# RECENT PROGRESS ON THE GROSS-PRASAD CONJECTURE

**Wee Teck Gan**

## 1 Introduction

The Gross-Prasad conjecture concerns a restriction or branching problem in the representation theory of real or p-adic Lie groups. It also has a global counterpart which is concerned with a family of period integrals of automorphic forms. The conjecture itself was proposed by Gross and Prasad in the context of special orthogonal groups in 2 papers [13,14] some twenty years ago. In a more recent paper [8], the conjecture was extended to all classical groups, i.e. orthogonal, unitary, symplectic and metaplectic groups. Though the conjecture has the same flavour for these different groups, each of the cases have their own peculiarities which make a uniform exposition somewhat difficult. As such, for the purpose of this expository article, we shall focus only on the case of unitary groups.

A motivating example is the following classical branching problem in the theory of compact Lie groups. Let $\pi$ be an irreducible finite dimensional representation of the compact unitary group $\mathrm{U}(n)$, and consider its restriction to the naturally embedded subgroup $\mathrm{U}(n-1)$. It is known that this restriction is multiplicity-free, but one may ask precisely which irreducible representations of $\mathrm{U}(n-1)$ occur in the restriction.

To give an answer to this question, we need to have names for the irreducible representations of $\mathrm{U}(n-1)$, so that we can say something like: "this one occurs but that one doesn't". Thus, it is useful to have a classification of the irreducible representations of $\mathrm{U}(n)$. Such a classification is provided by the Cartan-Weyl theory of highest weight, according to which the irreducible representations of $\mathrm{U}(n)$ are determined by their "highest weights" which are in natural bijection with sequences of integers

$$\underline{a} = (a_1 \leq a_2 \leq \cdots \leq a_n).$$

Now suppose that $\pi$ has highest weight $\underline{a}$. Then a beautiful classical theorem says that: an irreducible representation $\tau$ of $\mathrm{U}(n-1)$ with highest weight $\underline{b}$ occurs in the restriction of $\pi$ if and only if $\underline{a}$ and $\underline{b}$ are interlacing:

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq b_{n-1} \leq a_n.$$

W.T. Gan
Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076
E-mail: matgwt@math.nus.edu.sg

The Gross-Prasad conjecture considers the analogous restriction problem for the non-compact Lie groups $\mathrm{U}(p,q)$ and their p-adic analogs. As an example, consider the case when $n=2$, where we have seen that the representation $\pi_{\underline{a}}$ of $\mathrm{U}(2)$ contains $\tau_{\underline{b}}$ precisely when $a_1 \leq b_1 \leq a_2$. Consider instead the non-compact $\mathrm{U}(1,1)$ which is closely related to the group $\mathrm{SL}_2(\mathbb{R})$. Indeed, one has an isomorphism of real Lie groups

$$\mathrm{U}(1,1) \cong (\mathrm{SL}_2(\mathbb{R})^{\pm} \times S^1)/\Delta\mu_2.$$

Now let $\pi$ be an irreducible representation of $\mathrm{U}(1,1)$ (in an appropriate category); note that $\pi$ is typically infinite-dimensional but since $\mathrm{U}(1)$ is compact, the restriction of $\pi$ to $\mathrm{U}(1)$ is a direct sum of irreducible characters of $\mathrm{U}(1)$. It is known that this decomposition is multiplicity-free, so one is interested in determining precisely which characters of $\mathrm{U}(1)$ occur.

For this, it is again useful to have a classification of the irreducible representations of $\mathrm{U}(1,1)$. Such a classification has been known for a long time, and was the beginning of the systematic investigation of the infinite-dimensional representation theory of general reductive Lie groups, culminating in the work of Harish-Chandra, especially his construction and classification of the discrete series representations. These discrete series representations are the most fundamental representations, in the sense that every other irreducible representation can be built from them by a systematic procedure (parabolic induction and taking quotients).

For $\mathrm{U}(1,1)$, it turns out that the discrete series representations are essentially classified by a pair of integers $\underline{a} = (a_1 \leq a_2)$. More precisely, each such $\underline{a}$ gives rise to two discrete series representations $\pi_{\underline{a}}^+$ and $\pi_{\underline{a}}^-$. Then one can show that a irreducible representation $\tau_{\underline{b}}$ of $\mathrm{U}(1)$ occurs in the restriction of $\pi_{\underline{a}}^+$ (respectively $\pi_{\underline{a}}^-$) if and only if

$$b_1 > a_2 \quad \text{(respectively } b_1 < a_1\text{)}$$

i.e. $\pi_{\underline{a}}^{\pm} := \pi_{\underline{a}}^+ + \pi_{\underline{a}}^-$ contains $\tau_{\underline{b}}$ (with multiplicity one) if and only if $\underline{a}$ and $\underline{b}$ do not interlace!

Let us draw some lessons from this simple example:

(a) to address the branching problem, it is useful, even necessary, to have some classification of the irreducible representations of a real or p-adic Lie group. A conjectural classification exists and is called the local Langlands conjecture.

(b) it is useful to group certain representations of different but closely related groups together. In the example above, we see that if one groups together the representations $\pi_{\underline{a}}$ of $\mathrm{U}(2)$ and $\pi_{\underline{a}}^{\pm}$ of $\mathrm{U}(1,1)$, then the branching problem has a nice uniform answer:

$$\dim \mathrm{Hom}_{\mathrm{U}(1)}(\pi_{\underline{a}}, \tau_{\underline{b}}) + \dim \mathrm{Hom}_{\mathrm{U}(1)}(\pi_{\underline{a}}^{\pm}, \tau_{\underline{b}}) = 1$$

for any $\underline{a}$ and $\underline{b}$. That the local Langlands conjecture can be expanded to allow for such a classification was first suggested by Vogan.

(c) there is a simple recipe for deciding which of the three spaces $\mathrm{Hom}_{\mathrm{U}(1)}(\pi_{\underline{a}}, \tau_{\underline{b}})$ or $\mathrm{Hom}_{\mathrm{U}(1)}(\pi_{\underline{a}}^{\epsilon}, \tau_{\underline{b}})$ is nonzero, given by an appropriate interlacing condition. In the general case, we would like a similar such recipe. However, it will turn out that this is a delicate issue and formulating the precise condition is the most subtle part of the Gross-Prasad conjecture.

Let us give a summary of the paper. In Section 2, we formulate the branching problem precisely, and recall some basic results, such as the multiplicity-freeness of the restriction. This multiplicity-freeness result is proved only surprisingly recently, by the work of Aizenbud-Gouretvitch-Rallis-Schiffman [3], Waldspurger [33], Sun-Zhu [27] and Sun [26] . We shall introduce the local Langlands conjecture, in its refined form due to Vogan [28], in Section 3, where we use Vogan's notion of "pure inner forms". Then we shall state the local and global GP conjecture in Section 4. In the global setting, we also mention a refinement due to Ichino-Ikeda [19]. Finally, we describe some recent progress on the GP conjecture in the remaining sections, highlighting the work of Waldspurger [29–32] and Beuzart-Plesis [4] in the local case and the work of Jacquet-Rallis [20], Wei Zhang [38,39], Yifeng Liu [23] and Hang Xue [34,35] in the global case. We conclude with listing some outstanding problems in this story in the last section.

## 2 The Restriction Problem

Let $k$ be a field, not of characteristic 2. Let $\sigma$ be a non-trivial involution of $k$ having $k_0$ as the fixed field. Thus, $k$ is a quadratic extension of $k_0$ and $\sigma$ is the nontrivial element in the Galois group $\mathrm{Gal}(k/k_0)$. Let $\omega_{k/k_0}$ be the quadratic character of $k_0^\times$ associated to $k/k_0$ by local class field theory.

### 2.1 The spaces

Let $V$ be a finite dimensional vector space over $k$. Let

$$\langle -, - \rangle : V \times V \to k$$

be a non-degenerate, $\sigma$-sesquilinear form on $V$, which is $\epsilon$-symmetric (for $\epsilon = \pm 1$ in $k^\times$):

$$\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle$$
$$\langle u, v \rangle = \epsilon \cdot \langle v, u \rangle^\sigma .$$

### 2.2 The groups

Let $G(V) \subset \mathrm{GL}(V)$ be the algebraic subgroup of elements $T$ in $\mathrm{GL}(V)$ which preserve the form $\langle -, - \rangle$:

$$\langle Tv, Tw \rangle = \langle v, w \rangle .$$

Then $G(V)$ is a unitary group, defined over the field $k_0$.

If one takes $k$ to be the quadratic algebra $k_0 \times k_0$ with involution $\sigma(x,y) = (y,x)$ and $V$ a free $k$-module, then a non-degenerate form $\langle -, - \rangle$ identifies the $k = k_0 \times k_0$ module $V$ with the sum $V_0 + V_0^\vee$, where $V_0$ is a finite dimensional vector space over $k_0$ and $V_0^\vee$ is its dual. In this case $G(V)$ is isomorphic to the general linear group $\mathrm{GL}(V_0)$ over $k_0$. In our ensuing discussion, this split case can be handled concurrently, and is necessary for the global case.

