3}, \dfrac{3}{\alpha_0}, \dfrac{2}{2\alpha_1 - 1}$. *Let* $w(t) = \sum_{j=-\infty}^{\infty} w_j e^{ijt}$, $-\pi \leq t \leq \pi$, *where* $\sum_{j=-\infty}^{\infty} |j|^5 |w_j| < \infty$ *and* $w'(0) \neq 0$. *Then as* $h \to 0$,

$$\nu_{2\pi}(f_h; w) - \nu_{\mathbb{R}}(f) = O(h^2), \qquad \nu_{2\pi}(\widetilde{f}_h; w) - \nu_{\mathbb{R}}(f) = O(h^2). \tag{3.10}$$

Proof: Take $a = (a_j) \in \ell^2(\mathbb{Z})$, $a_j \in \mathbb{R}$. We shall make much use of the identity, for any integer n,

$$\sum_{j=-\infty}^{\infty} a_j a_{j+n} = \sum_{j=-\infty}^{\infty} a_j^2 - \frac{1}{2} \sum_{j=-\infty}^{\infty} (a_{j+n} - a_j)^2 . \tag{3.11}$$

Now by (2.9), for $g(t) = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$, $-\pi \leq t \leq \pi$,

$$\nu_{2\pi}(g; w) = \frac{A_1 A_2}{A_3}, \tag{3.12}$$

where for any $h > 0$,

$$A_1 := \sum_{j=-\infty}^{\infty} a_j^2 \sum_{\ell=-\infty}^{\infty} W_\ell \sum_{j=-\infty}^{\infty} a_j a_{j+\ell} - \left| \sum_{\ell=-\infty}^{\infty} w_\ell \sum_{j=-\infty}^{\infty} a_j a_{j+\ell} \right|^2 ,$$

$$A_2 := h \sum_{j=-\infty}^{\infty} a_j^2 \, h^3 \sum_{j=-\infty}^{\infty} j^2 a_j^2 - \left(h^2 \sum_{j=-\infty}^{\infty} j a_j^2 \right)^2 , \tag{3.13}$$

$$A_3 := \left(h \sum_{j=-\infty}^{\infty} a_j^2 \right)^2 \left| \sum_{\ell=-\infty}^{\infty} \ell w_\ell \, h \sum_{j=-\infty}^{\infty} a_j a_{j+\ell} \right|^2 .$$

Recalling (3.11) and noting that $\sum_{\ell=-\infty}^{\infty} W_\ell = |w(0)|^2$, $\sum_{\ell=-\infty}^{\infty} w_\ell = w(0)$, some calculation shows that for any $h > 0$,

$$\begin{aligned} A_1 \quad = \quad & -\frac{1}{2} \left(h \sum_{j=-\infty}^{\infty} a_j^2 \right) \sum_{\ell=-\infty}^{\infty} \ell^2 W_\ell \, h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{\ell^2} (a_{j+\ell} - a_j)^2 \\ & + \operatorname{Re} \left\{ w(0) \left(h \sum_{j=-\infty}^{\infty} a_j^2 \right) \sum_{\ell=-\infty}^{\infty} \ell^2 \overline{w_\ell} \, h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{\ell^2} (a_{j+\ell} - a_j)^2 \right\} \\ & - \frac{h^2}{4} \left| \sum_{\ell=-\infty}^{\infty} \ell^2 w_\ell \, h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{\ell^2} (a_{j+\ell} - a_j)^2 \right|^2 . \end{aligned} \tag{3.14}$$

Also, since $\sum_{\ell=-\infty}^{\infty} \ell w_\ell = -iw'(0)$, we have

$$A_3 = \left(h \sum_{j=-\infty}^{\infty} a_j^2 \right)^2 \left| w'(0)\, h \sum_{j=-\infty}^{\infty} a_j^2 - i\frac{h^2}{2} \sum_{\ell=-\infty}^{\infty} \ell^3 w_\ell\, h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{\ell^2} (a_{j+\ell} - a_j)^2 \right|^2 . \tag{3.15}$$

Now put $a_j = f(jh)$, $j \in \mathbb{Z}$. Then applying (3.3) and (3.7) to (3.14) and noting that

$$-\frac{1}{2} \sum_{\ell=-\infty}^{\infty} \ell^2 W_\ell + \mathrm{Re} \left\{ w(0) \sum_{\ell=-\infty}^{\infty} \ell^2 \overline{w_\ell} \right\} = |w'(0)|^2,$$

we can show that as $h \to 0$,

$$A_1 = |w'(0)|^2 \int_{-\infty}^{\infty} f^2 \int_{-\infty}^{\infty} f'^2 + O(h^2). \tag{3.16}$$

Also applying (3.3)–(3.5) to (3.13) gives

$$A_2 = \int_{-\infty}^{\infty} f^2 \int_{-\infty}^{\infty} x^2 f(x)^2\, dx - \left(\int_{-\infty}^{\infty} x f(x)^2\, dx \right)^2 + O(h^2). \tag{3.17}$$

Finally, applying (3.3) and (3.7) to (3.15) gives

$$A_3 = |w'(0)|^2 \left(\int_{-\infty}^{\infty} f^2 \right)^4 + O(h^2). \tag{3.18}$$

So comparing (3.12), (3.16)–(3.18) with (2.3) and recalling that f is real-valued gives the first result in (3.10).

Next suppose that (a_j) is given by (3.6). Then exactly as before we can apply (3.3)–(3.5) and (3.7) to (3.13)–(3.15) to deduce (3.16)–(3.18) and hence infer the second result in (3.10). ∎

Recalling that the two quantities in (2.11) are special cases of (2.9), we immediately have the following.

Corollary 3.1. *Assume the conditions of Theorem* 3.1 *and define, for* $h > 0$, $c_h = (c_j)$, $\widetilde{c}_h = (\widetilde{c}_j)$, *where*

$$c_j := f(jh), \quad j \in \mathbb{Z}, \qquad \widetilde{c}_j := \begin{cases} f(jh), & -N_h \le j \le N_h, \\ 0, & \textit{otherwise}. \end{cases}$$

Then as $h \to 0$,

$$\nu_{\ell^2}(c_h) - \nu_{\mathbb{R}}(f) = O(h^2), \qquad \nu_{\ell^2}(\widetilde{c}_h) - \nu_{\mathbb{R}}(f) = O(h^2),$$
$$\mu_{\ell^2}(c_h) - \nu_{\mathbb{R}}(f) = O(h^2), \qquad \mu_{\ell^2}(\widetilde{c}_h) - \nu_{\mathbb{R}}(f) = O(h^2).$$

Recalling that $\nu_{\mathbb{R}}(G) = \frac{1}{4}$ for the Gaussian function G, we can immediately deduce from Theorem 3.1 the following.

