THE LANGLANDS-WEISSMAN PROGRAM FOR BRYLINSKI-DELIGNE EXTENSIONS

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Abstract. We describe an evolving and conjectural extension of the Langlands program for a class of nonlinear covering groups of algebraic origin studied by Brylinski and Deligne. In particular, we describe the construction of an L-group extension of such a covering group (over a split reductive group) due to Weissman, study some of its properties and discuss a variant of it. Using this L-group extension, we describe a local Langlands correspondence for covering (split) tori and unramified genuine representations, using work of Savin, McNamara, Weissman and W.-W. Li. We then define the notion of automorphic (partial) L-functions attached to genuine automorphic representations of the covering groups of Brylinski and Deligne. Finally, we see how the L-group formalism explains certain anomalies in the representation theory of covering groups and examine some examples of Langlands functoriality such as base change.

Contents

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1. Introduction

One of the goals of the local Langlands program is to provide an arithmetic classification of the set of isomorphism classes of irreducible representations of a locally compact group $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a connected reductive group over a local field F. Analogously, if k is a number field with ring of adeles $\mathbb{A}$, the global Langlands program postulates a classification of automorphic representations of $\mathbb{G}(\mathbb{A})$ in terms of Galois representations. In this proposed arithmetic classification, which has been realised in several important instances, a key role is played by the L-group ${}^L\mathbb{G}$ of $\mathbb{G}$. This key notion was introduced by Langlands in his re-interpretation of the Satake isomorphism in the theory of spherical functions and used by him to introduce the notion of *automorphic L-functions*. One of the main goals of this paper is to do the same for a class of nonlinear covering groups of "algebraic origin" studied by Brylinski-Deligne [BD].

1.1. **Covering groups.**

The theory of the L-group is so far confined to the case when $\mathbb{G}$ is a connected reductive linear algebraic group. On the other hand, since Steinberg's beautiful paper [S], the structure theory of nonlinear covering groups of G (i.e. topological central extensions of G by finite groups) have been investigated by many mathematicians, notably Moore [Mo], Matsumoto [Ma], Deodhar [De], Deligne [D], Prasad-Raghunathan [PR1, PR2, PR3], and its relation to the reciprocity laws of abelian class field theory has been noted. In addition, nonlinear covering groups of G have repeatedly made their appearance in representation theory and the theory of automorphic forms. This goes way back to Jacobi's construction of his theta function, a holomorphic modular form of weight $1/2$, and a more recent instance is the work of Kubota [Ku] and the Shimura correspondence between integral and half integral weight modular forms. Both these examples concern automorphic forms and representations of the metaplectic group $\mathrm{Mp}_2(F)$, which is a nonlinear double cover of $\mathrm{SL}_2(F) = \mathrm{Sp}_2(F)$. As another example, the well-known Weil representation of $\mathrm{Mp}_{2n}(F)$ gives a representation theoretic incarnation of theta functions and has been a very useful tool in the construction of automorphic forms. Finally, much of Harish-Chandra's theory of local harmonic analysis and Langlands' theory of Eisenstein series continue to hold for such nonlinear covering groups (see [MW] and [L4]).

It is thus natural to wonder if the framework of the Langlands program can be extended to encompass the representation theory and the theory of automorphic forms of covering groups. There have been many attempts towards this end, such as Flicker [F], Kazhdan-Patterson [KP1, KP2], Flicker-Kazhdan [FK], Adams [A1, A2], Savin [Sa] among others. However, these attempts have tended to focus on the treatment of specific families of examples rather than a general theory. This is understandable, for what is lacking is a structure theory which is sufficiently functorial. For example, the classification of nonlinear covering groups given in [Mo, De, PR1, PR2] is given only when $\mathbb{G}$ is simply-connected and isotropic, in which case a universal cover exists.

1.2. **Brylinski-Deligne theory.**

A functorial structure theory was finally developed by Brylinski and Deligne [BD]. More precisely, Brylinski-Deligne considered the category of multiplicative $\mathbb{K}_2$-torsors on a connected reductive group $\mathbb{G}$ over F; these are extensions of $\mathbb{G}$ by the sheaf $\mathbb{K}_2$ of Quillen's K_2 group in the category of sheaves of groups on the big Zariski site of Spec(F):

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathbb{G}} \longrightarrow \mathbb{G} \longrightarrow 1.$$

In other words, Brylinski and Deligne started with an extension problem in the world of algebraic geometry. Some highlights of [BD] include:

- an elegant and functorial classification of this category in terms of enhanced root theoretic data, much like the classification of split connected reductive groups by their root data.
- the description of a functor from the category of multiplicative $\mathbb{K}_2$-torsors $\overline{\mathbb{G}}$ on $\mathbb{G}$ (together with an integer n such that $\#\mu_n(F) = n$, which determines the degree of the covering) to the category of topological central extensions $\overline{G}$ of G:

  $$1 \longrightarrow \mu_n \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

  These topological central extensions may be considered of "algebraic origin" and can be constructed using cocycles which are essentially algebraic in nature.
- though this construction does not exhaust all topological central extensions, it captures a sufficiently large class of such extensions, and essentially all interesting examples which have been investigated so far; for example, it captures all such coverings of G when $\mathbb{G}$ is split and simply-connected.

We shall give a more detailed discussion of the salient features of the Brylinski-Deligne theory in §2 and §3. Hence, the paper [BD] provides a structure theory which is essentially algebraic and categorical, and may be perceived as a natural extension of Steinberg's original treatment [S] from the split simply connected case to general reductive groups.

1.3. **Dual and L-groups.** One should expect that such a natural structure theory would elucidate the study of representations and automorphic forms of the Brylinski-Deligne covering groups $\overline{G}$, henceforth referred to as BD covering groups. Indeed, Brylinski and Deligne wrote in the introduction of [BD]: "We hope that for k a global field, this will prove useful in the study of metaplectic automorphic forms, that is, the harmonic analysis of functions on $\tilde{G}(\mathbb{A})/G(k)$".

The first person to fully appreciate this is probably our colleague M. Weissman. In a series of papers [W1, W2, HW], Weissman systematically exploited the Brylinski-Deligne theory to study the representation theory of covering tori, the unramified representations and the depth zero representations. This was followed by the work of several authors who discovered a "Langlands dual group" $\overline{G}^\vee$ for a BD covering group $\overline{G}$ (with $\mathbb{G}$ split) from different considerations. These include the work of Finkelberg-Lysenko [FL] and Reich [Re] in the framework of the geometric Langland program and the work of McNamara [Mc2, Mc3] who established a Satake isomorphism and interpreted it in terms of the dual group $\overline{G}^\vee$. The dual group $\overline{G}^\vee$ was constructed by making a metaplectic modification of the root datum of $\mathbb{G}$.

In [W3], Weissman built upon [Mc2] and gave a construction of the "L-group" ${}^L\overline{G}$ of a *split* BD covering group $\overline{G}$. The construction in [W3] is quite involved, and couched in the language of Hopf algebras. Moreover, with hindsight, it gives the correct notion only for a subclass of BD covering groups. In a foundational paper [W7], Weissman gives a simpler and completely general revised construction of the L-group for an arbitrary quasi-split BD covering group (not necessarily split), using the framework of étale gerbes, thus laying the groundwork for an extension of the Langlands program to the setting of BD covering groups.

1.4. **The L-group extension.** We shall describe in §4 Weissman's construction of the L-group of $\overline{G}$ for split $\mathbb{G}$ (given in the letter [W4]), where one could be more down-to-earth and avoid the notion of gerbes. The fact that this more down-to-earth construction is equivalent to the more sophisticated one in [W7] is shown in [W8]. At this point, let us note that since $\mathbb{G}$ is split, one is inclined to simply take ${}^L\overline{G}$ as the direct product $\overline{G}^\vee \times W_F$, where W_F denotes the Weil group of F. At least, this is what one is conditioned to do by the theory of L-groups for linear reductive groups. However, Weissman realised that such an approach would be overly naive.

