WAVELET FRAMES AND SHIFT-INVARIANT SUBSPACES OF PERIODIC FUNCTIONS

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Abstract

A general approach based on polyphase splines, with analysis in the frequency domain, is developed for studying wavelet frames of periodic functions of one or higher dimensions. Characterizations of frames for shift-invariant subspaces of periodic functions and results on the structure of these subspaces are obtained. Starting from any multiresolution analysis, a constructive proof is provided for the existence of a normalized tight wavelet frame. The construction gives the minimum number of wavelets required. As an illustration of the approach developed, the one-dimensional dyadic case is further discussed in detail, concluding with a concrete example of trigonometric polynomial wavelet frames.

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1. Introduction

The theory of frames was first introduced in [9], primarily for the study of nonharmonic Fourier series but also with an extension to a general Hilbert space setting. For a complex separable Hilbert space H and a countable index set I, we say that a sequence of elements $\{x_\nu\}_{\nu\in I}$ in H is a *frame* for H if there exist positive constants A and B such that for every $x \in H$,

$$A\,\|x\|^2 \le \sum_{\nu\in I} |\langle x, x_\nu \rangle|^2 \le B\,\|x\|^2, \tag{1.1}$$

where $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm of H respectively. The constants A and B, which are not unique, are called *bounds* of the frame. (In some literature, the sharpest possible constants of A and B in (1.1) are known as the lower and upper frame bounds respectively.) If $A = B = 1$, then $\{x_\nu\}_{\nu\in I}$ is a *normalized tight frame*. Normalized tight frames are a generalization of orthonormal bases. For details on basic concepts of frames, see for instance [5, 9, 17].

In recent years, due to the development of wavelet analysis, there is again much interest in the topic of frames. Many papers, for example [1, 2, 6, 7, 8, 16], have been published on wavelet frames for $L^2(\mathbb{R}^s)$, the space of square-integrable complex-valued functions over the Euclidean space $\mathbb{R}^s$, where s equals 1 or a higher integer. There are also various efforts in applying wavelet frames to different aspects of signal and image processing.

Motivated by the fact that many signals in practice are periodic, we develop in this paper a general theory of wavelet frames for $L^2([0,2\pi)^s)$, the space of s-dimensional 2π-periodic square-integrable complex-valued functions, where s is any positive integer. For studies of frames of periodic functions in other contexts, see [4, 14]. Here we build on the approach for constructing periodic wavelet bases in [10] to obtain periodic wavelet frames. The construction begins with a multiresolution analysis that has a more relaxed condition than in [10], making the overall process more tractable and flexible. Further, wavelets with very narrow bandwidths can be constructed, and they are more efficient and effective in the processing of narrow band signals corrupted by noise. Fixing notations, the inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$ of $L^2([0,2\pi)^s)$ are given by $\langle f, g \rangle := \frac{1}{(2\pi)^s}\int_{[0,2\pi)^s} f(x)\overline{g(x)}\,dx$, where $f, g \in L^2([0,2\pi)^s)$, and $\|\cdot\| := \langle \cdot,\cdot \rangle^{1/2}$. The Fourier series of a function $f \in L^2([0,2\pi)^s)$ is written as $\sum_{n\in\mathbb{Z}^s} \widehat{f}(n)e^{in\cdot}$, where $\widehat{f}(n) := \langle f, e^{in\cdot} \rangle$, $n \in \mathbb{Z}^s$, are its Fourier coefficients.

Let M be an $s \times s$ matrix with integer entries such that all its eigenvalues lie outside the unit circle. We set

$$D := M^T, \qquad d := |\det(M)| = |\det(D)|.$$

For $k \ge 0$, let $\mathcal{L}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathcal{R}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/D^k\mathbb{Z}^s$. Then $d^k = |\mathcal{L}_k| = |\mathcal{R}_k|$,

$$\mathbb{Z}^s = \bigcup_{\ell\in\mathcal{L}_k} (\ell + M^k\mathbb{Z}^s) = \bigcup_{j\in\mathcal{R}_k} (j + D^k\mathbb{Z}^s), \tag{1.2}$$

and for any distinct $\ell_1, \ell_2 \in \mathcal{L}_k$, $j_1, j_2 \in \mathcal{R}_k$,

$$(\ell_1 + M^k\mathbb{Z}^s) \cap (\ell_2 + M^k\mathbb{Z}^s) = \emptyset = (j_1 + D^k\mathbb{Z}^s) \cap (j_2 + D^k\mathbb{Z}^s).$$

For positive integers r and ρ, let $\mathcal{S}(M^k)^{r\times\rho}$ denote the class of periodic sequences of $r\times\rho$ complex matrices of period M^k, that is, $H_k(\ell+M^kp)=H_k(\ell)$ for all $H_k\in\mathcal{S}(M^k)^{r\times\rho}$, $\ell,p\in\mathbb{Z}^s$. When $r=\rho=1$, we write $\mathcal{S}(M^k)^{1\times 1}$ simply as $\mathcal{S}(M^k)$. For $H_k\in\mathcal{S}(M^k)^{r\times\rho}$, we define the finite Fourier transform of H_k by

$$\widehat{H}_k(j):=\sum_{\ell\in\mathcal{L}_k}H_k(\ell)e^{-ij\cdot(2\pi M^{-k}\ell)},\quad j\in\mathcal{R}_k.$$

Then $\widehat{H}_k\in\mathcal{S}(D^k)^{r\times\rho}$, and the sequence H_k can be recovered from $\widehat{H}_k$ by

$$H_k(\ell)=\frac{1}{|\mathcal{R}_k|}\sum_{j\in\mathcal{R}_k}\widehat{H}_k(j)e^{i\ell\cdot(2\pi D^{-k}j)},\quad \ell\in\mathcal{L}_k.$$

For $k\geq 0$ and $\ell\in\mathbb{Z}^s$, define the $2\pi M^{-k}\ell$*-shift operator* $T_k^\ell:L^2([0,2\pi)^s)\longrightarrow L^2([0,2\pi)^s)$ by

$$T_k^\ell f:=f(\cdot-2\pi M^{-k}\ell),\quad f\in L^2([0,2\pi)^s). \tag{1.3}$$

We seek functions ϕ_0^m, $m=1,2,\ldots,r$, and ψ_k^m, $k\geq 0$, $m=1,2,\ldots,\rho$, in $L^2([0,2\pi)^s)$, where r and ρ are some positive integers, such that the collection $\{\,\phi_0^m:m=1,2,\ldots,r\,\}\cup\{\,T_k^\ell\psi_k^m:k\geq 0,\ m=1,2,\ldots,\rho,\ \ell\in\mathcal{L}_k\,\}$ forms a frame, known as a *wavelet frame*, for the space $L^2([0,2\pi)^s)$. The functions ψ_k^m are called *periodic wavelets*, or simply *wavelets*. In this paper, we provide a general approach to construct normalized tight wavelet frames. The periodic wavelets here are nonstationary in the sense that the functions ψ_k^m, $k\geq 1$, need not be dilates of ψ_0^m, contrary to the usual situation of wavelets in the space $L^2(\mathbb{R}^s)$. Further, they are not necessarily functions obtained from periodizing appropriate wavelets in $L^2(\mathbb{R}^s)$.

Our construction is based on a multiresolution analysis of $L^2([0,2\pi)^s)$, which comprises a sequence of nesting finite-dimensional subspaces $\{V_k\}_{k\geq 0}$. Each of these subspaces is spanned by the $2\pi M^{-k}\ell$-shifts of r functions $\phi_k^1,\ldots,\phi_k^r$, where $\ell\in\mathcal{L}_k$. In Section 2, we study the structure of such shift-invariant subspaces V_k. In particular, we provide characterizations of $\{\,T_k^\ell\phi_k^m:m=1,2,\ldots,r,\ \ell\in\mathcal{L}_k\,\}$ forming a frame for V_k. To this end, we adapt the method of polyphase splines introduced in [10] for the study of wavelet bases of $L^2([0,2\pi)^s)$, which essentially carries out the analysis in the frequency domain. The beauty of polyphase splines lies in the fact that they provide a useful alternative spanning set for the shift-invariant subspace V_k. As V_k is finite dimensional, the polyphase splines facilitate the use of appropriate linear algebraic tools in analyzing its structure. In addition to characterizing frames, we also show that for any such shift-invariant subspace V_k, it is always possible to find functions $\theta_k^1,\ldots,\theta_k^r$ which give a normalized tight frame for V_k. Further, the subspace V_k can be decomposed into an orthogonal direct sum of shift-invariant subspaces each generated by one function.

We turn to the construction of normalized tight wavelet frames for $L^2([0,2\pi)^s)$ in Section 3. Beginning with a multiresolution analysis $\{V_k\}_{k\geq 0}$ of $L^2([0,2\pi)^s)$, the wavelet construction problem entails the finding of appropriate $\psi_k^m\in L^2([0,2\pi)^s)$ such that for every $k\geq 0$, $\{\,T_k^\ell\psi_k^m:m=1,2,\ldots,\rho,\ \ell\in\mathcal{L}_k\,\}$ forms a normalized tight frame for W_k, the orthogonal complement of V_k in V_{k+1}. Unlike the basis case often encountered in wavelet analysis, this problem need not have a solution when $\rho=r(d-1)$ wavelets are used. Fortunately, a solution always exists with $\rho=rd$ wavelets and the main result of the section provides a constructive proof for it. Polyphase splines again play a central role. A careful examination of the proof reveals that the construction actually gives the minimum number of nontrivial wavelets which generate a normalized tight frame for W_k.

In Section 4, we illustrate the general approach proposed by focusing on the case when $s = 1$, $M = 2$ and $r = 1$, the one-dimensional dyadic scenario most commonly studied in wavelet analysis. The results of the earlier sections now take a much simpler form, and are given in terms of the orthogonal splines in [13] which are a special case of polyphase splines. We end the paper with an example of a multiresolution analysis $\{V_k\}_{k\geq 0}$ where each V_k is generated by a trigonometric polynomial, and an analysis of whether one or two wavelets are needed for a normalized tight frame of its corresponding W_k.

2. Shift-invariant subspaces

Let r be a fixed positive integer. Throughout this section, for $k \geq 0$, we consider the $2\pi M^{-k}\ell$-shift invariant subspace V_k of $L^2([0, 2\pi)^s)$ of the form

$$V_k := \langle \{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \} \rangle, \tag{2.1}$$

the linear span of the collection $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$, where ϕ_k^m, $m = 1, 2, \ldots, r$, are functions in $L^2([0, 2\pi)^s)$, and the operators T_k^ℓ are as defined in (1.3). By (1.2), we see that the subspace V_k is indeed invariant under the operators T_k^ℓ, $\ell \in \mathbb{Z}^s$. The smallest number of nontrivial generators $\phi_k^1, \ldots, \phi_k^r$ of V_k is called the *length* of V_k and denoted by $\mathrm{len}(V_k)$. Since $\mathrm{len}(V_k)$ is finite, we say that V_k is a *finitely generated shift-invariant* (FSI) subspace of $L^2([0, 2\pi)^s)$. If $\mathrm{len}(V_k) = 1$, then V_k is called a *principal shift-invariant* (PSI) subspace of $L^2([0, 2\pi)^s)$. These terminologies are analogously defined as those in [3, 15] for shift-invariant subspaces of the space $L^2(\mathbb{R}^s)$, where the shift operator is with respect to translates over the integer lattice $\mathbb{Z}^s$.