Corollary 3.2. *Define for* $-\pi \le t \le \pi$,

$$g_h(t) = \sum_{j=-\infty}^{\infty} G(jh) e^{ijt}, \qquad \widetilde{g}_h(t) = \sum_{j=-N_h}^{N_h} G(jh) e^{ijt}, \tag{3.19}$$

where $N_h^{-1} = O(h^{1+\beta})$ *as* $h \to 0$ *for some* $\beta > 0$. *Then with* w *as in Theorem* 3.1, *as* $h \to 0$,

$$\nu_{2\pi}(g_h; w) = \frac{1}{4} + O(h^2), \qquad \nu_{2\pi}(\widetilde{g}_h; w) = \frac{1}{4} + O(h^2). \tag{3.20}$$

Moreover with $c_h = (c_j)$, $\widetilde{c}_h = (\widetilde{c}_j)$, *where for* $h > 0$,

$$c_j := G(jh), \quad j \in \mathbb{Z}, \qquad \widetilde{c}_j := \begin{cases} G(jh), & -N_h \le j \le N_h, \\ 0, & \textit{otherwise}, \end{cases}$$

we have, as $h \to 0$,

$$\nu_{\ell^2}(c_h) = \frac{1}{4} + O(h^2), \qquad \nu_{\ell^2}(\widetilde{c}_h) = \frac{1}{4} + O(h^2),$$
$$\mu_{\ell^2}(c_h) = \frac{1}{4} + O(h^2), \qquad \mu_{\ell^2}(\widetilde{c}_h) = \frac{1}{4} + O(h^2).$$

We note that (3.20) is related to Theorem 2.2 as follows. For $k = 1, 2, \ldots$, with p_k as in Theorem 2.2, we have

$$2^k p_k(t) = (1 + e^{it})^k (1 + e^{-it})^k = \sum_{j=-k}^{k} \binom{2k}{k+j} e^{ijt},$$

and by Stirling's formula,

$$\binom{2k}{k+j} \approx 2^{2k} \sqrt{\frac{2}{k}}\, G\left(j \sqrt{\frac{2}{k}} \right) \tag{3.21}$$

as $k \to \infty$. While (3.21) gives asymptotic behavior of the coefficients of $(1+z)^{2k}$, in [2] a stronger form of (3.21) is given for coefficients of polynomials whose zeros lie in a large region of the complex plane. This work can be used to derive asymptotic equality for uncertainty principles for a broad class of trigonometric polynomials which we shall not discuss here.

Next we consider the uncertainty product $\mu_\pi(f; w)$ in (2.13).

Theorem 3.2. *Take an even function* f *in* $\mathcal{S}(\alpha_0)$ *with* $f^{(\ell)}$ *in* $\mathcal{S}(\alpha_\ell)$, $\ell = 1, 2, 3$, *where* $\alpha_\ell > \frac{3}{2}$, $\ell = 0, 1, 2, 3$. *For* $h > 0$, *define* f_h, $\widetilde{f}_h$ *as in* (3.9), *where* $N_h^{-1} = O(h^{1+\beta})$ *as* $h \to 0$, *with* $\beta \ge \frac{2}{2\alpha_0 - 3}, \frac{3}{\alpha_0 - 1}, \frac{2}{2\alpha_1 - 3}$. *Let* $w(t) = \sum_{j=-\infty}^{\infty} w_j e^{ijt}$, $-\pi \le t \le \pi$, *be a non-negative, even function with at most a finite number of zeros such that* $\sum_{j=-\infty}^{\infty} |j|^5 |w_j| < \infty$, $w(0) \ne 0$ *and* $\frac{t(\pi - t)w'(t)}{w(t)}$, $0 \le t \le \pi$, *lies in* $\mathcal{L}^\infty[0, \pi]$. *Then as* $h \to 0$,

$$\mu_\pi(f_h; w) - \nu_{\mathbb{R}}(f) = O(h^2), \qquad \mu_\pi(\widetilde{f}_h; w) - \nu_{\mathbb{R}}(f) = O(h^2). \tag{3.22}$$

Proof: Note that from (2.13) that for an even, real-valued function g on $[-\pi, \pi]$,

$$\mu_\pi(g; w) = \frac{B_1 B_2 B_3}{B_4 B_5^2}, \tag{3.23}$$

where for any $h > 0$,

$$B_1 := h^3 \int_{-\pi}^{\pi} g'(t)^2 w(t)\, dt, \qquad B_2 := h^{-1} \int_{-\pi}^{\pi} (1 - \cos t) g(t)^2 w(t)\, dt,$$
$$B_3 := h \int_{-\pi}^{\pi} (1 + \cos t) g(t)^2 w(t)\, dt, \qquad B_4 := h \int_{-\pi}^{\pi} g(t)^2 w(t)\, dt,$$
$$B_5 := h \int_{-\pi}^{\pi} g(t)^2 \left(\cos t\, w(t) + \sin t\, w'(t) \right)\, dt.$$

As in (2.8), we write $g(t) = \sum_{j=-\infty}^{\infty} a_j e^{ijt}$, $-\pi \le t \le \pi$. In this case, for $j \in \mathbb{Z}$, a_j and w_j are real and $a_{-j} = a_j$, $w_{-j} = w_j$. Then by (3.11),

$$B_1 = w(0)h^3 \sum_{j=-\infty}^{\infty} j^2 a_j^2 - \frac{h^2}{2} \sum_{\ell=-\infty}^{\infty} \ell^2 w_\ell \, h^{-1} \sum_{j=-\infty}^{\infty} \frac{1}{\ell^2} \left((j+\ell)ha_{j+\ell} - jha_j\right)^2. \tag{3.24}$$

Now put $a_j = f(jh)$, $j \in \mathbb{Z}$. Applying (3.7) to the function $xf(x)$, the last term in (3.24) is $O(h^2)$ and so by (3.5),

$$B_1 = w(0) \int_{-\infty}^{\infty} x^2 f(x)^2 \, dx + O(h^2).$$

Similarly, we can apply (3.11) to derive formulas for B_2, B_3, B_4, B_5, and utilizing (3.3) and (3.7) gives

$$B_2 = \frac{1}{2} w(0) \int_{-\infty}^{\infty} f'^2 + O(h^2), \qquad B_3 = 2w(0) \int_{-\infty}^{\infty} f^2 + O(h^2),$$

$$B_4 = w(0) \int_{-\infty}^{\infty} f^2 + O(h^2), \qquad B_5 = w(0) \int_{-\infty}^{\infty} f^2 + O(h^2).$$

Hence by (3.23),

$$\mu_\pi(f_h; w) = \frac{\displaystyle\int_{-\infty}^{\infty} x^2 f(x)^2 \, dx \int_{-\infty}^{\infty} f'^2}{\left(\displaystyle\int_{-\infty}^{\infty} f^2\right)^2} + O(h^2).$$

Recalling (2.3) and noting that f is even and real-valued, this gives the first result in (3.22).

Next suppose that (a_j) is given by (3.6). Then exactly as before we can use (3.3), (3.5) and (3.7) to deduce the second result in (3.22). ■

Since $\nu_{\mathbb{R}}(G) = \frac{1}{4}$, we immediately have the following.

