Arithmetic Geometry
and
Automorphic Forms

A Regularized Siegel-Weil formula for Exceptional Groups

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to Steve Kudla, on the occasion of his 60th birthday

Abstract

We obtain a regularized Siegel-Weil formula for the dual pair $PGL_3 \times G_2$ in a split exceptional group H of type E_6. The key step is the regularization of the integral over PGL_3 of a "theta function" arising from the minimal representation of H. The formula then identifies the regularized theta integral as the leading term in the Laurent expansion (at a specific point) of an Eisenstein series on G_2.

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1 Introduction

Amongst the many beautifully written papers of Kudla, some of the most influential must be his series of papers [KR1-3] with Rallis on the Siegel-Weil formula. We begin with a brief recollection of the subject matter of these papers.

If $(Sp(U), O(V))$ is a dual reductive pair in $Sp(U \otimes V)$, the classical Siegel-Weil formula expresses the integral over $O(V)$ of a theta function on $Sp(U \otimes V)$ (coming from the Weil representation) as a special value of a corresponding Eisenstein series on $Sp(U)$. This formulation of the formula first appeared in the paper [We] of Weil, where the formula was established under some convergence conditions which ensure that the theta integral and the infinite sum defining the Eisenstein series are both absolutely convergent.

In [KR1] and [KR2], Kudla and Ralls extended the Siegel-Weil formula to all cases where the theta integral is still convergent but the Eisenstein series needs

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to be defined by analytic continuation. Later, in [KR3], they proved a *regularized* Siegel-Weil formula, even when the theta integral does not converge. One of the key contributions of [KR3] is thus the definition of a natural regularization of the divergent theta integral. The regularized Siegel-Weil formula then identifies this regularized theta integral with the leading term in the Laurent expansion of an associated Eisenstein series; this is the so-called first term identity. The first term identity of the regularized Siegel-Weil formula was subsequently extended to almost all classical dual pairs (with the exception of the quaternion unitary groups) by A. Ichino in [I1-3].

This brings us to the subject matter of the present paper. With the discovery of the minimal representation of a general group H (which is the analog of the Weil representation of the metaplectic group), one can ask for a Siegel-Weil formula for a general dual pair

$$G \times G' \subset H.$$

In [G1], we established such a formula for the dual pair $G_2 \times G$ in the quarternionic form of E_8, where G_2 is split and G is an anisotropic form of F_4. This is an analog of the original formula of Weil [We], in the sense that the theta integral over G and the infinite sum defining the Eisenstein series on G' are both absoultely convergent. In [G2], we we considered the dual pair $G_2 \times PD^\times$ in a rank 2 form of E_6, where D is a degree 3 division algebra. This is the analog of [KR1], in the sense that the theta integral is convergent but the Eisenstein series is defined by meromorphic continuation but is analyitc at the point of interest for the standard sections arising from the Weil representation. In both [G1] and [G2], the theta integral is convergent because the group G over which the integral is taken is anisotropic.

In this paper, we shall establish a *regularized* Siegel-Weil formula for the dual pair

$$G_2 \times PGL_3 \subset E_6,$$

which is the split version of [G2]. In particular, we shall explain how the theta integral can be regularized in a general setting. The reason for sticking with a particular dual pair is one of convenience: unlike the case of classical groups, where there is an extremely rich and uniform underlying theory of quadratic or Hermitian forms, the geometry involved in the setting of exceptional groups is special to each case and it would be hard to give a uniform exposition.

Let us state our results more precisely. Let F be a number field with adèle ring $\mathbb{A}$, and let H be the split adjoint group of type E_6 defined over F. Then $H(\mathbb{A})$ has a minimal representation $\Pi = \otimes_v \Pi_v$. In [GRS], a $H(\mathbb{A})$-equivariant embedding

$$\theta : \Pi \hookrightarrow \mathcal{A}_2(H)$$

of Π into the space of square-integrable automorphic forms on H was constructed. Up to scaling, this is the unique such embedding. Now given the dual pair $PGL_3 \times G_2 \subset H$, we want to consider the theta integral

$$I(\theta(f))(g) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \theta(f)(gh)\, dh, \qquad \text{for } g \in G_2(\mathbb{A}).$$

This is the theta lift of the trivial representation of PGL_3 and it should be an automorphic form on G_2. Unfortunately, the integral diverges for general $f \in \Pi$ since $PGL_3(F)\backslash PGL_3(\mathbb{A})$ is non-compact and the integrand $\theta(f)$ may not be sufficiently slowly increasing. In order to regularize the theta integral, one would like to do the following:

(1) find a non-zero $(PGL_3 \times G_2)$-submodule Π_0 of Π such that for any $f \in \Pi_0$, $\theta(f)$ is rapidly decreasing on $PGL_3(F)\backslash PGL_3(\mathbb{A})$;

(2) find a $(PGL_3 \times G_2)$-equivariant map $p : \Pi \longrightarrow \Pi_0$. Further, we want this map to be obtained in the following way. For a fixed finite (or archimedean) place v_0, we want to find an element z in the component of the Bernstein center associated to the trivial representation of $PGL_3(F_{v_0})$ (or the center of the universal enveloping algebra if v_0 is archimedean) such that p is simply given by the action of z.

If we can achieve the above, then we may define the regularized integral

$$I_{REG}(f)(g) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \theta(z \cdot f)(gh)\, dh.$$

Note that $I_{REG} : \Pi \longrightarrow \mathcal{A}(G_2)$ is a $PGL_3(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant map, since z commutes with both the actions of PGL_3 and G_2.

Let us explain briefly how the above regularization steps are carried out. We want to find some conditions on $f \in \Pi$ such that for any $g \in G_2(\mathbb{A})$, the function $h \mapsto \theta(f)(gh)$ is rapidly decreasing on $PGL_3(F)\backslash PGL_3(\mathbb{A})$. For this, we consider the Fourier expansion of $\theta(f)$ along an abelian unipotent radical of a maximal parabolic of H:

$$\theta(f)(gh) = \sum_{\psi} \theta(f)_{\psi}(gh).$$

Now it turns out that as a function on a Siegel domain of PGL_3, $\theta(f)_{\psi}(gh)$ is rapidly decreasing unless ψ lies in certain *degenerate* orbits relative to $PGL_3 \times G_2$. Thus, if $\theta(f)_{\psi}|_{PGL_3 \times G_2}$ is zero for these degenerate ψ's, then $\theta(f)$ will be rapidly decreasing on a Siegel domain of PGL_3. Further, because the Fourier coefficient of the minimal representation is the product of corresponding local functionals, the vanishing of these degenerate global Fourier coefficients can be ensured by requiring local vanishing at a fixed place v_0. For the details, we refer the reader to Sections 3 and 4.

In this way, we find a non-zero $(PGL_3 \times G_2)$-submodule Π_{0,v_0} of Π_{v_0} such that for any $f \in \Pi_0 = (\otimes_{v \neq v_0} \Pi_v) \otimes \Pi_{0,v_0}$, $\theta(f)$ is rapidly decreasing as a function of $PGL_3(F)\backslash PGL_3(\mathbb{A})$. Further, one can determine the constituents of the representation $\Pi_{v_0}/\Pi_{0,v_0}$. It turns out that those constituents of $\Pi_{v_0}/\Pi_{0,v_0}$ which belong to the same Bernstein component as the trivial representation lie in a proper Zariski closed subset of the variety of such representations. Thus, one can find an element z as in (2) above which kills all the constituents of Π/Π_0.

Having explained how to regularize the theta integral, our next problem is to identify the image of the map I_{REG}, or more specifically to identify the automorphic form $I_{REG}(f)$ for each given $f \in \Pi$. For this, we observe that there is another natural $PGL_3(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant map

$$E : \Pi \longrightarrow \mathcal{A}(G_2),$$

which is defined as follows. The minimal representation embeds as a submodule of a degenerate principal series $Ind_{P_H}^{H} \delta_{P_H}^{2/11}$, where P_H is the Heisenberg parabolic subgroup of H. By restriction of functions from H to G_2, one obtains a *surjective* PGL_3-invariant and G_2-equivariant map

$$\Pi \longrightarrow Ind_P^{G_2} \delta_P^{2/3}$$

where P is the Heisenberg parabolic of G_2. The Eisenstein series attached to this family of degenerate principal series on G_2 has a pole of order at most 2 at the point of interest. Thus, by considering the leading term of its Laurent expansion there, we obtain the desired map

$$E : \Pi \longrightarrow \mathcal{A}(G_2).$$

The regularized Siegel-Weil formula is simply the assertion that $I_{REG} = E$, at least after a suitable scaling.

To establish this identity, we shall consider the regularized theta lift of the Eisenstein series $E(\varphi, s, h)$ of PGL_3 associated to the degenerate principal series representation

$$I_Q(s) = Ind_Q^{PGL_3} |det|^{\frac{1}{2}+s} \quad \text{(unnormalized induction)}.$$

Here Q is a maximal parabolic subgroup of PGL_3 with unipotent radical U and Levi subgroup $L \cong GL_2$. In other words, we consider the integral

$$I_{REG}(f, \varphi, s)(g) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \theta(z \cdot f)(gh) \cdot E(\varphi, s, h)\, dh.$$

Since $E(\varphi, s, h)$ has a pole of order ≤ 1 at $s = 1/2$ with residue a constant function, we are interested in identifying the residue of $I_{REG}(f, \varphi, s)(g)$ at $s = 1/2$.

As in the classical case treated in [KR3], one may unfold this integral when $Re(s)$ is sufficiently large and obtain a corresponding Eisenstein series on G_2:

$$I_{REG}(f, \varphi, s)(g) = P_z(s) \cdot E(\Phi(f, \varphi, s), g),$$

where $P_z(s)$ is an explicit function depending on z, and $\Phi(f,\varphi,s)$ is a element of the degenerate principal series

$$I_P(s) = Ind_P^{G_2}|det|^{\frac{3}{2}+s} \quad \text{(unnormalized induction)}.$$

Here, P is the Heisenberg parabolic of G_2 with unipotent radical N and Levi subgroup $M \cong GL_2$. Further, the map

$$\Phi(-,-,s) : \Pi \otimes I_Q(s) \longrightarrow I_P(s)$$

is $(PGL_3 \times G_2)$-equivariant wherever it is holomorphic (for example when $Re(s) >> 0$).