Our objective is to study the structure of the $2\pi M^{-k}\ell$-shift invariant subspace V_k in (2.1). In particular, we are interested in characterizations of $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ forming a frame for V_k, which is

$$A \, \|f\|^2 \leq \sum_{m=1}^{r} \sum_{\ell \in \mathcal{L}_k} | \langle f, T_k^\ell \phi_k^m \rangle |^2 \leq B \, \|f\|^2, \quad f \in V_k, \tag{2.2}$$

for some positive constants A and B. (It is a normalized tight frame if we are able to choose $A = B = 1$.) Since V_k is a finite-dimensional subspace, the constants A and B always exist and the collection $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ is indeed a frame for V_k, see for instance [5, Proposition 1.1.2]. However, the bounds A and B may not remain the same for all the subspaces V_k as k varies. The focus here is frames with *common* bounds for the subspaces V_k, $k \geq 0$.

We approach the problem from the frequency domain. To this end, it is convenient to express V_k in (2.1) as

$$V_k = \{ f \in L^2([0, 2\pi)^s) : \widehat{f}(n) = \sum_{m=1}^{r} \widehat{\alpha}_k^m(n) \widehat{\phi}_k^m(n) \text{ for all } n \in \mathbb{Z}^s \text{ and some } \widehat{\alpha}_k^1, \ldots, \widehat{\alpha}_k^r \in \mathcal{S}(D^k) \}, \tag{2.3}$$

which follows from the definition of the finite Fourier transform. The main tool for our study is the polyphase splines introduced in [10]. For $k \geq 0$, we define *polyphase splines* $v_{k,j}^m$ in $L^2([0,2\pi)^s)$ by

$$v_{k,j}^m(x) := \sum_{p \in \mathbb{Z}^s} \widehat{\phi}_k^m(j + D^k p) e^{i(j+D^k p)\cdot x}, \quad x \in \mathbb{R}^s, \tag{2.4}$$

for $m = 1, 2, \ldots, r$, $j \in \mathcal{R}_k$, where $\widehat{\phi}_k^m(n)$, $n \in \mathbb{Z}^s$, are the Fourier coefficients of ϕ_k^m. Polyphase splines were used in [10, 11] to characterize and analyze linear independence, orthonormality and biorthogonality of sets spanned by $2\pi M^{-k}\ell$-shifts of functions in $L^2([0,2\pi)^s)$. We shall now see that they can also be used effectively in the study of frames.

Let us recall some useful facts about the polyphase splines. First of all, for $k \geq 0$, $j \in \mathcal{R}_k$,

$$\widehat{v}_{k,j}^m(n) = \begin{cases} \widehat{\phi}_k^m(j + D^k p), & \text{if } n = j + D^k p \text{ for some } p \in \mathbb{Z}^s, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Then it follows from Parseval's identity that

$$\langle v_{k,j}^m, v_{k,\ell}^\mu \rangle = \sum_{n \in \mathbb{Z}^s} \widehat{v}_{k,j}^m(n) \overline{\widehat{v}_{k,\ell}^\mu(n)} = 0 \quad \text{if } j \neq \ell, \tag{2.6}$$

for all $m, \mu = 1, 2, \ldots, r$.

Following [10], for each $j \in \mathcal{R}_k$, let

$$M_k(j) := \left(\langle v_{k,j}^m, v_{k,j}^\mu \rangle \right)_{m,\mu=1}^r. \tag{2.7}$$

Lemma 2.1. *The collection* $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ *is a frame for* V_k *with bounds* A *and* B *if and only if*

$$A \sum_{j \in \mathcal{R}_k} \widehat{\alpha}_k(j) M_k(j) \widehat{\alpha}_k(j)^* \leq d^k \sum_{j \in \mathcal{R}_k} \widehat{\alpha}_k(j) M_k(j) M_k(j)^* \, \widehat{\alpha}_k(j)^* \leq B \sum_{j \in \mathcal{R}_k} \widehat{\alpha}_k(j) M_k(j) \widehat{\alpha}_k(j)^* \tag{2.8}$$

for every $\widehat{\alpha}_k \in \mathcal{S}(D^k)^{1\times r}$.

Proof. We shall show that the frame condition (2.2) is equivalent to (2.8). Fix an arbitrary $f \in V_k$. Then there exist $\alpha_k^m \in \mathcal{S}(M^k)$, $m = 1, 2, \ldots, r$, such that

$$f = \sum_{m=1}^r \sum_{\ell \in \mathcal{L}_k} \alpha_k^m(\ell) \, T_k^\ell \phi_k^m. \tag{2.9}$$

For $m = 1, 2, \ldots, r$, since $\phi_k^m = \sum_{j \in \mathcal{R}_k} v_{k,j}^m$, we have

$$T_k^\ell \phi_k^m = \sum_{j \in \mathcal{R}_k} e^{-ij\cdot(2\pi M^{-k}\ell)} \, v_{k,j}^m, \quad \ell \in \mathcal{L}_k. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$f = \sum_{m=1}^r \sum_{j \in \mathcal{R}_k} \widehat{\alpha}_k^m(j) \, v_{k,j}^m, \tag{2.11}$$

where $\widehat{\alpha}_k^m(j) = \sum_{\ell \in \mathcal{L}_k} \alpha_k^m(\ell) \, e^{-ij\cdot(2\pi M^{-k}\ell)}$, $\widehat{\alpha}_k^m \in \mathcal{S}(D^k)$. By (2.6), we deduce from (2.11) that

$$\|f\|^2 = \sum_{j \in \mathcal{R}_k} \widehat{\alpha}_k(j) \, M_k(j) \, \widehat{\alpha}_k(j)^*, \tag{2.12}$$

where $\widehat{\alpha}_k(j) := (\widehat{\alpha}_k^1(j), \dots, \widehat{\alpha}_k^r(j))$. Using (2.10), (2.11) and the relation $\sum_{\ell\in\mathcal{L}_k} e^{i(j-\nu)\cdot(2\pi M^{-k}\ell)} = |\mathcal{L}_k|\,\delta_{\nu j}$ for $j,\nu \in \mathcal{R}_k$, we obtain

$$\sum_{m=1}^{r}\sum_{\ell\in\mathcal{L}_k} |\langle f, T_k^\ell \phi_k^m \rangle|^2 = d^k \sum_{j\in\mathcal{R}_k} \widehat{\alpha}_k(j)\, M_k(j)\, M_k(j)^*\, \widehat{\alpha}(j)^* \tag{2.13}$$

since $|\mathcal{L}_k| = d^k$. Now (2.12) and (2.13) show that (2.8) and (2.2) are equivalent, and the lemma follows. ■

Note that in the proof of the above lemma, we have shown that $V_k \subseteq \langle \{ v_{k,j}^m : m = 1,2,\dots,r,\ j \in \mathcal{R}_k \} \rangle$. In fact, we have

$$V_k = \langle \{ v_{k,j}^m : m = 1,2,\dots,r,\ j \in \mathcal{R}_k \} \rangle. \tag{2.14}$$

To see this, for each $m = 1,2,\dots,r$ and $j \in \mathcal{R}_k$, we define $\widehat{\alpha}_k^1, \dots, \widehat{\alpha}_k^r \in \mathcal{S}(D^k)$ by $\widehat{\alpha}_k^\mu \equiv 0$ for $\mu \neq m$, and $\widehat{\alpha}_k^m(\nu + D^k p) := \delta_{\nu j}$, $\nu \in \mathcal{R}_k$, $p \in \mathbb{Z}^s$. Then (2.5) implies that $\widehat{v}_{k,j}^m(n) = \sum_{\mu=1}^r \widehat{\alpha}_k^\mu(n)\widehat{\phi}_k^\mu(n)$ for all $n \in \mathbb{Z}^s$. By (2.3), we conclude that $v_{k,j}^m \in V_k$, and (2.14) follows. Further, we observe from (2.6) that

$$V_k = \bigoplus_{j\in\mathcal{R}_k}^{\perp} \langle \{ v_{k,j}^m : m = 1,2,\dots,r \} \rangle, \tag{2.15}$$

where the notation $\oplus^\perp$ denotes orthogonal direct sum. In other words, polyphase splines give an alternative set of functions that span V_k and possess partial orthogonality in the sense of (2.6).

Let $\mathcal{G}_k$ be the Gram matrix of $\{ T_k^\ell \phi_k^m : m = 1,2,\dots,r,\ \ell \in \mathcal{L}_k \}$. More specifically, writing as in [10],

$$\mathcal{G}_k := (\Phi_{m\mu})_{m,\mu=1}^r,$$

where

$$\Phi_{m\mu} := \big(\langle T_k^\ell \phi_k^m, T_k^\zeta \phi_k^\mu \rangle \big)_{\ell,\zeta\in\mathcal{L}_k}, \quad m,\mu = 1,2,\dots,r.$$

Based on Lemma 2.1, we obtain the following theorem which gives different characterizations of $\{ T_k^\ell \phi_k^m : m = 1,2,\dots,r,\ \ell \in \mathcal{L}_k \}$ being a frame for V_k having bounds A and B.

Theorem 2.1. *Let A and B be positive constants. Then the following are equivalent.*

(i) *The collection $\{ T_k^\ell \phi_k^m : m = 1,2,\dots,r,\ \ell \in \mathcal{L}_k \}$ is a frame for V_k with bounds A and B.*

(ii) *For each $j \in \mathcal{R}_k$, $\dfrac{A}{d^k} \le \lambda_j \le \dfrac{B}{d^k}$ for every nonzero eigenvalues λ_j of $M_k(j)$.*

(iii) *Every nonzero eigenvalue λ of $\mathcal{G}_k$ satisfies $A \le \lambda \le B$.*

(iv) *The collection $\{ \sqrt{d^k}\, v_{k,j}^m : m = 1,2,\dots,r,\ j \in \mathcal{R}_k \}$ is a frame for V_k with bounds A and B.*

Proof. We first prove that (i) and (ii) are equivalent. Suppose that (i) holds. Fix $\nu \in \mathcal{R}_k$ and let λ_ν be a nonzero eigenvalue of $M_k(\nu)$. Choose $\widehat{\alpha}_k(\nu) \in \mathbb{C}^r$ such that $\widehat{\alpha}_k(\nu)^*$ is an eigenvector of $M_k(\nu)$ corresponding to the eigenvalue λ_ν. For $j \in \mathcal{R}_k$ and $j \neq \nu$, set $\widehat{\alpha}_k(j)$ to be the zero vector in $\mathbb{C}^r$. Then Lemma 2.1 implies that

$$A\,\widehat{\alpha}_k(\nu) M_k(\nu) \widehat{\alpha}_k(\nu)^* \le d^k\, \widehat{\alpha}_k(\nu) M_k(\nu) M_k(\nu)^*\, \widehat{\alpha}_k(\nu)^* \le B\,\widehat{\alpha}_k(\nu) M_k(\nu) \widehat{\alpha}_k(\nu)^*. \tag{2.16}$$

Since $M_k(\nu)$ is Hermitian and $M_k(\nu)\widehat{\alpha}_k(\nu)^* = \lambda_\nu \widehat{\alpha}_k(\nu)^*$, it follows from (2.16) that

$$A\,\lambda_\nu \|\widehat{\alpha}_k(\nu)\|^2 \le d^k\, \lambda_\nu^2 \|\widehat{\alpha}_k(\nu)\|^2 \le B\,\lambda_\nu \|\widehat{\alpha}_k(\nu)\|^2.$$

Hence $\frac{A}{d^k} \le \lambda_\nu \le \frac{B}{d^k}$.