Corollary 3.3. *With g_h, $\widetilde{g}_h$ as in Corollary 3.2 and w as in Theorem 3.2, we have as $h \to 0$,*

$$\mu_\pi(g_h; w) = \frac{1}{4} + O(h^2), \qquad \mu_\pi(\widetilde{g}_h; w) = \frac{1}{4} + O(h^2).$$

Corollaries 3.2 and 3.3 imply the following. For any $c > 0$ and $0 < \epsilon < 1$, define for $k = 1, 2, \ldots$,

$$p_k(t) = \sum_{j=-k}^{k} G\left(\frac{cj}{k^{1-\epsilon}}\right) e^{ijt}, \quad -\pi \le t \le \pi.$$

Then as $k \to \infty$,

$$\nu_{2\pi}(p_k; w_1) = \frac{1}{4} + O\left(\frac{1}{k^{2(1-\epsilon)}}\right), \qquad \mu_\pi(p_k; w_2) = \frac{1}{4} + O\left(\frac{1}{k^{2(1-\epsilon)}}\right),$$

where w_1 and w_2 are functions as in Theorem 3.1 and Theorem 3.2 respectively. Hence for $k = 1, 2, \ldots$, by considering a trigonometric polynomial of degree k that minimizes $\nu_{2\pi}(p; w_1)$ over all trigonometric polynomials p of degree k, we can construct a sequence (q_k) of trigonometric polynomials of degree k which gives $\nu_{2\pi}(q_k; w_1) = \frac{1}{4} + O\left(\frac{1}{k^2}\right)$ as $k \to \infty$. This improves the order $\frac{1}{4} + O\left(\frac{1}{k}\right)$ from Theorem 2.2 for the special case when $w_1(t) = e^{it}$. Similarly, we can obtain a sequence (q_k) of trigonometric polynomials of degree k that yields $\mu_\pi(q_k; w_2) = \frac{1}{4} + O\left(\frac{1}{k^2}\right)$ as $k \to \infty$.

We note that Theorem 3.2 does not apply when $w(t) = w_\alpha(t) = c_\alpha |\sin t|^\alpha$, $-\pi \le t \le \pi$, for $\alpha > 0$, since then $w_\alpha(0) = 0$. In our final theorem of this section, we shall instead relate the uncertainty product $\mu_\pi(f; w_\alpha)$, for $\alpha > 0$, with the uncertainty product $\mu_{\mathbb{R}}(f; \omega_\alpha)$ in (2.20) where ω_α is as in (2.21). For $\beta > 0$, we shall denote by $\mathcal{S}^+(\beta)$ the set of all continuous functions $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfying $|f(x)| \le C(1+x)^{-\beta}$, $x \ge 0$, for some constant C.

Theorem 3.3. *Take $\alpha > 0$ and $f \in \mathcal{S}^+(\alpha_0)$ with $f^{(\ell)} \in \mathcal{S}^+(\alpha_\ell)$, $\ell = 1, 2, 3$, where $\alpha_0, \alpha_1, \alpha_2 > \frac{3+\alpha}{2}$ and $\alpha_3 > \frac{1+\alpha}{2}$. Let P_j, $j = 0, 1, \ldots$ be the ultraspherical polynomials of index $\frac{1}{2}(\alpha - 1)$, normalized by $P_j(1) = 1$, $j = 0, 1, \ldots$. For $h > 0$, define for $0 \le t \le \pi$,*

$$f_{h,\alpha}(t) = \sum_{j=0}^{\infty} h_j f\left(\left(j + \frac{\alpha}{2}\right) h\right) P_j(\cos t), \qquad \widetilde{f}_{h,\alpha}(t) = \sum_{j=0}^{N_h} h_j f\left(\left(j + \frac{\alpha}{2}\right) h\right) P_j(\cos t),$$

where $N_h^{-1} = O(h^{1+\beta})$ as $h \to 0$, with $\beta \ge \frac{2}{2\alpha_0 - 3 - \alpha}, \frac{3}{2\alpha_0 - \alpha}, \frac{2}{2\alpha_1 - 1 - \alpha}$. Then as $h \to 0$,

$$\mu_\pi(f_{h,\alpha}; w_\alpha) - \mu_{\mathbb{R}}(f; \omega_\alpha) = O(h^{\min(\alpha,2)}), \qquad \mu_\pi(\widetilde{f}_{h,\alpha}; w_\alpha) - \mu_{\mathbb{R}}(f; \omega_\alpha) = O(h^{\min(\alpha,2)}). \tag{3.25}$$

Before proving Theorem 3.3 we shall need two preliminary results. From the proofs of Lemmas 3.1 and 3.3 we can derive the following.

Lemma 3.4. *Take f in $\mathcal{S}^+(\alpha_0)$ with f'' in $\mathcal{S}^+(\alpha_2)$ for some $\alpha_0, \alpha_2 > 1$. Then for $h > 0$, $a_j = f(jh)$, $j \ge 0$,*

$$\frac{h}{2} a_0 + h \sum_{j=1}^{\infty} a_j - \int_0^\infty f = O(h^2), \tag{3.26}$$

as $h \to 0$. Moreover if $N_h^{-1} = O\left(h^{1+\frac{2}{\alpha_0 - 1}}\right)$, then as $h \to 0$, (3.26) holds with

$$a_j = \begin{cases} f(jh), & 0 \le j \le N_h, \\ 0, & \text{otherwise.} \end{cases} \tag{3.27}$$

Now take $\alpha \ge 0$, $f \in \mathcal{S}^+(\alpha_0)$ with $f^{(\ell)} \in \mathcal{S}^+(\alpha_\ell)$, $\ell = 1, 2, 3$, where $\alpha_0 > \frac{\alpha}{2}$ and $\alpha_1, \alpha_2, \alpha_3 > \frac{1+\alpha}{2}$. Then for $h > 0$, $a_j = f(jh)$, $j \ge 0$,

$$h^{-1} \sum_{j=0}^{\infty} \left(\left(j + \frac{1}{2}\right) h\right)^\alpha (a_{j+1} - a_j)^2 - \int_0^\infty x^\alpha f'(x)^2 \, dx = O\left(h^{\min(\alpha+1,\, 2)}\right), \tag{3.28}$$

as $h \to 0$. Moreover if $N_h^{-1} = O\left(h^{1+\beta}\right)$, $\beta \ge \frac{3}{2\alpha_0 - \alpha}, \frac{2}{2\alpha_1 - 1 - \alpha}$, then for $h > 0$, (3.28) holds as $h \to 0$ with (a_j) given by (3.27).

Our next result concerns asymptotic behavior of h_j and λ_j given in (2.15) and (2.16).

Lemma 3.5. *For $\alpha, h > 0$, $x_j := \left(j + \frac{\alpha}{2}\right) h$, $j = 1, 2, \ldots$,*

$$h^\alpha h_j = \frac{2 x_j^\alpha}{\Gamma(\alpha + 1)} + \begin{cases} O(h^\alpha), & 0 < \alpha \le 2, \\ x_j^{\alpha - 2} O(h^2), & \alpha \ge 2, \end{cases} \tag{3.29}$$

$$h^{\alpha+2}h_j j(j+\alpha) = \frac{2x_j^{\alpha+2}}{\Gamma(\alpha+1)} + x_j^{\alpha} O(h^2), \tag{3.30}$$

$$h^{\alpha}\lambda_j h_j = \frac{\left(x_j + \frac{1}{2}h\right)^{\alpha}}{\Gamma(\alpha+1)} + \begin{cases} O(h^{\alpha}), & 0 < \alpha \le 2, \\ x_j^{\alpha-2} O(h^2), & \alpha \ge 2, \end{cases} \tag{3.31}$$

as $h \to 0$, uniformly over $j \ge 1$.

