Theta Correspondence: Recent Progress and Applications

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Abstract. We describe some recent progress in the theory of theta correspondence over both local and global fields. We also discuss applications of these recent developments to the local Langlands conjecture, the Gross-Prasad conjecture and the theory of automorphic forms for the metaplectic groups.

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1. History

In this paper, we shall report on some recent progress in the theory of theta correspondence, as well as some applications to number theory and representation theory. The use of theta correspondence has a long history, but its status as a theory was formally initiated by R. Howe in the influential paper [28] written in the 1970's but only published much later. This built upon the work of A. Weil [91] in the 1960's which provided a representation theoretic treatment of theta functions via his construction of the so-called Weil representations. In this introduction, we give a brief account of the historical development since the late 1970's; we apologise for omitting the contributions of many people.

As a theory, theta correspondence has its own share of internal problems which needed to be addressed, but from the onset, it was perceived mainly as a tool for constructing representations and automorphic forms. In particular, it gives natural constructions of certain instances of Langlands functorial lifting. In this vein, one of the first successes is Waldspurger's complete and elegant description [84, 85] of the Shimura correspondence between cuspidal representations of PGL_2 and the metaplectic double cover Mp_2. Another is Howe and Piatetski Shapiro's construction [29] of nontempered cuspidal representations on U_3 and Sp_4, contradicting the naive Ramanujan-Petersson conjecture. Such construction of nontempered cuspidal representations was later extended by J.S. Li [49, 50, 51] and C. Moeglin [61] to the general setting, resulting in the construction of many interesting examples of unitary representations and square-integrable automorphic forms.

The 1980's saw many key developments in the theory of theta correspondence. Firstly, motivated by Waldspurger's work, Rallis initiated a program [76, 77, 78] aimed at determining the cuspidality and nonvanishing of global theta liftings. This led to two important series of work. One is the work of Piatetski-Shapiro and Rallis [73] on the doubling zeta integral, which is a Rankin-Selberg integral representing the standard L-function of classical groups. Another is the work of Kudla and Rallis [41, 42] on the Siegel-Weil formula, culminating in their paper [43]. Combining these two series of work resulted in the Rallis inner product formula in some instances. This characterises the nonvanishing of global theta liftings in terms of the analytic properties of the standard L-functions. In the course of their work, Kudla and Rallis were led to a local conjecture [44] about the nonvanishing of the local theta correspondence. They made significant progress towards this so-called conservation relation conjecture, proving it in many cases.

Secondly, starting with Kudla's paper [38], the local theta correspondence over p-adic fields was systematically investigated, culminating in Waldspurger's proof of the so-called Howe duality conjecture when $p \neq 2$ [86]. This Howe duality conjecture was shown by Howe himself [28] in the archimedean case, and for unramified groups in the nonarchimedean case [63]. Following this, significant understanding of the archimedean theta correspondence was obtained in the work of Adams-Barbasch [2, 3, 4] and Moeglin [60]. In particular, Adams-Barbasch determined the local theta correspondence over $\mathbb{C}$ completely [3], and also extended the local Shimura-Waldspurger correspondence from $\mathrm{Mp}_2(\mathbb{R})$ to $\mathrm{Mp}_{2n}(\mathbb{R})$ for general n [4]. In the p-adic case, the analogous results were conjectured but left open. In particular, Adams [1] formulated a conjecture on the functoriality of the theta correspondence in the language of A-packets, and D. Prasad [74, 75] formulated some precise conjectures describing the local theta correspondence in the (almost) equal rank case in terms of the local Langlands correspondence.

Since the mid-1990's, significant work continued to be done in classical theta correspondence, such as by Robetrs [79], Moeglin [62], Muić [66, 67, 68, 69], Muić-Savin [70], S.Y. Pan [71] and Ginzburg-Jiang-Soudry [21]. However, as many of the early pioneers turned their attention to other worthy endeavours, the field became relatively quiet compared with the flurry of activities in the 80's and early 90's and many of the problems highlighted some twenty years ago lie dormant and unresolved.

It was not until about 6 or 7 years ago that a new generation of researchers revisited these problems and it is a pleasure and privilege to report on the recent resolution of many of these problems here. This brings a certain degree of closure to the developments from 30 years ago, but we shall also highlight some exciting future directions.

2. Theta Correspondence

In this section, we describe the basic setup and questions in the theory of theta correspondence.

2.1. Dual Pairs. For simplicity, let F be a field of characteristic 0, and let $E = F$ or an étale quadratic F-algebra, with $\mathrm{Aut}(E/F) = \langle c \rangle$. With $\epsilon = \pm$, let V be a finite-dimensional ϵ-Hermitian space over E and W an $-\epsilon$-Hermitian space. Then $V \otimes_E W$ inherits a natural symplectic form over F and one has a natural map of isometry groups

$$\mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes_E W).$$

The images of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ are mutual commutants of each other, and such a pair of groups is called a *reductive dual pair*.

For ease of exposition, we shall henceforth focus on the case when E/F is a quadratic field extension, V is Hermitian and W is skew-Hermitian, so that $\mathrm{U}(V)$ and $\mathrm{U}(W)$ are unitary groups.

2.2. Invariants of spaces. The spaces V and W have a natural invariant known as the discriminant:

$$\mathrm{disc}V \in F^\times/\mathrm{N}_{E/F}(E^\times) \quad \text{and} \quad \mathrm{discW} \in \delta^{\dim W} \cdot F^\times/\mathrm{N}_{E/F}(E^\times)$$

where δ is a fixed trace zero element in $E^\times$. When F is a local field, it is convenient to encode the discriminant in a sign $\pm$:

$$\epsilon(V) = \omega_{E/F}(\mathrm{disc}V) \quad \text{and} \quad \epsilon(W) = \omega_{E/F}(\delta^{-\dim W} \cdot \mathrm{disc}W)$$

where $\omega_{E/F}$ is the nontrivial quadratic character of $F^\times/N_{E/F}(E^\times)$. Note that $\epsilon(W)$ depends on the choice of δ. Moreover, if F is nonarchimedean, Hermitian spaces are classified by $\dim(V)$ and $\epsilon(V)$; likewise for skew-Hermitian spaces.

2.3. Weil representation. Assume that F is a local field. The symplectic group $\mathrm{Sp}(V \otimes_E W)$ has a nonlinear S^1-cover $\mathrm{Mp}(V \otimes_E W)$ known as the metaplectic group. This metaplectic group has a distinguished representation ω_ψ depending on a nontrivial additive character ψ of F. If the embedding i can be lifted to a homomorphism

$$\tilde{i} : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Mp}(V \otimes_E W),$$

then we obtain a representation $\omega_\psi \circ \tilde{i}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$.

For the case of unitary groups considered here, a splitting can be specified [39] by picking two characters χ_V and χ_W of $E^\times$ such that

$$\chi_V|_{F^\times} = \omega_{E/F}^{\dim V} \quad \text{and} \quad \chi_W|_{F^\times} = \omega_{E/F}^{\dim W}.$$

Thus, $\mathrm{U}(V) \times \mathrm{U}(W)$ has a Weil representation $\omega_{V,W,\psi}$ that depends on ψ and the splitting data (χ_V, χ_W), which we shall suppress from the notation.