Indeed, the key insight of [W3] is that the construction of the L-group of a BD covering group should be the functorial construction of an extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1, \tag{1.1}$$

and an L-parameter for $\overline{G}$ should be a splitting of this short exact sequence. The point is that, *even if ${}^L\overline{G}$ is isomorphic to the direct product $\overline{G}^\vee \times W_F$*, it is not supposed to be equipped with a canonical isomorphism to $\overline{G}^\vee \times W_F$. This reflects the fact that there is no canonical irreducible genuine representation of $\overline{G}$, and hence there should not be any canonical L-parameter. Hence it would not be appropriate to say that the L-group of $\overline{G}$ "is" the direct product $\overline{G}^\vee$ with W_F.

As Weissman is the first person to make use of the full power of the Brylinski-Deligne structure theory for the purpose of representation theory and is the one who introduced the L-group extension, we shall call this evolving area the Langlands-Weissman program for BD extensions. As the adjective "evolving" is supposed to suggest, we caution the reader that the construction in [W4, W7] may not be the final word on the L-group.

1.5. **Results of this paper.** Against this backdrop, the purpose of this paper is to supplement the viewpoint of [W3, W4, W7] concerning the L-group ${}^L\overline{G}$ in several ways. In particular, we shall study some properties of the L-group extension, suggest a variant of it and provide supporting evidence for its essential correctness. We summarise our results here:

(i) (The L-group extension) Firstly, we show that the L-group extension of a split BD covering group constructed in [W3, W4, W7] is a split extension (see Proposition 6.9), but the L-group is *not* isomorphic to the direct product $\overline{G}^\vee \times W_F$ in general. This phenomenon can be seen already in the following simple family of BD covering groups

$$\overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/i_\eta(F^\times) \cong \mathrm{GL}_2(F)/NE_\eta^\times$$

where $\eta \in F^\times/F^{\times 2}$, with corresponding quadratic étale algebra E_η, and

$$i_\eta(t) = (t, (\eta, t)_2),$$

with $(-,-)_2$ denoting the quadratic Hilbert symbol. Observe that the first projection defines a topological central extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_\eta \longrightarrow \mathrm{PGL}_2(F) \longrightarrow 1.$$

Then it turns out that $\overline{G}_\eta^\vee \cong \mathrm{SL}_2(\mathbb{C})$ and

$${}^L\overline{G}_\eta \cong \mathrm{SL}_2(\mathbb{C}) \rtimes_\eta W_F$$

where the action of W_F on $\mathrm{SL}_2(\mathbb{C})$ is by the conjugation action via the map $W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ given by

$$w \mapsto \begin{pmatrix} \chi_\eta(w) & \\ & 1 \end{pmatrix},$$

with χ_η the quadratic character of W_F determined by η.

This family of BD covering groups is quite instructive, as it illustrates several interesting phenomena. For example, one can show that the covering splits over the hyperspecial maximal compact subgroup $\mathrm{PGL}_2(\mathcal{O}_F)$ if and only if $\eta \in \mathcal{O}_F^\times$ (see 4.6). This seems to contradict [Mc2, Thm. 4.2] (see [Mc3] where a corrected version is given) where it was claimed that a BD covering for a split $\mathbb{G}$ is always split over a hyperspecial maximal compact subgroup K. In general, we treat this issue of splitting over K in §4; see especially Theorem 4.2.

Though the L-group extension may not be split, the inner class of $\mathrm{Aut}(\overline{G}^\vee)$ induced by the conjugation action of W_F (via any choice of splitting) contains the trivial automorphism. Such a situation is familiar from the theory of endoscopy for linear algebraic groups, where in the definition of an endoscopic datum $(H, \mathcal{H}, s, \xi)$, the group $\mathcal{H}$ is a split extension of W_F by $H^\vee$ but not necessarily isomorphic to the L-group of H.

(ii) (Distinguished splittings) Secondly, we would like to argue that the L-group of a split BD covering group should be (the isomorphic class of) the extension (1.1), *together with a finite set of distinguished splittings* (which we will define and which give rise to isomorphisms ${}^L\overline{G} \cong \overline{G}^\vee \times W_F$ if they exist). If the degree of the covering is $n = 1$, so that $\overline{G} = G$, then the set of distinguished splittings is a singleton, so that ${}^L\overline{G}$ "is" $\overline{G}^\vee \times W_F$ in this case. For a subclass of BD covering groups, we show in Theorem 6.6 that the L-group short exact sequence (1.1) has such a family of distinguished splittings; this completes the results of [W3] where the cases for odd n and $n = 2$ were treated.

(iii) (Distinguished genuine characters) We show that the distinguished splittings of ${}^L\overline{G}$ are in natural bijection with a family of distinguished genuine characters of the center $Z(\overline{T})$ of the covering torus $\overline{T}$ (where T is a maximal F-split torus of G). These distinguished genuine characters of $Z(\overline{T})$ are "as close to being trivial characters as possible", and are invariant under the natural Weyl group action for *certain* $\overline{G}$ (see Theorem 6.8). Thus, they serve as natural base-points for the definition of principal series representations, extending the results of Savin [Sa] to general n and these $\overline{G}$. We also give an explicit construction of such distinguished genuine characters of $Z(\overline{T})$, using the Weil index and the n-th Hilbert symbol.

We point out, however, that for general $\overline{G}$, there may not exist distinguished splittings of ${}^L\overline{G}$ or Weyl-invariant genuine characters of $Z(\overline{T})$. For example, for the covering groups $\overline{G}_\eta$ discussed in (i), Weyl-invariant genuine characters exist if and only if $(\eta, -1)_2 = 1$.

(iv) (LLC) The fact that the extension ${}^L\overline{G}$ is split allows one to define the set of L-parameters for $\overline{G}$ as the set of splittings of ${}^L\overline{G}$ modulo the conjugation action of $\overline{G}^\vee$. In particular, one can formulate a conjectural (coarse) local Langland correspondence (LLC). This is done in §11. We verify this conjectural LLC in three cases:

- (LLC for covering tori) When $\mathbb{G} = \mathbb{T}$ is a split torus, the above passage between distinguished splittings of ${}^L\overline{T}$ and distinguished genuine characters of $Z(\overline{T})$ extends to give the local Langlands correspondence for covering tori, a problem first investigated in [W1] and also treated in [W7]. This is contained in Theorem 8.2.
- (Unramified LLC) For general split $\mathbb{G}$, a Satake isomorphism has been shown by McNamara [Mc2], W.-W. Li [L2] and Weissman [W7] (relative to a splitting s of $\overline{G}$ over K). In Theorem 9.4, we formulate the Satake isomorphism in terms of ${}^L\overline{G}$ and show in Theorem 9.8 that s-unramified representations of $\overline{G}$ are naturally parametrised by "s-unramified splittings" of ${}^L\overline{G}$. In particular, this gives the notion of Satake parameters (relative to s), which is the main ingredient in the definition of automorphic L-functions.
- (Metaplectic groups) When $\overline{G} = \mathrm{Mp}_{2n}(F)$ is the degree 2 cover of $\mathrm{Sp}_{2n}(F)$, we use the results of [AB, GS] (established using the theory of theta correspondences and the LLC for odd special orthogonal groups) to deduce the LLC for Mp_{2n} in terms of the L-group ${}^L\overline{G}$ considered here. In particular, this LLC for $\mathrm{Mp}_{2n}(F)$ (see Theorem 11.1) is independent of the choice of a nontrivial additive character ψ of F.

Using the results of Ban-Jantzen [BJ] (on the Langlands classification for covering groups) and W.-W. Li [L4] (on the theory of R-groups), one can reduce the coarse LLC to the discrete series case.