Since we know that $I_{REG}(f,\varphi,s)$ has a pole of order at most 1 at $s = 1/2$, the right hand side of the above identity should as well and we would like to identify its residue there. It turns out that $P_z(s)$ has a zero of order 2 at $s = 1/2$, whereas as we noted above, the Eisenstein series on G_2 associated to a standard or flat section of $I_P(s)$ can have a pole of order at most 2 at $s = 1/2$. Thus, it would appear that the right hand side of the above identity is actually holomorphic at $s = 1/2$.

The catch, however, is that the function $s \mapsto \Phi(f,\varphi,s)$ is *not* a standard section of $I_P(s)$. It is a meromorphic section of $I_P(s)$ and has a pole of order 1 at $s = 1/2$. More precisely, we shall show that $\Phi(f,\varphi,s)/\Lambda(s+1/2)\Lambda(s+3/2)\Lambda(2s+1)$ is holomorphic at $s = 1/2$, where $\Lambda(s)$ is the complete zeta function of F. Moreover, the quotient is non-vanishing there if f and φ are the spherical vectors and gives the spherical standard section of $I_P(\frac{1}{2})$ (up to an exponential function of s). This shows that the right hand side of the identity does attain a pole of order 1.

With this, our main result is:

Theorem 1.1. *For any archimedean place v, let Π_v^0 be the $(PGL_3 \times G_2)$-submodule of Π_v generated by the spherical vector. Then, for any $f \in \Pi^0 = (\otimes_{v|\infty}\Pi_v^0) \otimes \Pi_f$, we have*

$$I_{REG}(f) = c \cdot E(f),$$

for some nonzero scalar c.

The reason for considering a submodule of the minimal representation at archimedean places is the lack of sufficient knowledge of the theta lift of the trivial representation at these places. At the finite places, however, we know by [GS1] that the big theta lift of the trivial representation of PGL_3 is precisely the degenerate principal series representation $I_P(1/2)$.

We give some brief comments about the relation of this theorem to the main results of [KR3]. In [KR3], the dual pair considered is $Sp(U) \times O(V)$ (with $\dim V$ even) and there are 2 cases:

(i) $2 \leq \dim V \leq \frac{\dim U}{2} + 1$;

(ii) $\frac{\dim U}{2} + 1 < \dim V \leq \dim U$.

The distinguishing feature of these two cases is that the regularizing element z (or equivalently, the associated function $P_z(s)$) vanishes on the trivial representation of $O(V)$ in Case (ii). The case we discussed in this introduction is thus analogous to Case (ii) of [KR3]. Indeed in our case, $P_z(s)$ has a zero of order 2 at the point of interest $s = 1/2$.

Finally, we note that one can only expect to regularize the theta lift from G to G' (at least in the manner of [KR3] and this paper) provided the rank of G is at most the rank of G'. The reason is that since the regularized theta kernel is rapidly decreasing, one can lift any automorphic form on G. In particular, one can consider the regularized lift of the family of generic unramified Eisenstein series. This regularized lift is a function of $rank(G)$-number-of complex variables whose iterated-residue at a particular point is the regularized theta integral $I_{REG}(f)$. Thus it should be a non-zero function, which implies that almost all unramified generic representations of G has non-zero theta lift to G'. For this to happen, one typically needs $rank(G) \leq rank(G')$.

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2 The Minimal Representation

In this section, let F be a non-archimedean local field (instead of writing F_v), with ring of integers $\mathcal{O}_F$, local uniformizer ϖ and residue field $\mathbb{F}_q$. Let H be an exceptional group which is split and adjoint of type E defined over F. Because H is split, it can be canonically defined over $\mathbb{Z}$ and this furnishes us with a Chevalley basis of $Lie(H)$ and a hyperspecial maximal compact subgroup $K = H(\mathcal{O}_F)$. Though we shall only be interested in the case of E_6 in this paper, it is natural for us to give a uniform treatment for all groups of type E, since the information provided here will be necessary if one is interested in establishing a regularized Siegel-Weil formula for other exceptional dual pairs.

We begin by giving an exposition of the basic properties of the minimal representation Π of $H(F)$. Many of these results can be found in [Sa], [MS] and [GS2]. A key result that we shall discuss is an explicit formula for the spherical vector in the minimal representation. In the p-adic case, this formula is due in various cases to Kazhdan-Polishchuk [KP] and Savin-Woodbury [SW]. The archimedean analog, which we shall also require, is due to Dvorsky-Sahi [DS] and we shall discuss this at the end of the section.

2.1 The parabolic Q_H.

Just as the Weil representation, we can describe the restriction of Π to certain parabolic subgroups of H very neatly.

Let $Q_H = L_H \cdot U_H$ be a maximal parabolic subgroup of H such that U_H is abelian. Such a parabolic exists if H is of type E_6 or E_7; the Levi subgroup is of semisimple type D_5 and E_6 respectively. Let Ω_Q be the minimal non-trivial $L_H(F)$-orbit on the set of unitary characters of $U_H(F)$, which can be non-canonically identified with the opposite unipotent radical $\overline{U}_H(F)$. Indeed, Ω_Q is the orbit of the highest weight vector for the adjoint action of L_H on $\overline{U}_H$. Then it was shown by Savin [S1] that there is a $Q_H(F)$-equivariant embedding

$$i_Q : \Pi \hookrightarrow C^\infty(\Omega_Q).$$

Here the action of $Q_H(F)$ on $C^\infty(\Omega_Q)$ is given by:

$$\begin{cases} (l \cdot f)(\chi) = \delta_{Q_H}(l)^r \cdot f(l^{-1} \cdot \chi) \\ (u \cdot f)(\chi) = \chi(u) \cdot f(\chi) \end{cases}$$

where the value of r is given by the following table.

H	E_6	E_7
r	$1/4$	$2/9$

The image of i_Q contains the subspace $C_c^\infty(\Omega_Q)$ of smooth compactly supported functions, and we have an exact sequence of $Q_H(F)$-modules:

$$0 \longrightarrow C_c^\infty(\Omega_Q) \longrightarrow \Pi \longrightarrow \Pi_{U_H} \longrightarrow 0$$

Moreover, as a representation of $L_H(F)$, we have

$$\Pi_{U_H} \cong \delta_{Q_H}^r \oplus \delta_{Q_H}^{r/2} \cdot \Pi_{L_H}$$

where $\Pi(L_H)$ is the minimal representation of $L_H(F)$ with the center acting trivially. From this, one sees that there is an injectiion

$$j_Q : \Pi \hookrightarrow Ind_{Q_H}^H \delta_{Q_H}^r.$$

Indeed, Π is the unique K-spherical constituent of this degenerate prioncipal series. Further, one deduces from the exact sequence above that for a non-trivial character χ of $U_H(F)$,

$$\dim \mathrm{Hom}_{U_H(F)}(\Pi, \mathbb{C}_\chi) = \begin{cases} 1, & \text{if } \chi \in \Omega_Q; \\ 0, & \text{otherwise.} \end{cases}$$

When $\chi \in \Omega_Q$, a non-zero element of this Hom space is given by:

$$L_\chi^0(f) = i_Q(f)(\chi).$$

Let R_χ be the stabilizer of $\chi \in \Omega_Q$ in the derived subgroup of M_H, so that R_χ acts naturally on $\mathrm{Hom}_{U_H(F)}(\Pi, \mathbb{C}_\chi)$. From the action of M_H on $C^\infty(\Omega_Q)$ given above, it is easy to see that R_χ acts trivially on $\mathrm{Hom}_{U_H(F)}(\Pi, \mathbb{C}_\chi)$.

2.2 The spherical vector.

It is unfortunate that we do not have a characterization of the image of i_Q in $C^\infty(\Omega_Q)$. However, because of the Iwasawa decomposition, Π is generated as a Q_H-module by a sphercial vector f_0 for the hyperspecical maximal compact subgroup K. In this subsection, we shall describe a recent result of Savin-Woodbury [SW] which describes the function $i_Q(f_0)$ explicitly. For many purposes, this is as good as having a precise characterization of Π as a subspace of $C^\infty(\Omega_Q)$.

For ease of notation, we shall write f_0 in place of $i_Q(f_0)$. The fact that our H is defined over $\mathbb{Z}$ gives us an integral structure on $\overline{U}_H$ and thus a lattice $\Lambda = \overline{U}_H(\mathcal{O}) \subset \overline{U}_H(F)$. Set

$$\Omega_Q(n) = \Omega_Q \cap (\varpi^n\Lambda \setminus \varpi^{n+1}\Lambda)$$

so that

$$\Omega_Q = \bigcup_{n\in\mathbb{Z}} \Omega_Q(n).$$

The $U_H(\mathcal{O})$-invariance of f_0 immediately implies that f_0 vanishes on $\Omega_Q(n)$ if $n < 0$. The $L_H(\mathcal{O})$-invariance of f_0 and the fact that Ω_Q is the orbit of a highest weight vector imply that f_0 is constant on each $\Omega_Q(n)$. Thus, if we fix an element $\chi_0 \in \Omega_Q(0)$, then f_0 is completely described by $f_0(\varpi^n\chi_0)$ for $n \geq 0$. The following is the result of Savin-Woodbury [SW]:

Theorem 2.1. *(i) f_0 is non-zero on $\Omega_Q(0)$.*

(ii) If we normalize f_0 so that it takes value 1 on $\Omega_Q(0)$, then

$$f_0(\varpi^n\chi_0) = \sum_{k=0}^{n} p^{k\alpha} = \frac{p^{\alpha(n+1)} - 1}{p^\alpha - 1}$$

where $\alpha = 2$ or 3 if $H = E_6$ or E_7 respectively.