## 2.3 Pairs of spaces

Now suppose that $W \subset V$ is a nondegenerate subspace, with $V = W \oplus W^{\perp}$, satisfying:

$$\dim W^{\perp} = \begin{cases} 1 \text{ if } \epsilon = 1; \\ 0 \text{ if } \epsilon = -1. \end{cases}$$

One thus has a natural embedding $G(W) \hookrightarrow G(V)$ with $G(W)$ acting trivially on $W^{\perp}$. We set

$$G = G(V) \times G(W) \quad \text{and} \quad H = \Delta G(W) \subset G.$$

## 2.4 Pure inner forms

We shall assume henceforth that $k$ is a local field of characteristic 0. A pure inner form of $G(V)$ is a form of $G(V)$ constructed from an element of the Galois cohomology set $H^1(k_0, G(V))$. For the case at hand, the pure inner forms are easily described and are given by the groups $G(V')$ as $V'$ ranges over similar type of spaces as $V$ with $\dim V' = \dim V$.

More concretely, when $k$ is $p$-adic, there are two Hermitian or skew-Hermitian spaces of a given dimension, so that $G(V)$ has a unique pure inner form $G(V')$ (other than itself). When $\dim V$ is odd, the groups $G(V)$ and $G(V')$ are quasi-split isomorphic even though the spaces $V$ and $V'$ are not. When $\dim V$ is even, we take the convention that $G(V)$ is quasi-split whereas $G(V')$ is not.

When $k = \mathbb{C}$ and $k_0 = \mathbb{R}$, the pure inner forms of $G(V)$ are precisely the groups $\mathrm{U}(p, q)$ with $p + q = \dim_k V$.

Now given a pair of spaces $W \subset V$, we have the notion of *relevant pure inner forms*. A pair $W' \subset V'$ is a relevant pure inner form of $W \subset V$ if $W'$ and $V'$ are pure inner forms of $W$ and $V$ respectively, and $V/W \cong V'/W'$. Then, in the p-adic case, $W \subset V$ has a unique relevant pure inner form (other than itself).

## 2.5 The restriction problem

Now we can state the restriction problem. Let $\pi = \pi_1 \boxtimes \pi_2$ be an irreducible smooth representation of $G(k_0) = G(V) \times G(W)$. When $\epsilon = 1$, we are interested in determining

$$\mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C}) \cong \mathrm{Hom}_{G(W)}(\pi_1, \pi_2^{\vee}). \tag{2.1}$$

We shall call this the *Bessel case* of the local GP conjecture.

When $\epsilon = -1$, one needs an extra data to state the restriction problem. Since $W$ is skew-Hermitian, the space $\mathrm{Res}_{k/k_0}(W)$ inherits the structure of a symplectic space, so that

$$\mathrm{U}(W) \subset \mathrm{Sp}(\mathrm{Res}_{k/k_0}(W)).$$

The metaplectic group $\mathrm{Mp}(\mathrm{Res}_{k/k_0}(W))$ (which is an $S^1$-extension of $\mathrm{Sp}(\mathrm{Res}_{k/k_0}(W))$) has a Weil representation $\omega_{\psi_0}$ associated to a nontrivial additive character $\psi_0$ of $k_0$. It is known that the metaplectic covering splits over the subgroup $\mathrm{U}(W)$ but the splitting is not unique since $\mathrm{U}(W)$ has nontrivial unitary characters. However, a splitting can be specified by a pair $(\psi_0, \chi)$ where $\chi$ is a character of $k^\times$ such that $\chi|_{k_0^\times} = \omega_{k/k_0}$. For such a splitting $i_{W,\psi_0,\chi}$, we obtain a representation

$$\omega_{W,\psi_0,\chi} := \omega_{\psi_0} \circ i_{W,\psi_0,\chi}$$

of $\mathrm{U}(W)$; we call this a Weil representation of $\mathrm{U}(W)$.

Then one is interested in determining

$$\mathrm{Hom}_{H(k_0)}(\pi, \omega_{W,\psi_0,\chi}). \tag{2.2}$$

We call this the *Fourier-Jacobi* case of the local GP conjecture. To unify notations in the two cases, we set

$$\nu = \nu_{\psi_0,\chi} = \begin{cases} \mathbb{C} \text{ if } \epsilon = 1; \\ \omega_{W,\psi_0,\chi} \text{ if } \epsilon = -1. \end{cases}$$

We note that [8] considers pairs of spaces $W \subset V$ with arbitrary $\dim W^\perp$, and formulates a restriction problem in this general setting, whereas we have restricted ourselves to the case of $\dim W^\perp \leq 1$ in this article for simplicity.

### 2.6 Multiplicity-freeness

In a number of recent papers, beginning with [3] and followed by [27] and [26], the following fundamental theorem was shown:

**Theorem 2.3** *The space* $\mathrm{Hom}_{H(k_0)}(\pi, \nu)$ *is at most one-dimensional.*

Thus, the remaining question is whether this Hom space is 0 or 1-dimensional.

The case when $k = k_0 \times k_0$ is particularly simple. One has:

**Proposition 2.4** *When* $k = k_0 \times k_0$*, the above Hom space is 1 -dimensional when* $\pi$ *is generic, i.e. has a Whittaker model.*

The local GP conjecture gives a precise criterion for the Hom spaces above to be nonzero. However, to state the precise criterion requires substantial preparation and groundwork.

### 2.7 Periods

We now consider the global situation, where $F$ is a number field with ring of addles $\mathbb{A}$ and $E/F$ is a quadratic field extension. Hence the spaces $W \subset V$ are Hermitian or skew-Hermitian spaces over $E$ and the associated groups $G$ and $H$ are defined over $F$. Let $\mathcal{A}_{cusp}(G)$ denote the space of cuspidal automorphic forms of $G(\mathbb{A})$. When $\epsilon = 1$, there is a natural $H(\mathbb{A})$-invariant linear functional on $\mathcal{A}_{cusp}(G)$ defined by

$$\mathcal{P}_H(f) = \int_{H(F)\backslash H(\mathbb{A})} f(h) \cdot dh.$$

This map is called the $H$-period integral.

When $\epsilon = -1$, the Weil representation $\omega_{W,\psi_0,\chi}$ admits an automorphic realization via the formation of theta series. Then one considers

$$\mathcal{P}_H : \mathcal{A}(G) \boxtimes \overline{\omega_{W,\psi_0,\chi}} \longrightarrow \mathbb{C}$$

given by

$$\mathcal{P}_H(f \otimes \overline{\theta_\varphi}) = \int_{H(F)\backslash H(\mathbb{A})} f(h) \cdot \overline{\theta_\varphi(h)}\, dh.$$

Now let

$$\pi = \pi_1 \boxtimes \pi_2 \subset \mathcal{A}_{cusp}(G)$$

be a cuspidal representation of $G(\mathbb{A})$. Then the restriction of $\mathcal{P}_H$ to $\pi$ defines an element in

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \overline{\nu}, \mathbb{C}).$$

By Theorem 2.3, one knows that these adelic Hom spaces have dimension at most 1, and that the dimension is 1 precisely when the relevant local Hom spaces are nonzero for all places $v$ of $F$. Moreover, it is clear that the nonvanishing of these Hom spaces is necessary for the nonvanishing of $\mathcal{P}_H$.

The global GP conjecture gives a precise criterion for the nonvanishing of the globally-defined linear functional $\mathcal{P}_H$.

## 3 Local Langlands Correspondence

To understand the restriction problem described in the previous section, it will be useful to have a classification of the irreducible representations of $G(k_0)$ in the local case, and a classification of the cuspidal representations of $G(\mathbb{A})$ in the global case. The Langlands program provides such a classification, known as the (local or global) Langlands correspondence. On one hand, the Langlands correspondence can be viewed as a generalisation of the Cartan-Weyl theory of highest weights which classifies the irreducible representations of a connected compact Lie group. On the other hand, it can be considered as a profound generalisation of class field theory, which classifies the abelian extensions of a local or number field. In this section, we briefly review the salient features of the Langlands correspondence.