2.4. Local theta correspondonce. We will write $\mathrm{Irr}(\mathrm{U}(W))$ for the set of equivalence classes of irreducible smooth representations of $\mathrm{U}(W)$. For $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, one considers the maximal π-isotypic quotient of $\omega_{V,W,\psi}$:

$$\omega_{V,W,\psi} \twoheadrightarrow \pi \boxtimes \Theta(\pi)$$

where $\Theta(\pi)$ is some smooth representation of $\mathrm{U}(V)$. We shall denote by $\theta(\pi)$ the maximal semisimple quotient of $\Theta(\pi)$. The goal of local theta correspondence is to determine the representations $\Theta(\pi)$ and $\theta(\pi)$ as much as possible. For example, the *Howe duality conjecture* states that if $\Theta(\pi)$ is nonzero, then it has a unique irreducible quotient, i.e. $\theta(\pi)$ is irreducible.

Note that the case when $E = F \times F$ is also necessary for global applications. In this case, the dual pair is $\mathrm{GL}_m \times \mathrm{GL}_n$. The study of this local theta correspondence is essentially completed in the paper [58] of A. Minguez. For example, the Howe duality conjecture was completely resolved. Hence we shall say no more about this case in this paper.

2.5. Theta functions. We turn now to the global setting. Thus, let k be a global field with ring of adeles $\mathbb{A}$, and let K/k be a quadratic field extension. Let V and W be a Hermitian and skew-Hermitian space over K, and fix a pair of Hecke characters χ_V and χ_W of $\mathbb{A}_K^\times$ as before. For a nontrivial additive character $\psi = \otimes_v \psi_v$ of $k\backslash\mathbb{A}$, the adelic group $\mathrm{U}(V_\mathbb{A}) \times \mathrm{U}(W_\mathbb{A})$ possesses an abstract Weil representation $\omega_{V,W,\psi} = \otimes_v \omega_{V,W,\psi_v}$. It was shown by Weil that there is a natural equivariant map

$$\theta : \omega_{V,W,\psi} \longrightarrow \mathcal{A}(\mathrm{U}(V) \times \mathrm{U}(W)),$$

where the latter space denotes the space of automorphic forms on the dual pair. This map, called the "formation of theta functions", gives an automorphic realisation of $\omega_{V,W,\psi}$.

2.6. Global theta correspondence. One may use the functions $\theta(\phi)$ for $\phi \in \omega_{V,W,\psi}$ as kernel functions for the transfer of automorphic forms from $\mathrm{U}(W_\mathbb{A})$ to $\mathrm{U}(V_\mathbb{A})$. More precisely, if $f \in \mathcal{A}(\mathrm{U}(W))$, we set

$$\theta(\phi, f)(g) = \int_{\mathrm{U}(W_k)\backslash \mathrm{U}(W_\mathbb{A})} \theta(\phi)(gh) \cdot \overline{f(h)}\, dh,$$

where dh stands for the Tamagawa measure. This integral converges if f is a cusp form. Thus, if $\pi \subset \mathcal{A}(\mathrm{U}(W))$ is a cuspidal representation, then we obtain an equivariant map

$$\theta : \omega_{V,W,\psi} \otimes \overline{\pi} \longrightarrow \mathcal{A}(\mathrm{U}(V)).$$

The image of this map is denoted by $\Theta(\pi)$ and is called the global theta lift of π. Observe that one has, by definition, an equivariant map

$$\omega_{V,W,\psi} \twoheadrightarrow \pi \boxtimes \Theta(\pi).$$

The basic questions in global theta correspondence are whether the representation $\Theta(\pi)$ is cuspidal and whether it is nonzero. Note that if $\Theta(\pi)$ is nonzero and cuspidal, then it is semisimple, in which case $\Theta(\pi)$ is a quotient of the abstract representation $\otimes_v \Theta(\pi_v)$. Thus, if the Howe duality conjecture holds, then $\Theta(\pi) \cong \otimes_v \theta(\pi_v)$. In particular, the question of "what is $\Theta(\pi)$?" is essentially a local one.

3. Local Developments

In this section, we discuss some recent developments concerning the local theta correspondence over a nonarchimedean local field F of residual characteristic $p > 0$. The following theorem, known since 1990, summarizes some basic results of Howe, Kudla, Mœglin-Vignéras-Waldspurger and Waldspurger (see [28], [38], [63], [86]).

Theorem 3.1. (i) *For any $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, the representation $\Theta(\pi)$ is either zero or of finite length.*

(ii) *If π is supercuspidal, then $\Theta(\pi)$ is either zero or irreducible (and thus is equal to $\theta(\pi)$). Moreover, for any irreducible supercuspidal π and π',*

$$\Theta(\pi) \cong \Theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

(iii) *If $p \neq 2$, the Howe duality conjecture holds.*

3.1. Local Problems.

The main remaining problems in local theta correspondence are thus:

(a) Establish the Howe duality conjecture when the residual characteristic of F is $p = 2$.

(b) Determine when $\Theta(\pi)$ is nonzero, in terms of some basic invariants of π.

(c) When $\Theta(\pi)$ is nonzero, understand the representation $\theta(\pi)$ as much as possible, either by computing some of its invariants or determining it precisely, such as in terms of the local Langlands correspondence.

For (a), we have the following two results. The first is due to Li-Sun-Tian [52] whereas the second is a recent result of the author and S. Takeda [19]:

Theorem 3.2. *The representation $\theta(\pi)$ is multiplicity-free.*

Theorem 3.3. *(i) If π is tempered, then $\theta(\pi)$ is either zero or irreducible. Moreover, for any irreducible tempered π and π',*

$$\theta(\pi) \cong \theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

(ii) If $|\dim V - \dim W| \leq 1$, the Howe duality conjecture holds (for any residue characteristic p).

In the rest of this section, we shall focus on problems (b) and (c).

3.2. Witt towers and first occurrence. Rallis observed that it is fruitful to consider theta correspondence in a family. Let V_0 be an anisotropic Hermitian space over E, and for $r \geq 0$, let

$$V_r = V_0 \oplus \mathbb{H}^r$$

where $\mathbb{H}$ is the hyperbolic plane. The collection $\{V_r \mid r \geq 0\}$ is called a Witt tower of spaces. We note that any given space V is a member of a unique Witt tower of spaces $\{V_r\}$, where V_0 is the anisotropic kernel of V.

One can then consider a family of theta correspondences associated to the tower of reductive dual pairs $(\mathrm{U}(W), \mathrm{U}(V_r))$. For $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, the smallest non-negative integer r_0 such that $\Theta_{V_{r_0},W}(\pi) \neq 0$ is called the *first occurrence index* of π for the Witt tower $\{V_r\}$. By [63, p. 67], such an r_0 exists and $r_0 \leq \dim W$. Moreover, $\Theta_{V_r,W}(\pi) \neq 0$ for all $r \geq r_0$. Thus one way to rephrase problem (b) is to determine the first occurrence index of π in any given Witt tower.

3.3. Conservation relation. Harris-Kudla-Sweet [27] and Kudla-Rallis [44] discovered that the first occurrence indices of π for two different Witt towers $\{V_r\}$ and $\{V_r'\}$ are not independent of each other. More precisely, one considers two Witt towers $\{V_r\}$ and $\{V_r'\}$ such that

$$\dim V_0 \equiv \dim V_0' \mod 2 \quad \text{but} \quad \epsilon(V_0) \neq \epsilon(V_0').$$

Hence we may consider the first occurrence indices r_0 for the tower $\{V_r\}$ and r_0' for the tower $\{V_r'\}$. The following is a basic theorem in the subject:

Theorem 3.4. *For any $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, with first occurrence indices r_0 and r_0' in two related Witt towers, we have*

$$\dim V_{r_0} + \dim V'_{r_0'} = 2\dim W + 2.$$

This theorem was called the conservation relation conjecture of Kudla-Rallis. Kudla-Rallis [44] and Gong-Grenie [26] showed the inequality $\geq$ in the statement of the theorem, and also established the reverse inequality $\leq$ in many cases, for example for all supercuspidal representations. A simple and completely different proof of the theorem in the supercuspidal case was also discovered by A. Minguez [59]. Finally, a recent paper of Sun-Zhu [82] established the theorem in full. A corollary is the following dichotomy statement:

Corollary 3.5. *Let V and V' be two spaces in the Witt towers $\{V_r\}$ and $\{V_r'\}$ such that*

$$\dim V + \dim V' = 2 \cdot \dim W.$$

For any $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, exactly one of the theta lifts $\Theta_{V,W}(\pi)$ and $\Theta_{V',W}(\pi)$ is nonzero.