This theorem will be very important for our analysis of the section $\Phi(f, \varphi, s)$ alluded to in the introduction. It gives us precise control on the asympototics of the functions in Π near $0 \in \overline{U}_H(F)$:

Corollary 2.2. *For any $f \in \Pi$, there is a constant C such that*

$$|i_Q(f)(\varpi^n\chi_0)| \leq Cq^{\alpha n} \quad \text{as } n \to \infty.$$

2.3 The Heisenberg parabolic P_H.

Let $P_H = M_H \cdot N_H$ be the Heisenberg parabolic of H so that N_H is a Heisenberg group with one-dimensional center Z_H. Such a parabolic exists (and is unique) for all H of type E. We have the analog of the above discussion for P_H.

Let Ω_P be the minimal non-trivial orbit of $M_H(F)$ on the set of unitary characters of $N_H(F)$, which can be non-canonically identified with $\overline{V}_H(F) =$

$\overline{N}_H(F)/\overline{Z}_H(F)$. Then it was shown in [MS] that there is a $P_H(F)$-equivariant embedding

$$i_P : \Pi_{Z_H} \hookrightarrow C^\infty(\Omega_P),$$

where $P_H(F)$ acts on $C^\infty(\Omega_P)$ by:

$$\begin{cases} (m \cdot f)(\chi) = \delta_{P_H}(m)^s \cdot f(m^{-1} \cdot \chi) \\ (n \cdot f)(\chi) = \chi(n) \cdot f(\chi) \end{cases}$$

and the value of s is given by the following table.

H	E_6	E_7	E_8
s	2/11	3/17	5/29
t	3/22	2/17	3/29

The image of i_P contains $C_c^\infty(\Omega_P)$, and we have an exact sequence of $P_H(F)$-modules:

$$0 \longrightarrow C_c^\infty(\Omega_P) \longrightarrow \Pi_{Z_H} \longrightarrow \Pi_{N_H} \longrightarrow 0.$$

Moreover, as a representation of $M_H(F)$, we have:

$$\Pi_{Z_H} \cong \delta_{P_H}^s \oplus \delta_{P_H}^t \cdot \Pi(M_H)$$

where the value of t is given in the above table and $\Pi(M_H)$ is the minimal representation of $M_H(F)$ with center acting trivially. From this, one sees that there is an injection

$$j_P : \Pi \hookrightarrow Ind_{P_H}^H \delta_{P_H}^s.$$

Further, for each non-trivial character χ of $N_H(F)$, one has

$$\dim \mathrm{Hom}_{N_H(F)}(\Pi, \mathbb{C}_\chi) = \begin{cases} 1, & \text{if } \chi \in \Omega_P; \\ 0, & \text{otherwise.} \end{cases}$$

As above, a non-zero element of this Hom space for $\chi \in \Omega_P$ is given by

$$L_\chi^0(f) = i_P(f)(\chi).$$

The above gives a rather complete description of Π_{Z_H} as a $P_H(F)$-module. On the other hand, if ψ is a non-trivial character of $Z_H(F)$, then as a representation of the derived group $P_{H,der}$ of P_H,

$$\Pi_{Z_H,\psi} \cong \omega_\psi$$

where ω_ψ is the Weil representation $Mp(N_H/Z_H) \ltimes N_H$ restricted to $P_{H,der}(F)$. Thus, we have a short exact sequence of $P_H(F)$-modules:

$$0 \longrightarrow ind_{P_{H,der}}^{P_H} \omega_\psi \longrightarrow \Pi \longrightarrow \Pi_{Z_H} \longrightarrow 0.$$

In fact, one has

$$ind_{P_{H,der}}^{P_H} \omega_\psi \subset \Pi \subset Ind_{P_{H,der}}^{P_H} \omega_\psi.$$

The latter space can be realized as the space of P_H-smooth functions on $F^\times$ taking values in ω_ψ, while the former is the subspace of compactly supported such functions.

2.4 The spherical vector.

As above, we do not have a clean characterization of the subspace Π of $Ind_{P_{H,der}}^{P_H}\omega_\psi$, but we do have a precise description of the spherical vector f_0 in Π. This explicit description is due to Kazhdan-Polishchuk [KP]. We will not state their formula here, but rather we describe the image of f_0 in $\Pi_{Z_H} \subset C^\infty(\Omega_P)$.

The integral structure on H gives an $\mathcal{O}$-lattice $\Lambda = \overline{V}_H(\mathcal{O})$ in $\overline{V}_H(F)$. Let us set

$$\Omega_P(n) = \Omega_P \cap (\varpi^n\Lambda \smallsetminus \varpi^{n+1}\Lambda).$$

Write $\bar{f}_0$ for the image of the spherical vector in Π_{Z_H}. Now $\bar{f}_0$ is constant on each $\Omega_P(n)$, so that if χ_0 is a fixed element of $\Omega_P(0)$, then $\bar{f}_0$ is determined by $\bar{f}_0(\varpi^n\chi_0)$ for all $n \in \mathbb{Z}$. Here is the result:

Theorem 2.3. *$\bar{f}_0$ is zero on $\Omega_P(n)$ if $n < 0$. For $n \geq 0$,*

$$\bar{f}_0(\varpi^n\chi_0) = \sum_{i=0}^{n} p^{di} = \frac{p^{d(n+1)}-1}{p^d-1},$$

where $d = 1, 2$ or 4 for $H = E_6$, E_7 or E_8 respectively. In particular, for any $f \in \Pi_{Z_H}$, there is a constant C such that

$$|i_P(f)(\varpi^n\chi_0)| \leq Cq^{dn} \quad \text{as } n \to \infty.$$

2.5 Archimedean case.

We conclude this section with the archimedean analog of the above discussion. Hence, in this subsection, we let $F = \mathbb{R}$ or $\mathbb{C}$ and we regard $H(F)$ as a real Lie group. Let $K \subset H(F)$ be a maximal compact subgroup of $H(F)$ with the property that $K \cap L_H(F)$ is a maximal compact subgroup of $L_H(F)$. Here, we recall that $Q_H = L_H \cdot U_H$ is a maximal parabolic subgroup of H with abelian unipotent radical U_H. One can introduce a K-invariant length function on $\overline{U}_H$ as follows. Let θ be the Cartan involution associated to K, so that if $y \in \overline{U}_H \cong Lie(\overline{U}_H)$, then $\theta(y) \in U_H \cong Lie(U_H)$. We set

$$|y| = \sqrt{-\langle y, \theta(y)\rangle},$$

where $\langle -, - \rangle$ is the Killing form of $Lie(H(F))$. Now we have the following result of Dvorsky-Sahi [DS]:

Theorem 2.4. *The unitary completion of the minimal representation Π can be realized on the Hilbert space $L^2(\Omega_Q, \mu_{\Omega_Q})$ (where μ_{Ω_Q} is a natural L_H-equivariant measure on Ω_Q). The spherical vector f_0 is given by:*

$$f_0(\chi) = |\chi|^{-\frac{\alpha}{2}} \cdot K_{\frac{\alpha}{2}}(|\chi|).$$

Here K_r is the r-th Bessel function which is the unique solution (up to scaling) of the differential equation

$$f'' + z^{-1}f' - (1 + \frac{r^2}{z^2})f = 0$$

which decays exponentially as $z \to \infty$. Moreover, the value of α is given by the following table.

	E_6	E_7
$\mathbb{R}$	2	3
$\mathbb{C}$	4	6

For any smooth $f \in \Pi$, there is a constant C such that

$$|f(\chi)| \leq C \cdot |\chi|^{-\alpha}.$$

In the archimedean case, it was shown in [GRS] that for $\chi \in \Omega_Q$,

$$\dim \mathrm{Hom}_{R_\chi(F) \cdot U_H(F)}(\Pi, \mathbb{C}_\chi) \leq 1.$$

In terms of the realization of Π given in the above theorem, an element in this Hom space is given by the evaluation at χ, as in the non-archimedean case.

3 Dual Pairs and Geometry of Orbits

In this section, we consider various dual reductive pairs $G \times G_2 \subset H$ and study the geometry of the minimal representation relative to these dual pairs. In particular, we shall focus on the geometry related to the parabolic Q_H; this is needed for regularizing the theta lift of the trivial representation of PGL_3 to G_2. In later sections, we shall deal with the geometry related to the parabolic P_H, which is needed for the lifting from G_2 to G.

3.1 Dual Pairs.

We may consider various dual reductive pairs in H. These have the form $G \times G_2$, where G is listed below.

H	E_6	E_7	E_8
G	PGL_3	$PGSp_6$	F_4

In the case of E_6 or E_7, we have

$$(G \times G_2) \cap Q_H = Q \times G_2,$$

where $Q = L \cdot U$ is a maximal parabolic subgroup of G with abelian unipotent radical U and Levi subgroup $L \cong GL_2$ or GL_3 respectively. We choose these identifications of L so that $\delta_Q = |det|$ or $|det|^2$ respectively. Using the realization of Π via i_Q, we can describe the action of $Q \times G_2$ on Π:

$$\begin{cases} (g \cdot f)(\chi) = f(g^{-1} \cdot \chi) \ \text{ for } g \in G_2(F); \\ (u \cdot f)(\chi) = \chi(u) \cdot f(\chi) \ \text{ for } u \in U(F); \\ (l \cdot f)(\chi) = |det(l)|^2 \cdot f(l^{-1} \cdot \chi), \ \text{ for } l \in L(F). \end{cases}$$

We shall only deal with the dual pair $PGL_3 \times G_2$ in this paper. In particular, we shall be interested in defining a natural regularization of the theta lift from PGL_3 to G_2. As we explain in the last paragraph of the introduction, for the other cases above, we can only expect to regularize the theta lift from G_2 to G (rather than from G to G_2). The geometry involved for these will be somewhat different and so we will not treat these other cases here.

3.2 Geometry of Orbits.

In the rest of the paper, G will stand for PGL_3. For the purpose of regularization, it will be important for us to understand the geometry of the $L \times G_2$-orbits on Ω_Q. More precisely, we shall define a closed subset $\Omega_{Q,deg} \subset \Omega_Q$ which is stable under $L \times G_2$. This is the set of degenerate orbits and we shall be interested in the geometry of the $(L \times G_2)$-orbits in $\Omega_{Q,deg}$.