### 3.1 Weil-Deligne group

We first introduce the parametrizing set. For a local field $k$, let $W_k$ denote the Weil group of $k$. When $k$ is a p-adic field, one has a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_k & \longrightarrow & \mathrm{Gal}(\overline{k}/k) & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & I_k & \longrightarrow & W_k & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

where $I_k$ is the inertia group of $\mathrm{Gal}(\overline{k}/k)$, and $\widehat{\mathbb{Z}}$ is the absolute Galois group of the residue field of $k$, equipped with a canonical generator (the geometric Frobenius element $\mathrm{Frob}_k$). This exhibits the Weil group $W_k$ as a dense subgroup of the absolute Galois group of $k$. When $k$ is archimedean, we have

$$W_k = \begin{cases} \mathbb{C}^\times \text{ if } k = \mathbb{C}; \\ \mathbb{C}^\times \cup \mathbb{C}^\times \cdot j, \text{ if } k = \mathbb{R}, \end{cases}$$

where $j^2 = -1 \in \mathbb{C}^\times$ and $j \cdot z \cdot j^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. Set the Weil-Deligne group to be

$$WD_k = \begin{cases} W_k \text{ if } k \text{ is archimedean;} \\ W_k \times \mathrm{SL}_2(\mathbb{C}), \text{ if } k \text{ is p-adic.} \end{cases}$$

### 3.2 L-parameters

Now let ${}^L G(V)$ denote the L-group of $G(V)$ over $k_0$, so that

$${}^L G(V) = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(k/k_0),$$

with $n = \dim_k V$ and $\sigma$ acting on $\mathrm{GL}_n(\mathbb{C})$ as a pinned outer automorphism. By an L-parameter (or Langlands parameter) of $G(V)$, we mean a $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of admissible homomorphisms

$$\phi : WD_{k_0} \longrightarrow {}^L G(V)$$

such that the composite of $\phi$ with the projection onto $\mathrm{Gal}(k/k_0)$ is the natural projection of $WD_{k_0}$ to $\mathrm{Gal}(k/k_0)$.

The need to work with the semi-direct product $\mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(k/k_0)$ is quite a nuisance, but the following useful result was shown in [8]:

**Proposition 3.1** *Restriction to $W_k$ defines a bijection between the set of L-parameters for $G(V)$ and the set of equivalence classes of Frobenius semisimple, conjugate-self-dual representations*

$$\phi : WD_k \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

*of sign* $(-1)^{n-1}$.

This means that L-parameters for unitary groups are essentially local Galois representations with some conjugate-self-duality property.

### 3.3 Component groups

A invariant that one can attach to an L-parameter $\phi$ is its component group

$$S_\phi = \pi_0(Z_{\mathrm{GL}_n(\mathbb{C})}(\phi)^{\mathrm{Gal}(k/k_0)})$$

where we regard $\phi$ as a map $WD_{k_0} \longrightarrow {}^L G(V)$ here. Thus, $S_\phi$ is a finite group which can be described more explicitly as follows. Regarding $\phi$ now as a representation of $WD_k$, let us decompose $\phi$ into its irreducible components:

$$\phi = \bigoplus_i n_i \cdot \phi_i.$$

Then $S_\phi$ is an elementary abelian 2-group, i.e. a vector space over $\mathbb{Z}/2\mathbb{Z}$, which is equipped with a canonical basis:

$$S_\phi = \prod_i \mathbb{Z}/2\mathbb{Z} \cdot e_i$$

where the product runs over all indices $i$ such that $\phi_i$ is conjugate-self-dual of sign $(-1)^{n-1}$ (i.e. of same sign as $\phi$).

3.4 Local Langlands conjecture

We can now formulate the local Langlands conjecture for the groups $G(V)$:

**Local Langlands Conjecture (LLC)**

There is a natural bijection

$$\bigsqcup_{V'} \mathrm{Irr}(G(V')) \longleftrightarrow \Phi(G(V))$$

where the union on the LHS runs over all pure inner forms $V'$ of $V$ and the set $\Phi(G(V))$ is the set of isomorphism classes of pairs $(\phi, \eta)$ where $\phi$ is an L-parameter of $G(V)$ and $\eta \in \mathrm{Irr}(S_\phi)$.

Given an L-parameter $\phi$ for $G(V)$, we let $\Pi_\phi$ be the finite set of irreducible representations of $G(V')$, with $V'$ running over all pure inner forms of $V$, which corresponds to $(\phi, \eta)$ for some $\eta$. This is called the L-packet with L-parameter $\phi$. So

$$\bigsqcup_{V'} \mathrm{Irr}(G(V')) = \bigsqcup_{\phi} \Pi_\phi.$$

and an irreducible representation $\pi$ of $G(V)$ (or its pure inner form) is of the form

$$\pi = \pi(\phi, \eta)$$

for a unique pair $(\phi, \eta)$ as above. We shall frequently write $\pi = \pi(\eta)$ if $\phi$ is fixed and understood.

3.5 Status

The LLC has been established for the group $\mathrm{GL}(n)$ by Harris-Taylor [16] and Henniart [18]. For the unitary groups $G(V)$, the LLC was established in the recent paper of Mok [24] when $G(V)$ is quasi-split, following closely the book of Arthur [1] where the symplectic and orthogonal groups were treated. The results of [1] and [24] are at the moment conditional on the stabilisation of the twisted trace formula, but substantial efforts are currently being made towards this stabilisation, and one can be optimistic that in the coming months, the results will be unconditional. With the stabilisation at hand, one can also expect that the results for non-quasi-split unitary groups will also follow.

*For the purpose of this article, we shall assume that the LLC has been established.*

3.6 L-factors and $\epsilon$-factors

Given an L-parameter $\phi$ of $G(V)$, one can associate some arithmetic invariants. More precisely, if

$$\rho : {}^L G(V) \longrightarrow \mathrm{GL}(U)$$

is a complex representation, then we may form the local Artin L-factor over $k_0$:

$$L_{k_0}(s, \rho \circ \phi) = \frac{1}{\det(1 - q^{-s}(\rho \circ \phi)(Frob_{k_0})|U^{I_{k_0}})},$$

where $q$ is the cardinality of the residue field of $k_0$.

Similarly, if we are given

$$\rho : \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(U),$$

then we have a composite $\rho \circ \phi : WD_k \longrightarrow \mathrm{GL}(U)$ and we can form the analogous local Artin L-factor $L_k(s, \rho \circ \phi)$ over $k$; here we have identified $\phi$ with its restriction to $WD_k$.

Further, one can associate a local epsilon factor $\epsilon(s, \rho \circ \phi, \psi)$ which is a nowhere zero entire function of $s$ depending on $\phi$, $\rho$ and an additive character $\psi$ of $k$. This local epsilon factor is quite a subtle invariant, which satisfies a list of properties. While it is not hard to show that this list of properties characterize the local epsilon factor, the issue of existence is not trivial at all. Indeed, the existence of this invariant is due independently to Deligne [5] and Langlands.

We shall mention only one key property of the local epsilon factors that we need. If $\rho \circ \phi$ is a conjugate symplectic representation of $W_k$, and $\psi$ is a nontrivial additive character of $k/k_0$, then $\epsilon(1/2, \rho \circ \phi, \psi) = \pm 1$. Moreover, this sign depends only on the $Nk^{\times}$-orbit of $\psi$. Indeed, if $\dim \rho \circ \phi$ is even, then this sign is independent of the choice of $\psi$.

This sign will play an important role in the local GP conjecture.

### 3.7 Characterization by Whittaker datum

Perhaps some explanation is needed for the meaning of the adjective "natural" in the statement of the LLC. In what sense is the bijection in the LLC natural?

One possibility is that one could characterise the bijection postulated in the LLC by requiring that it preserves certain natural invariants that one can attach to both sides. This is the case for $\mathrm{GL}(n)$ where the local $L$-factors and local $\epsilon$-factors of pairs are used to characterise the correspondence; such a characterisation is due to Henniart [17].

For the unitary groups $G(V)$, the proof of the LLC given in [24] characterises the LLC in a different way: via a family of character identities arising in the theory of endoscopy. This elaborate theory requires one to normalize certain "transfer factors". By the work of Kottwitz-Shelstad [22] and the recent work of Kaletha [21], one can fix a normalisation of the transfer factors by fixing a "Whittaker datum" for $G(V)$. Let us explain briefly what this means.

The group $G(V)$ being quasi-split, one can choose a Borel subgroup $B = T \cdot U$ defined over $k_0$, with unipotent radical $U$. A Whittaker datum on $G(V)$ is a character $\chi : U(k_0) \longrightarrow S^1$ which is in general position, i.e. whose $T(k_0)$-orbit is open, and two such characters are equivalent if they are in the same $T(k_0)$-orbit.