These results place some constraints on the first occurrence indices but fall short of determining these indices. To go further, we need to introduce some basic invariants of π.

3.4. Local doubling zeta integral. The proof of Theorem 3.4 uses as a key tool the local doubling zeta integral, which was discovered by Piatetski-Shapiro and Rallis [73]. Analogous to the local zeta integral in Tate's thesis, the doubling zeta integral can be used to define the standard γ-factors for a pair (π, χ), with $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ and χ a character of $E^\times$. Though this family of zeta integrals was discovered in the mid-1980's, the precise treatment and definition of the local factor $\gamma(s, \pi, \chi, \psi)$ was only carried out by Lapid-Rallis [48] in 2003. From the γ-factors, one can then define the local L-factor $L(s, \pi, \chi)$ and the local ϵ-factor $\epsilon(s, \pi, \chi, \psi)$ following a standard procedure of Shahidi.

However, there is another way to define $L(s, \pi, \chi)$ and $\epsilon(s, \pi, \chi, \psi)$ from a family of zeta integrals: one could define $L(s, \pi, \chi)$ as the GCD of the family of zeta integrals as the data varies. The two ways of defining these local L-factors have complementary strengths, and one would really like them to give the same L-factors and ϵ-factors. This is finally proved in a recent paper [95] of S. Yamana, thus bringing the theory of the doubling zeta integral to a definitive conclusion.

3.5. Epsilon dichotomy. The local factors defined by the doubling zeta integral are very useful for the study of theta correspondence. As an example, in the context of Corollary 3.5, the following result [27, 14] determines exactly which of $\Theta_{V,W}(\pi)$ and $\Theta_{V',W}(\pi)$ is nonzero in the equal rank case.

Theorem 3.6. *Assume that* $\dim V = \dim W$. *Let* $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ *with central character* ω_π. *Then* $\Theta_{V,W}(\pi) \neq 0$ *if and only if*

$$\epsilon(\tfrac{1}{2}, \pi, \chi_V^{-1}, \psi) = \omega_\pi(-1) \cdot \chi_V(\delta)^{\dim W} \cdot \epsilon(V) \cdot \epsilon(W).$$

3.6. Poles of local γ-factors. As another example, the location of poles of the local γ-factors provides information on the first occurrence index.

Theorem 3.7. *Suppose that V and V' are two spaces in two related Witt towers such that* $\dim V = \dim V'$. *Assume that* $l := \dim W - \dim V > 0$. *Let π be an irreducible tempered representation of* $\mathrm{U}(W)$.

(i) *If one of $\Theta_{V,W}(\pi)$ and $\Theta_{V',W}(\pi)$ is nonzero, then $\gamma(s, \pi, \chi_V^{-1}, \psi)$ has a pole at* $s = \frac{l+1}{2}$.

(ii) *Suppose that either π is supercuspidal, or $l = 1$ and π is square integrable. Then the converse of* (i) *also holds.*

Corollary 3.8. *Let π be an irreducible tempered representation of* $\mathrm{U}(W)$. *Assume that $\gamma(s, \pi, \chi_V^{-1}, \psi)$ is holomorphic in* $\mathrm{Re}(s) \geq 1/2$. *Then we have: the first occurrence indices of π in the two Witt towers are given by:*

$$\begin{cases} \dim V_{r_0} = \dim V_{r_0'} = \dim W + 1 & \text{if } \dim W \neq \dim V_0 \mod 2, \\ \{\dim V_{r_0}, \dim V_{r_0'}\} = \{\dim W, \dim W + 2\} & \text{if } \dim W = \dim V_0 \mod 2. \end{cases}$$

Moreover, if $\dim V$ is the smaller of the two elements in the second case, then $\epsilon(V)$ is determined by Theorem 3.6.

Thus, one has a precise determination of the first occurrence indices in the tempered case when the relevant local γ-factor is entire in $\mathrm{Re}(s) \geq 1/2$. If π is supercuspidal, we shall see in a moment that one can determine the first occurrence indices precisely in general.

3.7. Prasad's conjecture. Consider the (almost) equal rank case when $\dim V - \dim W = 0$ or 1. For $\pi \in \mathrm{Irr}(\mathrm{U}(W))$, D. Prasad [74, 75] has given precise conjectures describing $\Theta_{V,W}(\pi)$ in terms of the local Langlands correspondence (LLC). We briefly recall the statement of the LLC.

The LLC for unitary groups postulates that each $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ is classified by two invariants (ϕ, η):

(a) $\phi = WD_E \longrightarrow \mathrm{GL}_n(\mathbb{C})$ (where WD_E is the Weil-Deligne group of E and $n = \dim W$) is a conjugate-dual representation of WD_E of sign $(-1)^{n-1}$ (see [13]);

(b) η is a collection of signs. Namely, if we decompose $\phi = \oplus_i m_i \phi_i$, with ϕ_i irreducible and I_ϕ denotes the set of indices such that ϕ_i is also conjugate-dual of sign $(-1)^{n-1}$, then $\eta = (\eta_i)$ is a collection of signs indexed by I_ϕ, satisfying

$$\epsilon(W) = \prod_{i \in I_\phi} \eta_i^{m_i}. \tag{1}$$

The LLC for quasi-split unitary groups has been proved by C. P. Mok [65], following Arthur's book [6] for the symplectic and orthogonal groups. The case of non-quasi-split unitary groups is the ongoing work of several people.

If $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ has L-parameter (ϕ, η), then Prasad's conjecture determines the L-parameter $(\theta(\phi), \theta(\eta))$ of $\theta(\pi) \in \mathrm{Irr}(\mathrm{U}(V))$ when it is nonzero.

Theorem 3.9. *Suppose that* $\dim V - \dim W = 0$ *or* 1. *Let* $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ *and consider* $\theta(\pi)$ *on* $\mathrm{U}(V)$. *Then we have:*

(i) If $\dim V = \dim W$, *then* $\theta(\pi)$ *is nonzero when the condition in Theorem 3.6 holds. Moreover,* $\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_W$, *so that* $I_\phi = I_{\theta(\phi)}$ *and*

$$\theta(\eta)_i/\eta_i = \epsilon(1/2, \phi_i \otimes \chi_V^{-1}, \psi(2 \cdot \mathrm{Tr}_{E/F}-)).$$

(ii) If $\dim V = \dim W + 1$, *then set* $\theta(\phi) = (\phi \otimes \chi_V^{-1}\chi_W) \oplus \chi_W$, *so that* $\#I_{\theta(\phi)} = \#I_\phi$ *or* $\#I_\phi + 1$ *depending on whether* ϕ *contains* χ_V *as a summand or not.*

(a) if ϕ *does not contain* χ_V, *then* $\theta(\pi) \neq 0$. *Its L-parameter is given by the* $\theta(\phi)$ *defined above, and*

$$\theta(\eta)_i = \eta_i \quad \text{for all } i \in I_\phi.$$

The extra sign in $\theta(\eta)$ *is associated to* χ_V *and is determined by the analog of the requirement (1) for the space* V, *so that*

$$\theta(\eta)_{\chi_V} = \epsilon(W) \cdot \epsilon(V).$$

(b) if ϕ contains χ_V (so that χ_V contributes to $I(\phi)$), then $\theta(\pi)$ is nonzero if and only if $\epsilon(V) = \epsilon(W) \cdot \eta_{\chi_V}$. Its L-parameter is given by $\theta(\phi)$ defined above and

$$\theta(\eta)_i = \eta_i \quad \textit{for all } i \in I_\phi.$$

This resolves problem (c) completely in the (almost) equal rank case, and is shown in a recent preprint [15] of the author with Ichino.