Conversely, suppose that (ii) holds. For each $j \in \mathcal{R}_k$, since $M_k(j)$ is Hermitian, there exists an $r \times r$ unitary matrix $U_k(j)$ such that

$$M_k(j) = U_k(j)^* D_k(j) U_k(j), \tag{2.17}$$

where $D_k(j) := \operatorname{diag}(\,\lambda_k^1(j), \ldots, \lambda_k^r(j)\,)$. As $M_k(j)$ is positive semi-definite, it follows from (ii) that $A\,\lambda_k^m(j) \le d^k\,(\lambda_k^m(j))^2 \le B\,\lambda_k^m(j)$ for all $m = 1, 2, \ldots, r$. Consequently, for any $(c_1, \ldots, c_r) \in \mathbb{C}^r$,

$$A \sum_{m=1}^{r} \lambda_k^m(j)\,|c_m|^2 \le d^k \sum_{m=1}^{r} (\lambda_k^m(j))^2\,|c_m|^2 \le B \sum_{m=1}^{r} \lambda_k^m(j)\,|c_m|^2,$$

from which we deduce that

$$\begin{aligned} A\,(U_k(j)\widehat{\alpha}_k(j)^*)^* D_k(j)(U_k(j)\widehat{\alpha}_k(j)^*) &\le d^k\,(U_k(j)\widehat{\alpha}_k(j)^*)^* D_k(j)^2 (U_k(j)\widehat{\alpha}_k(j)^*) \\ &\le B\,(U_k(j)\widehat{\alpha}_k(j)^*)^* D_k(j)(U_k(j)\widehat{\alpha}_k(j)^*) \end{aligned}$$

for any $\widehat{\alpha}_k \in \mathcal{S}(D^k)^{1\times r}$. By (2.17), the above inequalities amount to

$$A\,\widehat{\alpha}_k(j) M_k(j) \widehat{\alpha}_k(j)^* \le d^k\,\widehat{\alpha}_k(j) M_k(j) M_k(j)^*\,\widehat{\alpha}_k(j)^* \le B\,\widehat{\alpha}_k(j) M_k(j) \widehat{\alpha}_k(j)^*.$$

Since $j \in \mathcal{R}_k$ is arbitrary, we conclude that (2.8) holds, and by Lemma 2.1, this gives (i).

For the equivalence of (ii) and (iii), we note that by [10], $M_k(j) = \frac{1}{d^k}(\gamma_{m\mu}^j)_{m,\mu=1}^r$, where $\gamma_{m\mu}^j$, $j \in \mathcal{R}_k$, are the eigenvalues of $\Phi_{m\mu}$. Therefore it follows from [10, Lemma 3.1] that the eigenvalues of $\mathcal{G}_k$ are precisely the eigenvalues of all the $d^k M_k(j)$, $j \in \mathcal{R}_k$, and this leads to the equivalence of (ii) and (iii).

Finally, we show that (i) and (iv) are equivalent. Similar to the proof of Lemma 2.1, by using (2.11), we see that for any $f \in V_k$,

$$\sum_{\mu=1}^{r} \sum_{\nu \in \mathcal{R}_k} |\,\langle\, f, \sqrt{d^k}\, v_{k,\nu}^\mu \,\rangle\,|^2 = d^k \sum_{\nu \in \mathcal{R}_k} \widehat{\alpha}_k(\nu)\, M_k(\nu)\, M_k(\nu)^*\, \widehat{\alpha}(\nu)^*.$$

Now applying (2.13) gives the result. ∎

Using (2.15) and (2.1), Theorem 2.1 implies that the dimension of V_k is exactly the total number of nonzero eigenvalues of all the $M_k(j)$, which is also the total number of nonzero eigenvalues of $\mathcal{G}_k$.

The following theorem gives a characterization of a special class of tight frames for V_k in terms of polyphase splines.

Theorem 2.2. *The collection* $\{\, T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \,\}$ *is a normalized tight frame for* V_k *and* $\langle\, T_k^\ell \phi_k^m, T_k^\zeta \phi_k^\mu \,\rangle = 0$ *for all* $m, \mu = 1, 2, \ldots, r$, $m \ne \mu$, *and* $\ell, \zeta \in \mathcal{L}_k$, *if and only if for all* $j \in \mathcal{R}_k$, $\langle\, v_{k,j}^m, v_{k,j}^\mu \,\rangle = 0$ *if* $m \ne \mu$ *and* $\|v_{k,j}^m\|^2 = 0$ *or* $\frac{1}{d^k}$, *for all* $m, \mu = 1, 2, \ldots, r$; *that is,* $M_k(j)$ *is a diagonal matrix with diagonal entries* 0 *or* $\frac{1}{d^k}$ *for all* $j \in \mathcal{R}_k$.

Proof. As in the proof of [10, Lemma 3.1], there exists an $d^k \times d^k$ unitary matrix F such that $\Phi_{m\mu} = F^*\Gamma_{m\mu}F$, where $\Gamma_{m\mu} := \operatorname{diag}(\gamma^j_{m\mu})_{j\in\mathcal{R}_k}$, and $\gamma^j_{m\mu}$ are the eigenvalues of $\Phi_{m\mu}$. Since $\gamma^j_{m\mu} = d^k\langle v^m_{k,j}, v^\mu_{k,j}\rangle$ as noted in [10], the result now follows from Theorem 2.1. ■

The special type of normalized tight frames in Theorem 2.2 is particularly interesting. Indeed, its corresponding nonzero polyphase splines lead to an orthonormal basis of V_k. Specifically, it follows from (2.6), (2.14) and Theorem 2.2 that the collection $\{\sqrt{d^k}v^m_{k,j} : m \in \{1,2,\dots,r\},\ j \in \mathcal{R}_k,\ v^m_{k,j} \neq 0\}$ forms an orthonormal basis for V_k. However, the original collection $\{T^\ell_k\phi^m_k : m = 1,2,\dots,r,\ \ell \in \mathcal{L}_k\}$ is only partially orthogonal as $\langle T^\ell_k\phi^m_k, T^\zeta_k\phi^m_k\rangle$ could be nonzero even when ℓ and ζ are different. The above discussion gives another illustration of the strength of the polyphase spline approach. The change of functions in the method converts a normalized tight frame with certain partial orthogonality into an orthonormal basis after removing all the redundancies in the representation. This could simplify the analysis involved in studying V_k significantly. Of course, in the first place, one must begin with a normalized tight frame of the type described in Theorem 2.2. The following theorem shows that we can *always* find such a tight frame for V_k.

Theorem 2.3. *There exist functions $\theta^1_k,\dots,\theta^r_k$ in V_k such that $\{T^\ell_k\theta^m_k : m = 1,2,\dots,r,\ \ell\in\mathcal{L}_k\}$ forms a normalized tight frame for V_k, and for all $m,\mu = 1,2,\dots,r$ and $\ell,\zeta\in\mathcal{L}_k$,*

$$\langle T^\ell_k\theta^m_k, T^\zeta_k\theta^\mu_k\rangle = 0 \quad \text{if } m\neq\mu. \tag{2.18}$$

Proof. Fix $j\in\mathcal{R}_k$. Consider $M_k(j)$ as defined in (2.7). Since $M_k(j)$ is positive semi-definite and Hermitian, there exists an $r\times r$ unitary matrix $U_k(j)$ such that

$$U_k(j)M_k(j)U_k(j)^* = \operatorname{diag}(\lambda^1_k(j),\dots,\lambda^r_k(j)), \tag{2.19}$$

where $\lambda^1_k(j),\dots,\lambda^r_k(j)$ are the eigenvalues of $M_k(j)$ which are always nonnegative. For $m = 1,2,\dots,r$, define $\beta^m_k(j)$ by

$$\beta^m_k(j) := \begin{cases} \dfrac{1}{\sqrt{d^k\lambda^m_k(j)}}, & \text{if } \lambda^m_k(j)\neq 0,\\ 1, & \text{if } \lambda^m_k(j) = 0.\end{cases}$$

If we set $D_k(j) := \operatorname{diag}(\beta^1_k(j),\dots,\beta^r_k(j))$ and $E_k(j) := D_k(j)U_k(j)$, then we see from (2.19) that

$$E_k(j)M_k(j)E_k(j)^* = \operatorname{diag}(\delta^1_k(j),\dots,\delta^r_k(j)), \tag{2.20}$$

where $\delta^m_k(j) = 0$ or $\dfrac{1}{d^k}$, $m = 1,2,\dots,r$. Write $E_k(j) = (a_{m,\mu}(j))^r_{m,\mu=1}$. For $m = 1,2,\dots,r$, we define

$$w^m_{k,j} := \sum_{\mu=1}^r a_{m,\mu}(j)v^\mu_{k,j}. \tag{2.21}$$

By (2.15), $w^m_{k,j}\in V_k$ for $m = 1,2,\dots,r$. Since $E_k(j)$ is invertible, it follows from (2.21) that $\{w^m_{k,j} : m = 1,2,\dots,r\}$ is also a spanning set for $\langle\{v^m_{k,j} : m = 1,2,\dots,r\}\rangle$, and hence $V_k = \langle\{w^m_{k,j} : m = 1,2,\dots,r,\ j\in\mathcal{R}_k\}\rangle$.

For $m = 1, 2, \ldots, r$, we set

$$\theta_k^m := \sum_{j \in \mathcal{R}_k} w_{k,j}^m. \tag{2.22}$$

Then $\theta_k^1, \ldots, \theta_k^r$ lie in V_k, and the functions $w_{k,j}^m$, $m = 1, 2, \ldots, r$, $j \in \mathcal{R}_k$, are their corresponding polyphase splines. Now (2.20) and (2.21) imply that the matrix

$$(\langle w_{k,j}^m, w_{k,j}^\mu \rangle)_{m,\mu=1}^r = E_k(j) M_k(j) E_k(j)^* \tag{2.23}$$

is diagonal with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$. The result follows from Theorem 2.2. ■

In the proof of Theorem 2.3, the unitary matrices $U_k(j)$, $j \in \mathcal{R}_k$, can be appropriately chosen so that for $r_k := \operatorname{len}(V_k)$, the functions θ_k^m, $m = r_k + 1, r_k + 2, \ldots, r$, are the zero function. In other words, it is possible to use only r_k functions, instead of r functions, to generate a normalized tight frame for V_k that possesses the partial orthogonality property of (2.18). This is made explicit in the following corollary.

Corollary 2.1. *The length of V_k is given by*

$$\operatorname{len}(V_k) = \max_{j \in \mathcal{R}_k} \{ \dim(\langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle) \}. \tag{2.24}$$

With $r_k = \operatorname{len}(V_k)$, there exist $\theta_k^1, \ldots, \theta_k^{r_k} \in V_k$ such that $\{ T_k^\ell \theta_k^m : m = 1, 2, \ldots, r_k,\ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for V_k, and for all $m, \mu = 1, 2, \ldots, r_k$ and $\ell, \zeta \in \mathcal{L}_k$, (2.18) *holds.*

Proof. Let $s_k := \max_{j \in \mathcal{R}_k} \{ \dim(\langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle) \}$. First note that for every $j \in \mathcal{R}_k$, the number of nonzero eigenvalues of $M_k(j)$, which equals $\dim(\langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle)$, is always no larger than s_k. Multiplying (2.19) from the left and from the right by appropriate $r \times r$ orthogonal matrices, we get $\lambda_k^m(j) = 0$ for $m = s_k + 1, s_k + 2, \ldots, r$, $j \in \mathcal{R}_k$. Then it follows from (2.20) and (2.23) that correspondingly $\|w_{k,j}^m\|^2 = 0$ for $m = s_k + 1, s_k + 2, \ldots, r$, $j \in \mathcal{R}_k$. Consequently, by (2.22), $\theta_k^m = 0$ for $m = s_k + 1, s_k + 2, \ldots, r$. Hence, $\langle \{ T_k^\ell \theta_k^m : m = 1, 2, \ldots, s_k, \ell \in \mathcal{L}_k \} \rangle = V_k$ which implies that $\operatorname{len}(V_k) \le s_k$.