If $\dim V$ is odd, then any two generic characters of $U(k_0)$ are equivalent, so the LLC for $G(V)$ is quite canonical. On the other hand, when $\dim V$ is even, there are two equivalence classes of Whittaker data. In this case, we have:

**Lemma 3.2** *Using the form $\langle -, - \rangle$ on $V$, one gets a natural identification*

$$\text{Whittaker datua for } G(V)$$

$$\updownarrow$$

$$\begin{cases} Nk^{\times}\text{-orbits on nontrivial } \psi : k/k_0 \longrightarrow S^1, & \text{if } V \text{ is Hermitian;} \\ Nk^{\times}\text{-orbits on nontrivial } \psi_0 : k_0 \longrightarrow S^1, & \text{if } V \text{ is skew-Hermitian.} \end{cases}$$

As an illustration of the difference between the Hermitian and skew-Hermitian case, consider the case when $\dim V = 2$. When $V$ is split, we may choose a basis $\{e, f\}$ of $V$ so that $\langle e, f\rangle = 1$. If $V$ is Hermitian, then $U(k_0) \cong \{x \in k : Tr(x) = 0\}$, so that generic characters of $U(k_0)$ are identified with characters of the trace zero elements of $k$, which are simply characters of $k/k_0$. On the other hand, if $V$ is skew-Hermitian, then $U(k_0) \cong k_0$ so that generic characters of $U(k_0)$ are simply characters of $k_0$.

### 3.8 Generic parameters

Having fixed a Whittaker datum $(U, \chi)$, the LLC is also fixed, and has the following property. We say that an L-parameter $\phi$ is *generic* if $L_{k_0}(s, Ad \circ \phi)$ is holomorphic at $s = 1$, where $Ad$ denotes the adjoint representation of ${}^L G(V)$. For a generic $\phi$, the corresponding L-packet $\Pi_\phi$ contains generic representations. Then, relative to the fixed Whittaker datum $(U, \chi)$, the trivial character of $S_\phi$ corresponds to a $(U, \chi)$-generic representation; moreover, this is the unique $(U, \chi)$-generic representation in $\Pi_\phi$.

### 3.9 Global L-function

Now suppose we are in the global situation and $\pi = \otimes_v \pi_v$ is an automorphic representation of $G(V)$. Let $\rho$ be a representation of ${}^L G(V)$ as above. Then under the LLC, each local representation $\pi_v$ has an L-parameter $\phi_v$ and so one has the local L-factor $L_{F_v}(s, \pi_v, \rho) := L_{F_v}(s, \rho \circ \phi_v)$. Thus, one may form the global L-function

$$L_F(s, \pi, \rho) = \prod_v L_{F_v}(s, \pi_v, \rho)$$

where the product converges absolutely for $Re(s) >> 0$. This is an instance of automorphic L-functions. One of the basic conjectures of the Langlands program is that such automorphic L-functions admit a meromorphic continuation to $\mathbb{C}$ and satisfy a standard functional equation relating its value at $s$ to its value at $1 - s$. One has an analogous L-function $L_E(s, \pi, \rho)$ over $E$ if $\rho$ is a representation of $\mathrm{GL}_n(\mathbb{C})$.

Now for the group $G = G(V) \times G(W)$, we note that the groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_{n-1}(\mathbb{C})$ come with a standard or tautological representation std. Thus, taking $\rho = \mathrm{std}_n \boxtimes \mathrm{std}_{n-1}$, we have the corresponding global L-function

$$L_E(s, \pi, \rho) =: L_E(s, \pi_1 \times \pi_2) \quad \text{if } \pi = \pi_1 \boxtimes \pi_2.$$

By the results of [A] and [M], and the theory of Rankin-Selberg L-functions on $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$, one knows that $L(s, \pi_1 \times \pi_2)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the expected functional equation.

## 4 The Conjecture

After introducing the LLC in the last section, we are now ready to state the Gross-Prasad (GP) conjecture.

4.1 Multiplicity One

Let $\phi = \phi_1 \times \phi_2$ be a generic L-parameter for $G = G(V) \times G(W)$, and let $\Pi_\phi$ be the associated Vogan L-packet. A representation $\pi \in \Pi_\phi$ is thus a representation of $G(V') \times G(W')$ where $V'$ and $W'$ are pure inner forms of $V$ and $W$ respectively. We call $\pi$ relevant if $W' \subset V'$ is relevant.

**Local Gross-Prasad I**

If $\phi$ is a generic L-parameter for $G$, then

$$\sum_{\text{relevant } \pi \in \Pi_\phi} \dim \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}) = 1.$$

Thus, one has multiplicity one in Vogan packets: this is the generalisation of the lesson (b) mentioned in the introduction. The next part of the conjecture pinpoints the unique relevant $\pi$ for which the associated Hom space is nonzero. This is the most delicate part of the local GP conjecture, and we shall consider the Bessel and Fourier-Jacobi case separately.

4.2 Distinguished character

We shall define a distinguished character on $S_\phi$.

- **Bessel case.** Suppose first that $\epsilon = 1$ so that $\dim W^\perp = 1$. We need to specify a character $\eta_{GP}$ of $S_\phi$, which determines the distinguished representation in $\Pi_\phi$. Suppose that $\phi = \phi_1 \times \phi_2$, with

$$S_{\phi_1} = \prod_i \mathbb{Z}/2\mathbb{Z} \cdot a_i \quad \text{and} \quad S_{\phi_2} = \prod_j \mathbb{Z}/2\mathbb{Z} \cdot b_j,$$

  so that $S_\phi = S_{\phi_1} \times S_{\phi_2}$. Then we need to specify $\eta_{GP}(a_i) = \pm 1$ and $\eta_{GP}(b_j) = \pm 1$. We fix a nontrivial character $\psi : k/k_0 \longrightarrow S^1$ which determines the LLC for the even unitary group in $G = G(V) \times G(W)$. If $\delta \in k_0^\times$ is the discriminant of the odd-dimensional space in the pair $(V, W)$, we consider the character $\psi_{-2\delta}(x) = \psi(-2\delta x)$. Then we set

$$\eta_{GP}(a_i) = \epsilon(1/2, \phi_{1,i} \otimes \phi_2, \psi_{-2\delta});$$

  and

$$\eta_{GP}(b_j) = \epsilon(1/2, \phi_1 \otimes \phi_{2,j}, \psi_{-2\delta}).$$

- **Fourier-Jacobi case.** Suppose now that $\epsilon = -1$ so that $W^\perp = 0$. In this case, we need to fix a character $\psi_0 : k_0 \longrightarrow S^1$ and a character $\chi$ of $k^\times$ with $\chi|_{k_0^\times} = \omega_{k/k_0}$ to specify the representation $\nu_{W,\psi_0,\chi}$. The recipe for the distinguished character $\eta_{GP}$ of $S_\phi$ depends on the parity of $\dim V$.
  - If $\dim V$ is odd, let $e = \mathrm{disc} V \in k_{Tr=0}$, well-defined up to $Nk^\times$, and define an additive character of $k/k_0$ by $\psi(x) = \psi_0(Tr(ex))$. We set

$$\begin{cases} \eta_{GP}(a_i) = \epsilon(1/2, \phi_{1,i} \otimes \phi_2 \otimes \chi^{-1}, \psi); \\ \eta_{GP}(b_j) = \epsilon(1/2, \phi_1 \otimes \phi_{2,j} \otimes \chi^{-1}, \psi). \end{cases}$$

- If $\dim V$ is even, the fixed character $\psi_0$ is needed to fix the LLC for $G(V) = G(W)$. We set

$$\begin{cases} \eta_{GP}(a_i) = \epsilon(1/2, \phi_{1,i} \otimes \phi_2 \otimes \chi^{-1}, \psi); \\ \eta_{GP}(b_j) = \epsilon(1/2, \phi_1 \otimes \phi_{2,j} \otimes \chi^{-1}, \psi), \end{cases}$$

where the epsilon characters are defined using any nontrivial additive character of $k/k_0$ (the result is independent of this choice).

### 4.3 Local Gross-Prasad

Having defined a distinguished character $\eta_{GP}$ of $S_\phi$, we obtain a representation $\pi(\eta_{GP}) \in \Pi_\phi$. It is not difficult to check that $\pi(\eta_{GP})$ is a representation of a relevant pure inner form $G' = G(V') \times G(W')$. Now we have:

**Local Gross-Prasad II**

Let $\eta_{GP}$ be the distinguished character of $S_\phi$ defined above. Then

$$\mathrm{Hom}_H(\pi(\phi,\eta) \otimes \overline{\nu}, \mathbb{C}) \neq 0 \quad \text{if and only if} \quad \eta = \eta_{GP}.$$

### 4.4 Global conjecture

Suppose now that we are in the global situation, with $\pi = \pi_1 \boxtimes \pi_2$ a cuspidal representation of $G(\mathbb{A}) = \mathrm{U}(V)(\mathbb{A}) \times \mathrm{U}(W)(\mathbb{A})$. It follows by [M] that $\pi$ occurs with multiplicity one in the cuspidal spectrum. The global conjecture says:

**Global Gross-Prasad Conjecture**

The following are equivalent:
(i) The period interval $\mathcal{P}_H$ is nonzero when restricted to $\pi$;
(ii) For all places $v$, the local Hom space $\mathrm{Hom}_{H(F_v)}(\pi_v, \nu_v) \neq 0$ and in addition,

$$L_E(1/2, \pi_1 \times \pi_2) \neq 0.$$

Indeed, after the local GP, there will be a unique abstract representation $\pi(\eta_{GP}) := \otimes_v \pi_v(\eta_{GP,v})$ in the global L-packet of $\pi$ which supports a nonzero abstract $H(\mathbb{A})$-invariant linear functional. This representation lives on a certain group $G'_{\mathbb{A}} = \prod_v G(V'_v)$. To even consider the period integral on $\pi(\eta_{GP})$, one must first ask whether the group $G'_{\mathbb{A}}$ arises from a space $V'$ over $E$, or equivalently whether the collection of local spaces $\{V'_v\}$ is coherent. A necessary and sufficient condition for this is that $\epsilon_E(1/2, \pi_1 \times \pi_2) = 1$. If this condition holds, one should next asks if the abstract representation $\pi(\eta_{GP})$ occurs in the automorphic discrete spectrum of $G(V')$. This is controlled by the Arthur multiplicity formula. Only after knowing that $\pi(\eta_{GP})$ is automorphic can the period integral have a chance to be nonzero on the global L-packet of $\pi$.