3.8. First occurrence of supercuspidals. Putting the above results together, we can now determine the first occurrence of a supercuspidal representation $\pi \in \mathrm{Irr}(\mathrm{U}(W))$ in terms of some basic invariants of π. Set

$$\kappa = \begin{cases} 0, \text{ if } \dim W \neq \dim V_0 \mod 2; \\ 1/2, \text{ if } \dim W = \dim V_0 \mod 2, \end{cases}$$

and let l_0 be defined by:

$$\frac{l_0 + 1}{2} := \max\left(\{\kappa\} \cup \{s_0 : \gamma(s, \pi, \chi_V^{-1}, \psi) \text{ has a pole at } s = s_0\}\right).$$

Then it is known that l_0 is an integer of the same parity as $\dim W - \dim V_0$ and $-1 \leq l_0 \leq \dim W$. Moreover,

$$\{\dim V_{r_0}, \dim V'_{r_0}\} = \{\dim W - l_0, \, \dim W + 2 + l_0\}.$$

If $l_0 = 0$ or -1, then the first occurrence indices were already given in Corollary 3.8. If $l_0 > 0$, then $\dim W + 2 + l_0 > \dim W - l_0$, and so we need to determine which of $\dim V_{r_0}$ and $\dim V_{r'_0}$ is smaller. If $\dim V$ is the smaller of the two, we shall specify V by giving the sign $\epsilon(V)$:

- If $\dim W \equiv \dim V_0 \mod 2$, then $\epsilon(V)$ is determined by Theorem 3.6.
- If $\dim W \neq \dim V_0 \mod 2$, then $\epsilon(V)$ is determined by Theorem 3.9(ii)(b).

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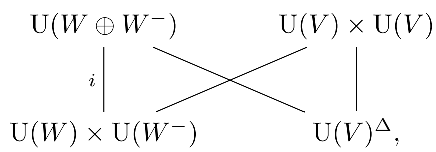
This resolves problem (b) for supercuspidal representations. The recent work of Moeglin [62] makes significant progress towards the general case.

4. Global Developments

In this section, we survey some global developments. Hence, k is a number field with ring of adeles $\mathbb{A}$. Let K/k be a quadratic field extension, and consider a dual pair $\mathrm{U}(W) \times \mathrm{U}(V)$ of unitary groups for K/k. Write $[\mathrm{U}(W)]$ to denote the space $\mathrm{U}(W_k)\backslash\mathrm{U}(W_{\mathbb{A}})$. If π is a cuspidal representation of $\mathrm{U}(W)$, we have its global theta lift $\Theta(\pi)$ on $\mathrm{U}(V)$. As in the local case, it is useful to consider a Witt tower $\{V_r\}$ and the associated global theta lifts $\Theta_{V_r}(\pi)$. It was shown by Rallis that there exists a minimal r_0 such that $\Theta_{V_{r_0}}(\pi) \neq 0$, in which case it is a cuspidal representation. The subsequent global theta lifts (for $r > r_0$) are noncuspidal and hence nonzero.

In view of this, the main question in global theta correspondence is to determine the nonvanishing of $\Theta_{V_r}(\pi)$ in terms of basic invariants of π. We may assume that $\Theta_{V_k}(\pi) = 0$ for $k < r$, so that $\Theta_{V_r}(\pi)$ is cuspidal. Write V for V_r henceforth.

4.1. Inner product. Rallis' approach [78] to answering this question is to compute the Petersson inner product $\langle \theta(\phi, f), \theta(\phi, f) \rangle$. The Rallis inner product formula relates this inner product to the special L-values of π. The mechanism for the Rallis inner product formula relies on the following see-saw diagram of dual pairs:



where W^- denotes the skew–Hermitian space obtained from W by multiplying the form by -1, so that $\mathrm{U}(W^-) = \mathrm{U}(W)$. Then one has:

$$\begin{aligned}
&\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle \\
&= \int_{[\mathrm{U}(V)]} \left(\int_{[\mathrm{U}(W)]} \theta(\phi_1)(g_1, h) \cdot \overline{f_1(g_1)} \, dg_1 \right) \cdot \left(\int_{[\mathrm{U}(W)]} \overline{\theta(\phi_2)(g_2, h)} \cdot f_2(g_2) \, dg_2 \right) dh \\
&= \int_{[\mathrm{U}(W) \times \mathrm{U}(W)]} \left(\int_{[\mathrm{U}(V)]} \theta(\phi_1)(g_1, h) \cdot \overline{\theta(\phi_2)(g_2, h)} \, dh \right) \cdot \overline{f_1(g_1)} \cdot f_2(g_2) \, dg_1 \, dg_2
\end{aligned} \tag{2}$$

where in the last equality, we have formally exchanged the integrals. This inner integral (if it converges) can be interpreted as the global theta lift of the constant function 1 of $\mathrm{U}(V)^\Delta$ to $\mathrm{U}(W \oplus W^-)$. The inner integral converges absolutely, so that the above exchange is valid, if one is in the Weil's convergent range:

$$r = 0 \quad \text{or} \quad \dim V - r > \dim W.$$

To proceed further, one would like to give a different interpretation of the inner integral. This is the content of the so-called Siegel-Weil formula: it identifies the inner integral with an Eisenstein series. Even in Weil's convergent range, this Siegel-Weil formula was achieved in a series of papers by Weil [92], Kudla-Rallis [41, 42], Ichino [34] and Yamana [93, 94], stretching over 40 years.

4.2. The Regularized Theta Integral. Henceforth, we shall consider life outside Weil's convergent range, so that $r > 0$ and $\dim V - r \leq \dim W$, in which case we have

$$0 < \dim V \leq 2 \dim W \quad \text{and} \quad r \leq \dim W.$$

Consider the Weil representation Ω of $\mathrm{U}(W \oplus W^-) \times \mathrm{U}(V)$. We are interested in the theta integral

$$I(\phi)(g) = \frac{1}{\tau(\mathrm{U}(V))} \cdot \int_{[\mathrm{U}(V)]} \Theta(\phi)(g,h)\, dh.$$

for $\phi \in \Omega$ and where $\tau(\mathrm{U}(V))$ denotes the Tamagawa number. The integral diverges, but under the above conditions, Kudla-Rallis [43] discovered a regularization of this theta integral.