On the other hand, since $r_k = \operatorname{len}(V_k)$, there exist r_k functions $\tau_k^1, \ldots, \tau_k^{r_k}$ such that $\langle \{ T_k^\ell \tau_k^m : m = 1, 2, \ldots, r_k,\ \ell \in \mathcal{L}_k \} \rangle = V_k$. Let $y_{k,j}^m$, $m = 1, 2, \ldots, r_k$, $j \in \mathcal{R}_k$, be their corresponding polyphase splines. Since $\tau_k^m \in V_k$ for $m = 1, 2, \ldots r_k$, we have by (2.3)

$$\langle \{ y_{k,j}^m : m = 1, 2, \ldots, r_k \} \rangle \subseteq \langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle, \quad j \in \mathcal{R}_k.$$

In fact, equality also holds due to the orthogonal decomposition (2.15). Thus for every $j \in \mathcal{R}_k$,

$$r_k \ge \dim(\langle \{ y_{k,j}^m : m = 1, 2, \ldots, r_k \} \rangle) = \dim(\langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle),$$

implying that $\operatorname{len}(V_k) = r_k \ge s_k$.

Combining the above observations gives the result. ■

Corollary 2.2. *The FSI subspace V_k can be decomposed into*

$$V_k = V_k^1 \oplus^\perp \cdots \oplus^\perp V_k^{r_k}, \tag{2.25}$$

where $r_k = \text{len}(V_k)$ *and for each* $m = 1, 2, \ldots, r_k$, V_k^m *is a PSI subspace of the form*

$$V_k^m := \langle \{ T_k^\ell \theta_k^m : \ell \in \mathcal{L}_k \} \rangle \tag{2.26}$$

for some θ_k^m *in* V_k.

Proof. With $\theta_k^1, \ldots, \theta_k^{r_k}$ as in Corollary 2.1, since $\{ T_k^\ell \theta_k^m : m = 1, 2, \ldots, r_k, \ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for V_k, it is well known (see for instance [2, Theorem 1.3.2]) that every $f \in V_k$ can be written as $f = \sum_{m=1}^{r_k} \sum_{\ell \in \mathcal{L}_k} \langle f, T_k^\ell \theta_k^m \rangle T_k^\ell \theta_k^m$. Defining V_k^m, $m = 1, 2, \ldots, r_k$, as in (2.26), it follows that $V_k = V_k^1 + \cdots + V_k^{r_k}$. Since (2.18) also holds, this sum is an orthogonal direct sum, giving (2.25). ■

The decomposition (2.25) shows that every FSI subspace V_k is an orthogonal direct sum of $\text{len}(V_k)$ PSI subspaces. Its proof also reveals that *any* $\theta_k^1, \ldots, \theta_k^r$ as in Theorem 2.3 will enable V_k to be decomposed into an orthogonal direct sum of PSI subspaces. The functions in Corollary 2.1 actually give the minimum number of such subspaces.

3. Tight wavelet frames

Our construction of a normalized tight wavelet frame for $L^2([0, 2\pi)^s)$ begins with a multiresolution analysis. We say that a sequence of subspaces $\{V_k\}_{k \geq 0}$ of $L^2([0, 2\pi)^s)$ is a *multiresolution analysis* (MRA) *of* $L^2([0, 2\pi)^s)$ *with multiplicity* r *and dilation matrix* M if it satisfies the following conditions:

- MRA1 For every $k \geq 0$, there exist functions $\phi_k^m \in V_k$, $m = 1, 2, \ldots, r$, such that $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ spans V_k.
- MRA2 For all $k \geq 0$, $V_k \subseteq V_{k+1}$.
- MRA3 $\overline{\bigcup_{k \geq 0} V_k} = L^2([0, 2\pi)^s)$.

The functions ϕ_k^m, $k \geq 0$, $m = 1, 2, \ldots, r$, are known as *scaling functions,* and $\phi_k := (\phi_k^1, \ldots, \phi_k^r)^T$, $k \geq 0$, are called *scaling vectors.* The scaling functions or scaling vectors are said to generate the MRA $\{V_k\}_{k \geq 0}$. Throughout this section, by the term MRA, we always refer to one with multiplicity r and dilation matrix M.

Comparing to the definition of a periodic multiresolution in [10], the difference of this more general definition lies in MRA1 where we allow the set $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ to be only a spanning set, instead of a basis, for V_k. Another possible generalization of the definition in [10] is to assume in MRA1 that for every $k \geq 0$, there exist functions $\phi_k^m \in V_k$, $m = 1, 2, \ldots, r$, for which $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r, \ \ell \in \mathcal{L}_k \}$ forms a frame for V_k with bounds A and B, where A and B are positive constants independent of k. This is the multidimensional periodic analog of a frame multiresolution analysis defined in [1]. However, by Theorem 2.3, it is actually equivalent to the condition MRA1 above.

The condition MRA1 ensures that for every $k \geq 0$, V_k is a $2\pi M^{-k}\ell$-shift invariant subspace. Such a subspace has been examined in more detail in the previous section. Let us now focus on the remaining two conditions of the MRA.

Proposition 3.1. *For $k \geq 0$, let V_k be as defined in* (2.1) *for some $\phi_k^m \in L^2([0,2\pi)^s)$, and let $v_{k,j}^m$ be the corresponding polyphase splines given by* (2.4)*. Then the following are equivalent for each $k \geq 0$.*

(i) $V_k \subseteq V_{k+1}$.

(ii) *There exists $H_{k+1} \in \mathcal{S}(M^{k+1})^{r\times r}$ such that*

$$\phi_k = \sum_{\ell\in\mathcal{L}_{k+1}} H_{k+1}(\ell) T_{k+1}^{\ell} \phi_{k+1}. \tag{3.1}$$

(iii) *There exists $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ such that*

$$\widehat{\phi}_k(n) = \widehat{H}_{k+1}(n)\widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}^s. \tag{3.2}$$

(iv) *There exists $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ such that*

$$v_{k,j} = \sum_{\ell\in\mathcal{R}_1} \widehat{H}_{k+1}(j + D^k\ell) v_{k+1,j+D^k\ell}, \quad j \in \mathcal{R}_k, \tag{3.3}$$

where $v_{k,j} := (v_{k,j}^1, \ldots, v_{k,j}^r)^T$.

Proof. It is clear from the definition of V_k in (2.1) that (i) implies (ii). By taking the finite Fourier transform of (3.1), we obtain (iii) from (ii). Further, (iii) leads to (iv), as a consequence of rewriting (2.4) using (3.2). Finally, (2.14) shows that (i) follows from (iv). ■

The equation (3.1) is known as the *periodic refinement equation*. Its equivalent formulation (3.3) in terms of polyphase splines will be particularly useful for our analysis. The equivalent conditions in Proposition 3.1 appear to be identical to those derived in [10]. However, in this case, for a given sequence of scaling vectors $\{\phi_k\}_{k\geq 0}$, the corresponding $\widehat{H}_{k+1}$, $k \geq 0$, need not be unique.

The following theorem provides a characterization for MRA3. Its proof is the same as that of [10, Corollary 3.2], which also applies under the present weaker assumption that $\{\, T_k^{\ell}\phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \,\}$ spans V_k instead of being a basis of V_k.

Theorem 3.1. *Suppose that $\{V_k\}_{k\geq 0}$ is a sequence of subspaces of $L^2([0,2\pi)^s)$ of the form* (2.1) *for which $V_k \subseteq V_{k+1}$ for every $k \geq 0$. Then $\overline{\bigcup_{k\geq 0} V_k} = L^2([0,2\pi)^s)$ if and only if the set*

$$\{\, n \in \mathbb{Z}^s : \widehat{\phi}_k^m(n) = 0 \text{ for all } k \geq 0, m = 1, 2, \ldots, r \,\}$$

is empty.

Starting from an MRA $\{V_k\}_{k\geq 0}$ of $L^2([0,2\pi)^s)$, we now consider the problem of constructing a normalized tight wavelet frame for $L^2([0,2\pi)^s)$. For each $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}, that is,

$$V_{k+1} = V_k \oplus^{\perp} W_k. \tag{3.4}$$

Then the space $L^2([0,2\pi)^s)$ can be decomposed into the orthogonal direct sum

$$L^2([0,2\pi)^s) = V_0 \oplus^{\perp} W_0 \oplus^{\perp} W_1 \oplus^{\perp} \cdots .$$

In other words, every $f \in L^2([0,2\pi)^s)$ can be written in the form

$$f = \sum_{k=-1}^{\infty} f_k, \tag{3.5}$$

where $f_{-1} \in V_0$, $f_k \in W_k$ for $k \geq 0$, with

$$\langle f_{k_1}, f_{k_2} \rangle = 0, \quad k_1 \neq k_2, \tag{3.6}$$

for $k_1, k_2 \geq -1$. Consequently,

$$\|f\|^2 = \sum_{k=-1}^{\infty} \|f_k\|^2. \tag{3.7}$$

We note that if $\{ \phi_0^m : m = 1,2,\ldots,r \}$ is a normalized tight frame for V_0 and if for every $k \geq 0$, there exist $\psi_k^1, \ldots, \psi_k^\rho \in L^2([0,2\pi)^s)$ such that $\{ T_k^\ell \psi_k^m : m = 1,2,\ldots,\rho,\ \ell \in \mathcal{L}_k \}$ is a normalized tight frame for W_k, where ρ is a positive integer, then the collection $\{ \phi_0^m : m = 1,2,\ldots,r \} \cup \{ T_k^\ell \psi_k^m : k \geq 0,\ m = 1,2,\ldots,\rho,\ \ell \in \mathcal{L}_k \}$ forms a normalized tight wavelet frame for $L^2([0,2\pi)^s)$. Indeed, every $f \in L^2([0,2\pi)^s)$ can be written in the form (3.5), where $f_{-1} \in V_0$, $f_k \in W_k$ for $k \geq 0$, satisfying (3.6). Thus

$$\sum_{m=1}^{r} |\langle f, \phi_0^m \rangle|^2 + \sum_{k=0}^{\infty} \sum_{m=1}^{\rho} \sum_{\ell \in \mathcal{L}_k} |\langle f, T_k^\ell \psi_k^m \rangle|^2 = \sum_{m=1}^{r} |\langle f_{-1}, \phi_0^m \rangle|^2 + \sum_{k=0}^{\infty} \sum_{m=1}^{\rho} \sum_{\ell \in \mathcal{L}_k} |\langle f_k, T_k^\ell \psi_k^m \rangle|^2.$$

On the other hand, by the tight frame condition of $\{ \phi_0^m : m = 1,2,\ldots,r \}$ and $\{ T_k^\ell \psi_k^m : m = 1,2,\ldots,\rho,\ \ell \in \mathcal{L}_k \}$ for $k \geq 0$, we have

$$\sum_{m=1}^{r} |\langle f_{-1}, \phi_0^m \rangle|^2 + \sum_{k=0}^{\infty} \sum_{m=1}^{\rho} \sum_{\ell \in \mathcal{L}_k} |\langle f_k, T_k^\ell \psi_k^m \rangle|^2 = \sum_{k=-1}^{\infty} \|f_k\|^2.$$

Then the result follows from (3.7).

In view of the above discussion, to obtain a normalized tight wavelet frame for $L^2([0,2\pi)^s)$, we may construct for every $k \geq 0$, functions $\psi_k^1, \ldots, \psi_k^\rho$ in W_k such that the set $\{ T_k^\ell \psi_k^m : m = 1,2,\ldots,\rho,\ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for W_k. A question of interest is the value of ρ, that is, the number of wavelets we construct. In the usual basis setup considered in [10], one needs a total of $r(d-1)$ wavelets. We shall see in due course that in general, this is insufficient for the setting here. Instead, rd wavelets will be constructed. However, in certain special situations which include the basis setup in [10], it turns out that r of these wavelets are the zero function, and we essentially have $r(d-1)$ wavelets from the construction. Illustration of all these will be provided by Example 4.1 in the next section.