4.5 The refined conjecture of Ichino-Ikeda

Ichino and Ikeda [19] have formulated a refinement of the global Gross-Prasad conjecture for tempered cuspidal representations on orthogonal groups. This takes the form of a precise identity comparing the period integral $\mathcal{P}_H$ with a locally defined $H$-invariant functional on $\pi$, with the special L-value $L(1/2, \pi_1 \times \pi_2)$ appearing as a constant of proportionality. Their refinement was subsequently extended to the Hermitian case by N. Harris [15].

More precisely, suppose that $\pi = \otimes_v \pi_v$ is a tempered cuspidal representation. The Petersson inner product $\langle -, - \rangle_{\mathrm{Pet}}$ on $\pi$ can be factored (non-canonically):

$$\langle -, - \rangle_{\mathrm{Pet}} = \prod_v \langle -, - \rangle_v.$$

In addition, the Tamagawa measures $dg$ and $dh$ on $G(\mathbb{A})$ and $H(\mathbb{A})$ admit decompositions $dg = \prod_v dg_v$ and $dh = \prod_v dh_v$. We fix such decompositions once and for all.

For each place $v$, we consider the functional on $\pi \otimes \overline{\pi}$ defined by

$$I_v^{\#}(f_1, f_2) = \int_{H(F_v)} \langle f_1, f_2 \rangle_v \, dh_v.$$

This integral converges when $\pi_v$ is tempered, and defines an element of $\mathrm{Hom}_{H_v \times H_v}(\pi_v \otimes \overline{\pi_v}, \mathbb{C})$. This latter space is at most 1-dimensional, as we know, and it was shown by Waldspurger that

$$I_v^{\#} \neq 0 \Longleftrightarrow \mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0.$$

We would like to take the product of the $I_v^{\#}$ over all $v$, but this Euler product would diverge. Indeed, for almost all $v$, where all the data involved are unramified, one can compute $I_v^{\#}(f_1, f_2)$ when $f_i$ are the spherical vectors used in the restricted tensor product decomposition of $\pi$ and $\overline{\pi}$. One gets:

$$I_v^{\#}(f_1, f_2) = \Delta_{G(V),v} \cdot \frac{L_{E_v}(1/2, \pi_1 \times \pi_2)}{L_{F_v}(1, \pi, Ad)},$$

where

$$\Delta_{G(V),v} = \prod_{k=1}^{\dim V} L(k, \omega_{E_v/F_v}^k).$$

Thus, though the Euler product diverges, it can be interpreted by meromorphic continuation of the L-functions which appear in this formula. Alternatively, one may normalise the local functionals by:

$$I_v(f_1, f_2) = \left( \Delta_{G(V),v} \cdot \frac{L_{E_v}(1/2, \pi_1 \times \pi_2)}{L_{F_v}(1, \pi, Ad)} \right)^{-1} \cdot I_v^{\#}(f_1, f_2).$$

Then we may set

$$I = \prod_v I_v \in \mathrm{Hom}_{H(\mathbb{A}) \times H(\mathbb{A})}(\pi \otimes \overline{\pi}, \mathbb{C}).$$

Since the period integral $\mathcal{P}_H \otimes \overline{\mathcal{P}_H}$ is another element in this Hom space, it must be a multiple of $I$.

**Refined Gross-Prasad conjecture**

One has:

$$\mathcal{P}_H \otimes \overline{\mathcal{P}_H} = \frac{1}{|S_\pi|} \cdot \Delta_{G(V)} \cdot \frac{L_E(1/2, \pi_1 \times \pi_2)}{L_F(1, \pi, Ad)} \cdot I,$$

where $S_\pi$ denotes the "global component group" of $\pi$ (which we have not really introduced before).

When $V$ is skew-Hermitian (i.e. in the Fourier-Jacobi case)., an analogous refined conjecture was formulated in a recent preprint of Hang Xue [35].

## 5 Recent Progress: Local Case

We now come to the more interesting part of the paper, namely the account of some definitive recent results concerning the above conjectures.

### 5.1 Proof of local conjecture for Bessel model.

In a stunning series of papers [29–32], Waldspurger has proved the Local GP conjecture for tempered representations of orthogonal groups over $p$-adic fields; the case of generic nontempered representations is then deduced from this by Moeglin-Waldspurger [25]. Shortly thereafter, his student Beuzart-Plessis adapted the arguments to the p-adic Hermitian case (for tempered representations) discussed in this paper. The results are proved under the same hypotheses needed to establish the LLC for $G(V)$ (i.e. stabilisation of the twisted trace formula, the LLC for inner forms etc). Thus, one might state the result as:

**Theorem 5.1** *Assume that $k$ is p-adic and the LLC (in the refined form due to Vogan) holds for $G = G(V) \times G(W)$. Then the local Gross-Prasad conjecture holds in the Bessel case.*

It would not be possible to give a respectable account of Waldspurger's proof here, but it is nonetheless useful to highlight the key idea. We will do so in a toy model which was studied in [9, §5].

### 5.2 Toy model over finite fields

Imagine for a moment that the field $k$ is a finite field, so that $G$ and $H$ are finite groups of Lie type. Then given an irreducible representation $\pi$ of $G(k_0)$, one has:

$$\dim \mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C}) = \langle \chi_\pi, 1 \rangle_H$$

where $\chi_\pi$ denotes the character of $\pi$ and the inner product on the RHS denotes the inner product of class functions on $H(k_0)$. This gives a character-theoretic way of computing the LHS, but how does this help?

We need another idea: base change to $\mathrm{GL}_n$. More precisely, consider the groups $G(V)(k_0) = \mathrm{U}_n(k_0)$ and $G(V)(k) = \mathrm{GL}_n(k)$. The Galois group $\mathrm{Gal}(k/k_0)$ acts on the latter, and one has the notion of $\sigma$-conjugacy classes on $G(V)(k)$, where $\sigma$ is the nontrivial element in $\mathrm{Gal}(k/k_0)$. It turns out that there is a natural bijection between

$$\{\text{conjugacy classes of } G(V)(k_0)\} \longleftrightarrow \{\sigma\text{-conjugacy classes of } G(V)(k)\},$$

thus inducing an isomorphism of the space of class functions and the space of $\sigma$-class functions. This isomorphism is in fact an isometry for the natural inner products on these two spaces.

In view of the above, one expects a relation between the irreducible representations of $G(V)(k_0)$ and the irreducible representations of $G(V)(k)$ which are invariant under $\mathrm{Gal}(k/k_0)$. Such $\sigma$-invariant representations $\Pi$ of $G(V)(k)$ can be extended to a representation $\tilde{\Pi}$ of $G(V)(k) \rtimes \mathrm{Gal}(k/k_0)$, and the restriction of $\chi_{\tilde{\Pi}}$ to the non-identity coset $G(V)(k) \cdot \sigma$ is a $\sigma$-class function. Now, if $\pi$ is a "sufficiently regular" representation of $G(V)(k_0)$, then there is a $\sigma$-invariant representation $\Pi$ of $G(V)(k)$ such that

$$\chi_\pi = \chi_{\tilde{\Pi}}|_{G(V)(k)\cdot\sigma}.$$

Thus, one has

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) = \langle \chi_\pi, 1 \rangle_{H(k_0)} = \langle \chi_{\pi_1}, \chi_{\pi_2}^\vee \rangle_{G(W)(k_0)} = \langle \chi_{\tilde{\Pi}_1}, \chi_{\tilde{\Pi}_2^\vee} \rangle_{G(W)(k)\cdot\sigma}.$$

Hence, one obtains the interesting identity:

$$\dim \mathrm{Hom}_{H(k)\rtimes \mathrm{Gal}(k/k_0)}(\tilde{\Pi}, \mathbb{C}) = \frac{1}{2} \cdot \dim \mathrm{Hom}_{H(k)}(\Pi, \mathbb{C}) + \frac{1}{2} \cdot \dim \mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C}).$$

In particular, if one understands the restriction problem for $\mathrm{GL}_n$ well, one can infer information about the restriction problem for unitary groups. For example, if we know that $\dim \mathrm{Hom}_{H(k)}(\Pi, \mathbb{C}) = 1$, then we deduce immediately that $\dim \mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C}) = 1$, since the LHS is equal to 0 or 1!