Since $G = PGL_3$, we have $L \cong GL_2$. In this case, as explained in [MS, Sect. 3, Pg. 99-100], the set Ω_Q can be described by:

$$\Omega_Q = \{(x,y) \in \mathbb{O}^2 : \mathbb{N}(x) = \mathbb{N}(y) = \overline{x} \cdot y = 0\}$$

where $\mathbb{O}$ is the (split) octonion algebra over F, with conjugate map $x \mapsto \overline{x}$ and norm map $\mathbb{N}(x) = x \cdot \overline{x}$. The action of $GL_2 \times G_2$ is given by:

$$\begin{cases} g \cdot (x,y) = (gx, gy), \text{ for } g \in G_2(F); \\ l \cdot (x,y) = (x,y)l^{-1}, \text{ for } l \in GL_2(F). \end{cases}$$

Let $\Omega_{Q,deg} \subset \Omega_Q$ be the subset consisting of those (x,y) such that $Fx + Fy$ has dimension 1 (as opposed to 2). It is clear $\Omega_{Q,deg}$ is stable under the action of $GL_2 \times G_2$. We proceed to describe the orbit structure. Let $\widehat{U}$ be the set of unitary characters of $U(F)$. By restriction, we have a $(L \times G_2)$-equivariant map

$$\tau_Q : \Omega_{Q,deg} \longrightarrow \widehat{U}.$$

In terms of the concrete description of Ω_Q above, the map τ_Q is given by

$$\tau_Q : (x,y) \mapsto (Tr(x), Tr(y))$$

where Tr is the trace map of $\mathbb{O}$.

Now the action of GL_2 on $\widehat{U}$ is isomorphic to the dual of the standard representation, and thus has 2 orbits on $\widehat{U}$: the zero orbit and its complement. Let Ω_0 and Ω_1 be the inverse image under τ_Q of the zero orbit and its complement respectively. Then we have:

Lemma 3.1. *(i) Ω_0 and Ω_1 are the only two $(GL_2 \times G_2)$-orbits in $\Omega_{Q,deg}$. Moreover, Ω_0 is a closed subset of $\Omega_{Q,deg}$.*

(ii) The orbit Ω_0 *has representative* $\chi_0 = (0, y)$ *where* $Tr(y) = 0 = \mathbb{N}(y)$*. Moreover,*

$$\Omega_0 = (G_2 \times GL_2) \times_{(P_1 \times B)} F^\times \chi_0$$

where P_1 *is a non-Heisenberg maximal parabolic of* G_2 *and* B *is the Borel subgroup of upper triangular matrices in* GL_2*. As a representation of* $Q \times G_2$*,*

$$C_c^\infty(\Omega_0) \cong Ind_{B \times P_1}^{GL_2 \times G_2} C_c^\infty(F^\times \chi_0)$$

with U *acting trivially. The action of* B *on* $C_c^\infty(F^\times)$ *is given by:*

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot f \right)(x) = f(xd) \cdot |ad|^2.$$

(iii) The orbit Ω_1 *has a representative* $\chi_1 = (0, 1 + y_0)$ *with* $Tr(y_0) = 0$ *and* $\mathbb{N}(y_0) = -1$*. Moreover,*

$$\Omega_1 = (G_2 \times GL_2) \times_{(SL_3 \times B)} F^\times \chi_1.$$

As a representation of $Q \times G_2$*, we have*

$$C_c^\infty(\Omega_1) \cong ind_{B \cdot U \times SL_3}^{Q \times G_2} C_c^\infty(F^\times \chi_1).$$

Here, SL_3 *acts trivially on* $C_c^\infty(F^\times)$*, the action of* B *is similar to that in (ii) and the action of* U *is:*

$$(u \cdot f)(x) = x(u) \cdot f(x).$$

Proof. The statements regarding Ω_0 were proved in [MS, Prop. 4.2]. On the other hand, using the action of GL_2, one can move any element of Ω_1 to an element of the form $(0, 1 + y_0)$ with some y_0 as in (ii). But the group $G_2(F)$ acts transitively on such y_0's, and the stabilizer of an element is isomorphic to $SL_3(F)$. This proves the statements concerning Ω_1. □

4 The Submodule Π_0

We continue with the notations of the previous section, so that F is a non-archimedean local field and $G = PGL_3$. The purpose of this section is to introduce a $(G \times G_2)$-submodule Π_0 of Π and to investigate the constituents of the quotient Π/Π_0.

4.1 The submodule Π_0 and a natural filtration.

In the last section, we have describe a subvariety $\Omega_{Q,deg} \subset \Omega_Q$; this is the subset of degenerate orbits. Let Ω_Q^0 be the open complement of $\Omega_{Q,deg}$ in Ω_Q. We set

$$\Pi_0 = \{f \in \Pi : \ (g,h) \cdot i_Q(f) \in C_c^\infty(\Omega_Q^0) \text{ for all } (g,h) \in G \times G_2\}.$$

Clearly, Π_0 is stable under the action of $G \times G_2$. Our next goal is to describe the constituents of Π/Π_0. It will follow from this description that Π_0 is non-zero.

The quotient Π/Π_0 has a natural filtration by $G \times G_2$-modules. Indeed, let

$$\pi : \Pi \longrightarrow \Pi_{U_H}$$

be the natural projection and consider the natural map obtained from π by Frobenius reciprocity:

$$\Pi \longrightarrow Ind_{Q \times G_2}^{G \times G_2} \Pi_{U_H}.$$

Let Π_1 be the kernel of this map. Then

$$\Pi_1 = \{ f \in \Pi : \ (g,h) \cdot i_Q(f) \in C_c^\infty(\Omega_Q) \text{ for all } (g,h) \in G \times G_2 \}.$$

It is thus clear that (via the embedding i_Q)

$$\Pi_1 \subset C_c^\infty(\Omega_Q) \subset \Pi.$$

Further, the natural map $\Pi_1 \longrightarrow C_c^\infty(\Omega_{Q,deg})$ given by restriction of functions is $(Q \times G_2)$-equivariant. Thus by Frobenius reciprocity, one has

$$\Pi_1 \longrightarrow Ind_{Q \times G_2}^{G \times G_2} C_c^\infty(\Omega_{Q,deg}).$$

By its very definition, the kernel of this map is Π_0, so that we have:

$$\Pi_0 \subset \Pi_1 \subset \Pi.$$

4.2 Subquotients of Π/Π_1.

With our knowledge of Π_{U_H}, we can describe the $G(F)$-constituents of the quotient Π/Π_1. It is clear that any $G(F)$-constituent of Π/Π_1 is a constituent of an induced representation of the form

$$I_Q(|det|^{3/2}) \quad \text{or} \quad I_Q(|det|^{1/2}\tau)$$

for some representation τ of the adjoint group $L/Z_L = PGL_2$. Note that the trivial representation is a constituent of $I_Q(|det|^{1/2})$.

4.3 Subquotients of Π_1/Π_0.

To describe the quotient Π_1/Π_0, we recall the orbit structure of $\Omega_{Q,deg}$ which gives a short exact sequence:

$$0 \longrightarrow Ind_{Q \times G_2}^{G \times G_2} C_c^\infty(\Omega_1) \longrightarrow Ind_{Q \times G_2}^{G \times G_2} C_c^\infty(\Omega_{Q,deg}) \longrightarrow Ind_{Q \times G_2}^{G \times G_2} C_c^\infty(\Omega_0) \longrightarrow 0.$$

This induces a corresponding filtration of Π_1/Π_0. Now the $G(F)$-constituents of $Ind_{Q \times G_2}^{G \times G_2} C_c^\infty(\Omega_0)$ can be easily read off from Lemma **??**(ii). They are of the form

$$I_Q(\pi(|-|, \chi))$$

where $\pi(|-|,\chi)$ is a principal series representation of GL_2 unitarily induced from the character $(|-|,\chi)$ of the diagonal torus (with χ arbitrary). Again, note that the trivial representation is a constituent of such an induced representation, when one takes χ to be trivial.

Finally, we need to investigate the constituents of $Ind_{Q\times G_2}^{G\times G_2} C_c^\infty(\Omega_1)$. This is slightly more complicated because the unipotent radical U of Q does not act trivially on $C_c^\infty(\Omega_1)$. To describe the answer, we need to introduce more notations. Let $Q' = L'\cdot U'$ be the other maximal parabolic of PGL_3. We choose an isomorphism $L' \cong GL_2$ so that $\delta_{Q'} = |det|^{-1/2}$. Concretely,

$$g \in GL_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \in PGL_3.$$

Let $B' = T'U'$ be the Borel subgroup of upper triangular matrices in L'.

Now by Lemma **??**(iii), one concludes that

$$Ind_{Q\times G_2}^{G\times G_2} C_c^\infty(\Omega_1) \cong ind_{B_G\times SL_3}^{G\times G_2} C_c^\infty(F^\times) \cong ind_{Q'\times SL_3}^{G\times G_2}\left(ind_{B'}^{L'} C_c^\infty(F^\times)\right).$$

Let us examine the representation $ind_{B'}^{L'} C_c^\infty(F^\times)$ more carefully. The action of B' is given by

$$\begin{cases} \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\cdot f\right)(x) = \frac{|a|^2}{|d|^4}\cdot f(x\cdot \frac{a}{d}), \\ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\cdot f\right)(x) = \psi(bx)\cdot f(x). \end{cases}$$

In particular, one sees that the center of L' acts by $z \mapsto |z|^{-2}$, and

$$ind_{B'}^{L'} C_c^\infty(F^\times) \cong |det|^{-1} \otimes ind_{U'}^{PGL_2}\psi.$$

So we see that the G-constituents of $C_c^\infty(\Omega_1)$ are constituents of

$$I_{Q'}(|det|^{-1/2}\tau)$$

with τ a representation of L' with trivial central character.

We summarize the above discussion in the following proposition:

Proposition 4.1. *Any irreducible G-constituent of Π/Π_0 is a constituent of one of the following representations:*

- $I_Q(|det|^{3/2})$*;*
- $I_Q(|det|^{1/2}\tau)$ *or* $I_{Q'}(|det|^{-1/2}\tau)$ *where τ has trivial central character;*
- $I_Q(\pi(|-|,\chi)$ *where χ is an arbitrary character.*

In particular, Π_0 is a non-zero submodule of Π.