More precisely, one can find an element z of the Bernstein center of $\mathrm{U}(V_v)$ at some place v of k such that $\Theta(z\cdot\phi)$ is rapidly decreasing as a function on $[\mathrm{U}(V)]$ and hence the integral $I(z\cdot\phi)$ converges. One considers the (spherical) Eisenstein series $E(s)$ associated to the family of degenerate principal series representations induced from the maximal parabolic subgroup of $\mathrm{U}(V)$ stabilising a maximal isotropic subspace of V. At the point $s = \rho_V = \frac{\dim V - r}{2}$,

$$\mathrm{Res}_{s=\rho_V} E(s) = \kappa$$

is a constant function. Moreover, one has $z \cdot E(s) = P_z(s) \cdot E(s)$ for some function $P_z(s)$. Now one sets

$$B(s,\phi) = \frac{1}{\kappa \cdot P_z(s) \cdot \tau(\mathrm{U}(V))} \cdot \int_{[\mathrm{U}(V)]} \Theta(z\cdot\phi)(g,h) \cdot E(s,h)\, dh.$$

This meromorphic function is the regularised theta integral and one is interested in its analytic behaviour at the point $s = \rho_V$.

The Laurent expansion of $B(s,\phi)$ at $s = \rho_V$ has the form

$$B(s,\phi) = \frac{B_{-1}(\phi)}{s-\rho_V} + B_0(\phi) + \cdots \quad \text{when } \dim V \le \dim W;$$

and

$$B(s,\phi) = \frac{B_{-2}(\phi)}{(s-\rho_V)^2} + \frac{B_{-1}(\phi)}{s-\rho_V} + \cdots \quad \text{when } \dim W < \dim V \le 2\dim W.$$

We shall refer to these two cases as the *first term range* and the *second term range* respectively. Each Laurent coefficient B_i gives a linear map

$$B_i : \omega \to \mathcal{A}(\mathrm{U}(W \oplus W^-))$$

and the one which is important for the inner product formula is the residue B_{-1}.

4.3. Siegel Eisenstein series. The purpose of the Siegel-Weil formula is to identify the automorphic forms $B_{-2}(\phi)$ and $B_{-1}(\phi)$ with the analogous Laurent coefficients of a Siegel-Eisenstein series $A(s,\phi)$ associated to ϕ.

More precisely, the diagonally embedded subspace $W^\Delta \subset W \oplus W^-$ is maximal isotropic, so that its stabiliser in $\mathrm{U}(W \oplus W^-)$ is a Siegel parabolic subgroup P, which has Levi factor $\mathrm{GL}(W^\Delta)$. Let

$$I_P(s) = \mathrm{Ind}_P^{\mathrm{U}(W\oplus W^-)} \chi_V |\det|^s$$

be the associated Siegel principal series representation. Now the Weil representation Ω can be realised on $\mathcal{S}(W^\nabla \otimes V)$ (where W^∇ is an isotropic complement to W^Δ), and the map $\phi \mapsto f_\phi$ with

$$f_\phi(g) = (\Omega(g)\phi)(0) \quad \text{for } g \in \mathrm{U}(W_{\mathbb{A}} \oplus W_{\mathbb{A}}^-)$$

defines a $\mathrm{U}(W \oplus W^-)$-equivariant and $\mathrm{U}(V)$-invariant map

$$\Omega \longrightarrow I_P(s_{V,W}) \quad \text{with } s_{V,W} := (\dim V - \dim W)/2.$$

One then sets

$$A(s,\phi) = E(s, f_\phi).$$

Observe that in the first term range, $s_{V,W} \leq 0$, whereas in the second term range, $s_{V,W} > 0$. If $s = s_{V,W} > 0$, the Laurent expansion of the Siegel-Eisenstein series $A(s,\phi)$ there has the form

$$A(s,\phi) = \frac{A_{-1}(\phi)}{s - s_{V,W}} + A_0(\phi) + \cdots$$

As for B_i, each A_i is a linear map $A_i : \Omega \to \mathcal{A}(\mathrm{U}(W \oplus W^-))$.

4.4. First term identity. Assume that we are in the first term range, so that $s_{V,W} \leq 0$. Let V' be the space in the same Witt tower as V such that

$$\dim V + \dim V' = 2 \dim W \quad (\text{so } \dim V' \geq \dim V).$$

The space V' is called the complementary space to V with respect to W and is such that $s_{V',W} \geq 0$. We shall write A_i' and B_i' for the relevant Laurent coefficients in the context of V'. Ikeda has defined in [35] a natural $\mathrm{U}(W \oplus W^-) \times \mathrm{U}(V)$-equivariant map

$$\mathrm{Ik} : \Omega' \longrightarrow \Omega,$$

where Ω' is the Weil representation for $\mathrm{U}(V') \times \mathrm{U}(W \oplus W^-)$. Then the first term identity established in [43, 32, 33, 35, 93] is the following identity:

Theorem 4.1. *Assume that we are in the first term range. Then for all $\phi \in \Omega$,*

$$c \cdot A'_{-1}(\phi') = A_0(\phi) = 2 \cdot B_{-1}(\phi),$$

where c is an explicit constant, $\phi' \in \Omega'$ is such that $\mathrm{Ik}(\pi_K \phi') = \phi$ and π_K is the projection onto the K-fixed space (with K a maximal compact subgroup of $\mathrm{U}(V'_{\mathbb{A}})$).

4.5. Second term identity. In a recent paper [16], the regularised Siegel-Weil formula is extended to the second term range. More precisely, we have:

Theorem 4.2 (**Siegel-Weil formula**)**.** *Suppose that $0 < r \leq \dim W$ and $\dim W < \dim V \leq \dim W + r$, so that we are in the second term range.*

(i) (First term identity) For all $\phi \in \Omega$*, one has*

$$A_{-1}(\phi) = B_{-2}(\phi).$$

(ii) (Second term identity) For all $\phi \in \Omega$*, one has*

$$A_0(\phi) = B_{-1}(\phi) - c \cdot \{B_0'(\mathrm{Ik}(\pi_K \phi))\} \mod \mathrm{Im} A_{-1}.$$

Here, c *is some explicit constant and* V' *is the complementary space to* V *with respect to* W *(so* $\dim V' < \dim V$ *here). Finally, the term* $\{...\}$ *on the RHS is interpreted to be* 0 *if* V' *is anisotropic.*

4.6. Rallis inner product formula. The Siegel-Weil formulas above and the theory of the doubling zeta integral, as completed by Yamana [95], enable one to establish the Rallis inner product formula. For the result in the first term range, we refer the reader to Yamana [95]. In the second term range, we have [16]:

Theorem 4.3. *Suppose that*

$$\dim W < \dim V \leq 2 \dim W \quad \textit{and} \quad r \leq \dim W$$

so that we are either in the second term range or the convergent range, depending on whether $\dim V \leq \dim W + r$ *or not. Let* π *be a cuspidal representation of* $\mathrm{U}(W)$ *and consider its global theta lift* $\Theta(\pi)$ *to* $\mathrm{U}(V)$*.*

(i) Assume that $\Theta(\pi)$ *is cuspidal. Then for* $\phi_1, \phi_2 \in \omega_{\psi,V,W}$ *and* $f_1, f_2 \in \pi$*,*

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle = [E:F] \cdot \mathrm{Val}_{s=s_{V,W}} \left(L(s + \frac{1}{2}, \pi \times \chi_V) \cdot Z^*(s, \phi_1 \otimes \overline{\phi_2}, f_1, f_2) \right),$$

where $s_{V,W} = (\dim V - \dim W)/2 > 0$*,* $L(s, \pi \times \chi_V)$ *is the standard L-function of* π*, and* $Z^*(s,-)$ *denotes the normalized doubling zeta integral.*

(ii) Assume further that for all places v *of* F*, the local theta lift* $\Theta_{n,r}(\pi_v)$ *is nonzero. Then* $L(s + \frac{1}{2}, \pi \times \chi_V)$ *is holomorphic at* $s = s_{V,W}$*, so that*

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle = [E:F] \cdot L(s_{V,W} + \frac{1}{2}, \pi \times \chi_V) \cdot Z^*(s_{V,W}, \phi_1 \otimes \overline{\phi_2}, f_1, f_2).$$

4.7. Nonvanishing of global theta lifts. As a consequence, we have the following local-global criterion for the nonvanishing of global theta lifts.