Proof: We first recall Stirling's series [18, p. 251] which gives

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} + O\left(\frac{1}{x^2}\right), \tag{3.32}$$

as $x \to \infty$. Now by (2.17), $\lambda_j h_j = \dfrac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+1)\Gamma(j+1)}$ and applying (3.32) gives

$$\Gamma(\alpha+1)\lambda_j h_j = \left(j + \frac{\alpha}{2} + \frac{1}{2}\right)^{\alpha} + \begin{cases} O(1), & 0 < \alpha \le 2, \\ O(j^{\alpha-2}), & \alpha \ge 2, \end{cases}$$

from which (3.31) follows. Similarly, on recalling (2.16) we can show that for $j \ge 1$,

$$\Gamma(\alpha+1)h_j = 2\left(j + \frac{\alpha}{2}\right)^{\alpha}\left(1 + O\left(\frac{1}{j^2}\right)\right).$$

From this we may derive (3.29) and (3.30). ■

Proof of Theorem 3.3: For real a_j, $j = 0, 1, \ldots$, we set $g(t) := \sum_{j=0}^{\infty} h_j a_j P_j(\cos t)$, $0 \le t \le \pi$, and see from (2.18) that

$$(\alpha+1)^2 \mu_\pi(g; w_\alpha) = \frac{\sum_{j=0}^{\infty} h_j j(j+\alpha) a_j^2 \left[\left(\sum_{j=0}^{\infty} h_j a_j^2\right)^2 - \left(\sum_{j=0}^{\infty} 2\lambda_j h_j a_j a_{j+1}\right)^2\right]}{\sum_{j=0}^{\infty} h_j a_j^2 \left(\sum_{j=0}^{\infty} 2\lambda_j h_j a_j a_{j+1}\right)^2}. \tag{3.33}$$

Now applying (2.17) and recalling that $\lambda_0 = 1$, we have

$$\begin{aligned} &\left(\sum_{j=0}^{\infty} h_j a_j^2\right)^2 - \left(\sum_{j=0}^{\infty} 2\lambda_j h_j a_j a_{j+1}\right)^2 \\ &= \sum_{j=0}^{\infty} \lambda_j h_j \left(a_{j+1} - a_j\right)^2 \left(2\sum_{j=0}^{\infty} h_j a_j^2 - \sum_{j=0}^{\infty} \lambda_j h_j \left(a_{j+1} - a_j\right)^2\right), \end{aligned}$$

and hence we can rewrite (3.33) as

$$(\alpha+1)^2 \mu_\pi(g; w_\alpha) = \frac{C_1 C_2 \left(2C_3 - h^2 C_2\right)}{C_3 \left(C_3 - h^2 C_2\right)^2}, \tag{3.34}$$

where for $h > 0$,

$$C_1 = h^{\alpha+3} \sum_{j=0}^{\infty} h_j j(j+\alpha) a_j^2, \quad C_2 = h^{\alpha-1} \sum_{j=0}^{\infty} \lambda_j h_j \left(a_{j+1} - a_j\right)^2, \quad C_3 = h^{\alpha+1} \sum_{j=0}^{\infty} h_j a_j^2.$$

Applying Lemma 3.5, putting $a_j = f(x_j)$, $j \geq 0$, followed by appropriate use of Lemma 3.4, gives

$$C_1 = \frac{2}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha+2} f(x)^2 \, dx + O(h^2), \tag{3.35}$$

$$C_2 = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} f'(x)^2 \, dx + O(h^{\min(\alpha,2)}), \tag{3.36}$$

$$C_3 = \frac{2}{\Gamma(\alpha+1)} \int_0^{\infty} x^{\alpha} f(x)^2 \, dx + O(h^{\min(\alpha,2)}). \tag{3.37}$$

Substituting (3.35)–(3.37) into (3.34) and recalling (2.20) yields the first result in (3.25). Similarly, if (a_j) is given by

$$a_j = \begin{cases} f(x_j), & 0 \leq j \leq N_h, \\ 0, & \text{otherwise}, \end{cases}$$

where $N_h^{-1} = O(h^{1+\beta})$ as $h \to 0$, with $\beta \geq \dfrac{2}{2\alpha_0 - 3 - \alpha}, \dfrac{3}{2\alpha_0 - \alpha}, \dfrac{2}{2\alpha_1 - 1 - \alpha}$, then we can apply Lemma 3.4 to obtain (3.35)–(3.37) and hence deduce the second result in (3.25). ■

Since $\mu_{\mathbb{R}}(G; \omega_\alpha) = \frac{1}{4}$, $\alpha \geq 0$, for the Gaussian function G, Theorem 3.3 immediately gives the following.

Corollary 3.4. *Take $\alpha > 0$ and let P_j, $j \geq 0$, be as in Theorem 3.3. Define, for $0 \leq t \leq \pi$,*

$$g_{h,\alpha}(t) = \sum_{j=0}^{\infty} h_j G\left(\left(j + \frac{\alpha}{2}\right) h\right) P_j(\cos t), \qquad \widetilde{g}_{h,\alpha}(t) = \sum_{j=0}^{N_h} h_j G\left(\left(j + \frac{\alpha}{2}\right) h\right) P_j(\cos t), \tag{3.38}$$

where $N_h^{-1} = O(h^{1+\beta})$ as $h \to 0$, for some $\beta > 0$. Then as $h \to 0$,

$$\mu_\pi(g_{h,\alpha}; w_\alpha) = \frac{1}{4} + O(h^{\min(\alpha,2)}), \qquad \mu_\pi(\widetilde{g}_{h,\alpha}; w_\alpha) = \frac{1}{4} + O(h^{\min(\alpha,2)}).$$

We note that it was shown in [14], by different methods, that $\lim_{h \to 0} \mu_\pi(g_{h,\alpha}; w_\alpha) = \frac{1}{4}$.

4. Multivariate uncertainty principles

We shall now extend the general uncertainty principle (2.2) to handle multiple pairs of symmetric or normal operators. This will give rise to uncertainty inequalities for various multivariate situations of interest.

Theorem 4.1. *Let $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ be symmetric or normal operators with domain and range in the same Hilbert space $\mathcal{H}$. Then for any nonzero v in $\mathcal{D}(A_j B_j) \cap \mathcal{D}(B_j A_j)$, $j = 1, \ldots, n$,*

$$\frac{1}{4} \left(\sum_{j=1}^{n} |\langle [A_j, B_j] v, v \rangle| \right)^2 \leq \left(\sum_{j=1}^{n} \Delta_v(A_j)^2 \right) \left(\sum_{j=1}^{n} \Delta_v(B_j)^2 \right), \tag{4.1}$$

where we use the notation, for an operator A on $\mathcal{H}$:

$$\Delta_v(A) := \left(\|Av\|^2 - \frac{|\langle Av, v \rangle|^2}{\|v\|^2} \right)^{1/2}.$$

Moreover equality holds in (4.1) *if and only if*

$$\frac{1}{4}|\langle[A_j,B_j]v,v\rangle|^2=\Delta_v(A_j)^2\Delta_v(B_j)^2,\quad j=1,\dots,n,\tag{4.2}$$

and there exists $\lambda\geq 0$ *such that*

$$\Delta_v(A_j)^2=\lambda\Delta_v(B_j)^2,\quad j=1,\dots,n.$$

Proof: Applying (2.2) for $j=1,\dots,n$, gives

$$\frac{1}{2}\sum_{j=1}^{n}|\langle[A_j,B_j]v,v\rangle|\leq\sum_{j=1}^{n}\Delta_v(A_j)\Delta_v(B_j).\tag{4.3}$$

Then (4.1) is a consequence of the Cauchy-Schwarz inequality. The conditions for equality follows from requiring equality in both (4.3) and in the Cauchy-Schwarz inequality. ■

Characterizations of equality in (4.2) for symmetric operators are provided in Theorem 2.1. The following result gives sufficient conditions for equality in (4.1).