Proof. Only the last statement remains to be proven. In [GS1], the local theta correspondence for $PGL_3 \times G_2$ was completely determined. In particular, every irreducible representation of PGL_3 occurs as a quotient of Π. But Π/Π_0 does not contain all representations; for example, it does not contain supercuspidal representations of PGL_3. Thus Π_0 is non-zero. □

4.4 Bernstein center.

A most important consequence of Prop. **??** is the existence of an element z_G in the Bernstein center of G with the following properties:

- z_G lies in the component of the Bernstein center associated to the trivial representation of G,
- z_G annihilates all the constituents of Π/Π_0,

The purpose of this subsection is to construct such an element z_G.

As a $\mathbb{C}$-algebra, the relevant component of the Bernstein center is given by

$$Z_G \cong \mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]^{S_3}/(x_1x_2x_3 - 1)$$

Here the action of S_3 is by permutation of the x_i's. The action of Z_G on an unramified principal series $I_{B_G}(\chi_1, \chi_1, \chi_3)$ (with $\chi_1\chi_2\chi_3 = 1$) is given by:

$$p(x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}) \mapsto p(|\chi_1(\varpi)|^{\pm 1}, |\chi_2(\varpi)|^{\pm 1}, |\chi_3(\varpi)|^{\pm 1}) \in \mathbb{C}$$

for $p \in Z_G$.

Now set

$$z_G = \prod_{i=1}^{3} (x_i - q)(x_i^{-1} - q).$$

We have:

Proposition 4.2. *(i) The element z_G annihilates Π/Π_0.*

(ii) Moreover, z_G acts on the unramified representation $I_Q(s)$ by the constant

$$P_z(s) = (q^{-(s+\frac{1}{2})} - q)(q^{s+\frac{1}{2}} - q)(q^{-(s-\frac{1}{2})} - q)(q^{s-\frac{1}{2}} - q)(q^{2s} - q)(q^{-2s} - q).$$

The function $P_z(s)$ has a double zero at $s = \frac{1}{2}$ (which corresponds to the trivial representation).

Proof. We use Prop. **??** to enumerate the unramified representations which occur in Π/Π_0 and check that they are annihilated by z_G:

- the representation $I_Q(|det|^{3/2})$ has parameter $(|-|, |-|^2, |-|^{-3})$, which is killed by z_G;

- the unramified constituents of $I_Q(|det|^{1/2}\tau)$ or $I_{Q'}(|det|^{-1/2}\tau)$ have parameters of the form $(|-|^{s+1/2}, |-|^{-s+1/2}, |-|^{-1})$ or the inverse of this. In either case, these are killed by z_G.
- the representation $I_Q(\pi(|-|, |-|^s))$ has parameter $(|-|, |-|^s, |-|^{-s-1})$, which is killed by z_G.

The unramified constituent of $I_Q(|det|^s)$ has parameter $(|-|^{s+\frac{1}{2}}, |-|^{s-\frac{1}{2}}, |-|^{-2s})$. From this, it is clear that z_G acts by the scalar $P_z(s)$ defined in the proposition and one sees that $P_z(s)$ does have a zero of order 2 at $s = 1/2$.

□

It is natural to ask if there is an element z_{G_2} in the Bernstein center of G_2 such that the element $z_G - z_{G_2}$ acts trivially on the minimal representation Π. This will be the non-archimedean analog of the correspondence of infinitesimal character in the archimedean case. A natural candidate for z_{G_2} can be constructed as follows.

Observe that the element z_G is not only S_3-invariant; it is invariant under the involution which sends all x_i to x_i^{-1}. The group W generated by S_3 and this involution is the Weyl group of G_2, and the component of the Bernstein center of G_2 associated to unramified representations is:

$$Z_{G_2} \cong \mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]^W.$$

Clearly, there is a natural (injective) algebra homomorphism

$$\theta : Z_{G_2} \longrightarrow Z_G.$$

Thus there is a unique element $z_{G_2} \in Z_{G_2}$ such that $\theta(z_{G_2}) = z_G$, and one may hope that $z_G - z_{G_2}$ is identically zero on the minimal representation Π. Indeed, one may hope that for any $z \in Z_{G_2}$, the element $\theta(z) - z$ is identically zero on Π. However, as we now explain, this is not the case.

More precisely, if St denotes the Steinberg representation of $G = PGL_3$, then it was shown in [GS1] that the small theta lift of St is the direct sum of two irreducible representations, one of which has nonzero Iwahori-fixed vectors and the other is supercuspidal. In other words, the theta correspondence does not respect Bernstein components. Since every element $z \in Z_{G_2}$ kills supercuspidal representations, but $\theta(z) \in Z_G$ may be nonzero on St, the element $\theta(z) - z$ cannot act as zero on Π. However, the specific element $z_G = \theta(z_{G_2})$ used in our regularization does in fact act as zero on St. So it is possible that $z_G - z_{G_2}$ is identically zero on Π, but we do not know how to show this.

5 Regularization of Theta Integral

Henceforth, let F be a number field with adèle ring $\mathbb{A}$. Let $\Pi = \otimes_v \Pi_v$ be the global minimal representation of $H(\mathbb{A})$. We have a $H(\mathbb{A})$-equivariant embedding

$$\theta : \Pi \hookrightarrow \mathcal{A}_2(H)$$

of Π into the space of square-integrable automorphic forms on H. Consider the dual pair $PGL_3 \times G_2$ in $H = E_6$ as in the previous section. In this section, we shall explain how to regularize the theta integral

$$I(f)(g) = \int_{G(F)\backslash G(\mathbb{A})} \theta(f)(gh)\, dh.$$

5.1 Convergence of theta integral.

Let us fix a non-archimedean place v_0 of F. In the previous section, we have defined a non-zero $(G \times G_2)$-submodule $\Pi_{v_0,0} \subset \Pi_{v_0}$. Set

$$\Pi_0 = \Pi_{v_0,0} \otimes \left(\bigotimes_{v \neq v_0} \Pi_v \right).$$

Then Π_0 is a non-zero $(G(\mathbb{A}) \times G_2(\mathbb{A}))$-submodule of Π. We shall first show that $\theta(f)$ is rapidly decreasing on $G(F)\backslash G(\mathbb{A})$ if $f \in \Pi_0$, so that the integral $I(f)$ is well-defined.

5.2 Fourier coefficients of Π.

Let $\Omega_Q(F)$ be the minimal non-trivial $L_H(F)$-orbit on the set of unitary characters of $U_H(F)\backslash U_H(\mathbb{A})$ (which can be non-canonically identified with the set $\overline{U}_H(F)$). The Zariski closure $\overline{\Omega}_Q(F)$ of $\Omega_Q(F)$ in $\overline{U}_H(F)$ is simply the union of $\Omega_Q(F)$ with the trivial character (which is identified with $0 \in \overline{U}_H(F)$). As in the previous section, we have the closed subvariety $\Omega_{Q,deg}(F)$ of $\Omega_Q(F)$ and its open complement $\Omega_Q^0(F)$.

For $f \in \Pi$, the Fourier expansion of $\theta(f)$ along the unipotent radical U_H of the maximal parabolic Q_H takes the form:

$$\theta(f)(h) = \sum_{\chi \in \overline{\Omega}_Q} \theta(f)_{U_H,\chi} = \theta(f)_{U_H}(h) + \sum_{\chi \in \Omega_Q} \theta(f)_{U_H,\chi}(h).$$

The constant term $\theta(f)_{U_H}$, when regarded as a function on $L_H(\mathbb{A})$, belongs to the automorphic representation $\delta_{Q_H}^r \oplus \delta_{Q_H}^{r/2} \Pi_{L_H}$ where r is given in the table in (**??**) and Π_{L_H} is the global minimal representation of $L_H(\mathbb{A})$ with trivial central character.

Lemma 5.1. *If $f \in \Pi_0$, then $\theta(f)_{U_H}$ and $\theta(f)_{U_H,\chi}$ for $\chi \in \Omega_{Q,deg}$ are both zero when restricted to $G \times G_2$.*

Proof. The claim for the constant term $\theta(f)_{U_H}$ is clear from the definition of Π_0, since $(\Pi_{0,v_0})_{U_H} = 0$. Consider now the non-trivial Fourier coefficient $\theta(f)_{U_H,\chi}$. Because of the local multiplicity one result

$$\dim \mathrm{Hom}_{R_H(F_v)\cdot U_H(F_v)}(\Pi_v, \mathbb{C}_\chi) = 1,$$

we deduce that if $f = \otimes f_v \in \otimes_v \Pi_v$, then

$$\theta(f)_{U_H,\chi}(h) = \prod_v L_v(h_v f_v)$$

for some non-zero element L_v of $\mathrm{Hom}_{R_H(F_v)\cdot U_H(F_v)}(\Pi_v, \mathbb{C}_\chi)$. Up to scaling, L_v is simply the canonical element $L^0_{\chi_v}$ introduced in (**??**). In particular, if $f_v \in \Pi_{v,0}$ for some finite place v, then $L_v(h_v f_v) = 0$ for any $h_v \in G(F_v) \times G_2(F_v)$. Hence, if $f \in \Pi_0$, then

$$\theta(f)_{U_H,\chi}(h) = 0$$

for any $\chi \in \Omega_{Q,deg}$ and any $h \in G(\mathbb{A}) \times G_2(\mathbb{A})$. □

5.3 Siegel Domains.

Before moving on, we briefly recall the notion of a Siegel domain for G. Choose a Borel subgroup $B = T \cdot U_0$ of G which gives rise to a set of simple roots Δ of G with respect to T. Fix a special maximal compact subgroup $K_G = \prod_v K_{G_v}$ of $G(\mathbb{A})$. With this data, we set

$$T_r = \{t \in T(F_\infty)^0 : |\alpha(t)| > r \text{ for all } \alpha \in \Delta\}.$$