(v) (Enlagred L-group) In Section 10, we suggest a slightly different treatment of the L-group given in [W3, W4, W7], by treating several closely related BD covering groups together. For example, the representation theory of all the groups $\overline{G}_\eta$ in (i) can clearly be treated together in terms of the representation theory of $\mathrm{GL}_2(F)$. Extending this instructive example, one can slightly enhance Weissman's construction in [W4, W7] to give an enlarged L-group ${}^L\overline{G}^\#$:

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G}^\# \longrightarrow W_F \times (T_{Q,n}^{sc})^\vee[n] \longrightarrow 1 \tag{1.2}$$

where $(T_{Q,n}^{sc})^\vee[n]$ is a finite group: it is the group of n-torsion points in the maximal split torus of the adjoint quotient of $\overline{G}^\vee$. One can write the above exact sequence as:

$$1 \longrightarrow \overline{G}^\# = \overline{G}^\vee \rtimes (T_{Q,n}^{sc})^\vee[n] \longrightarrow {}^L\overline{G}^\# \longrightarrow W_F \longrightarrow 1 \tag{1.3}$$

where the action of $(T_{Q,n}^{sc})^\vee[n]$ on $\overline{G}^\vee$ is via the canonical adjoint action of $(\overline{G}^\vee)_{ad}$ on $\overline{G}^\vee$. If one pulls back (1.2) using the map

$$\mathrm{id} \times \chi_\eta : W_F \longrightarrow W_F \times \mu_2,$$

one recovers the L-group extension ${}^L\overline{G}_\eta$. Thus, ${}^L\overline{G}^\#$ is an amalgam of all ${}^L\overline{G}_\eta$ and it always has a distinguished splitting. Using this enlarged L-group and exploiting z-extensions (see §2.9), we can reduce the LLC for BD covering groups to a distinguished subclass of such groups, corresponding to $\eta = 1$.

As an example, for the groups $\overline{G}_\eta$ discussed in (i), (1.2) is:

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C}) \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C})^\pm \xrightarrow{\det} W_F \times \mu_2 \longrightarrow 1,$$

whereas (1.3) is:

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C})^\pm \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C})^\pm \longrightarrow W_F \longrightarrow 1.$$

(vi) (Automorphic L-functions) Using the above results, we define in §13 the notion of "automorphic L-functions" associated to an automorphic representation of a global BD covering group and a representation R of ${}^L\overline{G}$. How can one describe the representations R of ${}^L\overline{G}$? One way is to exploit the distinguished splittings of ${}^L\overline{G}$ constructed above.

Indeed, a distinguished splitting $s_0 : W_F \longrightarrow {}^L\overline{G}^\#$ gives rise to the notion of *L-parameters relative to* s_0: for any splitting $s : W_F \longrightarrow {}^L\overline{G}^\#$, one sets

$$\phi_s(w) = s(w)/s_0(w)^{-1} \in \overline{G}^\# = \overline{G}^\vee \rtimes (T_{Q,n}^{sc})^\vee[n]$$

so that $\phi_s : W_F \longrightarrow \overline{G}^\#$. In particular, the Satake isomorphism furnishes the notion of *Satake parameters relative to* s_0. A distinguished splitting s_0 thus allows one to define the notion of the L-function (with respect to s_0) associated to a representation R of the (enlarged) dual group $\overline{G}^\#$ (whose representations are easy to write down in terms of those of the connected Lie groups $\overline{G}^\vee$). Together with the Satake isomorphism, we thus have the notion of the "automorphic L-function" associated to s_0 and R.

In fact, if R factors through the adjoint group of $\overline{G}^\#$, the resulting L-function is independent of the choice of s_0; in particular the adjoint L-function is canonically defined. Another instance is the Langlands-Shahidi type L-functions, as we explain in §13.4. In the PhD thesis [Ga] of the second author, it was shown that the constant terms of Eisenstein series on BD covering groups are expressed in terms of these Langlands-Shahidi type L-functions, which extends the results of Langlands' famous monograph [La] to the nonlinear setting of this paper. One would like to show, as in the linear case, that these automorphic L-functions are nice, i.e. have meromoprhic continuation and satisfy a functional equation of the usual type. The meromorphic continuation of the Langlands-Shahidi type L-functions has been shown by the second author [Ga], but is open at this moment in general.

(vii) (Anomalies) We explain how the L-group formalism explains certain anomalies which have been empirically observed in the representation theory of certain covering groups. One is the fact that representations in a given L-packet of $\mathrm{Mp}(2n)$ can have different central characters. Another is the observation that when restricting an irreducible genuine representation of a Kazhdan-Patterson cover of $\mathrm{GL}(n)$ to its cover over $\mathrm{SL}(n)$, the irreducible summands need not occur in the same L-packet. These phenomena can be neatly explained in terms of the L-group.

(viii) (Langlands Functoriality) Finally, we examine certain instances of Langlands functoriality. The L-group formalism already gives a suggestion of the "endoscopic groups" of a BD covering group $\overline{G}$. Indeed, if we let $G_{Q,n}$ be the quasi-split linear algebraic group whose L-group is isomorphic to the L-group of $\overline{G}$. Then it is natural to think of the endoscopic groups of $G_{Q,n}$ as the endoscopic groups of $\overline{G}$. This should be correct to a first approximation, though (as one sees from the work of W.-W. Li for $\mathrm{Mp}(2n)$) the notion of isomorphism of endoscopic data has to be modified. In any case, one then expects instances of endoscopic transfer, at least on the level of weak liftings. One supporting evidence is that one has an isomorphism of Iwahori-Hecke algebras of $\overline{G}$ and $G_{Q,n}$, as formulated in Theorem 15.1.

Another example we examine is the case of base change. It turns out that it is not so automatic to formulate this notion on the L-group side. We achieve this by appealing to a result of E. Bender [B]. We then show in Theorem 14.3 that, for an extension K/F of local fields, base change for covering tori is obtained by pulling back via a lifting of the usual norm map $\mathcal{N}_T : \mathbb{T}(K) \longrightarrow \mathbb{T}(F)$ to the level of covers, i.e. using a commutative diagram:

$$\begin{array}{ccc} \overline{T}_K & \xrightarrow{\mathcal{N}_{\overline{T}}} & \overline{T}_F \\ \downarrow & & \downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathcal{N}_T} & \mathbb{T}(F), \end{array}$$

with the map $\mathcal{N}_{\overline{T}}$ being constructed by the results of Bender [B]. These topics are contained in §14 and §15.

We conclude the paper with a number of examples, illustrating the above constructions and results, as well as highlighting a few basic questions which we feel are crucial for carrying the theory forward.

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2. **Brylinski-Deligne Extensions**

Let F be a field and $\mathbb{G}$ a connected reductive linear algebraic group over F. We shall assume that $\mathbb{G}$ is split over F in this paper. Fix a maximal split torus $\mathbb{T}$ contained in a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$. Let $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the cocharacter lattice and $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ the character lattice. Then one has the set of roots $\Phi \subset X$ and the set of coroots $\Phi^\vee \subset Y$ of $(\mathbb{G}, \mathbb{T})$ respectively. The Borel subgroup $\mathbb{B}$ determines the set of simple roots $\Delta \subset \Phi$ and the set of simple coroots $\Delta^\vee \in \Phi^\vee$. For each $\alpha \in \Phi$, one has the associated root subgroup $\mathbb{U}_\alpha \subset \mathbb{G}$ which is normalised by $\mathbb{T}$. We shall fix an épinglage or pinning for $(\mathbb{G}, \mathbb{T})$, so that for each $\alpha \in \Phi$, one has an isomorphism $x_\alpha : \mathbb{G}_a \longrightarrow \mathbb{U}_\alpha$.

Hence our initial data for this paper is a pinned connected split reductive group $(\mathbb{G}, \mathbb{T}, \mathbb{B}, x_\alpha)$ over F.

2.1. **Multiplicative K_2-torsors.**

The algebraic group $\mathbb{G}$ defines a sheaf of groups on the big Zariski site on $\mathrm{Spec}(F)$. Let $\mathbb{K}_2$ denote the sheaf of groups on $\mathrm{Spec}(F)$ associated to the K_2-group in Quillen's K-theory. Then a multiplicative K_2-torsor is an extension

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathbb{G}} \longrightarrow \mathbb{G} \longrightarrow 1$$

of sheaves of groups on $\mathrm{Spec}(F)$. We consider the category $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ of such extensions where the morphisms between objects are given by morphisms of extensions. Given two such central extensions, one can form their Baer sum: this equips $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ with the structure of a commutative Picard category.

In [BD], Brylinski and Deligne made a deep study of $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ and obtained an elegant classification of this category when $\mathbb{G}$ is a connected reductive group. We recall their results briefly in the case when $\mathbb{G}$ is split.