Theorem 4.2. *Let* $A_1,\dots,A_n$, $B_1,\dots,B_n$ *be symmetric operators with domain and range in the same Hilbert space* $\mathcal{H}$, *and let* $v\in\mathcal{D}(A_jB_j)\cap\mathcal{D}(B_jA_j)$, $j=1,\dots,n$, *be nonzero. Suppose that there exists a nonzero constant* $\mu\in\mathbb{R}$ *such that*

$$(A_j-i\mu B_j)v=0,\quad j=1,\dots,n.\tag{4.4}$$

Then equality holds for (4.1) *at* v.

Proof: Fix j, $1\leq j\leq n$. If v is an eigenvector of A_j or B_j, then by Theorem 2.1, equality holds in (4.2). Otherwise, (4.4) implies that v is an eigenvector (with eigenvalue 0) of the operator $A_j-i\mu B_j$, and by Theorem 2.1, equality holds in (4.2). Now by (4.4), $A_jv=i\mu B_jv$ and so

$$\Delta_v(A_j)^2=\|A_jv\|^2-\frac{|\langle A_jv,v\rangle|^2}{\|v\|^2}=\mu^2\|B_jv\|^2-\frac{\mu^2|\langle B_jv,v\rangle|^2}{\|v\|^2}=\mu^2\Delta_v(B_j)^2.$$

So by Theorem 4.1, equality holds for (4.1) at v. ■

We note that the condition (4.4) is of particular interest. For instance, when $n=1$, it plays a crucial role in Theorems 11 and 12 of [4] in identifying the boundaries of the geometric regions there.

Consider again the situation of Theorem 4.1 and let $\mathcal{H}=\mathcal{L}^2(\mathcal{X}^n,m)$, where

$$m(x_1,\dots,x_n)=m_0(x_1)\cdots m_0(x_n)\tag{4.5}$$

for some measure m_0 on the space $\mathcal{X}$. Let A and B be symmetric or normal operators on $\mathcal{L}^2(\mathcal{X},m_0)$. Suppose that for $j=1,\dots,n$, A_j and B_j are symmetric or normal operators on $\mathcal{L}^2(\mathcal{X}^n,m)$ such that whenever

$$f(x_1,\dots,x_n)=f_1(x_1)\cdots f_n(x_n),\quad f_j\in\mathcal{L}^2(\mathcal{X},m_0),\tag{4.6}$$

then

$$A_jf(x_1,\dots,x_n)=\left(\prod_{k=1,\,k\neq j}^{n}f_k(x_k)\right)Af_j(x_j),\tag{4.7}$$

$$B_jf(x_1,\dots,x_n)=\left(\prod_{k=1,\,k\neq j}^{n}f_k(x_k)\right)Bf_j(x_j).\tag{4.8}$$

Proposition 4.1. *Let $\mathcal{H} = \mathcal{L}^2(\mathcal{X}^n, m)$, where* (4.5) *holds. For $j = 1, \dots, n$, let A_j and B_j be linear operators on $\mathcal{H}$ which satisfy* (4.7) *and* (4.8) *whenever* (4.6) *holds. Consider $f \in \mathcal{D}(A_jB_j) \cap \mathcal{D}(B_jA_j)$, $j = 1, \dots, n$, such that $\sum_{j=1}^n |\langle [A_j, B_j]f, f\rangle| \neq 0$. If f is of the form*

$$f(x_1, \dots, x_n) = f_0(x_1) \cdots f_0(x_n), \tag{4.9}$$

for some f_0 in $\mathcal{L}^2(\mathcal{X}, m_0)$, then

$$\frac{\left(\sum_{j=1}^n \Delta_f(A_j)^2\right)\left(\sum_{j=1}^n \Delta_f(B_j)^2\right)}{\left(\sum_{j=1}^n |\langle [A_j, B_j]f, f\rangle|\right)^2} = \frac{\Delta_{f_0}(A)^2 \Delta_{f_0}(B)^2}{|\langle [A, B]f_0, f_0\rangle|^2}. \tag{4.10}$$

Proof: Take f to be of the form (4.9). Then $\|f\|^2 = \|f_0\|^{2n}$. For $j = 1, \dots, n$, (4.7) and (4.8) give, after some calculation,

$$\Delta_f(A_j)^2 = \|f_0\|^{2(n-1)} \Delta_{f_0}(A)^2, \qquad \Delta_f(B_j)^2 = \|f_0\|^{2(n-1)} \Delta_{f_0}(B)^2,$$
$$\langle [A_j, B_j]f, f\rangle = \|f_0\|^{2(n-1)} \langle [A, B]f_0, f_0\rangle.$$

The required identity (4.10) follows. ■

We shall now focus on specific cases of the uncertainty principle (4.1). These cases are multivariate extensions of the inequalities in Section 2. The domains of the respective operators involved are defined similarly, and to save a plethora of details, we shall not state them here.

Example 4.1. Take $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$ and for $j = 1, \dots, n$, the self-adjoint linear operators $A_jf(x) = x_jf(x)$, $B_jf(x) = i\frac{\partial f}{\partial x_j}(x)$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then Theorem 4.1 and Proposition 4.1 give the uncertainty principle

$$\frac{1}{4} \leq \frac{\left(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2\, dx - \frac{\left|\int_{\mathbb{R}^n} x|f(x)|^2\, dx\right|^2}{\int_{\mathbb{R}^n} |f|^2}\right)\left(\int_{\mathbb{R}^n} |\nabla f|^2 - \frac{\left|\int_{\mathbb{R}^n} (\nabla f)\overline{f}\right|^2}{\int_{\mathbb{R}^n} |f|^2}\right)}{n^2 \left(\int_{\mathbb{R}^n} |f|^2\right)^2},$$

for all nonzero $f \in \mathcal{D}(A_jB_j) \cap \mathcal{D}(B_jA_j)$, $j = 1, \dots, n$, where equality is attained by the multivariate Gaussian $f(x) := (2\pi)^{-n/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$, as is well known.