## 5.3 Work of Waldspurger and Beuzart-Plessis

We can now give an impressionistic sketch of the contributions of Waldspurger and Beuzart-Plessis. If $\pi$ is an irreducible representation, its character distribution $\chi_\pi$ is a locally integrable function on the regular elliptic set of $G(k_0)$. One would like to integrate this character function over $H(k_0)$ and relate this integral to $\dim \mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C})$.

The first innovation is thus to give a character-theoretic computation of the number dim $\mathrm{Hom}_{H(k_0)}(\pi, \mathbb{C})$. Of course, the naive integral above does not make sense, and one has to discover the appropriate expression. The expression discovered by Waldspurger is a sum over certain elliptic (not necessarily maximal) tori of weighted integrals involving the character $\chi_\pi$. One such torus is the trivial torus.

Now when one sums up the above expression for $\dim \mathrm{Hom}_H(\pi, \mathbb{C})$ over all relevant $\pi$'s in $\Pi_\phi$, one may exploit the character identities involved in the Jacquet-Langlands type transfer between a group and its inner forms. It turns out that the sum of terms corresponding to a fixed nontrivial torus cancels out and thus vanishes. Only the term corresponding to the trivial torus survive and this gives local GP I (multiplicity one in Vogan packet).

To obtain the more precise local GP II, one uses the character identities of twisted endoscopy to relate the problem to the analogous one on $\mathrm{GL}_n$, which one understands completely; this is similar to what was done in the toy model over finite fields, but it is decidedly more involved. In particular, to be able to detect the local epsilon factors, Waldspurger and Beuzart-Plessis needed to express the local epsilon factors (on $\mathrm{GL}_n$) in terms of certain character theoretic integrals. In this way, local GP conjecture II was deduced.

## 6 Recent Progress: Global Case

In this section, we give an account of recent results on the global GP conjecture.

### 6.1 Work of Ginzburg-Jiang-Rallis

Even before the GP conjectures were extended to all classical groups in [8], Ginzburg, Jiang and Rallis [10–12] have studied the question of the nonvanishing of the relevant global periods. In particular, they showed one direction of the global GP conjecture:

**Theorem 6.1** *Let $\pi = \pi_{\boxtimes}\pi_2$ be a cuspidal representation of $G(\mathbb{A})$ and assume that there is a weak lifting of $\pi$ to an automorphic representation $\Pi = \Pi_1 \boxtimes \Pi_2$ of $G(\mathbb{A}_E)$ (as given for example in [M] when $G$ is quasi-split over $F$). Then, in both the Bessel and Fourier-Jacobi cases, we have:*

$$\mathcal{P}_H \text{ nonzero on } \pi \Longrightarrow L_E(1/2, \pi_1 \times \pi_2) \neq 0.$$

It is not clear if their approach can be used to prove the converse direction.

### 6.2 Relative trace formula on unitary groups

The most general method of attack for the global GP conjecture in the Hermitian case (i.e. Bessel case) is a relative trace formula (RTF) which was developed by Jacquet-Rallis [20], almost concurrently as [8] was being written. We shall give a brief description of this influential approach.

Consider the action $R$ of $C_c^\infty(G(\mathbb{A}))$ on $L^2(G(F)\backslash G(\mathbb{A}))$ by right translation. For $f \in C_c^\infty(G(\mathbb{A}))$, the operator $R(f)$ is given by a kernel function

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

on $G(\mathbb{A}) \times G(\mathbb{A})$. In the usual trace formula, one is interested in computing $\mathrm{Tr}R(f)$ and when $G$ is anisotropic, this trace is computed as

$$\mathrm{Tr}R(f) = \int_{G(F)\backslash G(\mathbb{A})} K_f(x,x)\, dx.$$

When $G$ is $F$-isotropic, one needs to regularise the RHS, and this is the elaborate theory of the invariant trace formula developed by Arthur. One then develops two expression for this integral: a spectral side involving the representations appearing in the automorphic discrete spectrum, and a geometric side involving weighted orbital integrals. Thus, in principle (and also in practice), the distribution $f \mapsto \mathrm{Tr}R(f)$ contains the full information of the automorphic discrete spectrum of $G$.

In the relative trace formula, one is interested in detecting automorphic representations on which certain period integrals are nonzero. In the global GP conjecture, we are interested in the nonvanishing of the period $\mathcal{P}_H$. Thus it is reasonable to consider the integral

$$I(f) = \int_{(H(F)\backslash H(\mathbb{A}))^2} K_f(x,y)\, dx\, dy. \tag{6.1}$$

When some local conditions are placed on the test function $f = \otimes_v f_v$, the integral above actually converges, and one can try to obtain a spectral expansion of the integral, as well as a geometric one. Such a theory has been carried out by Wei Zhang [39], building upon the work of Jacquet-Rallis [20].

## 6.3 Spectral and geometric expansions

More precisely, we assume that:

- at some finite place $v_1$, $f_{v_1}$ is the matrix coefficient of a supercuspidal representation;
- for another finite place $v_2$, $f_{v_2}$ is supported on the "regular semisimple" elements (where the notion of regular semisimple is relative to the action of $H \times H$ on $G$).

Assuming $f$ is of this form, the spectral expansion of (6.1) takes the form

$$I(f) = \sum_{\pi} J_\pi(f)$$

where the sum runs over cuspidal representations of $G(\mathbb{A})$ and

$$J_\pi(f) = \sum_{\phi} \left( \int_{H(F)\backslash H(\mathbb{A})} (\pi(f)\phi)(x)\, dx \right) \cdot \left( \overline{\int_{H(F)\backslash H(\mathbb{A})} \phi(x)\, dx} \right)$$

is the Bessel distribution associated to $\pi$, with the sum running over an orthonormal basis of $\pi$.

On the other hand, one has a geometric expansion of (6.1) which is given by

$$I(f) = \sum_{\gamma} O(\gamma, f)$$

where the sum runs over "regular semisimple" $H(F) \times H(F)$-orbits on $G(F)$ and the orbital integral is given by

$$O(\gamma, f) = \int_{H(\mathbb{A}) \times H(\mathbb{A})} f(x^{-1}\gamma y)\, dx\, dy.$$

Hence one has an equality

$$\sum_{\pi} J_\pi(f) = \sum_{\gamma} O(\gamma, f).$$

Since $G(V)(\mathbb{A})$ has nontrivial centre, it is possible to incorporate a central character $\chi$ everywhere in the above discussion and consider the $\chi$-part of $K_f$ and $I(f)$; we omit the technical details here.

## 6.4 Relative trace formula on GL

As in the discussion over finite fields and local fields, we are seeking to relate the problem of periods on $G(V)(\mathbb{A})$ to the analogous problem on $G(V)(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_E) \times \mathrm{GL}_{n-1}(\mathbb{A}_E)$. Thus, we need to set up an analogous relative trace formula on $G(V)(\mathbb{A}_E)$ and compare it to the one on $G(V)(\mathbb{A})$ which we have above.

Thus for $f' = \otimes_v f'_v \in C_c^\infty(G(V)(\mathbb{A}_E))$, we have the kernel function $K_{f'}$ as above. We then consider the following integral of the kernel function:

$$I'(f') = \int_{x \in H(E)\backslash H(\mathbb{A}_E)} \int_{y \in H'(F)\backslash H'(\mathbb{A})} K_{f'}(x, y) \cdot \mu(y)\, dx,\, dy \qquad (6.2)$$

where $H' \cong \mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over $F$ and $\mu$ is an automorphic character of $H'$ given by

$$\mu = (\omega_{E/F}^{n-1} \circ \det) \otimes (\omega_{E/F}^{n} \circ \det).$$

Once again, we should have fixed a central character $\chi'$ and then consider the $\chi'$-part of $K_{f'}$ and $I'(f')$, but we will ignore this technical issue in this paper.

The period over $H(\mathbb{A}_E)$ is the analog of the Gross-Prasad period for general linear groups. On the other hand, the period over $H'(\mathbb{A})$ conjecturally detects those automorphic representations of $G(\mathbb{A}_E)$ which lies in the image of the base change from $G(\mathbb{A})$; this is apparently a conjecture of Flicker-Rallis. Thus, the distribution $I'$ is designed to capture automorphic representations of $G(\mathbb{A}_E)$ which comes from the unitary group $G(\mathbb{A})$ and at the same time supports a Gross-Prasad period.