2.2. **Split torus.**

Suppose $\mathbb{T}$ is a split torus, with cocharacter lattice $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ and character lattice $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. Then we have:

Proposition 2.1. *Let $\mathbb{T}$ be a split torus over F. The category $\mathbf{CExt}(\mathbb{T}, \mathbb{K}_2)$ is equivalent as a commutative Picard category (by an explicit functor) to the category whose objects are pairs $(Q, \mathcal{E})$, where*

- *Q is a $\mathbb{Z}$-valued quadratic form on Y, with associated symmetric bilinear form $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$;*
- *$\mathcal{E}$ is a central extension of groups*

$$1 \longrightarrow F^\times \longrightarrow \mathcal{E} \longrightarrow Y \longrightarrow 1$$

whose associated commutator map $[-,-] : Y \times Y \to F^\times$ *is given by*

$$[y_1, y_2] = (-1)^{B_Q(y_1,y_2)}.$$

The set of morphisms between $(Q, \mathcal{E})$ *and* $(Q', \mathcal{E}')$ *is empty unless* $Q = Q'$*, in which case it is given by the set of isomorphisms of extensions from* $\mathcal{E}$ *to* $\mathcal{E}'$*. The Picard structure is defined by*

$$(Q, \mathcal{E}) + (Q', \mathcal{E}') = (Q + Q', \text{Baer sum of } \mathcal{E} \text{ and } \mathcal{E}').$$

Observe that the isomorphism class of the extension $\mathcal{E}$ is completely determined by the commutator map and hence by the quadratic form Q. The extension $\mathcal{E}$ is obtained from $\overline{\mathbb{T}}$ as follows. Let $F(\!(\tau)\!)$ denote the field of Laurent series in the variable τ over F. Then one has

$$1 \longrightarrow \mathbb{K}_2(F(\!(\tau)\!)) \longrightarrow \overline{\mathbb{T}}(F(\!(\tau)\!)) \longrightarrow \mathbb{T}(F(\!(\tau)\!)) = Y \otimes_{\mathbb{Z}} F(\!(\tau)\!)^\times \longrightarrow 1.$$

The map $y \mapsto y(\tau)$ defines a group homomorphism $Y \longrightarrow \mathbb{T}(F(\!(\tau)\!))$. Pulling back by this morphism and pushing out by the residue map

$$\text{Res} : \mathbb{K}_2(F(\!(\tau)\!)) \longrightarrow \mathbb{K}_1(F) = F^\times$$

defined by

$$\text{Res}(f, g) = (-1)^{v(f) \cdot v(g)} \cdot \left(\frac{f^{v(g)}}{g^{v(f)}}(0) \right),$$

one obtains the desired extension $\mathcal{E}$.

2.3. **Simply-connected groups.** Suppose now that $\mathbb{G}$ is a split simply-connected semisimple group over F and recall that we have fixed the épinglage $(\mathbb{T}, \mathbb{B}, x_\alpha)$. Let $W = N(\mathbb{T})/\mathbb{T}$ be the corresponding Weyl group. Since $\mathbb{G}$ is simply-connected, the coroot lattice is equal to Y, so that the set of simple coroots $\Delta^\vee$ is a basis for Y.

Now we have:

Proposition 2.2. *The category* $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ *is equivalent (as commutative Picard categories) to the category whose objects are* W*-invariant* $\mathbb{Z}$*-valued quadratic form* Q *on* Y*, and whose only morphisms are the identity morphisms on each object.*

As a result of this proposition, whenever we are given a quadratic form Q on Y, Q gives rise to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}_Q$ on $\mathbb{G}$, unique up to unique isomorphism, which may be pulled back to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}_Q$ on $\mathbb{T}$ and hence gives rise to an extension $\mathcal{E}_Q$ of Y by $F^\times$. The automorphism group of the extension $\mathcal{E}_Q$ is $\text{Hom}(Y, F^\times)$. Following [BD, §11], one can rigidify $\mathcal{E}_Q$ by giving it an extra structure, as we now explain.

2.4. **Rigidifying $\mathcal{E}_Q$.** We continue to assume that $\mathbb{G}$ is simply-connected. We have already fixed the épinglage $\{x_\alpha : \alpha \in \Phi\}$ for $\mathbb{G}$, so that

$$x_\alpha : \mathbb{G}_a \longrightarrow \mathbb{U}_\alpha \subset \mathbb{G}.$$

Indeed, one has an embedding

$$i_\alpha : \text{SL}_2 \hookrightarrow \mathbb{G}$$

which restricts to $x_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices. By [BD], one has a canonical lifting

$$\tilde{x}_\alpha : \mathbb{G}_a \longrightarrow \overline{\mathbb{U}}_\alpha \subset \overline{\mathbb{G}}.$$

For $t \in \mathbb{G}_m$, we set

$$n_\alpha(t) = x_\alpha(t) \cdot x_{-\alpha}(-t^{-1}) \cdot x_\alpha(t) = i_\alpha \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \in N(\mathbb{T}_Q),$$

and

$$\tilde{n}_\alpha(t) = \tilde{x}_\alpha(t) \cdot \tilde{x}_{-\alpha}(-t^{-1}) \cdot \tilde{x}_\alpha(t) \in \overline{\mathbb{G}}_Q$$

Then one has a map

$$s_\alpha : \mathbb{T}_\alpha := \alpha^\vee(\mathbb{G}_m) \longrightarrow \overline{\mathbb{T}}_Q$$

given by

$$\alpha^\vee(t) \mapsto \tilde{n}_\alpha(t) \cdot \tilde{n}_\alpha(-1).$$

This is a section of $\overline{\mathbb{G}}_Q$ over $\mathbb{T}_\alpha$, which is trivial at the identity element. The section s_α is useful in describing the natural conjugation action of $N(\mathbb{T}_Q)$ on $\overline{\mathbb{T}}_Q$. By [BD, Prop. 11.3], one has the nice formula:

$$\tilde{n}_\alpha(1) \cdot \tilde{t} \cdot \tilde{n}_\alpha(1)^{-1} = \tilde{t} \cdot s_\alpha(\alpha^\vee(\alpha(t)^{-1}). \tag{2.3}$$

Moreover, the collection of sections $\{s_\alpha : \alpha \in \Delta\}$ provides a collection of elements $s_\alpha(\alpha^\vee(a)) \in \overline{\mathbb{T}}_Q$ with $a \in \mathbb{G}_m$, and $\overline{\mathbb{T}}_Q$ is generated by K_2 and the collection of $s_\alpha(\alpha^\vee(a))$.

Taking points in $F(\!(\tau)\!)$, we have the element

$$s_\alpha(\alpha^\vee(\tau)) \in \overline{\mathbb{T}}_Q(F(\!(\tau)\!)),$$

which gives rise (via the construction of $\mathcal{E}_Q$) to an element

$$s_Q(\alpha^\vee) \in \mathcal{E}_Q.$$

Then we rigidify $\mathcal{E}_Q$ by equipping it with the set $\{s_Q(\alpha^\vee) : \alpha^\vee \in \Delta^\vee\}$: there is a unique automorphism of $\mathcal{E}_Q$ which fixes all these elements.

In the following, we shall fix a choice of the data $(\overline{\mathbb{G}}_Q, \overline{\mathbb{T}}_Q, \mathcal{E}_Q)$ *for each* W*-invariant quadratic form* Q *on* Y *when* $\mathbb{G}$ *is split and simply-connected.* This is not a real choice, because any two choices are isomorphic by a unique isomorphism. The section s_Q constructed above provides $\mathcal{E}_Q$ with a system of generators: $\mathcal{E}_Q$ is generated by $s_Q(\alpha^\vee) \in \Delta^\vee$ and $a \in F^\times$ subject to the relations:

- $a \in F^\times$ is central;
- $[s_Q(\alpha^\vee), s_Q(\beta^\vee)] = (-1)^{B_Q(\alpha^\vee, \beta^\vee)}$ for $\alpha^\vee, \beta^\vee \in \Delta^\vee$.