Example 4.2. As for the univariate uncertainty principle (2.19), (2.20), we let $w \in \mathcal{C}(\mathbb{R})$ be a non-negative, even function with at most a finite number of zeros for which $\frac{x_0 w'(x_0)}{w(x_0)}$, $x_0 > 0$, is bounded on compact subsets of $[0, \infty)$. Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n, m)$, where m is given by (4.5) and m_0 has density function w. For $j = 1, \dots, n$, we define the self-adjoint operator $A_jf(x) = x_jf(x)$ and symmetric operator $B_jf(x) = i\frac{\partial f}{\partial x_j}(x) + \frac{i}{2}\frac{w'(x_j)}{w(x_j)}(f(x) - f(R_jx))$, where $x \in \mathbb{R}^n$ and

$$(R_jx)_k = \begin{cases} -x_k, & k = j, \\ x_k, & k \neq j. \end{cases} \tag{4.11}$$

We shall assume that f is even, i.e. $f(R_j x) = f(x)$, $j = 1, \dots, n$. Then Theorem 4.1 gives the uncertainty principle

$$\frac{1}{4} \le \frac{\displaystyle\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 \, dm(x) \int_{\mathbb{R}^n} |\nabla f|^2 \, dm}{\left(\displaystyle\sum_{j=1}^{n} \left| \int_{\mathbb{R}^n} |f(x)|^2 \frac{\partial}{\partial x_j} \left(x_j w(x_1) \cdots w(x_n) \right) dx \right| \right)^2}, \tag{4.12}$$

for all $f \in \mathcal{D}(A_j B_j) \cap \mathcal{D}(B_j A_j)$, $j = 1, \dots, n$, such that the denominator in (4.12) is nonzero. By Proposition 4.1, equality holds in (4.12) for the multivariate Gaussian provided that w satisfies the condition given after (2.23). For the special case when $w = w_\alpha$, $\alpha \ge 0$, given by (2.21), the above are essentially results in [13].

Example 4.3. We shall now extend to the multivariate setting the uncertainty principle (2.6), (2.5). As in that case we take w to be an absolutely continuous 2π-periodic function with $w' \in \mathcal{L}^\infty[-\pi, \pi] \backslash \{0\}$. Let $\mathcal{H} = \mathcal{L}^2([-\pi, \pi]^n)$ and for $j = 1, \dots, n$, define the linear operators $A_j f(t) = w(t_j) f(t)$, $B_j f(t) = i \dfrac{\partial f}{\partial t_j}(t)$, where $t = (t_1, \dots, t_n) \in [-\pi, \pi]^n$. Note that A_j is normal and B_j is self-adjoint.

Define $W : [-\pi, \pi]^n \longrightarrow \mathbb{C}^n$ by $W(t) = (w(t_1), \dots, w(t_n))$. Then Theorem 4.1 gives the uncertainty principle

$$\frac{1}{4} \le \frac{\left(\displaystyle\int_{[-\pi,\pi]^n} |Wf|^2 - \frac{\left| \displaystyle\int_{[-\pi,\pi]^n} W|f|^2 \right|^2}{\displaystyle\int_{[-\pi,\pi]^n} |f|^2} \right) \left(\displaystyle\int_{[-\pi,\pi]^n} |\nabla f|^2 - \frac{\left| \displaystyle\int_{[-\pi,\pi]^n} (\nabla f)\overline{f} \right|^2}{\displaystyle\int_{[-\pi,\pi]^n} |f|^2} \right)}{\left(\displaystyle\sum_{j=1}^{n} \left| \int_{[-\pi,\pi]^n} w'(t_j) |f(t)|^2 \, dt \right| \right)^2}, \tag{4.13}$$

for all $f \in \mathcal{D}(B_j)$, $j = 1, \dots, n$, such that $\displaystyle\sum_{j=1}^{n} \left| \int_{[-\pi,\pi]^n} w'(t_j) |f(t)|^2 \, dt \right| \ne 0$.

Again we may apply Proposition 4.1 to show that if f in (4.13) is of the form (4.9) for some $f_0 \in \mathcal{L}^2[-\pi, \pi]$, then the right-hand side of (4.13) equals $\nu_{2\pi}(f_0; w)$ as given in (2.5). Now assume that w satisfies the conditions of Theorem 3.1 and choose f_0 to be one of the functions in (3.19) of Corollary 3.2. Then by Corollary 3.2, the right-hand side of (4.13) equals $\dfrac{1}{4} + O(h^2)$, as $h \to 0$. For the special case of (4.13) when $w(t_0) = e^{it_0}$, we can also take

$$f_0(t_0) := (1 + \cos t_0)^k, \quad k = 1, 2, \dots, \tag{4.14}$$

then Theorem 2.2, with $\alpha = 0$, shows that the right-hand side of (4.13) equals $\dfrac{1}{4} + \dfrac{1}{2(4k-1)}$.

Example 4.4. As our final example, we extend to the multivariate case the uncertainty principle (2.12), (2.13). As there we let $w \in \mathcal{C}[-\pi, \pi]$ be a non-negative, even function with at most a finite number of zeros such that the function $\dfrac{t_0(\pi - t_0) w'(t_0)}{w(t_0)}$, $0 < t_0 < \pi$, lies in $\mathcal{L}^\infty[0, \pi]$. Let $\mathcal{H} = \mathcal{L}^2([-\pi, \pi]^2, m)$, where m is given by (4.5) and m_0 has density function w. For $j = 1, \dots, n$, we define the linear operators $A_j f(t) = e^{it_j} f(t)$, $B_j f(t) = i\dfrac{\partial f}{\partial t_j}(t) + \dfrac{i}{2}\dfrac{w'(t_j)}{w(t_j)}(f(t) - f(R_j t))$, where $t \in [-\pi,\pi]^n$ and $R_j t$ is given by (4.11). Here A_j is normal and B_j is symmetric. Let $\cos t := (\cos t_1, \dots, \cos t_n)$. Then Theorem 4.1 gives the uncertainty principle

$$\frac{1}{4} \le \frac{\left(n \displaystyle\int_{[0,\pi]^n} |f|^2\, dm - \frac{\left| \displaystyle\int_{[0,\pi]^n} \cos t\, |f(t)|^2\, dm(t) \right|^2}{\displaystyle\int_{[0,\pi]^n} |f|^2\, dm} \right) \displaystyle\int_{[0,\pi]^n} |\nabla f|^2\, dm}{\left(\displaystyle\sum_{j=1}^{n} \left| \int_{[0,\pi]^n} |f(t)|^2 \frac{\partial}{\partial t_j} \left(\sin t_j\, w(t_1) \cdots w(t_n)\right) dt \right| \right)^2}, \tag{4.15}$$

for all even functions $f \in \mathcal{D}(B_j)$, $j = 1, \dots, n$, such that the denominator in (4.15) is nonzero.

Once more applying Proposition 4.1 shows that if f in (4.15) is given by (4.9) for some even f_0 in $\mathcal{L}^2([-\pi,\pi], m_0)$, then the right-hand side of (4.15) equals $\mu_\pi(f_0; w)$ as given in (2.13). Now suppose that w satisfies the conditions of Theorem 3.2 and choose f_0 to be one of the functions in (3.19). Then by Corollary 3.3, the right-hand side of (4.15) equals $\dfrac{1}{4} + O(h^2)$, as $h \to 0$.

Finally we suppose that for some $\alpha \ge 0$, $w = w_\alpha$ as in (2.14). Then Theorem 2.2 shows that for f_0 as in (4.14) and f given by (4.9), the right-hand side of (4.15) equals $\dfrac{1}{4} + \dfrac{1}{2(4k+\alpha-1)}$. We note that for $\alpha > 0$, asymptotic equality in (4.15) can also be attained by choosing f_0 to be one of the functions in (3.38) of Corollary 3.4. In this case, the right-hand side of (4.15) equals $\dfrac{1}{4} + O(h^{\min(\alpha,2)})$, as $h \to 0$.