In any case, assuming that $f'$ satisfies the analogous condition as $f$, one has an equality

$$\sum_{\Pi} I_{\Pi}(f') = \sum_{\gamma'} O_{\gamma'}(f')$$

where the sum on the LHS runs over all cuspidal representations of $G(\mathbb{A}_E)$ and

$$I_{\Pi}(f') = \sum_{\phi} \left( \int_{H(E)\backslash H(\mathbb{A}_E)} (\Pi(f')\phi)(x)\, dx \right) \cdot \left( \overline{\int_{H'(F)\backslash H'(\mathbb{A})} \phi(x) \cdot \mu(x)\, dx} \right)$$

is the analogous Bessel distribution, and the sum on the RHS runs over all "regular semisimple" elements for the action of $H(E) \times H'(F)$ on $G(E)$ and

$$O(\gamma', f') = \int_{H(\mathbb{A}_E)} f(x^{-1}\gamma' y) \cdot \mu(y)\, dx\, dy$$

is the analogous orbital integral.

## 6.5 Transfer

Now one would like to compare the distribution $I$ and $I'$ by matching the geometric expansion of $I$ and $I'$. As in the theory of the usual trace formula, one first considers whether the "regular semisimple" orbits occurring in the geometric expansion of $I$ and $I'$ are in natural bijection. This matching of "regular semisimple" orbits was done by Jacquet-Rallis [20]. Thus, one is led to consider the comparison of orbital integrals on both sides. For this, one needs to introduce a transfer factor

$$\Delta_v : G(F_v)_{rs} \times G(E_v)_{rs} \longrightarrow \mathbb{C}$$

for each place $v$ which is nonzero only on matching pairs $(\gamma, \gamma')$ of regular semisimple elements. We may thus regard $\Delta_v$ as a function on $G(E_v)_{rs}$. Then the main properties of $\Delta_v$ are:

- (equivariance) For $(x, y) \in H(E_v) \times H'(F_v)$,

$$\Delta_v(xg'y) = \mu(y) \cdot \Delta_v(g').$$

- (product formula) For $\gamma' \in G(E)$,

$$\prod_v \Delta_v(\gamma') = 1.$$

It turns out that it is not hard to write down such a function. This allows us to make the following basic definition:

**Definition** We say that $f$ and $f'$ are matching test functions if

$$\Delta_v(\gamma') \cdot O(\gamma', f') = O(\gamma, f)$$

for every pair $(\gamma, \gamma')$ of matching regular semisimple orbits.

One is thus led to the following local problems:

- (Fundamental lemma): For almost all places $v$ of $F$, let $f_0$ and $f_0'$ be the unit element in the spherical Hecke algebra of $G(F_v)$ and $G(E_v)$ respectively. Then $f_0$ and $f_0'$ are matching functions.
- (Transfer) For every $f$, one can find a matching $f'$. Conversely, for every $f'$, one can find a matching $f$.

These two statements are straightforward when $v$ is a place of $F$ which splits in $E$. Thus, the main issue is in the inert case.

## 6.6 Work of Zhiwei Yun

In [36], Zhiwei Yun has verified the fundamental lemma over local function fields, using geometric techniques analogous to those used by B.C. Ngo in his thesis (which verified the Jacquet-Ye fundamental lemma for another relative trace formula). Having this, J. Gordon has shown (in the appendix to [36]) that this implies the desired fundamental lemma over p-adic fields, when $p$ is sufficiently large. Thus, we know that the fundamental lemma for unit elements hold.

## 6.7 Work of Wei Zhang

In a recent breakthrough paper [38], Wei Zhang has shown that (Transfer) also holds at any $p$-adic place. Let us give an impressionistic sketch of the main steps of the proof.

- Using a Cayley transform argument, one reduces the existence of transfer on the group level to the existence of an analogous transfer on the level of Lie algebras;
- By a local characterisation of orbital integrals, one is reduced to showing the existence of transfer in a neighbourhood of each semisimple (not necessarily regular) element;
- Using the theory of generalised Harish-Chandra descent, as developed by Aizenbud-Gourevitch [2], one is reduced to showing the existence of local transfer in a neighbourhood of the zero element (of another space);
- One shows that, modulo test functions which are killed by the infinitesimal version of $I$, the space of test functions is generated by those functions supported off the null cone and their various partial Fourier transforms. This requires an uncertainty principle type result of Aizenbud-Gourevitch on the nonexistence of certain equivariant distributions whose various Fourier transforms (including the trivial transform) are supported in the null cone.
- For test functions supported off the null cone, transfer is shown inductively.

- Finally, one shows that if $f$ and $f'$ are matching functions, then their partial Fourier transforms are also matching.

As a consequence of the fundamental lemma and the existence of transfer, one can compare the two RTF's and obtain an identity:

$$\sum_{\pi} J_{\pi}(f) = \sum_{\Pi} I_{\Pi}(f')$$

for matching test functions $f$ and $f'$ satisfying some local conditions as specified above. Using this, Wei Zhang showed in [38] the global GP conjecture subject to certain local conditions. This is the strongest result on the global GP conjecture known to date.

**Theorem 6.4** *Let $\pi = \pi_1 \boxtimes \pi_2$ be a cuspidal representation of $G = G(V) \times G(W)$. Suppose that*

- *all infinite places of $F$ are split in $E$;*
- *for two finite places of $F$ split in $E$, $\pi_v$ is supercuspidal.*

*Then the following are equivalent:*
(i) $L_E(1/2, \pi_1 \times \pi_2) \neq 0$.
(ii) *For some relevant pure inner form $W' \subset V'$, and some cuspidal representation $\pi'$ of $G' = G(V') \times G(W')$ which is nearly equivalent to $\pi$ (i.e. belongs to the same global L-packet as $\pi$), the period integral $\mathcal{P}_H$ is nonzero on $\pi'$.*

Indeed, in view of the local GP, the pure inner form $G'$ and its representation $\pi'$ is uniquely determined, since there is a unique relevant representation in each local L-packet which could support a nonzero abstract $H$-invariant functional.

In a sequel [39], Wei Zhang was able to refine the argument to derive the refined GP conjecture, as formulated by N. Harris [15].

**Theorem 6.5** *Let $\pi = \pi_1 \boxtimes \pi_2$ be a tempered cuspidal representation of $G = G(V) \times G(W)$. Suppose that*

- *all infinite places of $F$ are split in $E$;*
- *for at least one finite place of $F$ split in $E$, $\pi_v$ is supercuspidal;*
- *for every place $v$ of $F$ inert in $E$ such that $\pi_v$ is not unramified, either $H(F_v)$ is compact or $\pi_v$ is supercuspidal.*

*Then the refined GP conjecture holds for $\pi$, with $|S_{\pi}| = 4$.*

### 6.8 Work of Yifeng Liu and Hang Xue

We have devoted considerable attention to the Bessel case of the global GP. Let us consider the Fourier-Jacobi case now. Following the work of Jacquet-Rallis [20], Yifeng Liu [23] has developed an analogous relative trace formula in the skew-Hermitian case, which compares the Fourier-Jacobi period on unitary groups with the analogous period on general linear groups. He also showed that the fundamental lemma in this case can be reduced to that in the Bessel case.

Building upon this, Hang Xue [34] has recently achieved in his thesis work the analog of [38] for the Fourier-Jacobi case, thus establishing the global GP conjecture with some local conditions in the skew-Hermitian case. In a recent preprint [35], Xue has formulated the refined GP conjecture in the Fourier-Jacobi case and then verified it subject again to some local conditions, analogous to what was done in [39].

## 7 Outstanding Questions

After this lengthy discussion of recent progresses, it is perhaps time to take stock of the remaining questions concerning the GP conjecture. Here are some problems which come to mind:

(i) **Archimedean case:** Almost nothing is known about the local GP conjecture in the archimedean case. The methods developed by Waldspurger is character theoretic in nature and should apply in the archimedean case too, at least in principle. One naturally expects that there may be greater analytic difficulties in the archimedean case, but these should be regarded as more of a technical, rather than a fundamental, nature. Thus, it is reasonable to expect that someone with a strong analytic background could work through the proof of Waldspurger and Beuzart-Plessis and adapt the arguments to the archimedean case. Beuzart-Plessis has informed the author that he is currently working on this project.

(ii) **Local Fourier-Jacobi case.** What about the Fourier-Jacobi case of the local GP conjecture in the p-adic case? As explained in [6], the Bessel and Fourier-Jacobi periods are connected by the local theta correspondence, and if one knows enough about the local theta correspondence, the Fourier-Jacobi case of the local GP will also follow. In a recently completed preprint [7], A. Ichino and the author has established this link and thus complete the proof of the Fourier-Jacobi case of local GP in the p-adic case. The archimedean case still remains open.