Theorem 4.4. *Assume the same conditions on* (V, W) *as in Theorem 4.3. Let* π *be a cuspidal representation of* $\mathrm{U}(W)$ *and consider its global theta lift* $\Theta(\pi)$ *to* $\mathrm{U}(V)$*. Assume that* $\Theta(\pi)$ *is cuspidal.*

(i) If $\Theta(\pi)$ *is nonzero, then*

(a) for all places v*,* $\Theta(\pi_v) \neq 0$*, and*

(b) $L(s_{V,W} + \frac{1}{2}, \pi \times \chi_V) \neq 0$ *i.e. nonzero holomorphic.*

(ii) The converse to (i) holds if $K_v = k_v \times k_v$ *for all archimedean places* v *of* k*. More generally, under the conditions (a) and (b) in (i), there is a Hermitian space* V' *over* K *such that*

- $V' \otimes_k k_v \cong V \otimes_k k_v$ *for every finite or complex place of* k*;*
- *the global theta lift* $\Theta'(\pi)$ *of* π *to* $\mathrm{U}(V')$ *is nonzero.*

The reason for not having the converse to (i) in general is that, if $K_v/k_v = \mathbb{C}/\mathbb{R}$, we do not know the equivalence of the nonvanishing of the local theta correspondence and that of the normalised doubling zeta integral on an appropriate submodule of its domain.

5. Variations and extensions.

In this section, we want to mention some extensions of the theory of theta correspondence which have been pursued in the last 20 years.

5.1. Exceptional theta correspondence. There is no reason to confine oneself to dual pairs in the symplectic group. One could consider dual pairs in any connected reductive group G. For theta correspondence, however, one also needs the analog of the Weil representation. It turns out that the Weil representation is the "smallest" infinite-dimensional representation of the metaplectic group. This suggests that one should consider the analogous smallest representation of $G(F)$. Such a representation is called a minimal representation of $G(F)$.

There is a series of work devoted to the construction and classification of minimal representations of an arbitrary $G(F)$. Of these, one might mention various papers of Kazhdan, Savin and Torasso [36, 37, 80, 83]. In the global case, the automorphic realisation of the minimal representations have been constructed, largely using residues of Eisenstein series [22].

With the theory of minimal representations in place, one can start to consider theta correspondence. While the setup is the same as classical theta correspondence, one key difference is that one does not know the analog of the Howe duality conjecture for exceptional theta correspondence; in particular, one does not know the analog of Theorem 3.1. For work on exceptional theta correspondences, we may mention a series of papers by Savin and various collaborators [25, 31, 30, 57].

5.2. Singular theta lifting of Borcherds. In his 1994 ICM address [10], Borcherds described a singular theta lifting for classical dual pairs. In classical global theta correspondence, one integrates the theta kernel against cusp forms, and there is no issue with convergence. In Borcherd's context, one is trying to lift functions which blow up exponentially at the cusps, and Borcherds gave a regularisation of such a singular theta integral. An example of such a function is the classical j-function on the upper half plane. This theory is so far not representation theoretic in nature, but it allows Borcherds to construct many beautiful examples of automorphic forms which possess infinite product expansion [11], analogous to the classical η-function.

5.3. Arithmetic theta lifting of Kudla. Since the mid-1990's, Kudla [40] has pursued an arithmetic version of the theory of theta correspondence. This provides a lifting of automorphic forms to classes in the arithmetic Chow group of a Shimura variety. One goal of Kudla's program is to establish an arithmetic version of the Rallis inner product formula in the equal rank case, which involves the central derivative of the standard L-function instead of the central value. This will require an arithmetic Siegel-Weil formula. A low rank example was established in the book [45] of Kudla-Rapoport-Yang. In his PhD thesis [53, 54], Y. F. Liu formulated such a conjectural arithmetic Rallis inner product formula in the context of unitary groups of arbitrary rank.

5.4. Geometric theta lifting of Lysenko. The theory of theta correspondence was almost single-handedly extended to the framework of the Geometric Langlands Program by S. Lysenko [55, 56]. Together with V. Lafforgue [46, 47], the theory of minimal representations was also suitably geometrized.

6. Local Langlands Correspondence

In the rest of this report, we will discuss a number of applications of theta correspondence. The first such application is to the local Langlands conjecture (LLC). Unlike the earlier sections, we will no longer restrict ourselves to unitary groups.

6.1. LLC for $\mathbf{GSp_4}$. In [18] and [20], the local theta correspondence was used to establish the LLC for the group $\mathrm{GSp}_4(F)$ and its non-split inner form, where F is a p-adic field. This uses an extension of the theta correspondence from the setting of isometry dual pairs to the setting of similitude dual pairs.

Let us briefly explain how theta correspondence is used in the proof. Let W be the 4-dimensional symplectic space, V the split quadratic space of dimension 6 and trivial discriminant, and V' the anisotropic quadratic space of dimension 4 and trivial discriminant. These quadratic spaces belong to the two different related Witt towers, and we consider the similitude theta correspondence for $\mathrm{GSp}(W) \times \mathrm{GO}(V)$ and $\mathrm{GSp}(W) \times \mathrm{GO}(V')$. By the dichotomy statement in Corollary 3.5, we deduce that each $\pi \in \mathrm{Irr}(\mathrm{GSp}(W))$ has nonzero theta lift to exactly one of $\mathrm{GO}(V)$ or $\mathrm{GO}(V')$. Using this, one deduces an injection

$$\mathrm{Irr}(\mathrm{GSp}(W)) \hookrightarrow \mathrm{Irr}(\mathrm{GSO}(V)) \bigsqcup \mathrm{Irr}(\mathrm{GSO}(V')).$$

Now one notes that

$$\begin{cases} \mathrm{GSO}(V') \cong (\mathrm{GL}_2(F) \times D^\times)/\{(t, t^{-1}) : t \in F^\times\} \\ \mathrm{GSO}(V) \cong (\mathrm{GL}_4(F) \times F^\times)/\{(t, t^{-2}) : t \in F^\times\}, \end{cases}$$

where D is the quaternion division F-algebra. In particular, the LLC is known for these two groups, so one may assign L-parameters to representations of $\mathrm{GSp}_4(F)$.

6.2. LLC for G_2.

We now describe an ongoing work of the author with G. Savin on the LLC for the split exceptional group of type G_2 using the exceptional theta correspondence. Quite amazingly, it turns out that a similar strategy as in the GSp_4 case can be implemented.

More precisely, one has the two dual pairs

$$\mathrm{G}_2 \times \mathrm{PB}^\times \subset \mathrm{E}_6^{\mathrm{B}} \quad \text{and} \quad \mathrm{G}_2 \times \mathrm{PGSp}_6 \subset \mathrm{E}_7$$

where B denotes a degree 3 division F-algebra, $\mathrm{E}_6^{\mathrm{B}}$ is an inner form of type E_6 and F-rank 2 and E_7 is the split group of this type. Consider the local theta correspondence for these two dual pairs, a key result is the following analog of dichotomy and the Howe duality conjecture:

Theorem 6.1. *Each $\pi \in \mathrm{Irr}(\mathrm{G}_2)$ has nonzero theta lift to exactly one of* $\mathrm{PB}^\times$ *or* PGSp_6. *Moreover, the nonzero $\theta(\pi)$ is irreducible. In particular, one has an injection*

$$\mathrm{Irr}(\mathrm{G}_2) \hookrightarrow \mathrm{Irr}(PB^\times) \sqcup \mathrm{Irr}(\mathrm{PGSp}_6).$$

By the Jacquet-Langlands correspondence and the LLC for PGL_3 and Sp_6 (due to Arthur), one may then hope to assign L-parameters to $\pi \in \mathrm{Irr}(\mathrm{G}_2)$. This theorem will play a key role in our ongoing work to establish the full LLC for G_2.