In the following, we begin by taking $\rho = rd$. For each $k \geq 0$, let $u_{k,j}^m$ be the polyphase splines associated with ψ_k^m defined by

$$u_{k,j}^m(x) := \sum_{p \in \mathbb{Z}^s} \widehat{\psi}_k^m(j + D^k p) e^{i(j+D^k p)\cdot x}, \quad x \in \mathbb{R}^s, \tag{3.8}$$

for $m = 1,2,\ldots,rd$, $j \in \mathcal{R}_k$. Analogous to Proposition 3.1, we have the following result which we state without proof.

Proposition 3.2. *For $k \geq 0$, let V_k be as defined in* (2.1) *and*

$$W_k := \langle \{ T_k^\ell \psi_k^m : m = 1, 2, \dots, rd,\ \ell \in \mathcal{L}_k \} \rangle \tag{3.9}$$

for some $\phi_k^\mu, \psi_k^m \in L^2([0, 2\pi)^s)$, and let $v_{k,j}^\mu$ and $u_{k,j}^m$ be the corresponding polyphase splines given by (2.4) *and* (3.8)*. Then the following are equivalent for each $k \geq 0$.*

(i) $W_k \subseteq V_{k+1}$.

(ii) *There exists $G_{k+1} \in \mathcal{S}(M^{k+1})^{rd\times r}$ such that*

$$\psi_k = \sum_{\ell \in \mathcal{L}_{k+1}} G_{k+1}(\ell) T_{k+1}^\ell \phi_{k+1},$$

where $\psi_k := (\psi_k^1, \dots, \psi_k^{rd})^T$.

(iii) *There exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$ such that*

$$\widehat{\psi}_k(n) = \widehat{G}_{k+1}(n)\widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}^s.$$

(iv) *There exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$ such that*

$$u_{k,j} = \sum_{\ell \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k\ell) v_{k+1, j+D^k\ell}, \quad j \in \mathcal{R}_k, \tag{3.10}$$

where $u_{k,j} := (u_{k,j}^1, \dots, u_{k,j}^{rd})^T$.

As in Proposition 3.1, for $k \geq 0$ and given ϕ_{k+1} and ψ_k, the sequence $\widehat{G}_{k+1}$ need not be unique.

Suppose now that $\{V_k\}_{k\geq 0}$ is an MRA of $L^2([0, 2\pi)^s)$ generated by scaling vectors ϕ_k, $k \geq 0$, with associated $v_{k,j}$ and $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$. Fix $k \geq 0$. Let W_k be as defined in (3.9) for some $\psi_k^m \in L^2([0, 2\pi)^s)$ and let $u_{k,j}^m$ be the corresponding polyphase splines. Note that by (2.14), we have

$$W_k = \langle \{ u_{k,j}^m : m = 1, 2, \dots, rd,\ j \in \mathcal{R}_k \} \rangle.$$

Suppose that there exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$ such that (3.10) holds. Then it follows from Proposition 3.2 that $W_k \subseteq V_{k+1}$. We shall now derive a necessary and sufficient condition for W_k to be an orthogonal complement of V_k in V_{k+1}. For $j \in \mathcal{R}_k$, let

$$\widetilde{V}_{k+1,j} := \langle \{ v_{k+1, j+D^k\ell}^m : m = 1, 2, \dots, r,\ \ell \in \mathcal{R}_1 \} \rangle. \tag{3.11}$$

Observe that $0 \leq \dim(\widetilde{V}_{k+1,j}) \leq r|\mathcal{R}_1| = rd$ for all $j \in \mathcal{R}_k$. It is clear that $V_{k+1} = \bigoplus_{j\in\mathcal{R}_k}^{\perp} \widetilde{V}_{k+1,j}$. Thus $0 \leq \dim(V_{k+1}) \leq |\mathcal{R}_k|\, rd = rd^{k+1}$. We also note that $V_k = \bigoplus_{j\in\mathcal{R}_k}^{\perp} V_{k,j}$ and $W_k = \bigoplus_{j\in\mathcal{R}_k}^{\perp} W_{k,j}$, where for $j \in \mathcal{R}_k$,

$$V_{k,j} := \langle \{ v_{k,j}^m : m = 1, 2, \dots, r \} \rangle, \qquad W_{k,j} := \langle \{ u_{k,j}^m : m = 1, 2, \dots, rd \} \rangle. \tag{3.12}$$

It is easy to check that $V_{k+1} = V_k \oplus^\perp W_k$ if and only if

$$\widetilde{V}_{k+1,j} = V_{k,j} \oplus^\perp W_{k,j} \tag{3.13}$$

for all $j \in \mathcal{R}_k$. By (2.6), (2.7), (3.3) and (3.10), we deduce that (3.13) is equivalent to

$$\sum_{\ell \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k\ell) M_{k+1}(j + D^k\ell) \widehat{H}_{k+1}(j + D^k\ell)^* = 0 \tag{3.14}$$

and

$$\dim(\widetilde{V}_{k+1,j}) = \dim(V_{k,j}) + \dim(W_{k,j}) \tag{3.15}$$

for $j \in \mathcal{R}_k$.

For every $j \in \mathcal{R}_k$, define

$$N_k(j) := \big(\langle u_{k,j}^m, u_{k,j}^\mu \rangle \big)_{m,\mu=1}^{rd}.$$

Note that as a consequence of (3.3) and (3.10), we have

$$M_k(j) = \sum_{\ell \in \mathcal{R}_1} \widehat{H}_{k+1}(j + D^k\ell) M_{k+1}(j + D^k\ell) \widehat{H}_{k+1}(j + D^k\ell)^* \tag{3.16}$$

and

$$N_k(j) = \sum_{\ell \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k\ell) M_{k+1}(j + D^k\ell) \widehat{G}_{k+1}(j + D^k\ell)^*. \tag{3.17}$$

Now for the MRA $\{V_k\}_{k\geq 0}$, by Theorem 2.3, we may assume that for every $k \geq 0$, $\{ T_k^\ell \phi_k^m : m = 1, 2, \ldots, r,\ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for V_k and that $\langle T_k^\ell \phi_k^m, T_k^\zeta \phi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, r$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{L}_k$. Then Theorem 2.2 implies that $M_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$. Similarly, $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, rd,\ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for W_k and $\langle T_k^\ell \psi_k^m, T_k^\zeta \psi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, rd$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{L}_k$, if and only if $N_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$.

The above discussion shows that the problem of constructing normalized tight wavelet frames for $L^2([0, 2\pi)^s)$ can be formulated in terms of finding appropriate $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$. More precisely, for each $k \geq 0$, given that $M_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$, it suffices to find $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$ such that (3.14), (3.15) hold and $N_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$. Note that this construction not only gives a normalized tight frame $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, rd,\ \ell \in \mathcal{L}_k \}$ for W_k but also the additional property that $\langle T_k^\ell \psi_k^m, T_k^\zeta \psi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, rd$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{L}_k$.

Theorem 3.2. *Suppose that $\{V_k\}_{k\geq 0}$ is an MRA of $L^2([0, 2\pi)^s)$ with scaling vectors ϕ_k and associated $v_{k,j}$ and $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{r\times r}$ such that for each $k \geq 0$, $M_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$. Then for every $k \geq 0$, there exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd\times r}$ that satisfies the conditions (3.14) and (3.15), and that $N_k(j)$ is a diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$ for all $j \in \mathcal{R}_k$.*

Proof. Fix $k \geq 0$, $j \in \mathcal{R}_k$. Let S be the $rd \times rd$ matrix defined by

$$S := \operatorname{diag}\big(\sqrt{d^k M_{k+1}(j + D^k\ell_1)}, \ldots, \sqrt{d^k M_{k+1}(j + D^k\ell_d)} \big),$$

where $\ell_1, \ldots, \ell_d$ denote all the elements of $\mathcal{R}_1$. By the hypothesis, each $\sqrt{d^k M_{k+1}(j + D^k\ell_\nu)}$ is an $r \times r$ diagonal matrix with diagonal entries $\sqrt{d^k}\|v_{k+1,j+D^k\ell_\nu}^m\|$ which are 0 or $\dfrac{1}{\sqrt{d}}$ for $\nu = 1, 2, \ldots, d$, $m = 1, 2, \ldots, r$. Set

$$H := \big(\widehat{H}_{k+1}(j + D^k\ell_1) \,\big|\, \cdots \,\big|\, \widehat{H}_{k+1}(j + D^k\ell_d) \big),$$

and let A be the $rd \times r$ matrix given by $A := (HS)^* = SH^*$. Then it follows from (3.16) that

$$A^*A = d^k M_k(j). \tag{3.18}$$

Observe that $d^k M_k(j)$ is an $r \times r$ diagonal matrix with diagonal entries 0 or 1, and this implies that the nonzero columns of A form an orthonormal set of vectors in $\mathbb{C}^{rd}$.

Let q be the number of nonzero columns of A. First note that

$$\dim(V_{k,j}) = \dim(\langle \{ v_{k,j}^m : m = 1, 2, \ldots, r \} \rangle) = \operatorname{rank}(M_k(j)) = q, \tag{3.19}$$

where the last equality follows from (3.18). Now, by performing elementary column operations that interchange columns of A, we obtain a matrix where all the nonzero columns are in the first q columns. That is, there exists an $r \times r$ orthogonal matrix F such that

$$AF = \big(A_q \,\big|\, 0\big),$$

where A_q is an $rd \times q$ matrix such that all the columns are nonzero. Note that

$$A_q^* A_q = I_q, \tag{3.20}$$

where I_q denotes the $q \times q$ identity matrix.

Let $p := \operatorname{rank}(S)$. Observe that

$$\dim(\widetilde{V}_{k+1,j}) = \sum_{\ell \in \mathcal{R}_1} \dim(\langle \{ v_{k+1,j+D^k\ell}^m : m = 1, 2, \ldots, r \} \rangle) = \sum_{\ell \in \mathcal{R}_1} \operatorname{rank}(M_{k+1}(j + D^k \ell)) = p. \tag{3.21}$$

Since $V_{k,j} \subseteq \widetilde{V}_{k+1,j}$, it follows from (3.19) and (3.21) that $p \geq q$.