Here, if $F_\infty \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$, then for $t \in T(F_\infty)$, $|\alpha(t)| \in (\mathbb{R}^\times_+)^{r_1+r_2}$, and the inequality $|\alpha(t)| > r$ means that each component of $|\alpha(t)|$ is $> r$. If μ_α is the fundamental coweight associated to $\alpha \in \Delta$, then the set $\{\mu_\alpha\}$ is a basis of the cocharacter group $X_*(T)$ since G is adjoint here. Thus, the set $\{\mu_\alpha\}$ gives an isomorphism $T \cong \mathbb{G}_m^n$ so that

$$T_r = \{a = (a_i) \in (\mathbb{R}^\times_+)^{nr_1} \times (\mathbb{C}^\times)^{nr_2} : |a_i| > r\}.$$

A Siegel domain of G is a subset of $G(\mathbb{A})$ of the form:

$$S_r = \omega \cdot T_r \cdot K_G$$

where ω is a compact set of $U_0(\mathbb{A})$. By reduction theory, when ω is large enough,

$$G(\mathbb{A}) = G(F) \cdot S_r.$$

5.4 Rapid decrease of nondegenerate Fourier coefficients.

In this subsection, we prove the following crucial proposition:

Proposition 5.2. *If $f \in \Pi_0$, the function $\theta(f)$ is rapidly decreasing on $G(F)\backslash G(\mathbb{A})$.*

Proof. It suffices to consider the restriction of $\theta(f)$ to the Siegel domain for G introduced above. Consider the Fourier expansion of $\theta(f)$ along U_H. By Lemma **??**, we only need to examine $\theta(f)_{U_H,\chi}$ for $\chi \in \Omega^0_Q$. We claim that $\theta(f)_{U_H,\chi}$ is rapidly decreasing on S_r if $\chi \in \Omega^0_Q$. This is a consequence of the following basic general lemma:

Lemma 5.3. *Let G be a semisimple group defined over F with Borel subgroup $B = T \cdot U_0$. Assume for simplicity that G is adjoint, and let S_r be a Siegel domain of G defined as in the previous subsection. Suppose that H is a semisimple group containing G and U_H is an abelian unipotent subgroup of H which is normalized by the torus T If Φ_U denotes the set of roots of T in U_H, then we assume that Φ_U is contained in the set of positive roots of G.*

For a unitary character χ of $U_H(F)\backslash U_H(\mathbb{A})$, let

$$\Phi_\chi = \{\alpha \in \Phi_U : \chi|_{U_{H,\alpha}} \neq 1\}.$$

Suppose that χ satisfies the following condition:

(*) for each fundamental coweight μ_β, there exists $\alpha \in \Phi_\chi$ such that $\langle \mu_\beta, \alpha \rangle > 0$.

Then for any automorphic form f on H, the Fourier coefficient $f_{U_H,\chi}$ is rapidly decreasing on S_r. In fact, if Ψ^* denotes the set of χ's which satisfy the condition (*), then $\sum_{\chi \in \Psi^*} f_{U_H,\chi}$ is rapidly decreasing on S_r.

Proof. The argument is the same as that in [MW, Lemma I2.10, Pg. 30-34]. It uses repeated integration by parts and the fact that f has *uniform* moderate growth; we omit the rather tedious proof. □

It is easy to check that the conditions of the lemma hold for the case at hand. Thus we can apply this lemma to deduce that

$$\sum_{\chi \in \Omega_Q^0} \theta(f)_{U_H,\chi}$$

is rapidly decreasing on S_r. Thus for $f \in \Pi_0$, $\theta(f)$ is rapidly decreasing on S_r, as desired. □

5.5 Definition of regularized theta integral.

If z_G is the element of the Bernstein center of $G(F_{v_0})$ defined in the previous section, then we have:

$$z_G(\Pi) \subset \Pi_0.$$

This allows us to define the regularized theta integral.

Definition: For $f \in \Pi$, set

$$I_{REG}(f)(g) = \int_{G(F)\backslash G(\mathbb{A})} \theta(z_G \cdot f)(gh)\, dh.$$

The map $f \mapsto I_{REG}(f)$ is a $G(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant map from Π to $\mathcal{A}(G_2)$. Note that it is not at all clear that the map I_{REG} is non-zero.

In fact, since $\theta(z_G \cdot f)$ is rapidly decreasing on $G(F)\backslash G(\mathbb{A})$, one can integrate it against any automorphic form on G. In particular, if $E(\varphi, s, h)$ is the Eisenstein series associated to a standard section $\varphi_s \in I_Q(s)$, then we set

$$I_{REG}(f,\varphi,s)(g) = \int_{G(F)\backslash G(\mathbb{A})} \theta(z_G \cdot f)(gh) \cdot E(\varphi, s, h)\, dh.$$

This is a meromorphic function of s which has a pole of order at most 1 at $s = \frac{1}{2}$. If φ is the spherical vector φ_0 in $I_Q(s)$, then the residue of $I_{REG}(f, \varphi_0, s)(g)$ at $s = \frac{1}{2}$ is a scalar multiple of $I_{REG}(f)(g)$.

6 Regularized Siegel-Weil Formula

In this section, we shall determine the function $I(f, \varphi, s)$ as an automorphic form on G_2. As a consequence, we shall obtain the regularized Siegel-Weil formula.

6.1 Unfolding the Eisenstein series.

Suppose that $Re(s)$ is sufficiently large and assume for simplicity that $f \in \Pi_0$. We may unfold the Eisenstein series $E(\varphi, s, h)$ and compute:

$$\begin{aligned}
&\int_{G(F)\backslash G(\mathbb{A})} \theta(f)(gh) \cdot E(\varphi, s, h)\, dh \\
&= \int_{Q(F)\backslash G(\mathbb{A})} \theta(f)(gh) \cdot \varphi_s(h)\, dh \\
&= \int_{U(\mathbb{A})L(F)\backslash G(\mathbb{A})} \int_{U(F)\backslash U(\mathbb{A})} \theta(f)(ugh) \cdot \varphi_s(h)\, du\, dh \\
&= \int_{U(\mathbb{A})L(F)\backslash G(\mathbb{A})} \sum_{\chi \in \Omega_Q : \chi|_{U(\mathbb{A})} = 1} \theta(f)_{U_H,\chi}(gh) \cdot \varphi_s(h)\, dh \\
&= \int_{U(\mathbb{A})L(F)\backslash G(\mathbb{A})} \sum_{\chi \in \Omega_Q^0 : \chi|_{U(\mathbb{A})} = 1} \theta(f)_{U_H,\chi}(gh) \cdot \varphi_s(h)\, dh \\
&= \int_{U(\mathbb{A})L(F)\backslash G(\mathbb{A})} \sum_{\gamma_2 \in P(F)\backslash G_2(F)} \sum_{\gamma_1 \in L(F)} \theta(f)_{U_H,\chi_0}(\gamma_2 g, \gamma_1 h) \cdot \varphi_s(\gamma_1 h)\, dh \\
&= \sum_{\gamma_2 \in P(F)\backslash G_2(F)} \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \theta(f)_{U_H,\chi_0}(\gamma_2 gh) \cdot \varphi_s(h)\, dh
\end{aligned}$$

This computation is valid when $Re(s)$ is sufficiently large. In the course of the computation, we have used the fact that the set

$$\Omega_Q^{00} = \{\chi \in \Omega_Q^0 : \chi|_{U(\mathbb{A})} = 1\}$$

is acted upon transitively by $G_2(F) \times L(F)$ and the stabilizer of an appropriate element χ_0 is $\Delta GL_2 \cdot N(F) \hookrightarrow P(F) \times L(F)$, where GL_2 is embedded diagonally. This is [MS, Prop. 4.2, Pg. 101-102]. In particular, $\Omega_Q^{00} \cong G_2(F)/N(F)$.

For any $f \in \Pi$, let us set

$$\Phi(f,\varphi,s)(g) = \int_{U(\mathbb{A})\backslash G(\mathbb{A})} \theta(f)_{U_H,\chi_0}(gh)\cdot\varphi_s(h)\,dh.$$

It is not difficult to check that this integral is convergent if $Re(s) >> 0$. Further, in a right half plane of convergence,

$$\Phi(f,\varphi,s) \in I_P(s).$$

Thus, we see from the above computation that for $f \in \Pi$ and $Re(s) >> 0$,

$$I_{REG}(f,\varphi,s)(g) = E(\Phi(z_G\cdot f,\varphi,s),g) = P_z(s)\cdot E(\Phi(f,\varphi,s),g),$$

where $E(\Phi(f,\varphi,s),g)$ is the Eisenstein series associated to the section $\Phi(f,\varphi,s) \in I_P(s)$ and $P_z(s)$ is the meromorphic function giving the action of z_G^* (the adjoint operator of z_G) on $I_Q(s)$.

It is important to note that the section $\Phi(f,\varphi,s)$ is not necessarily a standard or flat section. However, for $Re(s) >> 0$, there are finitely many standard sections Φ_j and holomorphic functions $\alpha_j(s)$ such that

$$\Phi(f,\varphi,s) = \sum_j \alpha_j(s)\Phi_i.$$

Our next task is to show that $\Phi(f,\varphi,s)$ (or equivalently the coefficients $\alpha_j(s)$) admits a meromorphic continuation to the half plane $Re(s) > -\frac{1}{2}$. In fact, it extends to a meromorphic function on the whole complex plane, but we shall not need this fact here, since we are only interested in the analytic behaviour of $\Phi(f,\varphi,s)$ at $s = \frac{1}{2}$.

6.2 The meromorphic section $\Phi(f,\varphi,s)$.

Let us set:

$$\Phi_v(f_v,\varphi_v,s)(g_v) = \int_{U(F_v)\backslash G(F_v)} L_{\chi_0,v}(g_v h_v f_v)\cdot\varphi_s(h_v)\,dh_v.$$

Then

$$\Phi(f,\varphi,s) = \prod_v \Phi_v(f_v,\varphi_v,s).$$

We first examine the local factor $\Phi_v(f_v,\varphi_v,s)$.