7. Gross-Prasad Conjecture

One typical application of theta correspondence is that it relates certain periods on one member of a dual pair with certain periods on the other member. One such family of periods which has attracted much attention recently is the Gross-Prasad (GP) periods, which was considered by Gross and Prasad in the context of the special orthogonal groups in two papers [23, 24] some twenty years ago. They formulated precise conjectures for the nonvanishing of the GP periods. In a recent paper [13], these conjectures were extended to arbitrary classical groups. For ease of exposition, we shall consider the case of unitary groups.

7.1. GP periods.

Let V_{n+1} be a Hermitian space of dimension $n+1$ over E and W_n a skew-Hermitian space of dimension n over E. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1, so that we have a natural inclusion $\mathrm{U}(V_n) \hookrightarrow \mathrm{U}(V_{n+1})$. In particular, if we set

$$G_n = \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1}) \quad \text{or} \quad \mathrm{U}(W_n) \times \mathrm{U}(W_n)$$

and

$$H_n = \mathrm{U}(V_n) \quad \text{or} \quad \mathrm{U}(W_n),$$

then we have a diagonal embedding $\Delta : H_n \hookrightarrow G_n$.

In the Hermitian case, one is interested in determining $\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \mathbb{C})$ for $\pi \in \mathrm{Irr}(G_n)$. We shall call this the *Bessel* case of the GP conjecture. Indeed, what we have described is a special case: the general Bessel case deals with a pair of Hermitian spaces $V' \subset V$ such that $\dim V/V'$ is odd.

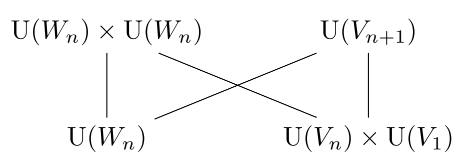
In the skew-Hermitian case, the restriction problem requires another piece of data: a Weil representation ω_{ψ,χ,W_n} of $\mathrm{U}(W_n)$, where χ is a character of $E^\times$ such that $\chi|_{F^\times} = \omega_{E/F}$. Then one is interested in determining $\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \omega_{\psi,\chi,W_n})$. We shall call this the *Fourier-Jacobi* case (FJ) of the GP conjecture. As before, the general FJ case deals with a pair of skew-Hermitian spaces $W' \subset W$ such that $\dim W/W'$ is even. To unify notation, we shall let $\nu = \mathbb{C}$ or ω_{ψ,χ,W_n} in the respective cases.

7.2. Gross-Prasad conjecture. It was shown in [5] and [81] that the above Hom spaces have dimension at most 1. Thus the main issue is to determine when the Hom space is nonzero. The Gross-Prasad conjecture gives an answer for this issue, formulated in the framework of the local Langlands correspondence. It can be loosely stated as follows:

1. Given a generic L-parameter ϕ for G_n there is a unique η such that the representation $\pi(\phi, \eta)$ satisfies $\mathrm{Hom}_{\Delta H_n}(\pi(\phi,\eta), \nu) \neq 0$.
2. There is a precise recipe, in terms of local ϵ-factor for the distinguished character η.

In a stunning series of papers [87], [88], [89], [90], Waldspurger has established the Bessel case of the GP conjecture for special orthogonal groups in the case of tempered L-parameters; the case of general generic L-parameters is then dealt with by Mœglin-Waldspurger [64]. Beuzart-Plessis [7], [8], [9] has since extended Waldspurger's techniques to settle the Bessel case of the GP conjecture for unitary groups in the tempered case.

7.3. Theta correspondence. Now the Bessel and Fourier-Jacobi cases of the GP conjecture are related by the local theta correspondence. More precisely, there is a see-saw diagram



and the associated see-saw identity reads:

$$\mathrm{Hom}_{\mathrm{U}(W_n)}(\Theta_{\psi,\chi,V_n,W_n}(\sigma) \otimes \omega_{\psi,\chi,V_1,W_n}, \pi) \cong \mathrm{Hom}_{\mathrm{U}(V_n)}(\Theta_{\psi,\chi,V_{n+1},W_n}(\pi), \sigma)$$

for $\pi \in \mathrm{Irr}(\mathrm{U}(W_n))$ and $\sigma \in \mathrm{Irr}(\mathrm{U}(V_n))$. Hence the left-hand side of the see-saw identity concerns the Fourier-Jacobi case (FJ) whereas the right-hand side concerns the Bessel case (B). It is thus apparent that precise knowledge of the local theta correspondence for unitary groups of (almost) equal rank will give the precise relation of (FJ) to (B).

In particular, as a consequence of the proof of Prasad's conjecture in Theorem 3.9, the FJ case of the GP conjecture was verified in [15] . Hence one has:

Theorem 7.1. *Assume the LLC for unitary groups. Then both the Bessel and FJ cases of the GP conjecture hold.*

8. Shimura-Waldspurger Correspondence

We will conclude by returning to the Shimura-Waldspurger (SW) correspondence for Mp_2, which in some sense initiated many of the developments discussed in this paper. In particular, we will discuss its extension to Mp_{2n}.

8.1. Local SW correspondence. Let F be a nonarchimedean local field. Let W be the $2n$-dimensional symplectic vector space, and let V^+ and V^- be the two $2n+1$-dimensional quadratic spaces with trivial discriminant, with V^+ split. Then one may consider the theta correspondence for $\mathrm{Mp}(W) \times \mathrm{O}(V^\epsilon)$. As a consequence of Theorem 3.6, the following was shown in [17]:

Theorem 8.1. *Fix a nontrivial additive character ψ of F. The theta correspondence with respect to ψ gives a bijection*

$$\mathrm{Irr}_\epsilon(\mathrm{Mp}W)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)),$$

where we consider genuine representations of $\mathrm{Mp}(W)$ *on the LHS. Assuming the LLC for* $\mathrm{SO}(V^\pm)$*, one then inherits an LLC for* $\mathrm{Mp}(W)$*. Moreover, this LLC satisfies a list of expected properties which characterise it uniquely.*

When F is archimedean, the analogous theorem was obtained by Adams-Barbasch [4] some 20 years ago, and described in Adams' 1994 ICM talk [2].

8.2. Global SW correspondence. Now assume that we are working over a number field k. It is natural to attempt to use the global theta correspondence to obtain a precise description of the automorphic discrete spectrum of $\mathrm{Mp}(W_\mathbb{A})$. For readers familiar with Waldspurger's work [84, 85] in the case when $\dim W = 2$, it will be apparent that there is an obstruction to this approach: the global theta lift $\Theta(\pi)$ of a cuspidal representation π of $\mathrm{Mp}(W_\mathbb{A})$ or $\mathrm{SO}(V_\mathbb{A})$ may be 0 and it is nonzero precisely when $L(1/2, \pi) \neq 0$.