Now we perform elementary row operations that interchange the rows of the diagonal matrix S to obtain a matrix in 'reduced row-echelon form', except that the leading entry in each of the first p rows is $\frac{1}{\sqrt{d}}$. This amounts to multiplying the matrix S on the left by an $rd \times rd$ orthogonal matrix E such that $\sqrt{d}(ES)$ is in reduced row-echelon form. Set $A' := EAF$, and note that

$$A' = E\big(A_q \,\big|\, 0\big) = \big(EA_q \,\big|\, 0\big) = \left(\begin{array}{c|c} A_{pq} & 0 \\ \hline 0 & 0 \end{array}\right),$$

where A_{pq} is an $p \times q$ matrix which consists of those rows of A_q that were formerly multiplied by the nonzero diagonal entries of S. It follows from (3.20) that $A_{pq}^* A_{pq} = I_q$. Therefore the q columns of A_{pq} form an orthonormal set of vectors in $\mathbb{C}^p$. Hence there exists an $p \times p$ unitary matrix Q such that

$$QA_{pq} = \left(\frac{I_q}{0}\right). \tag{3.22}$$

As Q is an extension of A_{pq}^*, we can replace its first q rows by A_{pq}^* and write

$$Q = \left(\frac{A_{pq}^*}{B}\right), \tag{3.23}$$

where B is an $(p - q) \times p$ matrix with orthonormal rows. Define an $rd \times rd$ matrix B' by

$$B' := \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right) E.$$

We now construct an $rd \times rd$ matrix G as follows. Suppose that the $(i_1, i_1), \ldots, (i_p, i_p)$ entries of S are the nonzero diagonal entries, where $1 \le i_1 < \cdots < i_p \le rd$. For each $\alpha = 1, 2, \ldots, p$, set the first $p - q$ entries of the i_αth column of G to be the αth column of the matrix $\sqrt{d}\, B$, and the remaining $rd - (p - q)$ entries to be zero. The entries of each of the remaining $rd - p$ columns of G can be any complex numbers. Since $\sqrt{d}(ES)$ is in reduced row-echelon form, it follows that $\sqrt{d}(SE^T)$ is in reduced column-echelon form. Thus we see that G satisfies

$$GSE^T = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right),$$

equivalently, $B' = GS$ as E is orthogonal. Now we define the $rd \times r$ matrices $\widehat{G}_{k+1}(j + D^k \ell)$, $\ell \in \mathcal{R}_1$, by

$$G = \big(\widehat{G}_{k+1}(j + D^k \ell_1) \,\big|\, \cdots \,\big|\, \widehat{G}_{k+1}(j + D^k \ell_d) \big).$$

Since Q is unitary, we have

$$QQ^* = \left(\begin{array}{c|c} A_{pq}^* A_{pq} & A_{pq}^* B^* \\ \hline BA_{pq} & BB^* \end{array}\right) = \left(\begin{array}{c|c} I_q & 0 \\ \hline 0 & I_{p-q} \end{array}\right).$$

Consequently, we see that

$$\begin{aligned} B'AF &= (B'E^T)(EAF) \\ &= \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} A_{pq} & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} BA_{pq} & 0 \\ \hline 0 & 0 \end{array}\right) = 0. \end{aligned}$$

Since F is invertible, this implies that $B'A = 0$, that is, $GSSH^* = 0$. Hence we obtain the condition (3.14). Next, again since E is orthogonal, we also have

$$\begin{aligned} GSS^*G^* &= B'(B')^* = (B'E^T)(B'E^T)^* \\ &= \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} B^* & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} BB^* & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} I_{p-q} & 0 \\ \hline 0 & 0 \end{array}\right), \end{aligned} \tag{3.24}$$

where the last equality follows from the properties of B. Thus by (3.17), we deduce that $N_k(j)$ is an $rd \times rd$ diagonal matrix with diagonal entries 0 or $\dfrac{1}{d^k}$. Finally, (3.24) also implies that

$$\dim(W_{k,j}) = \dim(\langle \{ u_{k,j}^m : m = 1, 2, \ldots, rd \} \rangle) = \operatorname{rank}(N_k(j)) = p - q,$$

and we conclude from (3.19) and (3.21) that (3.15) is satisfied.

In the above, for the special case when $q = 0$, as the matrices A_q and A_{pq} do not exist, we skip the steps involving them and take B to be any $p \times p$ unitary matrix. On the other hand, for the situation when $p = q$, the matrix B does not exist, and so in this case, we skip the part of the proof dealing with B.

Finally, by extending periodically the values of the matrices $\widehat{G}_{k+1}(j + D^k \ell)$, $j \in \mathcal{R}_k$, $\ell \in \mathcal{R}_1$, we obtain $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd \times r}$. This completes the proof of the theorem. ∎

The proof of Theorem 3.2 is constructive, and it provides an algorithmic procedure for finding the appropriate $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{rd \times r}$, $k \ge 0$.

A closer look at the above construction reveals that among the resulting wavelets $\psi_k^1, \ldots, \psi_k^{rd}$, some of them could be the zero function. Indeed, let

$$n_k := \max_{j \in \mathcal{R}_k} \{ \dim(\widetilde{V}_{k+1,j}) - \dim(V_{k,j}) \}. \tag{3.25}$$

By (3.17), (3.19), (3.21) and (3.24), for $m = n_k + 1, n_k + 2, \ldots, rd$, we have $\|u_{k,j}^m\|^2 = 0$ for every $j \in \mathcal{R}_k$, and so $\psi_k^m = \sum_{j \in \mathcal{R}_k} u_{k,j}^m = 0$. Note that $\dim(\widetilde{V}_{k+1,j})$ and $\dim(V_{k,j})$ are invariant under the change of spanning set provided by Theorem 2.3. Thus the same conclusion also holds when we begin with an arbitrary MRA of $L^2([0, 2\pi)^s)$, instead of one satisfying the hypothesis of Theorem 3.2. We summarize these observations as the following result.

Corollary 3.1. *Starting from an MRA $\{V_k\}_{k \geq 0}$ of $L^2([0, 2\pi)^s)$, for every $k \geq 0$, let n_k be as defined in (3.25) and W_k the orthogonal complement of V_k in V_{k+1}. Then there exist $\psi_k^1, \ldots, \psi_k^{n_k} \in L^2([0, 2\pi)^s)$ such that $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, n_k, \ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for W_k, with $\langle T_k^\ell \psi_k^m, T_k^\zeta \psi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, n_k$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{L}_k$. In addition, the collection $\{ \phi_0^m : m = 1, 2, \ldots, r \} \cup \{ T_k^\ell \psi_k^m : k \geq 0, \ m = 1, 2, \ldots, n_k, \ \ell \in \mathcal{L}_k \}$ is a normalized tight wavelet frame for $L^2([0, 2\pi)^s)$.*

Based on Corollary 3.1, further information on the minimum number of wavelets required to generate each of the wavelet subspace W_k can be obtained.

Theorem 3.3. *Suppose that $\{V_k\}_{k \geq 0}$ is an MRA of $L^2([0, 2\pi)^s)$ generated by scaling vectors ϕ_k, with associated $M_k(j)$ as in (2.7). For $k \geq 0$, let W_k be the orthogonal complement of V_k in V_{k+1}, ρ_k a positive integer such that $\rho_k \leq rd$, and n_k as defined in (3.25). Then the following are equivalent for every $k \geq 0$.*

(i) *The set*

$$\Gamma_k := \Big\{ j \in \mathcal{R}_k : \sum_{\ell \in \mathcal{R}_1} \operatorname{rank}(M_{k+1}(j + D^k \ell)) - \operatorname{rank}(M_k(j)) > \rho_k \Big\} \tag{3.26}$$

is empty.

(ii) *There holds $n_k \leq \rho_k$.*

(iii) *There exist $\psi_k^1, \ldots, \psi_k^{\rho_k} \in L^2([0, 2\pi)^s)$ such that $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, \rho_k, \ \ell \in \mathcal{L}_k \}$ forms a normalized tight frame for W_k with $\langle T_k^\ell \psi_k^m, T_k^\zeta \psi_k^\mu \rangle = 0$ for all $m, \mu = 1, 2, \ldots, \rho_k$, $m \neq \mu$, and $\ell, \zeta \in \mathcal{L}_k$.*

(iv) *There exist $\psi_k^1, \ldots, \psi_k^{\rho_k} \in L^2([0, 2\pi)^s)$ such that $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, \rho_k, \ \ell \in \mathcal{L}_k \}$ spans W_k.*

Proof. By (i), for every $j \in \mathcal{R}_k$,

$$\sum_{\ell \in \mathcal{R}_1} \operatorname{rank}(M_{k+1}(j + D^k \ell)) - \operatorname{rank}(M_k(j)) \leq \rho_k.$$

Using (3.19) and (3.21), it follows from (3.25) that (ii) holds. To see that (ii) implies (iii), we apply Corollary 3.1 to obtain the appropriate ψ_k^m, $m = 1, 2, \ldots, n_k$. For $m = n_k + 1, n_k + 2, \ldots, \rho_k$, we set $\psi_k^m := 0$.

It is also clear that (iv) is a consequence of (iii). Finally, to obtain (i) from (iv), suppose that there exists $j \in \Gamma_k$. Then by (3.12), (3.15), (3.19), (3.21) and (3.26),

$$\dim(W_{k,j}) = \dim(\langle \{ u_{k,j}^m : m = 1, 2, \ldots, \rho_k \} \rangle) > \rho_k,$$

a contradiction. ■

The above equivalent conditions indicate that $\{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{L}_k \}$ spans W_k if and only if $n_k \le \rho_k$. So n_k is the smallest possible value of ρ_k. In other words, our constructive proof of Theorem 3.2 actually gives the minimum number of nontrivial wavelets that generate W_k and $\mathrm{len}(W_k) = n_k$. The latter is also apparent from Corollary 2.1 since by (2.24), (3.12), (3.15) and (3.25),

$$\mathrm{len}(W_k) = \max_{j \in \mathcal{R}_k} \{ \dim(W_{k,j}) \} = \max_{j \in \mathcal{R}_k} \{ \dim(\widetilde{V}_{k+1,j}) - \dim(V_{k,j}) \} = n_k. \tag{3.27}$$

The arguments for (3.27) also show that when $V_{k+1} = V_k + W_k$ under the weaker condition $V_k \cap W_k = \{0\}$ instead of $V_k \perp W_k$, if $W_k = \langle \{ T_k^\ell \psi_k^m : m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{L}_k \} \rangle$, then ρ_k cannot be less than n_k. That is, it is not possible for W_k to be generated by a smaller number of wavelets even when V_{k+1} is only an algebraic direct sum of V_k and W_k. Indeed, similar to (3.13), for every $j \in \mathcal{R}_k$, we have $\widetilde{V}_{k+1,j} = V_{k,j} + W_{k,j}$ with $V_{k,j} \cap W_{k,j} = \{0\}$. Again (3.15) and (3.27) are applicable, giving $\rho_k \ge n_k$.

Returning to the setting of (3.4), as mentioned earlier, the case when $\rho_k = r(d-1)$ is of interest as it corresponds to the basis setup of MRA in [10]. For an MRA where the scaling functions ϕ_k^m, $m = 1, 2, \ldots, r$, generate a basis of V_k, n_k takes the value of $r(d-1)$ which equals ρ_k, and so the equivalent conditions of Theorem 3.3 always hold. However, if $\rho_k = r(d-1)$ but the scaling functions of the MRA do not generate a basis of V_k, it is possible that n_k is larger than $r(d-1)$, meaning that the shifts of $r(d-1)$ wavelets are not sufficient to span W_k. This possibility is reflected by the fact that the set Γ_k in (3.26) of Theorem 3.3 may be nonempty. For the special case of $r = 1$, (3.26) simplifies to

$$\Gamma_k = \left\{ j \in \mathcal{R}_k : v_{k,j} = 0,\ v_{k+1,j+D^k\ell} \ne 0 \text{ for all } \ell \in \mathcal{R}_1 \right\}. \tag{3.28}$$

This will be examine in greater detail in the next section for the one-dimensional setting of $s = 1$ and $M = D = 2$, which gives periodic analogs of results in [1]. In particular, having $r(d-1) = 1$ and $rd = 2$, Example 4.1 provides concrete situations in which the minimum number of wavelets generating W_k is one or two.

4. The one-dimensional dyadic PSI case

To illustrate the results in the earlier sections, we now focus on the one-dimensional dyadic PSI case, that is, when $s = 1$, $M = D = 2$ and $r = 1$. In addition, we shall examine in greater detail some of the results for this special case. Since $s = 1$ and $M = D = 2$, it follows that for $k \ge 0$, $\mathcal{L}_k = \mathcal{R}_k = \{ 0, 1, \ldots, 2^k - 1 \}$ and $T_k^\ell f = f(\cdot - \frac{2\pi\ell}{2^k})$, where $f \in L^2[0, 2\pi)$ and $\ell \in \mathbb{Z}$. Further, $r = 1$ implies that the multiresolution subspace V_k is a PSI subspace, which facilitates closer analysis.