Using the Iwasawa decomposition, we see that

$$\begin{aligned}
&\Phi_v(f_v,\varphi_v,s)(g_v)\\
&= \int_{K_v} \varphi_v(k_v)\cdot\left(\int_{l_v\in GL_2(F_v)} |det(l_v)|^2 i_Q(g_v k_v\cdot f_v)(l_v^{-1}\cdot\chi_0)\cdot|det(l_v)|^{\frac{1}{2}+s}\cdot\delta_P(l_v)^{-1}\,dl_v\right)dk_v\\
&= \int_{K_v} \varphi_v(k_v)\cdot\left(\int_{l_v\in GL_2(F_v)} |det(l_v)|^{\frac{3}{2}+s}\cdot i_Q(g_v k_v f_v)(l_v^{-1}\cdot\chi_0)\,dl_v\right)dk_v
\end{aligned}$$

Let us set

$$Z_v(f_v, s, g_v) = \int_{l_v \in GL_2(F_v)} |det(l_v)|^{\frac{3}{2}+s} \cdot i_Q(g_v f_v)(l_v^{-1}\chi_0)\, dl_v$$

so that

$$\Phi_v(f, \varphi, s)(g_v) = \int_{K_v} \varphi_v(k_v) \cdot Z_v(k_v \cdot f_v, s, g_v)\, dk_v.$$

In particular, the analytic properties of Φ_v is controlled by that of Z_v. The integral defining Z_v is similar to the zeta integral for the standard L-function of GL_2. However, instead of having a Schwarz-Bruhat function on $M_2(F_v)$ in the integrand, we have a function on $M_2(F_v) \smallsetminus \{0\}$ with polynomial singularities at 0.

The main result of this subsection is:

Proposition 6.1. *(i) The integral defining $Z_v(f_v, s, g_v)$ converges absolutely for $Re(s) > -\frac{1}{2}$. If $f_{0,v}$ is the K_v-spherical vector in Π_v, then*

$$Z_v(f_{0,v}, s, 1) = \alpha_v(s) \cdot \zeta_v(\frac{1}{2} + s) \cdot \zeta_v(\frac{3}{2} + s) \cdot \zeta_v(1 + 2s)$$

where

$$\alpha_v(s) = \begin{cases} 1, & \text{if } v \text{ is finite;} \\ 2^{2s-1} \cdot \pi^{2s+1}, & \text{if } v \text{ is real;} \\ 2^{4s-4} \cdot (2\pi)^{4s+3}, & \text{if } v \text{ is complex.} \end{cases}$$

In particular, $\alpha_v(s)$ is an exponential function of s and has no zeros or poles.

(ii) The integral defining $\Phi_v(f_v, \varphi_v, s)(g_v)$ converges absolutely for $Re(s) > -\frac{1}{2}$. In this half-plane of convergence, $\Phi_v(-, -, s_0)$ is a $G(F_v)$-invariant and $G_2(F_v)$-equivariant surjective *map from $\Pi_v \otimes I_{Q,v}(s_0) \longrightarrow I_{P,v}(s_0)$. If $\varphi_{0,v}$ is a spherical vector in $I_Q(s_0)$, then $\Phi_v(f_{0,v}, \varphi_{0,v}, s_0)$ is a spherical vector in $I_P(s_0)$.*

Proof. (i) To see that the integral defining Z_v converges when $Re(s) > -\frac{1}{2}$, we shall make use of the asymptotic behaviour of $i_Q(f_v)$ given in Corollary **??** and Theorem **??**.

Recall from §**??** that the element $\chi_0 \in \Omega_Q^{00}$ is given by $(x_0, y_0) \in \mathbb{O}^2$ and without loss of generality, we may assume that $\chi_0 \in \Omega_Q(0)$ (cf. §**??**). Moreover, the action of $l \in L(F_v) \cong GL_2(F_v)$ is given by:

$$l : (x_0, y_0) \mapsto (x_0, y_0) \cdot l^{-1}.$$

Let $B_v = T_v{\cdot}U_v$ be the Borel subgroup of upper triangular matrices in $GL_2(F_v)$. Then by the Iwasawa decomposition, the integral defining $Z_v(f_v, s, 1)$ is dominated (in absolute value) by

$$\int_{T_v^+} |\det(a)|^{\frac{3}{2}+s} \cdot |i_Q(f)((x_0, y_0) \cdot t)| \cdot \delta_{B_v}(t)^{-1} \cdot dt$$

where δ_{B_v} is the modulus character of B_v and T_v^+ is the subset of T_v consisting of elements

$$t = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad \text{with } |a| \leq |b|.$$

Now using the estimates given in Corollary **??** and Theorem **??**, we see that

$$|i_Q(f)((x_0, y_0) \cdot t)| = |i_Q(f)((ax_0, by_0))| \leq C \cdot |b|^{-2}$$

for some constant C. Thus the above integral is bounded above by a constant multiple of

$$\int\int |ab|^{\frac{3}{2}+s} \cdot |\frac{a}{b}|^{-1} \cdot |b|^{-2}\, da\, db = \int\int |a|^{\frac{1}{2}+s} \cdot |b|^{\frac{1}{2}+s}\, da\, db,$$

which converges for $Re(s) > -1/2$.

To prove (i), it remains to compute $Z_v(f_{0,v}, s, 1)$ when $Re(s) > -\frac{1}{2}$. We consider the archimedean and non-archimedean cases separately.

p-adic case: Using the Cartan decomposition and Theorem **??**, we have

$$Z_v(f_{0,v}, s, 1) = \sum_{a \geq b} |\det \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}|^{\frac{3}{2}+s} \cdot \frac{p^{2(b+1)} - 1}{p^2 - 1} \cdot Vol(GL_2(\mathcal{O}_v) \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} GL_2(\mathcal{O}_v)).$$

Now

$$Vol(GL_2(\mathcal{O}_v) \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} GL_2(\mathcal{O}_v)) = \begin{cases} 1, \text{ if } a = b; \\ p^{a-b}(1 + p^{-1}), \text{ if } a \neq b. \end{cases}$$

Using this, one sees that the sum for Z_v is essentially a number of geometric series which converges for $Re(s) > -\frac{1}{2}$. A pleasant computation, which we will leave to the reader, leads to the answer

$$Z_v(f_v, s, 1) = \zeta_v(\frac{1}{2} + s) \cdot \zeta_v(\frac{3}{2} + s) \cdot \zeta_v(1 + 2s).$$

Real case: Using the Cartan decomposition and Theorem **??**, we have

$$Z_v(f_{0,v}, s, 1) = \int_{a > b} |\det \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}|^{\frac{3}{2}+s} \cdot \frac{1}{\sqrt{e^{2a} + e^{2b}}} \cdot K_1(\sqrt{e^{2a} + e^{2b}}) \cdot \sinh(a - b)\, da\, db.$$

By the change of variables:

$$x = e^{a-b}, \qquad y = e^b,$$

we see that

$$Z_v(f_{0,v}, s, 1) = \int_{y=0}^{\infty} \int_{x=1}^{\infty} x^{\frac{3}{2}+s} \cdot y^{3+2s} \cdot \frac{1}{y\sqrt{1 + x^2}} \cdot K_1(y\sqrt{1 + x^2}) \cdot \frac{x - x^{-1}}{2} \frac{dx}{x} \frac{dy}{y}.$$

Replacing y by $z = y\sqrt{1+x^2}$ (with x fixed), we get:

$$Z_v(f_{0,v}, s, 1) = \left(\int_{z=0}^{\infty} z^{2+2s} K_1(z)\, \frac{dz}{z}\right) \cdot \left(\int_{x=1}^{\infty} \left(\frac{x}{1+x^2}\right)^{\frac{3}{2}+s} \cdot \frac{x - x^{-1}}{2}\, \frac{dx}{x}\right).$$

By [B, Lemma 1.9.1], we have: for $Re(s) > -\frac{1}{2}$,

$$\int_{z=0}^{\infty} z^{2+2s} K_1(z)\, \frac{dz}{z} = 2^{2s} \cdot \Gamma\left(\frac{3+2s}{2}\right) \cdot \Gamma\left(\frac{1+2s}{2}\right).$$

On the other hand, an exercise in calculus gives: for $Re(s) > -\frac{1}{2}$,

$$\int_{x=1}^{\infty} \left(\frac{x}{1+x^2}\right)^{\frac{3}{2}+s} \cdot \frac{x - x^{-1}}{2}\, \frac{dx}{x} = \frac{1}{2^{\frac{1}{2}+s}(1+2s)}.$$

Thus,

$$Z_v(f_{0,v}, s, 1) = 2^{s-\frac{1}{2}} \cdot \Gamma\left(\frac{1+2s}{2}\right)^2.$$

By the duplication formula for the gamma function, this is equal to

$$2^{2s-1}\pi^{2s+1} \cdot \zeta_v(\frac{1}{2}+s) \cdot \zeta_v(\frac{3}{2}+s) \cdot \zeta_v(1+2s).$$

Complex case: As in the real case, we use the Cartan decomposition and Theorem **??**. After a similar change of variables, we have

$$Z_v(f_{0,v}, s, 1) = \int_{y=0}^{\infty}\int_{x=1}^{\infty} x^{3+2s} \cdot y^{6+4s} \cdot \frac{1}{y^2(1+x^2)} \cdot K_2(y\sqrt{1+x^2}) \cdot \left(\frac{x - x^{-1}}{2}\right)^2 \frac{dx}{x}\, \frac{dy}{y}.$$

Replacing y by $z = y\sqrt{1+x^2}$ (with x fixed), we get:

$$Z_v(f_{0,v}, s, 1) = \left(\int_{z=0}^{\infty} z^{4+4s} K_2(z)\, \frac{dz}{z}\right) \cdot \left(\int_{x=1}^{\infty} \left(\frac{x}{1+x^2}\right)^{3+2s} \cdot \left(\frac{x - x^{-1}}{2}\right)^2 \frac{dx}{x}\right).$$

Both these integrals on the right coverge for $Re(s) > -\frac{1}{2}$. The first integral is equal to

$$2^{4s+2} \cdot \Gamma(3+2s) \cdot \Gamma(1+2s),$$

whereas the second can be computed as follows:

$$\begin{aligned}
&\int_{1}^{\infty} \left(\frac{x}{1+x^2}\right)^{3+2s} \cdot \left(\frac{x - x^{-1}}{2}\right)^2 \frac{dx}{x} \\
=&\frac{1}{2} \cdot \int_{0}^{\infty} \left(\frac{x}{1+x^2}\right)^{3+2s} \cdot \left(\frac{x - x^{-1}}{2}\right)^2 \frac{dx}{x} \\
=&\frac{1}{2^4} \cdot \int_{0}^{\infty} \frac{t^{\frac{1}{2}+s}}{(1+t)^{3+2s}} \cdot (t-1)^2\, \frac{dt}{t} \\
=&\int_{0}^{\infty} \frac{t^{\frac{3}{2}+s}}{(1+t)^{3+2s}} - 2\frac{t^{\frac{1}{2}+s}}{(1+t)^{3+2s}} + \frac{t^{-\frac{1}{2}+s}}{(1+t)^{3+2s}}\, dt.
\end{aligned}$$

Recalling that
$$\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,dt = \Gamma(a)\cdot\Gamma(b)\cdot\Gamma(a+b)^{-1},$$
we deduce that
$$\begin{aligned} Z_v(f_{0,v}, s, 1) &= 2^{4s-1}\cdot\Gamma(\frac{1}{2}+s)\cdot\Gamma(\frac{3}{2}+s)\cdot\Gamma(1+2s) \\ &= 2^{4s-4}(2\pi)^{4s+3}\cdot\zeta_v(\frac{1}{2}+s)\cdot\zeta_v(\frac{3}{2}+s)\cdot\zeta_v(1+2s). \end{aligned}$$

We have thus proven (i).