2.5. **General reductive groups.** Now let $\mathbb{G}$ be a split connected reductive group over F, with fixed épinglage $(\mathbb{T}, \mathbb{B}, x_\alpha)$. Let

$$i_{sc} : Y^{sc} := \mathbb{Z}[\Delta^\vee] \subset Y$$

be the inclusion of the coroot lattice Y^{sc} into Y, and let $X^{sc} \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual lattice of Y^{sc}. Then the quadruple $(X^{sc}, \Delta, Y^{sc}, \Delta^\vee)$ is the root datum of the simply-connected cover $\mathbb{G}^{sc}$ of the derived group of $\mathbb{G}$, and one has a natural map

$$q : \mathbb{G}^{sc} \to \mathbb{G}.$$

Let $\mathbb{T}^{sc}$ be the preimage of $\mathbb{T}$ in $\mathbb{G}^{sc}$, so that one has a commutative diagram

$$\begin{array}{ccc} \mathbb{T}^{sc} & \longrightarrow & \mathbb{G}^{sc} \\ \downarrow & & \downarrow q \\ \mathbb{T} & \longrightarrow & \mathbb{G}. \end{array}$$

The classification of the multiplicative $\mathbb{K}_2$-torsors on $\mathbb{G}$ is an amalgam of the two Propositions above. Given a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ of $\mathbb{G}$, the above commutative diagram induces by pullbacks a commutative digram of multiplicative $\mathbb{K}_2$-torsors:

$$\begin{array}{ccc} \overline{\mathbb{T}}^{sc} & \longrightarrow & \overline{\mathbb{G}}^{sc} \\ \downarrow & & \downarrow q \\ \overline{\mathbb{T}} & \longrightarrow & \overline{\mathbb{G}}. \end{array}$$

Then

- the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}$ gives a pair $(Q, \mathcal{E})$ by Proposition 2.1;
- the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}^{sc}$ corresponds to a quadratic form Q^{sc}, and it was shown in [BD] that Q^{sc} is simply the restriction of Q to Y^{sc}.
- we have fixed the data $(\overline{\mathbb{G}}_{Q^{sc}}, \overline{\mathbb{T}}_{Q^{sc}}, \mathcal{E}_{Q^{sc}})$ associated to Q^{sc}. Thus we have a canonical isomorphism
$$f : \overline{\mathbb{G}}_{Q^{sc}} \longrightarrow \overline{\mathbb{G}}^{sc},$$
restricting to an isomorphism
$$f : \overline{\mathbb{T}}_{Q^{sc}} \longrightarrow \overline{\mathbb{T}}^{sc}$$
which then induces an isomorphism
$$f : \mathcal{E}_{Q^{sc}} \longrightarrow \mathcal{E}^{sc} = q^*(\mathcal{E}).$$
This isomorphism is characterised as the unique one which sends the elements $s_{Q^{sc}}(\alpha^\vee) \in \mathcal{E}_{Q^{sc}}$ (for $\alpha^\vee \in \Delta^\vee$) to the corresponding elements $s(\alpha^\vee) \in q^*(\mathcal{E})$. In particular, we have a commutative diagram
$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & \mathcal{E}_{Q^{sc}} & \longrightarrow & Y^{sc} & \longrightarrow & 1 \\ & & \| & & \downarrow f & & \| & & \\ 1 & \longrightarrow & F^\times & \longrightarrow & \mathcal{E} & \longrightarrow & Y^{sc} & \longrightarrow & 1 \end{array}$$

We have thus attached to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ a triple $(Q, \mathcal{E}, f)$. Now we have:

Theorem 2.4. *The category* $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ *is equivalent (via the above construction) to the category* $\mathbf{BD}_{\mathbb{G}}$ *whose objects are triples* $(Q, \mathcal{E}, f)$*, where*

- $Q : Y \to \mathbb{Z}$ *is a* W*-invariant quadratic form;*
- $\mathcal{E}$ *is an extension of* Y *by* $F^{\times}$ *with commutator map* $[y_1, y_2] = (-1)^{B_Q(y_1,y_2)}$*;*
- $f : \mathcal{E}_{Q^{sc}} \cong q^*(\mathcal{E})$ *is an isomorphism of extensions of* Y^{sc} *by* $F^{\times}$*.*

The set of morphisms from $(Q, \mathcal{E}, f)$ *to* $(Q', \mathcal{E}', f')$ *is empty unless* $Q = Q'$*, in which case it consists of isomorphisms of extensions* $\phi : \mathcal{E} \longrightarrow \mathcal{E}'$ *such that* $f = f' \circ q^*(\phi)$*. In particular, the automorphism group of an object is* $\mathrm{Hom}(Y/Y^{sc}, F^{\times})$*.*

2.6. **Bisectors and Incarnation.** While the above results give a nice classification of multiplicative $\mathbb{K}_2$-torsors over split reductive groups $\mathbb{G}$ over F, it is sometimes useful and even necessary to work with explicit cocycles for computation. The paper [BD] does provide a category of nice algebraic cocycles, as explicated in [W3], and one may replace the category of triples $(Q, \mathcal{E}, f)$ by a slightly simpler category with more direct connections to cocycles.

Extending the treatment in [W3], we consider (not necessarily symmetric) $\mathbb{Z}$-valued bilinear forms D on Y satisfying:

$$D(y, y) = Q(y) \text{ for all } y \in Y,$$

so that

$$B_Q(y_1, y_2) = D(y_1, y_2) + D(y_2, y_1).$$

Such a D is called a bisector of Q. As shown in [W4], for any Q, there exists an associated bisector.

We consider a category

$$\mathbf{Bis}_{\mathbb{G}} = \bigcup_{Q} \mathbf{Bis}_{\mathbb{G},Q}$$

where the full subcategory $\mathbf{Bis}_{\mathbb{G},Q}$ consists of pairs (D, η) where D is a bisector of Q and

$$\eta : Y^{sc} \longrightarrow F^{\times}$$

is a group homomorphism. Given two pairs (D_1, η_1) and (D_2, η_2), the set of morphisms is the set of functions $\xi : Y \longrightarrow F^{\times}$ such that

(a) $\xi(y_1 + y_2) \cdot \xi(y_1)^{-1} \cdot \xi(y_2)^{-1} = (-1)^{D_1(y_1,y_2) - D_2(y_1,y_2)}$;
(b) $\xi(\alpha^{\vee}) = \eta_2(\alpha^{\vee})/\eta_1(\alpha^{\vee})$ for all $\alpha^{\vee} \in \Delta^{\vee}$.

and where the composition of morphisms is given by multiplication: $\xi_1 \circ \xi_2(y) = \xi_1(y) \cdot \xi_2(y)$. Note that in (b), ξ may not be a group homomorphism when restricted to Y^{sc}, but we are only requiring the identity in (b) to hold as functions of sets when restricted to $\Delta^{\vee}$. Moreover, as shown in [W3], given two bisectors D_1 and D_2 of Q, one can always find ξ such that (a) holds. Thus, up to isomorphism, there is no loss of generality in fixing D (for a fixed Q).

Then it was shown in [W3, §2.2] that there is an incarnation functor

$$\mathbf{Inc}_{\mathbb{G}} : \mathbf{Bis}_{\mathbb{G}} \longrightarrow \mathbf{BD}_{\mathbb{G}} \longrightarrow \mathbf{CExt}(\mathbb{G}, \mathbb{K}_2).$$

The second functor is a quasi-inverse to the functor in Theorem 2.4. On the level of objects, the first functor sends the pair (D, η) in $\mathbf{Bis}_{\mathbb{G},Q}$ to the triple $(Q, \mathcal{E}, f)$ defined as follows:

- $\mathcal{E} = Y \times F^\times$ with group law

$$(y_1, a_1) \cdot (y_2, a_2) = (y_1 + y_2, a_1 a_2 (-1)^{D(y_1,y_2)}).$$

- $f : \mathcal{E}_{Q^{sc}} \to \mathcal{E}$ is given by

$$f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)) \quad \text{for any } \alpha \in \Delta.$$

It was shown in [W3] that this functor is an essentially surjective functor. In fact, our definition of morphisms in $\mathbf{Bis}_{\mathbb{G}}$ differs slightly from that of [W3]. With our version here, this functor is fully faithful as well so that **Inc** is in fact an equivalence of categories.