For every $k \geq 0$, let $\phi_k \in L^2[0, 2\pi)$ and consider the shift-invariant subspace

$$V_k := \langle \{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \} \rangle. \tag{4.1}$$

In this special case, the polyphase splines defined in (2.4) reduce to *orthogonal splines* $v_{k,j}$, $j \in \mathcal{R}_k$, in $L^2[0, 2\pi)$ given by

$$v_{k,j}(x) := \sum_{p \in \mathbb{Z}} \widehat{\phi}_k(j + 2^k p) e^{i(j+2^k p)x}, \quad x \in \mathbb{R}.$$

In view of (2.6), we have the orthogonality condition that $\langle v_{k,j}, v_{k,\ell} \rangle = 0$ if $j \neq \ell$. Orthogonal splines were first introduced in [13] to characterize linear independence and orthonormality of the collection $\{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \}$, and subsequently motivated the definition of polyphase splines in [10]. The main difference between the situation on hand and the basis setting of [13] is that here some of the functions $v_{k,j}$, $j \in \mathcal{R}_k$, may be the zero function. In this connection, we define

$$\mathcal{N}_k := \{ j \in \mathcal{R}_k : v_{k,j} = 0 \}. \tag{4.2}$$

Then the functions $v_{k,j}$, $j \in \mathcal{R}_k \backslash \mathcal{N}_k$, are nontrivial and orthogonal. By (2.14), the subspace V_k in (4.1) can be written as $V_k = \langle \{ v_{k,j} : j \in \mathcal{R}_k \backslash \mathcal{N}_k \} \rangle$, and so $\dim(V_k) = 2^k - |\mathcal{N}_k|$.

The following is an immediate consequence of Theorem 2.1, and it provides a characterization of the frame condition (2.2) for this case.

Corollary 4.1. *The collection* $\{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \}$ *is a frame for* V_k *with bounds* A *and* B *if and only if*

$$\frac{A}{2^k} \leq \|v_{k,j}\|^2 \leq \frac{B}{2^k}, \quad j \in \mathcal{R}_k \backslash \mathcal{N}_k. \tag{4.3}$$

If the conditions are satisfied, the collection $\{ \sqrt{2^k} v_{k,j} : j \in \mathcal{R}_k \backslash \mathcal{N}_k \}$ *is an orthogonal basis and hence also a frame for* V_k *with bounds* A *and* B*.*

In the situation when $\mathcal{N}_k$ is empty, $\{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \}$ forms a basis of V_k which is the setting in [13]. Corollary 4.1 is the periodic analog of [1, Theorem 3.4] which characterizes frames for subspaces generated by integer translates of a function in $L^2(\mathbb{R})$.

Now, consider an MRA $\{V_k\}_{k \geq 0}$ of $L^2[0, 2\pi)$ generated by scaling functions ϕ_k, $k \geq 0$. Then it follows from Proposition 3.1 that there exists $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ for which

$$\widehat{\phi}_k(n) = \widehat{h}_{k+1}(n) \widehat{\phi}_{k+1}(n), \quad n \in \mathbb{Z}, \tag{4.4}$$

or equivalently,

$$v_{k,j} = \widehat{h}_{k+1}(j) v_{k+1,j} + \widehat{h}_{k+1}(j + 2^k) v_{k+1,j+2^k}, \quad j \in \mathcal{R}_k. \tag{4.5}$$

As noted in the last section, $\widehat{h}_{k+1}$ need not be unique. However, for this special case here, we are able to provide further information on the values of $\widehat{h}_{k+1}(j)$, $j \in \mathcal{R}_{k+1}$.

Proposition 4.1. *Suppose that* $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ *satisfies* (4.5)*. Then for* $j \in \mathcal{R}_{k+1} \backslash \mathcal{N}_{k+1}$, $\widehat{h}_{k+1}(j)$ *is uniquely determined by*

$$\widehat{h}_{k+1}(j) = \frac{\langle v_{k, j \bmod 2^k}, v_{k+1,j} \rangle}{\|v_{k+1,j}\|^2}, \tag{4.6}$$

but all sequences $\widehat{p}_{k+1} \in \mathcal{S}(2^{k+1})$ *that coincide with* $\widehat{h}_{k+1}$ *over the set* $\mathcal{R}_{k+1}\backslash\mathcal{N}_{k+1}$ *also satisfy* (4.5). *Consequently, a sequence* $\widehat{h}_{k+1}$ *for which* (4.5) *holds is unique if and only if* $\mathcal{N}_{k+1} = \emptyset$.

Proof. The unique values of $\widehat{h}_{k+1}(j)$, $j \in \mathcal{R}_{k+1}\backslash\mathcal{N}_{k+1}$, in (4.6) follow from (4.5). Let $\widehat{p}_{k+1} \in \mathcal{S}(2^{k+1})$ be a sequence that agrees with $\widehat{h}_{k+1}$ on $\mathcal{R}_{k+1}\backslash\mathcal{N}_{k+1}$. Since $v_{k+1,j} = 0$ for $j \in \mathcal{N}_{k+1}$, it is easy to see that $\widehat{p}_{k+1}$ satisfies (4.5). This also gives the necessary and sufficient condition $\mathcal{N}_{k+1} = \emptyset$ for the uniqueness of $\widehat{h}_{k+1}$ satisfying (4.5). ■

Now, consider the wavelet construction problem based on the MRA $\{V_k\}_{k\geq 0}$ generated by the scaling functions ϕ_k, $k \geq 0$, with associated $v_{k,j}$. We are not yet able to apply directly the procedure in the proof of Theorem 3.2 as $\|v_{k,j}\|^2$ need not be 0 or $\frac{1}{2^k}$ for all $j \in \mathcal{R}_k$. Thus following the proof of Theorem 2.3, we define the function $\theta_k \in V_k$ whose orthogonal splines $w_{k,j}$, $j \in \mathcal{R}_k$, are given by

$$w_{k,j} := \begin{cases} \dfrac{v_{k,j}}{\sqrt{2^k}\|v_{k,j}\|}, & \text{if } j \in \mathcal{R}_k\backslash\mathcal{N}_k, \\ 0, & \text{if } j \in \mathcal{N}_k. \end{cases} \tag{4.7}$$

Then $\|w_{k,j}\|^2$ equals 0 or $\frac{1}{2^k}$ for all $j \in \mathcal{R}_k$, and the set $\{\, j \in \mathcal{R}_k : w_{k,j} = 0 \,\}$ coincides with $\mathcal{N}_k$. Further, with $\theta_k = \sum_{j\in\mathcal{R}_k} w_{k,j}$, the collection $\{\, T_k^\ell \theta_k : \ell \in \mathcal{L}_k \,\}$ forms a normalized tight frame for V_k. Since $V_k \subseteq V_{k+1}$, it follows from Proposition 3.1 that there exists $\widehat{c}_{k+1} \in \mathcal{S}(2^{k+1})$ such that

$$w_{k,j} = \widehat{c}_{k+1}(j) w_{k+1,j} + \widehat{c}_{k+1}(j + 2^k) w_{k+1,j+2^k}, \quad j \in \mathcal{R}_k.$$

With the original orthogonal splines $v_{k,j}$, we also have (4.5) for some $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$. It should be noted that for every $j \in \mathcal{R}_{k+1}\backslash\mathcal{N}_{k+1}$, $\widehat{h}_{k+1}(j) = 0$ if and only if $\widehat{c}_{k+1}(j) = 0$. Indeed, it follows from (4.6) in Proposition 4.1 that for $j \bmod 2^k \in \mathcal{N}_k$, since $v_{k,j \bmod 2^k} = w_{k,j \bmod 2^k} = 0$, both $\widehat{h}_{k+1}(j)$ and $\widehat{c}_{k+1}(j)$ are zero. On the other hand, for $j \bmod 2^k \in \mathcal{R}_k\backslash\mathcal{N}_k$, we have

$$\widehat{c}_{k+1}(j) = \frac{\sqrt{2}\|v_{k+1,j}\|}{\|v_{k,j \bmod 2^k}\|} \widehat{h}_{k+1}(j) \tag{4.8}$$

which is nonzero.

We are now ready to apply Theorem 3.2 to the MRA $\{V_k\}_{k\geq 0}$ through the new scaling functions θ_k, $k \geq 0$, whose corresponding $\|w_{k,j}\|^2$ equals 0 or $\frac{1}{2^k}$ for all $j \in \mathcal{R}_k$. First, for $k \geq 0$, we introduce the sets

$$\mathcal{E}_k^{0,0} := \{\, j \in \mathcal{R}_k : v_{k+1,j} = 0, v_{k+1,j+2^k} = 0 \,\}, \quad \mathcal{E}_k^{1,0} := \{\, j \in \mathcal{R}_k : v_{k+1,j} \neq 0, v_{k+1,j+2^k} = 0 \,\},$$

$$\mathcal{E}_k^{0,1} := \{\, j \in \mathcal{R}_k : v_{k+1,j} = 0, v_{k+1,j+2^k} \neq 0 \,\}, \quad \mathcal{E}_k^{1,1} := \{\, j \in \mathcal{R}_k : v_{k+1,j} \neq 0, v_{k+1,j+2^k} \neq 0 \,\},$$

whose disjoint union is exactly $\mathcal{R}_k$. These sets are unchanged when defined for the new orthogonal splines $w_{k+1,j}$. We further note that in the event that $\mathcal{N}_k = \emptyset$, we have $\mathcal{E}_k^{0,0} = \mathcal{E}_k^{1,0} = \mathcal{E}_k^{0,1} = \emptyset$ and $\mathcal{E}_k^{1,1} = \mathcal{R}_k$.

We obtain the following results from the constructive proof of Theorem 3.2, the relations (4.7) and (4.8), and the observation that for every $j \in \mathcal{R}_{k+1}\backslash\mathcal{N}_{k+1}$, $\widehat{h}_{k+1}(j) = 0$ if and only if $\widehat{c}_{k+1}(j) = 0$.

The wavelets ψ_k^1, ψ_k^2 are given by

$$(\psi_k^1, \psi_k^2)^T := \sum_{j \in \mathcal{R}_k} (u_{k,j}^1, u_{k,j}^2)^T,$$

where

$$(u_{k,j}^1, u_{k,j}^2)^T := (\widehat{g}_{k+1}^1(j), \widehat{g}_{k+1}^2(j))^T v_{k+1,j} + (\widehat{g}_{k+1}^1(j+2^k), \widehat{g}_{k+1}^2(j+2^k))^T v_{k+1,j+2^k}, \quad j \in \mathcal{R}_k,$$

$\widehat{g}_{k+1}^1, \widehat{g}_{k+1}^2 \in \mathcal{S}(2^{k+1})$, and $v_{k+1,j}$, $j \in \mathcal{R}_{k+1}$, are the orthogonal splines corresponding to the original scaling function ϕ_{k+1}.