(ii) The first and last statements of (ii) are clear from (i). It is also clear that $\Phi_v(-,-,s_0)$ is a $G(F_v)$-invariant and $G_2(F_v)$-equivariant map
$$\Pi_v \otimes I_{Q,v}(s_0) \longrightarrow I_{P,v}(s_0)$$
when $Re(s_0) > -\frac{1}{2}$. The surjectivity of this map follows from the fact that when $Re(s) > -\frac{1}{2}$, $I_{P,v}(s_0)$ is generated by the spherical vector which is in the image of the map since $\Phi_v(f_{0,v}, \varphi_{0,v}, s_0)$ is non-zero. The proposition is proven. □

Corollary 6.2. *For each $f \in \Pi$, the global integral $\Phi(f,\varphi,s)$ converges when $Re(s) > \frac{1}{2}$ and has meromorphic continuation to $Re(s) > -\frac{1}{2}$ (at least). It has a pole of order at most 1 at $s = \frac{1}{2}$. This pole is attained when f and φ are both spherical vectors, in which case $Res_{s=\frac{1}{2}}\Phi(f,\varphi,s)$ is a spherical vector of $I_P(\frac{1}{2})$.*

6.3 Eisenstein series.

We now consider the Eisenstein series associated to the family of degenerate principal series $I_P(s)$. This Eisenstein series has been studied by Ginzburg-Jiang [GJ] who showed:

Proposition 6.3. *If $\Phi_s \in I_P(s)$ is a standard section, then $E(\Phi,s,g)$ has at most a pole of order 2 st $s = \frac{1}{2}$. This double pole is attained by the spherical section.*

In view of the proposition, we have the following Laurent expansion:
$$E(\Phi,s,g) = \frac{A_{-2}(\Phi)(g)}{(s-\frac{1}{2})^2} + \frac{A_{-1}(\Phi)(g)}{s-\frac{1}{2}} + \ldots.$$
In particular, we have a non-zero $G_2(\mathbb{A})$-equivariant map
$$A_{-2} : I_P(\frac{1}{2}) \longrightarrow \mathcal{A}(G_2).$$
It was shown in [GJ] that the image of A_{-2} is contained in the space $\mathcal{A}_2(G_2)$ of square-integrable automorphic forms. To understand the image of A_{-2} as a representation, we need to know the structure of $I_P(\frac{1}{2})$.

6.4 The degenerate principal series $I_P(\frac{1}{2})$.

We begin with the structure of the local principal series $I_{P,v}(\frac{1}{2})$:

Proposition 6.4. *(i) If v is p-adic, then $I_{P,v}(\frac{1}{2})$ has a filtration*

$$0 \subset I_{0,v} \subset I_{1,v} \subset I_{2,v} = I_{P,v}(\frac{1}{2})$$

such that

$$\begin{cases} I_{0,v} \cong \pi_{I,v} \\ I_{1,v}/I_{0,v} \cong J_{\beta,v}(St, \frac{1}{2}) \\ I_{2,v}/I_{1,v} \cong J_{\beta,v}(\pi(1,1),1). \end{cases}$$

Moreover, $\pi_{I,v}$ is the unique irreducible submodule and is a discrete series representation with 1-dimensional space of Iwahori-fixed vectors. The representation $J_{\beta,v}(\pi(1,1),1)$ is the unique irreducible quotient.

(ii) If v is real, then there is a non-split exact sequence

$$0 \longrightarrow \pi_\kappa \longrightarrow I_{P,v}(\tfrac{1}{2}) \longrightarrow J_{\beta,v}(\pi(1,1),1) \longrightarrow 0.$$

(iii) If v is complex, $I_{P,v}(\frac{1}{2}) = J_{\beta,v}(\pi(1,1),1)$ is irreducible.

(iv) In each case, the unique irreducible quotient $J_{\beta,v}(\pi(1,1),\frac{1}{2})$ is spherical.

As a result of Props. **??** and **??**, we have:

Corollary 6.5. *The image of A_{-2} is the globally spherical representation $J_\beta(\pi(1,1),1)$.*

6.5 The regularized Siegel-Weil formula.

We are now in a position to state the regularized Siegel-Weil formula. Consider the Laurent expansion of the following functions at $s = \frac{1}{2}$:

$$\begin{cases} P_z(s) = c(z)(s-\frac{1}{2})^2 + \ldots\ldots \\ \\ \Phi(f,\varphi_0,s) = \frac{\Phi_{-1}(f)}{s-\frac{1}{2}} + \ldots\ldots \end{cases}$$

Here, $\Phi_{-1}(f) \in I_P(\frac{1}{2})$ and can be uniquely extended to a standard section of $I_P(s)$. We have shown that:

$$I_{REG}(f,\varphi_0,s)(g) = P_z(s)\cdot E(\Phi(f,\varphi_0,s),s,g).$$

Taking the leading coefficient of the Laurent expansion of both sides, we have:

Proposition 6.6. *For any $f \in \Pi$,*

$$I_{REG}(f) = c(z)\cdot A_{-2}(\Phi_{-1}(f)).$$

In particular, I_{REG} is a nonzero map. Indeed, it is nonzero on the spherical vector f_0 of Π.

Note that the map $\Phi_{-1} : \Pi \longrightarrow I_P(1/2)$ is $G_2(\mathbb{A})$-equivariant but is not $PGL_3(\mathbb{A})$-invariant. However, the above identity shows that the map $A_{-2} \circ \Phi_{-1}$ is $PGL_3(\mathbb{A})$-invariant.

On the other hand, we have another natural map

$$E : \Pi \hookrightarrow \mathcal{A}(G)$$

defined as follows. Composing the $H(\mathbb{A})$-equivariant inclusion

$$j_P : \Pi \hookrightarrow Ind_{P_H}^{H} \delta_{P_H}^{2/11}$$

with the restriction of functions to $G_2(\mathbb{A})$, one obtains a $G(\mathbb{A})$-invariant and $G_2(\mathbb{A})$-equivariant surjective map

$$\Xi : \Pi \longrightarrow I_P(1/2),$$

which sends the spherical vector f_0 to a nonzero spherical vector in $I_P(1/2)$. On further composing this map with A_{-2} gives the desired map

$$E = A_{-2} \circ \Xi.$$

Hence we would like to show that

$$I_{REG} = \lambda \cdot E$$

for some nonzero scalar λ, which is the sought-after Siegel-Weil formula. Observe that both I_{REG} and E are nonzero elements of

$$\mathrm{Hom}_{PGL_3 \times G_2}(\Pi, \mathbf{1} \boxtimes \mathrm{Im} A_{-2}).$$

Thus, if the above Hom space is 1-dimensional, we would be done. Unfortunately, because of the lack of knowledge of the archimedean theta correspondence, we are unable to show this. However, we do have:

Lemma 6.7. *Fix a vector $f_\infty \in \Pi_\infty$ and consider the two maps on Π_f given by*

$$f \mapsto E(f_\infty \otimes f) \quad \text{and} \quad f \mapsto I_{REG}(f_\infty \otimes f).$$

Then these two maps are scalar multiples of each other (with the scalar possibly depending on f_∞ and possibly zero).

Proof. By [GS1], one knows that for each finite place v, the big theta lift of the trivial representation of $G(F_v)$ is the representation $I_{P_v}(1/2)$ which has unique irreducible quotient $\theta_v(\mathbf{1})$. Hence,

$$\dim \mathrm{Hom}_{G(F_v) \times G_2(F_v)}(\Pi_v, \mathbf{1} \boxtimes (\mathrm{Im} A_{-2})_v) = 1.$$

Consequently, we have

$$\dim \mathrm{Hom}_{G(\mathbb{A}_f) \times G_2(\mathbb{A}_f)}(\Pi_f, \mathbf{1} \boxtimes (\mathrm{Im} A_{-2})_f) = 1.$$

Since the two maps of interest are elements of this Hom space, we deduce that they are scalar multiples of each other. $\square$

On the other hand, since the two maps E and I_{REG} are nonzero on the spherical vector f_0, we see that there is a nonzero scalar λ such that

$$I_{REG}(f) = \lambda \cdot E(f)$$

for any $f \in \Pi^0$, where Π^0 is the $G(\mathbb{A}) \times G_2(\mathbb{A})$-span of the spherical vector f_0. Together with the lemma, this implies that

$$I_{REG}(f) = \lambda \cdot E(f)$$

for all $f \in \Pi^0_\infty \otimes \Pi_f$, where Π^0_∞ is the $(G(F_\infty) \times G_2(F_\infty))$-span of the spherical vector $f_{0,\infty}$. To conclude, we have:

Theorem 6.8. *There is a nonzero scalar λ such that for $f \in \Pi^0_\infty \otimes \Pi_f$, we have*

$$I_{REG}(f) = \lambda \cdot E(f).$$

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