5. Uncertainty principles on spheres

We now turn to uncertainty principles on the n-sphere $\mathbb{S}^n := \{x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : |x| = 1\}$ with normalized surface measure σ. Let $\mathcal{L}^2(\mathbb{S}^n)$ be the space of complex-valued square-integrable functions on $\mathbb{S}^n$ with the inner product $\langle f, g\rangle := \displaystyle\int_{\mathbb{S}^n} f\overline{g}\, d\sigma$.

We begin with an uncertainty principle on $\mathbb{S}^n$ for radial functions. Suppose that f is a complex-valued function on $\mathbb{S}^n$ which is radial, i.e. for some unit vector u, $f(x)$ depends only on $x \cdot u$, $x \in \mathbb{S}^n$. For $x \in \mathbb{S}^n$ we may put $x \cdot u = \cos t$, $0 \le t \le \pi$, and write $f(x) = g(t)$ for a function g on $[0,\pi]$. The frequency variance of a radial function f in $\mathcal{C}^2(\mathbb{S}^n)$ is given by $V(f) := -\displaystyle\int_{\mathbb{S}^n} (\Delta_s f)\overline{f}\, d\sigma$, where Δ_s is the Laplace-Beltrami operator on $\mathbb{S}^n$ with eigenvalues $-k(k+n-1)$, $k \ge 0$, see [14]. Let $w = w_{n-1}$ as in (2.14). It was shown in [14] that the uncertainty principle (2.12) for the function g in $\mathcal{L}^2_w[-\pi,\pi]$ and (2.13) give the following uncertainty principle on $\mathbb{S}^n$. For all radial functions $f \in \mathcal{C}^2(\mathbb{S}^n)$ such that $\displaystyle\int_{\mathbb{S}^n} x|f(x)|^2\, d\sigma(x) \ne 0$,

$$\frac{1}{4} \le \frac{V(f)}{n^2 \displaystyle\int_{\mathbb{S}^n} |f|^2\, d\sigma} \left[\left(\frac{\displaystyle\int_{\mathbb{S}^n} |f|^2\, d\sigma}{\left| \displaystyle\int_{\mathbb{S}^n} x|f(x)|^2\, d\sigma(x) \right|} \right)^2 - 1 \right]. \tag{5.1}$$

For $n = 2$, (5.1) was derived for more general functions f on $\mathbb{S}^2$ in [8].

Recall from the discussion preceeding Theorem 2.2 that it is not possible for equality to hold in (2.12). As (5.1) is obtained from (2.12), equality is not attained in (5.1) either. Since the right-hand side of (5.1) equals $\mu_\pi(g;w_\alpha)$ in (2.13) for $\alpha = n-1$, $n = 2,3,\ldots$, Theorem 2.2 shows that $f_k(x) := (1+x\cdot u)^k$, $x \in \mathbb{S}^n$, gives asymptotic equality in (5.1), i.e.

$$\frac{V(f_k)}{n^2\int_{\mathbb{S}^n} f_k^2\,d\sigma}\left[\left(\frac{\int_{\mathbb{S}^n} f_k^2\,d\sigma}{\left|\int_{\mathbb{S}^n} x f_k(x)^2\,d\sigma(x)\right|}\right)^2 - 1\right] = \frac{1}{4} + \frac{1}{2(4k+n-2)}.$$

Asymptotic equality in (5.1) can also be attained by radial functions on $\mathbb{S}^n$ corresponding to the functions in (3.38) of Corollary 3.4 with $\cos t = x\cdot u$, $x \in \mathbb{S}^n$.

For the rest of the paper, we deal with general functions on $\mathbb{S}^n$. We denote by $\nabla_{\mathbb{S}^n} = (D_0,\ldots,D_n)$ the operator defined as follows. Let f be a complex-valued $\mathcal{C}^1$ function on $\mathbb{S}^n$ and take $x \in \mathbb{S}^n$. Then the component of $\nabla_{\mathbb{S}^n} f(x)$ normal to the sphere at x is zero, while any component of $\nabla_{\mathbb{S}^n} f(x)$ tangential to the sphere at x is equal to the corresponding component of $\nabla f(x)$, i.e. $x\cdot\nabla_{\mathbb{S}^n} f(x) = 0$ and for $y \in \mathbb{R}^{n+1}\backslash\{0\}$ with $y\cdot x = 0$, $y\cdot\nabla_{\mathbb{S}^n} f(x) = y\cdot\nabla f(x)$. Now define $F : \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow \mathbb{R}$ by $F(x) = f\left(\frac{x}{|x|}\right)$. Since the component of $\nabla F(x)$ normal to the sphere at x is zero, we have $\nabla_{\mathbb{S}^n} f(x) = \nabla F(x)$. In particular this gives for $j,k = 0,\ldots,n$,

$$D_j(x_k) = \begin{cases} 1 - x_j^2, & j = k, \\ -x_j x_k, & j \neq k. \end{cases} \tag{5.2}$$

Next we derive an 'integration by parts' formula for the operator D_j. Take j, $0 \le j \le n$. For $0 \le t \le \pi$, let S_t denote the $(n-1)$-sphere $\{x \in \mathbb{S}^n : x_j = \cos t\}$ of radius $\sin t$. Note that S_t lies in a hyperplane orthogonal to the jth coordinate axis and so for $x \in \mathbb{S}^n$ with $x_j = \cos t_0$, $D_j f(x) = -\sin t_0\,\partial_j f(x)$, where $\partial_j f(x) := \left.\frac{d}{dt} f(y(t))\right|_{t=t_0}$, $y(t)_j := \cos t$ and $y(t)_k := \frac{x_k \sin t}{\sin t_0}$ for $k \neq j$. Denoting by σ_m the normalized surface measure on $\mathbb{S}^m$, $m = n-1, n$, we have

$$\begin{aligned}
\int_{\mathbb{S}^n} D_j f\,d\sigma_n &= -\int_{\mathbb{S}^n} \sin t\,\partial_j f(x)\,d\sigma_n(x) \\
&= -\int_0^\pi \sin t\,\sin^{n-1} t \int_{S_t} \partial_j f\,d\sigma_{n-1}\,dt \\
&= -\int_0^\pi \sin^n t\,\frac{d}{dt}\int_{S_t} f\,d\sigma_{n-1}\,dt \\
&= \int_0^\pi n\cos t\,\sin^{n-1} t \int_{S_t} f\,d\sigma_{n-1}\,dt \\
&= n\int_{\mathbb{S}^n} \cos t\, f(x)\,d\sigma_n(x) = n\int_{\mathbb{S}^n} x_j f(x)\,d\sigma_n(x).
\end{aligned}$$

Henceforward we write $\sigma = \sigma_n$. Applying the above to the function fg, where $f,g \in \mathcal{C}^1(\mathbb{S}^n)$, gives

$$\int_{\mathbb{S}^n} (D_j f) g\,d\sigma = n\int_{\mathbb{S}^n} x_j f(x) g(x)\,d\sigma(x) - \int_{\mathbb{S}^n} f(D_j g)\,d\sigma. \tag{5.3}$$

Theorem 5.1. *For all* $f \in \mathcal{C}^1(\mathbb{S}^n)\backslash\{0\}$,

$$\frac{n^2}{4}\frac{\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^4}{\left(\int_{\mathbb{S}^n}|f|^2\,d\sigma\right)^2} \leq \left(\int_{\mathbb{S}^n}|f|^2\,d\sigma - \frac{\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2}{\int_{\mathbb{S}^n}|f|^2\,d\sigma}\right) \times\left(\int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n} f|^2\,d\sigma - \frac{\left|\int_{\mathbb{S}^n}(\nabla_{\mathbb{S}^n} f)\,\overline{f}\,d\sigma\right|^2}{\int_{\mathbb{S}^n}|f|^2\,d\sigma}\right). \tag{5.4}$$