Moreover, one can choose **Inc** so that if $\overline{\mathbb{G}}$ is the BD extension corresponding to (D, η), then $\overline{\mathbb{T}}$ can be described explicitly using D. Namely, if

$$D = \sum_i x_1^i \otimes x_2^i \in X \otimes X,$$

then one may regard $\overline{\mathbb{T}} = \mathbb{T} \times K_2$ with group law:

$$(t_1, 1) \cdot (t_2, 1) = (t_1 t_2, \prod_i \{x_1^i(t_1), x_2^i(t_2)\}).$$

The associated extension $\mathcal{E}$ is then described as above in terms of the bisector D. Further, we have the following explicit description for the section $q \circ s_\alpha$:

Proposition 2.5. *In terms of the above realisation of $\overline{T}$ by D, the section*

$$q \circ s_\alpha : \alpha^\vee(\mathbb{G}_m) \longrightarrow \overline{\mathbb{T}} \quad \textit{for } \alpha \in \Delta$$

is given by

$$q \circ s(\alpha^\vee(a)) = (\alpha^\vee(a), \{\eta(\alpha^\vee), a\}).$$

Proof. Suppose that $\overline{\mathbb{G}}$ is incarnated by (D, η) and corresponds to the triple $(Q, \mathcal{E}, f)$ where $\mathcal{E}$ and f are described by (D, η) as above. For fixed $\alpha \in \Delta$, one may write

$$q \circ s_\alpha(a) = (\alpha^\vee(a), \aleph_\alpha(a)),$$

where $\aleph_\alpha \in \mathrm{Hom}_{\mathrm{Zar}}(\mathbb{G}_m, \mathbb{K}_2)$ is a homomorphism of sheaves of abelian groups for the big Zariski site. By [BD, §3.7-3.8], or in more details [Bl, Thm. 1.1], we have

$$F^\times = \mathbb{K}_1(F) \cong \mathrm{Hom}_{\mathrm{Zar}}(\mathbb{G}_m, \mathbb{K}_2)$$

where the isomorphism is given by

$$b \mapsto (a \mapsto \{b, a\}).$$

Hence, there exists some $\lambda_\alpha \in F^\times$ such that

$$\aleph_\alpha(a) = \{\lambda_\alpha, a\} \quad \text{for } a \in \mathbb{G}_m.$$

From this, it follows from the definition of f that

$$f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \lambda_\alpha) \in \mathcal{E}.$$

By hypothesis, however, we have:

$$f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)).$$

Hence we deduce that

$$\lambda_\alpha = \eta(\alpha^\vee)$$

and thus

$$q \circ s_\alpha(a) = (\alpha^\vee(a), \aleph_\alpha(a)) = (\alpha^\vee(a), \{\eta(\alpha^\vee), a\}),$$

as desired. □

Thus, the category $\mathbf{Bis}_{\mathbb{G}}$ provides a particularly nice and explicit family of cocycles for BD extensions, and the essential surjectivity of **Inc** says that every BD extension possesses such a cocycle, at least on the maximal torus $\mathbb{T}$. This will be useful for computations.

2.7. **Fair bisectors.** In [W3], Weissman singled out a property of bisectors which he called *fair*. By definition, a bisector D is fair if it satisfies the following:

- for any $\alpha \in \Delta$ such that $Q(\alpha^\vee) \equiv 0 \mod 2$, $D(\alpha^\vee, y) \equiv D(y, \alpha^\vee) \equiv 0 \mod 2$ for all $y \in Y$.

Weissman showed that for any Q, $\mathbf{Bis}_{\mathbb{G},Q}$ contains a fair bisector. We shall henceforth fix a fair bisector for a given Q. The value of fairness will be apparent later on.

At the moment, we simply note that the objects $(D, 1) \in \mathbf{Bis}_{\mathbb{G},Q}$ with D fair and η the trivial homomorphism) are quite special (as we shall see). Thus, we have a distinguished class of multiplicative $\mathbb{K}_2$-torsors with invariant Q.

When $Q = 0$, for example, the bisector $D = 0$ is fair, and $(D, 1)$ gives the split extension $\mathbb{G} \times \mathbb{K}_2$. In some sense, the $\mathbb{K}_2$-torsor with invariants $(D, 1)$ should be regarded as "closest to being a split extension among those with invariants (D, η)". As we shall illustrate in the rest of this section, the general BD extensions can often be described in terms of these distinguished BD extensions.

2.8. **The case $Q = 0$.** Let us consider the example when $Q = 0$. Then we may take the bisector $D = 0$ and regard the objects of $\mathbf{Bis}_Q$ as the set of homomorphisms $\eta : Y^{sc} \to F^\times$. Let $\overline{\mathbb{G}}_\eta$ be the corresponding multiplicative $\mathbb{K}_2$-torsor on $\mathbb{G}$. Then $\overline{\mathbb{G}}_{\eta_1}$ and $\overline{\mathbb{G}}_{\eta_2}$ are isomorphic precisely when η_1/η_2 can be extended to a homomorphism of Y to $F^\times$.

How can we characterize the distinguished BD-extension in $\mathbf{Bis}_Q$ using the BD data $(Q, \mathcal{E}, f)$ when $D = 0$? Since $D = 0$, $\mathcal{E}$ is an abelian group and hence is a split extension: $\mathcal{E} = Y \times F^\times$. Each $\eta \in \mathrm{Hom}(Y^{sc}, F^\times)$ gives a map

$$f_\eta : \mathcal{E}_{Q^{sc}} \longrightarrow \mathcal{E}$$

defined by

$$f_\eta(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)) \quad \text{for } \alpha \in \Delta.$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, F^\times)$ to the short exact sequence $Y^{sc} \to Y \to Y/Y^{sc}$, we obtain as part of the long exact sequence:

$$\mathrm{Hom}_{\mathbb{Z}}(Y, F^\times) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(Y^{sc}, F^\times) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathbb{Z}}(Y/Y^{sc}, F^\times) \longrightarrow 0.$$

For any $\eta \in \mathrm{Hom}(Y^{sc}, F^\times)$, its image under δ is the extension

$$1 \longrightarrow F^\times \longrightarrow (Y \times F^\times)/f_\eta(Y^{sc}) \longrightarrow Y/Y^{sc} \longrightarrow 1$$

The long exact sequence thus implies that this extension is split precisely when η is equivalent to 1. This gives a way of characterising the distinguished isomorphism class in $\mathbf{Bis}_Q$ when $D = 0$.

How can we construct the other BD extensions in $\mathbf{Bis}_Q$ when $D = 0$? We assume further that $\mathbb{G}$ is semisimple and let $q : \mathbb{G}^{sc} \longrightarrow \mathbb{G}$ be the natural isogeny with kernel $Z = \mathrm{Tor}_{\mathbb{Z}}(Y/Y^{sc}, \mathbb{G}_m) \hookrightarrow \mathbb{T}^{sc} = Y^{sc} \otimes_{\mathbb{Z}} \mathbb{G}_m$. Then $q^*(\overline{\mathbb{G}}_\eta)$ is isomorphic to the split extension $\mathbb{G}^{sc} \times \mathbb{K}_2$ by a unique isomorphism. We may thus construct $\overline{\mathbb{G}}_\eta$ by starting with $\mathbb{G}^{sc} \times \mathbb{K}_2$ and then considering a quotient of this by a suitable embedding $Z \hookrightarrow \mathbb{G}^{sc} \times \mathbb{K}_2$. For this, we note that η induces a map

$$i_\eta : Z = \mathrm{Tor}_{\mathbb{Z}}(Y/Y^{sc}, \mathbb{G}_m) \hookrightarrow Y^{sc} \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow F^\times \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow \mathbb{K}_2.$$

Then we have

$$\overline{\mathbb{G}}_\eta = (\overline{\mathbb{G}} \times \mathbb{K}_2)/\{(z, i_\eta(z)) : z \in Z\}.$$

2.9. ***z*-extensions.** We consider another example which will play a crucial role later on, namely when Y/Y^{sc} is a free abelian group. In this case, for any $(D, \eta) \in \mathbf{Bis}_Q$, η can be extended to a homomorphism of Y, and so any two (D, η_1) and (D, η_2) are isomorphic. This means that there is a unique isomorphism class of objects in $\mathbf{Bis}_Q$, just like the case when $\mathbb{G}$ is simply-connected (where $Y = Y^{sc}$). However, the automorphism group $\mathrm{Hom}(Y/Y^{sc}, F^\times)$ of an object is not trivial (unless $Y = Y^{sc}$).