1. For $j \in \mathcal{E}_k^{0,0}$, the values $\widehat{g}_{k+1}^m(j)$ and $\widehat{g}_{k+1}^m(j+2^k)$, $m = 1, 2$, are arbitrary complex numbers, but
$$u_{k,j}^1 = u_{k,j}^2 = 0.$$
2. For $j \in \mathcal{E}_k^{1,0}$, the values $\widehat{g}_{k+1}^m(j+2^k)$, $m = 1, 2$, are arbitrary complex numbers. If $\widehat{h}_{k+1}(j) \neq 0$, we have $\widehat{g}_{k+1}^1(j) = \widehat{g}_{k+1}^2(j) = 0$, implying that
$$u_{k,j}^1 = u_{k,j}^2 = 0.$$
If $\widehat{h}_{k+1}(j) = 0$, we have $\widehat{g}_{k+1}^1(j) = \sqrt{2} e^{i\gamma}$ for some $\gamma \in \mathbb{R}$ and $\widehat{g}_{k+1}^2(j) = 0$, and so
$$u_{k,j}^1 = \frac{e^{i\gamma} v_{k+1,j}}{\sqrt{2^k} \|v_{k+1,j}\|}, \qquad u_{k,j}^2 = 0.$$
3. For $j \in \mathcal{E}_k^{0,1}$, the values $\widehat{g}_{k+1}^m(j)$, $m = 1, 2$, are arbitrary complex numbers. If $\widehat{h}_{k+1}(j+2^k) \neq 0$, we have $\widehat{g}_{k+1}^1(j+2^k) = \widehat{g}_{k+1}^2(j+2^k) = 0$, implying that
$$u_{k,j}^1 = u_{k,j}^2 = 0.$$
If $\widehat{h}_{k+1}(j+2^k) = 0$, we have $\widehat{g}_{k+1}^1(j+2^k) = \sqrt{2} e^{i\gamma}$ for some $\gamma \in \mathbb{R}$ and $\widehat{g}_{k+1}^2(j+2^k) = 0$, and so
$$u_{k,j}^1 = \frac{e^{i\gamma} v_{k+1,j+2^k}}{\sqrt{2^k} \|v_{k+1,j+2^k}\|}, \qquad u_{k,j}^2 = 0.$$
4. For $j \in \mathcal{E}_k^{1,1}$, if $v_{k,j} \neq 0$,
$$\widehat{g}_{k+1}^1(j) = \frac{\sqrt{2} e^{i\gamma} \|v_{k+1,j+2^k}\| \overline{\widehat{h}_{k+1}(j+2^k)}}{\|v_{k,j}\|}, \qquad \widehat{g}_{k+1}^1(j+2^k) = -\frac{\sqrt{2} e^{i\gamma} \|v_{k+1,j}\| \overline{\widehat{h}_{k+1}(j)}}{\|v_{k,j}\|}$$
for some $\gamma \in \mathbb{R}$ and $\widehat{g}_{k+1}^2(j) = \widehat{g}_{k+1}^2(j+2^k) = 0$, implying that
$$u_{k,j}^1 = \frac{e^{i\gamma} \|v_{k+1,j+2^k}\| \overline{\widehat{h}_{k+1}(j+2^k)}}{\sqrt{2^k} \|v_{k,j}\| \, \|v_{k+1,j}\|} v_{k+1,j} - \frac{e^{i\gamma} \|v_{k+1,j}\| \overline{\widehat{h}_{k+1}(j)}}{\sqrt{2^k} \|v_{k,j}\| \, \|v_{k+1,j+2^k}\|} v_{k+1,j+2^k}, \qquad u_{k,j}^2 = 0.$$
If $v_{k,j} = 0$, we have
$$\begin{pmatrix} \widehat{g}_{k+1}^1(j) & \widehat{g}_{k+1}^1(j+2^k) \\ \widehat{g}_{k+1}^2(j) & \widehat{g}_{k+1}^2(j+2^k) \end{pmatrix} = \sqrt{2} B = \sqrt{2} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$
where B is an arbitrary 2×2 unitary matrix, and so
$$u_{k,j}^m = \frac{b_{m1}}{\sqrt{2^k} \|v_{k+1,j}\|} v_{k+1,j} + \frac{b_{m2}}{\sqrt{2^k} \|v_{k+1,j+2^k}\|} v_{k+1,j+2^k}, \quad m = 1, 2.$$

It is interesting to note that $u_{k,j}^2$ is always the zero function except when $j \in \mathcal{E}_k^{1,1}$ with $v_{k,j} = 0$, that is, $j \in \Gamma_k$ since (3.28) reduces to

$$\Gamma_k = \{ j \in \mathcal{R}_k : v_{k,j} = 0, v_{k+1,j} \neq 0, v_{k+1,j+2^k} \neq 0 \} \tag{4.9}$$

here. Thus $\psi_k^2 = \sum_{j\in\mathcal{R}_k} u_{k,j}^2 = \sum_{j\in\Gamma_k} u_{k,j}^2$ which is the zero function if and only if Γ_k is empty, illustrating the conclusion of Theorem 3.3. In other words, for $k \geq 0$, two nonzero wavelets ψ_k^1, ψ_k^2, instead of only one, are needed to generate the corresponding wavelet subspace W_k whenever the set Γ_k is nonempty.

We also note that by selecting $\gamma \in \mathbb{R}$ and the matrix B in the above appropriately, we obtain periodic analogs of normalized tight wavelet frames constructed by the approach in [1, 2, 12]. Our construction here originates from solving the matrix extension problem in Section 3 under the most general periodic setting. However, it is difficult to extend directly the approach in [1, 2, 12] to more general scenarios.

Example 4.1. (Trigonometric polynomial wavelet frames) Let $\{L_k\}_{k\geq 0}$ be a strictly increasing sequence of nonnegative integers satisfying $L_k \leq 2^{k-1}$. For $k \geq 0$, define $\phi_k \in L^2[0, 2\pi)$ by

$$\widehat{\phi}_k(n) := \begin{cases} 1, & \text{if } n = -L_k, \ldots, L_k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Then the trigonometric polynomials ϕ_k, $k \geq 0$, generate an MRA $\{V_k\}_{k\geq 0}$ of $L^2[0, 2\pi)$. To see this, it suffices to verify the conditions MRA2 and MRA3. Using (4.10), for $k \geq 0$ we see that (4.4) holds for some $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ whose values on the subset $\{ -L_{k+1}, \ldots, L_{k+1} \}$ of $\{ -2^k, \ldots, 2^k \}$ are uniquely given by

$$\widehat{h}_{k+1}(j) = \begin{cases} 1, & \text{if } j \in \{ -L_k, \ldots, L_k \}, \\ 0, & \text{if } j \in \{ -L_{k+1}, \ldots, L_{k+1} \} \backslash \{ -L_k, \ldots, L_k \}. \end{cases}$$

By Proposition 3.1, $V_k \subseteq V_{k+1}$ for $k \geq 0$. Further, since $\{L_k\}_{k\geq 0}$ is strictly increasing, it follows that $\lim_{k\to\infty} L_k = \infty$. In view of (4.10), it is impossible to find an integer n_0 such that $\widehat{\phi}_k(n_0) = 0$ for all $k \geq 0$. Applying Theorem 3.1, we conclude that $\overline{\bigcup_{k\geq 0} V_k} = L^2[0, 2\pi)$. Hence, $\{V_k\}_{k\geq 0}$ is an MRA of $L^2[0, 2\pi)$.

Let us analyze whether one or two wavelets are needed to generate each of the corresponding wavelet subspaces W_k. In particular, for every $k \geq 0$, we examine whether the set Γ_k as defined in (4.9) is empty. For $k = 0, 1, 2$ and the case when $k \geq 3$ with $L_k = 2^{k-1}$, the set $\mathcal{N}_k$ given by (4.2) is empty, and so Γ_k is empty. Next, consider $k \geq 3$ with $L_k < 2^{k-1}$. Then we have $\mathcal{N}_k = \{ L_k + 1, \ldots, 2^k - L_k - 1 \}$. Consequently, we conclude that for $L_{k+1} < 2^{k-1}$, $\Gamma_k = \emptyset$; and for $2^{k-1} \leq L_{k+1} \leq 2^k$,

$$\Gamma_k = \{ L_k + 1, \ldots, 2^k - L_k - 1 \} \cap \{ 2^k - L_{k+1}, \ldots, L_{k+1} \} \neq \emptyset.$$

Hence in this case, $\Gamma_k = \emptyset$ if and only if $L_{k+1} < 2^{k-1}$.

For every $k \geq 0$, to ensure that ψ_k^1 is sufficient to generate a normalized tight frame for W_k, the set Γ_k must be empty. In this connection, we may select $L_0 = 0$ and $L_k = 2^{k-1}$ for $k \geq 1$, or the combination $L_0 = 0$, $L_k = 2^{k-1}$ for $k = 1, 2, 3$, $L_4 = 5$ or 6, and $L_{k-1} < L_k < 2^{k-2}$ for $k \geq 5$. The former refers to the case when $\{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \}$ forms a basis of V_k for every $k \geq 0$ as $\mathcal{N}_k = \emptyset$, while the latter is on the situation when $\{ T_k^\ell \phi_k : \ell \in \mathcal{L}_k \}$ is only a spanning set for V_k for $k \geq 4$ since $\mathcal{N}_k \neq \emptyset$.

On the other hand, if for some $k \geq 3$ the set Γ_k is nonempty, then two nonzero wavelets ψ_k^1, ψ_k^2 are needed to generate a normalized tight frame for W_k. This situation occurs for all $k \geq 3$ when we take $L_k = k$ for $k = 0, 1, 2, 3$, and $2^{k-2} \leq L_k < 2^{k-1}$ for $k \geq 4$.

In summary, depending on the sequence $\{L_k\}_{k\geq 0}$, the collection $\{\phi_0\} \cup \{ T_k^\ell \psi_k^1 : k \geq 0,\ \ell \in \mathcal{L}_k \}$ or $\{\phi_0\} \cup \{ T_k^\ell \psi_k^1, T_k^\ell \psi_k^2 : k \geq 0,\ \ell \in \mathcal{L}_k \}$ forms a trigonometric polynomial normalized tight wavelet frame for $L^2[0, 2\pi)$. We also note that the case of $L_0 = 0$ and $L_k = 2^{k-1}$ for all $k \geq 1$, leads to a trigonometric polynomial orthonormal basis of $L^2[0, 2\pi)$.

As a final note, we mention briefly how we can extend the approach in this section to construct trigonometric polynomial wavelet frames for $L^2([0, 2\pi)^s)$, $s > 1$. Let $\{\mathcal{J}_k\}_{k\geq 0}$ be an increasing sequence of subsets of $\mathbb{Z}^s$ such that $\mathcal{J}_k \subseteq \{ -2^{k-1}, \ldots, 2^{k-1} \}^s$ and for every $n \in \mathbb{Z}^s$, $\lim_{k\to\infty} \chi_{\mathcal{J}_k}(n) = 1$, where $\chi_{\mathcal{J}_k}$ denotes the characteristic function of $\mathcal{J}_k$. For $k \geq 0$, define $\phi_k \in L^2([0, 2\pi)^s)$ by

$$\widehat{\phi}_k(n) := \chi_{\mathcal{J}_k}(n), \quad n \in \mathbb{Z}^s.$$

Similar to Example 4.1, we see that the trigonometric polynomials ϕ_k, $k \geq 0$, generate an MRA of $L^2([0, 2\pi)^s)$ with $r = 1$ and $M = D = 2I_s$. The constructive proof of Theorem 3.2 can be applied to obtain the corresponding trigonometric polynomial normalized tight wavelet frame for $L^2([0, 2\pi)^s)$. As $d = 2^s$, more cases need to be considered here than for the space $L^2[0, 2\pi)$. More specifically, for every $j \in \mathcal{R}_k$, $p = \dim(\widetilde{V}_{k+1,j})$ in (3.21) ranges from 0 to 2^s, while $q = \dim(V_{k,j})$ in (3.19) remains as 0 or 1 with $p \geq q$. For nonzero values of p, when $q = 0$, the matrix B in (3.23) can be any $p \times p$ unitary matrix. When $q = 1$, among others, we may use a Householder matrix to construct the $p \times p$ unitary matrix Q in (3.22), which gives the $(p - 1) \times p$ matrix B in (3.23).

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