Proof: Let $j = 0, \ldots, n$ and consider the operators $A_j f(x) = x_j f(x)$, $B_j f(x) = iD_j f(x) - \frac{in}{2}x_j f(x)$ on $\mathcal{L}^2(\mathbb{S}^n)$, where $x = (x_0, \ldots, x_n) \in \mathbb{S}^n$. The operator A_j is bounded and self-adjoint. We take the domain of B_j to be $\mathcal{D}(B_j) = \mathcal{C}^1(\mathbb{S}^n)$. Since $\mathcal{C}^1(\mathbb{S}^n)$ contains the spherical harmonics which are dense in $\mathcal{L}^2(\mathbb{S}^n)$, $\mathcal{D}(B_j)$ is dense in $\mathcal{L}^2(\mathbb{S}^n)$. By (5.3), B_j is a symmetric operator. Thus we can apply Theorem 4.1. Note that in this case, $\mathcal{D}(A_jB_j)\cap\mathcal{D}(B_jA_j) = \mathcal{C}^1(\mathbb{S}^n)$ for all $j = 0, \ldots, n$.

Now, take $f \in \mathcal{C}^1(\mathbb{S}^n)\backslash\{0\}$. Then

$$\sum_{j=0}^{n}\Delta_f(A_j)^2 = \int_{\mathbb{S}^n}|f|^2\,d\sigma - \frac{\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2}{\int_{\mathbb{S}^n}|f|^2\,d\sigma} \tag{5.5}$$

since $|x| = 1$. For $j = 0, \ldots, n$, we may use (5.3) and then (5.2) to derive

$$\|B_j f\|^2 = \int_{\mathbb{S}^n}|D_j f|^2\,d\sigma + \frac{n}{2}\int_{\mathbb{S}^n}|f|^2\,d\sigma - \frac{n(n+2)}{4}\int_{\mathbb{S}^n} x_j^2|f(x)|^2\,d\sigma(x), \tag{5.6}$$

and similarly we may employ (5.3) to obtain

$$|\langle B_j f, f\rangle|^2 = \left|\int_{\mathbb{S}^n}(D_j f)\overline{f}\,d\sigma\right|^2 - \frac{n^2}{4}\left(\int_{\mathbb{S}^n} x_j|f(x)|^2\,d\sigma(x)\right)^2. \tag{5.7}$$

Then applying (5.6) and (5.7) gives, after some calculation,

$$\sum_{j=0}^{n}\Delta_f(B_j)^2 = \int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n} f|^2\,d\sigma + \frac{n^2}{4}\int_{\mathbb{S}^n}|f|^2\,d\sigma - \frac{\left|\int_{\mathbb{S}^n}(\nabla_{\mathbb{S}^n} f)\overline{f}\,d\sigma\right|^2 - \frac{n^2}{4}\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2}{\int_{\mathbb{S}^n}|f|^2\,d\sigma}. \tag{5.8}$$

Finally, by (5.2),

$$\sum_{j=0}^{n}|\langle[A_j, B_j]f, f\rangle| = \sum_{j=0}^{n}\left(\int_{\mathbb{S}^n}|f|^2\,d\sigma - \int_{\mathbb{S}^n} x_j^2|f(x)|^2\,d\sigma(x)\right) = n\int_{\mathbb{S}^n}|f|^2\,d\sigma, \tag{5.9}$$

and applying Theorem 4.1 with (5.5), (5.8) and (5.9) yields, after some rearrangement, the required result. ∎

Corollary 5.1. *For all* $f \in \mathcal{C}^1(\mathbb{S}^n)\backslash\{0\}$,

$$\frac{n^2}{4}\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2 \le \left(\int_{\mathbb{S}^n}|f|^2\,d\sigma - \frac{\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2}{\int_{\mathbb{S}^n}|f|^2\,d\sigma}\right)\int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n} f|^2\,d\sigma. \tag{5.10}$$

Proof: For $j = 0,\dots,n$, by (5.3), $\mathrm{Re}\left\{\int_{\mathbb{S}^n}(D_j f)\overline{f}\,d\sigma\right\} = \frac{n}{2}\int_{\mathbb{S}^n} x_j|f(x)|^2\,d\sigma(x)$, and so

$$\left|\int_{\mathbb{S}^n}(\nabla_{\mathbb{S}^n} f)\,\overline{f}\,d\sigma\right|^2 \ge \sum_{j=0}^{n}\frac{n^2}{4}\left|\int_{\mathbb{S}^n} x_j|f(x)|^2\,d\sigma(x)\right|^2 = \frac{n^2}{4}\left|\int_{\mathbb{S}^n} x|f(x)|^2\,d\sigma(x)\right|^2. \tag{5.11}$$

Combining (5.4) and (5.11), and rearranging appropriately, we obtain (5.10). ■

When restricted to real-valued functions, the inequality (5.10) includes several known uncertainty principles on $\mathbb{S}^n$. Indeed, for $n = 1$, (5.10) gives (2.7) for real-valued functions.

For $n = 2$, recall the following inequality in [8] for $\int_{\mathbb{S}^2}|f|^2\,d\sigma = 1$:

$$\left|\int_{\mathbb{S}^2} x|f(x)|^2\,d\sigma(x)\right|^2 \le \left(1 - \left|\int_{\mathbb{S}^2} x|f(x)|^2\,d\sigma(x)\right|^2\right)\int_{\mathbb{S}^2}|\Omega f - a(f)f|^2\,d\sigma, \tag{5.12}$$

where $\Omega f(x) := -ix \times \nabla f(x)$ and $a(f) := \int_{\mathbb{S}^2}(\Omega f)\overline{f}\,d\sigma$. If f is real-valued, we see from (5.2) and (5.3) that for $j,k = 0,\dots,n$, $j \ne k$,

$$\int_{\mathbb{S}^2} x_j(D_k f(x))f(x)\,d\sigma(x) = \frac{3}{2}\int_{\mathbb{S}^2} x_j x_k f(x)^2\,d\sigma(x) = \int_{\mathbb{S}^2} x_k(D_j f(x))f(x)\,d\sigma(x),$$

and thus $a(f) = 0$. Also for $x \in \mathbb{S}^2$, since $|x \times \nabla f(x)| = |x \times \nabla_{\mathbb{S}^2} f(x)|$ and $x \cdot \nabla_{\mathbb{S}^2} f(x) = 0$, $|\Omega f(x)| = |x|\,|\nabla_{\mathbb{S}^2} f(x)| = |\nabla_{\mathbb{S}^2} f(x)|$. Hence (5.10) gives (5.12) for real-valued f.

Finally we suppose that f is radial, i.e. for some $u \in \mathbb{S}^n$, $f(x)$ depends only on $x \cdot u$, $x \in \mathbb{S}^n$. Then for $x \in \mathbb{S}^n$ we may put $f(x) = g(t)$ for some function g on $[0,\pi]$. In this case, (5.10) implies (2.12) when $w = w_{n-1}$ as in (2.14).

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