As we shall see later, some questions about a BD extension can be reduced to the case when Y/Y^{sc} is free. This is achieved via the consideration of $z-$extensions. More precisely, it is not hard to see that given any $\mathbb{G}$, one can find a central extension of connected reductive groups:

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \xrightarrow{\pi} \mathbb{G} \longrightarrow 1$$

where

- $\mathbb{H}$ is such that $Y_{\mathbb{H}}/Y_{\mathbb{H}}^{sc}$ is free;
- $\pi_* : Y_{\mathbb{H}}^{sc} \longrightarrow Y_{\mathbb{G}}$ is an isomorphism;
- Z is a split torus which is central in $\mathbb{H}$.

Such an extension is called a z-extension.

Given such a z-extension, and a BD extension $\overline{G}_\eta$ with invariant (D, η), we obtain a BD extension $\mathbb{H}_\eta := \pi^*(\overline{G}_\eta)$ on $\mathbb{H}$ with BD invariant $(D \circ \pi, \eta \circ \pi)$, so that one has

$$1 \longrightarrow Z_\eta = Z \longrightarrow \overline{\mathbb{H}}_\eta \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

By our discussion above, we may choose an isomorphism $\xi : \overline{\mathbb{H}}_1 \cong \overline{\mathbb{H}}_\eta$. Thus, via ξ, we have

$$1 \longrightarrow \xi^{-1}(Z_\eta) \longrightarrow \overline{\mathbb{H}}_1 \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

This shows that for any given bisector D, all the BD extensions $\overline{\mathbb{G}}_\eta$ can be described as the quotient of a fixed BD extension $\overline{\mathbb{H}}_1$ (with $\eta = 1$) by a suitable splitting of the split torus Z.

2.10. **Running example.** Let us illustrate the above discussion using a simple example, where

$$\mathbb{G} = \mathrm{PGL}_2, \quad Y = \mathbb{Z} \supset Y^{sc} = 2\mathbb{Z} \quad \text{and} \quad D = 0.$$

Then we take the z-extension to be

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_2 \xrightarrow{\pi} \mathrm{PGL}_2 \longrightarrow 1.$$

The distinguished BD-extension with $\eta = 1$ is the split extension $\overline{\mathbb{G}}_1 = \mathrm{PGL}_2 \times \mathbb{K}_2$ and its pullback to GL_2 is the split extension $\overline{\mathbb{H}} = \pi^*(\overline{\mathbb{G}}_1) = \mathrm{GL}_2 \times \mathbb{K}_2$. For any $\eta \in \mathrm{Hom}(Y^{sc}, F^\times) \cong F^\times$, the BD extension $\overline{\mathbb{G}}_\eta$ can then be described as

$$\overline{\mathbb{G}}_\eta = (\mathrm{GL}_2 \times \mathbb{K}_2)/\{(z, i_\eta(z)) : z \in \mathbb{G}_m\},$$

where

$$i_\eta(z) = \{\eta, z\} \in \mathbb{K}_2.$$

This is a rather trivial family of BD extensions since their pullback to $\mathbb{G}^{sc} = \mathrm{SL}_2$ is split. Nonetheless, we shall use them as our running examples, as they already exhibit various properties we want to highlight in this paper.

3. Topological Covering Groups

In this section, we will pass from the algebro-geometric world of multiplicative $\mathbb{K}_2$-torsors to the world of topological central extensions. Let F be a local field. If F is nonarchimedean, let $\mathcal{O}$ denote its ring of integers with residue field κ.

3.1. **BD covering groups.** Start with a multiplicative $\mathbb{K}_2$-torsor $\mathbb{K}_2 \to \overline{\mathbb{G}} \to \mathbb{G}$, with associated BD data $(Q, \mathcal{E}, f)$ or bisector data (D, η). By taking F-points, we obtain (since $H^1(F, \mathbb{K}_2) = 0$) a short exact sequence of abstract groups

$$1 \longrightarrow \mathbb{K}_2(F) \longrightarrow \overline{\mathbb{G}}(F) \longrightarrow G = \mathbb{G}(F) \longrightarrow 1.$$

Now let $\mu(F)$ denote the set of roots of unity contained in the local field $F \neq \mathbb{C}$; when $F = \mathbb{C}$, we let $\mu(F)$ be the trivial group. Then the Hilbert symbol gives a map

$$(-,-)_F : \mathbb{K}_2(F) \longrightarrow \mu(F).$$

For any n dividing $\#\mu(F)$, one has the n-th Hilbert symbol

$$(-,-)_n = (-,-)_F^{\#\mu(F)/n} : \mathbb{K}_2(F) \longrightarrow \mu_n(F).$$

Pushing the above exact sequence out by the n-th Hilbert symbol, one obtains a short exact sequence of locally compact topological groups

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

We shall call this the BD covering group associated to the BD data $(Q, \mathcal{E}, f, n)$, or to the bisector data (D, η, n).

Since we are considering degree n covers, it will be useful to refine certain notions taking into account the extra data n:

- for a bisector data (D, η), we write η_n for the composite

$$\eta_n : Y^{sc} \longrightarrow F^\times \longrightarrow F^\times / F^{\times n}.$$

- with D fixed, we say that η_n and η'_n are equivalent if η_n/η'_n extends to a homomorphism $Y \longrightarrow F^\times/F^{\times n}$.

3.2. **Canonical unipotent section.** Because a BD extension is uniquely split over any unipotent subgroup, one has unique splittings:

$$\tilde{x}_\alpha : F \longrightarrow \overline{U}_\alpha \quad \text{for each } \alpha \in \Phi.$$

Indeed, as shown in [MW, Appendix I] and [L2], there is a unique section

$$i : \{\text{all unipotent elements of } G\} \longrightarrow \overline{G}$$

satisfying:

- for each unipotent subgroup $\mathbb{U} \subset \mathbb{G}$, the restriction of i to $U = \mathbb{U}(F)$ is a group homomorphism;
- the map i is G-equivariant.

For example, for each $\alpha \in \Phi$, we have seen that there is a homomorphism

$$\tilde{x}_\alpha : \mathbb{G}_a \longrightarrow \overline{\mathbb{G}}$$

which induces a homomorphism

$$\tilde{x}_\alpha : F \longrightarrow \overline{G}$$

lifting the inclusion $x_\alpha : F \hookrightarrow G$. Then one has $\tilde{x}_\alpha = i \circ x_\alpha$.

3.3. **Covering torus $\overline{T}$.** We may consider the pullback of $\overline{G}$ to the maximal split torus T:

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{T} \longrightarrow T \longrightarrow 1.$$

As we observed in the last section, the bisector D furnishes a cocycle for the multiplicative $\mathbb{K}_2$-torsor on $\mathbb{T}$. Thus the covering torus $\overline{T}$ also inherits a nice cocycle, giving us a rather concrete description of $\overline{T}$.

More precisely, $\overline{T} = T \times_D \mu_n(F)$ is generated by elements $\zeta \in \mu_n(F)$ and $y(a)$, for $y \in Y$ and $a \in F^\times$, subject to the relations:

- the elements ζ are central;
- $[y_1(a), y_2(b)] = (a,b)_n^{B_Q(y_1,y_2)}$ for all $y_1, y_2 \in Y$ and $a, b \in F^\times$;
- $y_1(a) \cdot y_2(a) = (y_1 + y_2)(a) \cdot (a,a)_n^{D(y_1,y_2)}$;
- $y(a) \cdot y(b) = y(ab) \cdot (a,b)_n^{Q(y)}$.