(iii) **Global conjecture for orthogonal groups.** The Jacquet-Rallis trace formula has been very successful for the Hermitian and skew-Hermitian case, but for the orthogonal groups, which is the original case of the GP conjecture, one still does not have a strategy which works in all cases. It will be very interesting to have a new approach. The main difficulty in formulating a relative trace formula which compares the orthogonal groups and the general linear groups is that there is no convenient characterisation, in terms of periods, of automorphic representations of $\mathrm{GL}(n)$ which are lifted from even orthogonal groups.

(iv) **Arithmetic Case.** If $\epsilon_E(1/2, \pi_1 \times \pi_2) = -1$, the unique representation in the global L-packet of $\pi$ which supports an abstract $H(\mathbb{A})$-invariant period lives on a group $\prod_v G(V_v')$ which is incoherent, i.e. there is no Hermitian space over $E$ whose localisation agrees with $\{V_v'\}$. In this case, $L_E(1/2, \pi_1 \times \pi_2) = 0$ and the period integral $\mathcal{P}_H$ is automatically zero on each automorphic $\pi'$ in the global L-packet of $\pi$.

It turns out that something even more interesting happens in this case. Namely, under some conditions, one may construct another "period integral" on $\pi$ coming from a height pairing on a certain Shimura variety associated to $G$, and the nonvanishing of this arithmetic period integral is then governed by the nonvanishing of the central derivative $L_E'(1/2, \pi_1 \times \pi_2)$. The precise conjecture is given in [8, §27].

Now as in the usual GP conjecture, one expects a refinement of this arithmetic GP conjecture in the form of an exact formula relating the arithmetic period integral to the locally defined functional $I$ in the refined GP. Such a refinement has been proposed by Wei Zhang. For the group $\mathrm{U}(2) \times \mathrm{U}(1)$, it specializes to the generalised Gross–Zagier formula, which was shown in the recent book [37] of Yuan-Zhang-Zhang in the parallel weight two case.

More amazingly, Wei Zhang has suggested that the relative trace formula of Jacquet-Rallis could be used to attack this refined arithmetic GP conjecture. This new application of the relative trace formula is an extremely exciting development, and it remains to be seen if it can be carried out in this case, and perhaps even in other scenarios.

**Acknowledgements** This paper is a written account of my lecture delivered at the annual meeting of the Vietnam Institute for Advanced Study in Mathematics held on July 20-21, 2013. I thank Bao Chau Ngo for his invitation to deliver a lecture and the local organisers, especially Professor Ho Hai Phung, for their warm hospitality and travel support.

## References

1. J. Arthur, J.: The endoscopic classification of representations: orthogonal and symplectic groups. To appear as a Colloquium Publication of the American Mathematical Society (2013)
2. Aizenbud, A., Gourevitch, D.: Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem. With an appendix by the authors and Eitan Sayag, Duke Math. J. **149**(3), 509-567 (2009)
3. Aizenbud, A., Gourevitch, D., Rallis, S., Schiffmann, G.: Multiplicity One Theorems. Ann. of Math. (2) **172**(2), 1407- 1434 (2010)
4. Beuzart-Plessis, R.: La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires. Preprint (2012)
5. Deligne, P.: Les constantes des équations fonctionnelles des fonctions $L$, Modular Functions on one variable, II, pp 501-597, Lecture Notes in Mathematics, vol. 349, Springer, Berlin 1973
6. Gan, W.T.: Representations of metaplectic groups. Fifth International Congress of Chinese Mathematicians. Part 1, 2, 155-170, AMS/IP Stud. Adv. Math., 51, pt. 1, 2, Amer. Math. Soc., Providence, RI (2012)
7. Gan, W.T., Ichino, A.: The Gross-Prasad conjecture and local theta correspondence. Preprint.
8. Gan, W.T., Gross, B.H., Prasad, D.: Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups. Asterisque **346**, 1-110 (2012)
9. Gan, W.T., Gross, B.H., Prasad, D.: Restrictions of representations of classical groups: examples. Asterisque **346**, 111-170 (2012)
10. Ginzburg, D. Jiang, D., Rallis, S.: On the nonvanishing of the central value of the Rankin-Selberg $L$-functions. J. Amer. Math. Soc. **17**(3), 679–722 (2004)
11. Ginzburg, D., Jiang, D., Rallis, S.: On the nonvanishing of the central value of the Rankin-Selberg $L$-functions, II. Automorphic Representations, $L$-functions and Applications: Progress and Prospects, 157-191, Ohio State Univ. Math. Res. Inst. Publ. 11, de Gruyter, Berlin (2005)
12. Ginzburg, D., Jiang, D., Rallis, S.: Models for certain residual representations of unitary groups. Automorphic forms and L-functions I. Global aspects, 125-146, Contemp. Math., 488, Amer. Math. Soc., Providence, RI (2009)
13. Gross, B., Prasad, D.: On the decomposition of a representation of $\mathrm{SO}_n$ when restricted to $\mathrm{SO}_{n-1}$. Canad. J. Math. **44**(5), 974-1002 (1992)
14. Gross, B., Prasad, D.: On irreducible representations of $\mathrm{SO}_{2n+1}\times\mathrm{SO}_{2m}$. Canad. J. Math. **46**(5), 930-950 (1994)
15. Harris, N.: The Refined Gross-Prasad Conjecture for Unitary Groups. To appear in IMRN, available at http://arxiv.org/abs/1201.0518
16. Harris, M., Taylor, R.: The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich, Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ (2001)
17. Henniart, G.: Caractérisation de la correspondance de Langlands locale par les facteurs ? de paires. Invent. Math. **113**(2), 339350 (1993)
18. Henniart, G.: Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps $p$-adique. Invent. Math. **139**(2), 439–455 (2010)
19. Ichino, A., Ikeda, T.: On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. Geometric and Functional Analysis **19**(5), 1378-1425 (2010)
20. Jacquet, H., Rallis, S.: On the Gross-Prasad conjecture for unitary groups. On certain L-functions, 205-264, Clay Math. Proc., 13, Amer. Math. Soc., Providence, RI (2011)
21. Kaletha, T.: Rigid inner forms of real and p-adic groups. Preprint, available at http://arxiv.org/abs/1304.3292
22. Kottwitz, R., Shelstad, D.: Foundations of twisted endoscopy. Astérisque **255** (1999)
23. Liu, Y.F.: Relative trace formulae toward Bessel and Fourier-Jacobi periods of unitary groups. Preprint (2010), available at http://math.mit.edu/ liuyf/Bessel.pdf
24. Mok, C.P.: Endoscopic classification of representations of quasi-split unitary groups. To appear in Memoirs of AMS

25. Moeglin, C., Waldspurger, J.-L.: La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général, Asterisuqe **347** (2012)
26. Binyong Sun: Multiplicity one theorems for Fourier-Jacobi models. Amer. J. Math. **134**(6), 1655-1678 (2012)
27. Binyong Sun, Chen-Bo Zhu: Multiplicity one theorems: the Archimdean case. Ann. of Math. (2) **175**(1), 23-44 (2012)
28. Vogan, D.: The local Langlands conjecture. Contemporary Maths, AMS **145**, 305-379 (1993)
29. WaldspurgerJ.-L.: Une Formule intégrale reliée à la conjecture locale de Gross-Prasad. Compositio Math. **146**, 1180-1290 (2010)
30. Waldspurger, J.-L.: Une Formule intégrale reliée à la conjecture locale de Gross-Prasad, 2ème partie: extension aux représentations tempérées. Asterisque **346**, 171-311
31. Waldspurger, J.-L.: Calcul d'une valeur d'un facteur epsilon par une formule intégrale. Asterisque **347** (2012)
32. Waldspurger, J.-L.: La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spciaux orthogonaux. Asterisque **347** (2012)
33. Waldspurger, J.-L.: Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann. Asterisuqe **346**, 313-318 (2012)
34. Xue, H.: The Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$. Preprint (2012), available at http://math.columbia.edu/ xuehang/ggp.pdf
35. Xue, H.: Fourier–Jacobi periods and the central value of Rankin–Selberg L-function. Preprint (2013)
36. Yun, Z.W.: The fundamental lemma of Jacquet and Rallis. With an appendix by Julia Gordon, Duke Math. J. **156**(2), 167-27 (2011)
37. Yuan, X., Zhang, S.W., Zhang, W.: Heights of CM points I: Gross-Zagier formula. Preprint, available at *http://www.math.columbia.edu/ szhang/papers/HCMI.pdf*
38. Zhang, W.: Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups. To appear in Annals of Math, available at http://www.math.columbia.edu/ wzhang/math/online/transfer.pdf
39. Zhang, W.: Automorphic period and the central value of Rankin-Selberg L-function. To appear in J. Amer. Math. Soc., available at http://arxiv.org/abs/1208.6280