This obstruction already occurs when $\dim W = 2$, and was not easy to overcome. Waldspurger had initially alluded to results of Flicker proved by the trace formula. Nowadays, one could appeal to a result of Friedberg-Hoffstein [12], stating that if $\epsilon(1/2, \pi) = 1$, then there exists a quadratic Hecke character χ such that

$L(1/2, \pi \times \chi) \neq 0$. When $\dim W > 2$, however, the analogous analytic result does not seem to be forthcoming and may be very hard. We are going to suggest a new approach in the higher rank case, but before that, we would like to describe the analog of Arthur's conjecture for Mp_{2n}.

8.3. Arthur's conjecture for $\mathbf{Mp}_{2n}$.

For a fixed additive automorphic character ψ, one expects that

$$L^2_{disc} = \bigoplus_{\Psi} L^2_{\Psi,\psi}$$

where

$$\Psi = \bigoplus_i \Psi_i = \bigoplus_i \Pi_i \boxtimes S_{r_i}$$

is a global discrete A-parameter for Mp_{2n}; it is also an A-parameter for SO_{2n+1}. Here, S_{r_i} is the r_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and Π_i is a cuspidal representation of GL_{n_i} such that

$$\begin{cases} L(s, \Pi_i, \wedge^2) \text{ has a pole at } s = 1, \text{ if } r_i \text{ is odd;} \\ L(s, \Pi_i, \mathrm{Sym}^2) \text{ has a pole at } s = 1, \text{ if } r_i \text{ is even.} \end{cases}$$

Moreover, we have $\sum_i n_i r_i = 2n$ and the summands Ψ_i are mutually distinct.

For a given Ψ, one inherits the following additional data:

- for each v, one inherits a local A-parameter

$$\Psi_v = \bigoplus_i \Psi_{i,v} = \bigoplus_i \Pi_{i,v} \boxtimes S_{r_i}.$$

  By the LLC for GL_N, we may regard each $\Pi_{i,v}$ as an n_i-dimensional representation of the Weil-Deligne group WD_{k_v}. Hence, we may regard Ψ_v as a $2n$-dimensional representation of $WD_{k_v} \times \mathrm{SL}_2(\mathbb{C})$.

- one has a "global component group"

$$A_\Psi = \bigoplus_i \mathbb{Z}/2\mathbb{Z} \cdot a_i$$

  which is a $\mathbb{Z}/2\mathbb{Z}$-vector space equipped with a distinguished basis indexed by the Ψ_i's. Similarly, for each v, we have the local component group A_{Ψ_v} which is defined as the component group of the centralizer of the image of Ψ_v, thought of as a representation of $WD_{k_v} \times \mathrm{SL}_2(\mathbb{C})$. There is a natural diagonal map

$$\Delta : A_\Psi \longrightarrow \prod_v A_{\Psi_v}.$$

- For each v, one has a local A-packet associated to Ψ_v and ψ_v:

$$\Pi_{\Psi_v,\psi_v} = \{\sigma_{\eta_v} : \eta_v \in \mathrm{Irr}(A_{\Psi_v})\},$$

consisting of unitary representations (possibly zero, possibly reducible) of $\mathrm{Mp}_{2n}(k_v)$ indexed by the set of irreducible characters of A_{Ψ_v}. On taking tensor products of these local A-packets, we obtain a global A-packet

$$A_{\Psi,\psi} = \{\sigma_\eta : \eta = \otimes_v \eta_v \in \mathrm{Irr}(\prod_v A_{\Psi_v})\}$$

consisting of abstract unitary representations $\sigma_\eta = \otimes_v \sigma_{\eta_v}$ of $\mathrm{Mp}_{2n}(\mathbb{A})$ indexed by the irreducible characters $\eta = \otimes_v \eta_v$ of $\prod_v A_{\Psi_v}$.

- Arthur has attached to Ψ a quadratic character (possibly trivial) ϵ_Ψ of A_Ψ, This character plays an important role in the multiplicity formula for the automorphic discrete spectrum of SO_{2n+1}. For Mp_{2n}, we need to define a modification of ϵ_Ψ.

More precisely, consider the L-parameter $\Phi_\Psi = \bigoplus_i \Phi_{\Psi_i}$ associated to Ψ, with

$$\Phi_{\Psi_i} = \bigoplus_{k=0}^{r_i-1} \Pi_i \cdot |-|^{(r_i-1-2k)/2}.$$

Then define $\eta_\Psi \in \mathrm{Irr} A_\Psi$ by

$$\eta_\Psi(a_i) = \epsilon(1/2, \Phi_{\Psi_i}) = \begin{cases} \epsilon(1/2, \Pi_i), & \text{if } L(s, \Pi_i, \wedge^2) \text{ has a pole at } s = 1; \\ 1, & \text{if } L(s, \Pi_i, \mathrm{Sym}^2) \text{ has a pole at } s = 1. \end{cases}$$

The modified quadratic character of A_Ψ in the metaplectic case is

$$\tilde{\epsilon}_\Psi = \epsilon_\Psi \cdot \eta_\Psi.$$

We can now state the conjecture.

Arthur Conjecture for Mp_{2n}

There is a decomposition

$$L^2_{disc}(\mathrm{Mp}_{2n}) = \bigoplus_\Psi L^2_{\Psi,\psi}$$

where the sum runs over equivalence classes of discrete A-parameters of Mp_{2n}. For each such Ψ,

$$L^2_{\Psi,\psi} \cong \bigoplus_{\eta \in \mathrm{Irr}(\prod_v A_{\Psi_v}) : \Delta^*(\eta) = \tilde{\epsilon}_\Psi} \sigma_\eta$$

8.4. A new approach. In an ongoing work, we are developing a new approach for the Arthur conjecture described above. Namely, by results of Arthur [6], one now has a classification of the automorphic discrete spectrum of SO_{2r+1} for all r. Instead of trying to construct the automorphic discrete spectrum of Mp_{2n} by theta lifting from SO_{2n+1}, one could attempt to use theta liftings from SO_{2r+1} for $r \geq n$. Let us illustrate this in the case when $\dim W = 2$.

Let π be a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A}) = \mathrm{SO}(V_{\mathbb{A}}^+)$. Then π gives rise to a near equivalence class in the automorphic discrete spectrum of Mp_2. If $L(1/2, \pi) \neq 0$, this near equivalence class can be exhausted by the global theta lifts of π and its Jacquet-Langlands transfer to inner forms of PGL_2. When $L(1/2, \pi) = 0$, we consider the A-parameter

$$\psi = \pi \boxtimes S_1 \oplus 1 \boxtimes S_2 \quad \text{for } \mathrm{SO}_5.$$

This is a so-called Saito-Kurokawa A-parameter. By Arthur, ψ indexes a near equivalence class in the automorphic discrete spectrum of SO_5. In a well-known paper [72], Piatetski-Shapiro gave a construction of the Saito-Kurokawa representations by theta lifting from Mp_2, using Waldspurger's results as initial data. However, *one can turn the table around.*

Namely, taking the Saito-Kurokawa near equivalence classes as given by Arthur, one can consider their theta lift back to Mp_2. By the Rallis inner product formula, such a theta lift is nonzero if the partial L-function

$$L^S(s, \Phi_\psi) = L^S(s, \pi) \cdot \zeta(s + \frac{1}{2}) \cdot \zeta(s - \frac{1}{2})$$

has a pole at $s = 3/2$, or equivalently if $L^S(3/2, \pi) \neq 0$. Now this is certainly much easier to ensure than the nonvanishing at $s = 1/2$! In this way, one can construct the desired near equivalence class for Mp_2 associated to π and by studying the local theta correspondence in detail, one can recover Waldspurger's results from 30 years ago.

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