As the second relation shows, $\overline{T}$ is not necessarily an abelian group. Moreover, the sections $q \circ s_\alpha$ for $\alpha \in \Delta$ takes the form

$$q \circ s_\alpha(\alpha^\vee(a)) = \alpha^\vee(a) \cdot (\eta(\alpha^\vee), a)_n.$$

3.4. **The torus $T_{Q,n}$.** The center $Z(\overline{T})$ of $\overline{T}$ is generated by $\mu_n(F)$ and those elements $y(a)$ such that

$$B_Q(y,z) \in n\mathbb{Z} \quad \text{for all } z \in Y.$$

Thus, we define:

$$Y_{Q,n} := Y \cap nY^*,$$

where $Y^* \subset Y \otimes_{\mathbb{Z}} \mathbb{Q}$ is the dual lattice of Y relative to B_Q. Then the center of $\overline{T}$ is generated by $\mu_n(F)$ and the elements $y(a)$ for all $y \in Y_{Q,n}$ and $a \in F^\times$. It is clear that $nY \subset Y_{Q,n}$.

Let $\mathbb{T}_{Q,n}$ be the split torus with cocharacter group $Y_{Q,n}$ and $T_{Q,n} := \mathbb{T}_{Q,n}(F)$. The inclusion $Y_{Q,n} \hookrightarrow Y$ gives an isogeny of tori

$$h : T_{Q,n} \longrightarrow T.$$

We may pullback the covering $\overline{T}$ using h, thus obtaining a covering torus $\overline{T}_{Q,n}$:

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{T}_{Q,n} \longrightarrow T_{Q,n} \longrightarrow 1.$$

Then $\overline{T}_{Q,n}$ is generated by $\zeta \in \mu_n(F)$ and elements $y(a)$ with $y \in Y_{Q,n}$ with the same relations as those for $\overline{T}$. However, the second relation now becomes

$$[y_1(a), y_2(b)] = 1, \quad \text{for all } y_1, y_2 \in Y_{Q,n} \text{ and } a, b \in F^\times.$$

Thus $\overline{T}_{Q,n}$ is an abelian group. Moreover, it follows from the definition of pullbacks that there is a canonical homomorphism $\mathrm{Ker}(h) \to \overline{T}_{Q,n}$, so that one has a short exact sequence of topological groups

$$1 \longrightarrow \mathrm{Ker}(h) \longrightarrow \overline{T}_{Q,n} \longrightarrow Z(\overline{T}) \longrightarrow 1.$$

In particular, to give a character of $Z(\overline{T})$ is to give a character of $\overline{T}_{Q,n}$ trivial on the subgroup $\mathrm{Ker}(h)$.

3.5. **The kernel of h.** We need to have a better handle of $\mathrm{Ker}(h)$. The inclusions $nY \longrightarrow Y_{Q,n} \longrightarrow Y$ give rise to isogenies

$$T \xrightarrow{\;g\;} T_{Q,n} \xrightarrow{\;h\;} T$$

so that $h \circ g$ is the n-power map on T. We have:

Lemma 3.1. *The kernel of h is contained in the image of g. Indeed,* $\mathrm{Ker}(h) = g(T[n])$.

Proof. By the elementary divisor theorem, we may pick a basis $\{e_i\}$ of Y such that a basis of $Y_{Q,n}$ is given by $\{k_i e_i\}$ for some positive integers k_i. Such bases allow us to identify the maps

$$T = (F^\times)^r \xrightarrow{\;g\;} T_{Q,n} = (F^\times)^r \xrightarrow{\;h\;} T = (F^\times)^r$$

explicitly as

$$g(t_i) = (t_i^{n/k_i}) \quad \text{and} \quad h(t_i) = (t_i^{k_i}).$$

Thus,

$$\mathrm{Ker}(h) = \{(\zeta_i) : \zeta_i^{k_i} = 1\} = g(T[n]) = \{(\zeta_i^{n/k_i}) : \zeta_i^n = 1\}.$$

□

Using the generators and relations for $\overline{T}_{Q,n}$, it is easy to see that the map

$$y(a) \mapsto (ny)(a) \in \overline{T}_{Q,n}$$

gives a group homomorphism $\tilde{g} : T \longrightarrow \overline{T}_{Q,n}$. The lemma implies that a character of $\overline{T}_{Q,n}$ trivial on the image of $\tilde{g}$ necessarily factors through to a character of $Z(\overline{T})$.

4. **Tame Case**

Suppose now that F is a p-adic field with residue field κ. Assume that p does not divide n: we shall call this the tame case. The main question we want to consider in this section is whether a tame BD cover $\overline{G}$ necessarily splits over a hyperspecial maximal compact subgroup K of G. Note that the $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ over F may not be defined over $\mathcal{O}_F$. Otherwise the splitting of K into $\overline{G}$ is canonical, as discussed in [W7, §7].

Since we have fixed a Chevalley system of épinglage for $\mathbb{G}$, we have its associated maximal compact subgroup K generated by $x_\alpha(\mathcal{O}_F)$ for all $\alpha \in \Phi$ and the maximal compact subgroup $\mathbb{T}(\mathcal{O}) = Y \otimes_{\mathbb{Z}} \mathcal{O}^\times$ of T. In particular $K = \underline{\mathbb{G}}(\mathcal{O})$ for a smooth reductive group $\underline{\mathbb{G}}$ over $\mathcal{O}$. To ease notation, we shall simply write $\mathbb{G}$ for $\underline{\mathbb{G}}$ in what follows. Let $\mathbb{G}_\kappa$ denote the special fiber $\mathbb{G} \times_{\mathcal{O}} \kappa$ of $\mathbb{G}$. One has a natural reduction map

$$\mathbb{G}(\mathcal{O}) \longrightarrow G_\kappa := \mathbb{G}_\kappa(\kappa),$$

whose kernel is a pro-p group. Restricting the BD cover to K, one has a topological central extension

$$1 \longrightarrow \mu_n \longrightarrow \overline{\mathbb{G}}(\mathcal{O}) \longrightarrow \mathbb{G}(\mathcal{O}) \longrightarrow 1.$$

Here, observe that we have abused notation and write $\overline{\mathbb{G}}(\mathcal{O})$ for the inverse image of $\mathbb{G}(\mathcal{O})$ in $\overline{G}$. We would like to determine if this extension splits.

4.1. **The tame extension.**

All extensions in the tame case arise in the following way from the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$. The prime-to-p part of $\mu(F)$ is naturally isomorphic to $\kappa^\times$, and there is a tame symbol $\mathbb{K}_2(F) \longrightarrow \kappa^\times$ defined by

$$\{a, b\} \mapsto \text{ the image of } (-1)^{\mathrm{ord}(a)\cdot \mathrm{ord}(b)} \cdot \frac{a^{\mathrm{ord}(b)}}{b^{\mathrm{ord}(a)}} \text{ in } \kappa^\times.$$

Pushing out by this tame symbol gives the tame extension

$$1 \longrightarrow \kappa^\times \longrightarrow \overline{G}^{\mathrm{tame}} \longrightarrow G \longrightarrow 1.$$

Hence any degree n BD extension $\overline{G}$ with $(n, p) = 1$ is obtained as a pushout of $\overline{G}^{\mathrm{tame}}$.

4.2. **Residual extension.**

We shall consider first the case of the tame extension $\overline{G}^{\mathrm{tame}}$ so that $n = \#\kappa^\times$. It was shown in [BD, §12] and [W2] that there is an extension of reductive algebraic groups over κ:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{\mathbb{G}}_\kappa \longrightarrow \mathbb{G}_\kappa \longrightarrow 1$$

with the following property:

- for any unramified extension F' of F with ring of integers $\mathcal{O}'$ and residue field κ', the tame extension
$$1 \longrightarrow \kappa'^{\times} \longrightarrow \overline{\mathbb{G}}(\mathcal{O}') \longrightarrow \underline{\mathbb{G}}(\mathcal{O}') \longrightarrow 1$$
is the pullback of the extension
$$1 \longrightarrow \mathbb{G}_m(\kappa') = \kappa'^{\times} \longrightarrow \tilde{G}_{\kappa'} = \tilde{\mathbb{G}}_{\kappa}(\kappa') \longrightarrow G_{\kappa'} = \mathbb{G}_{\kappa}(\kappa') \longrightarrow 1$$
with respect to the reduction map $\mathbb{G}(\mathcal{O}') \longrightarrow G_{\kappa'}$.

We call this extension of algebraic groups over κ the residual extension.

4.3. **Classification.** In [W2], algebraic extensions of $\mathbb{G}_{\kappa}$ by $\mathbb{G}_m$ were classified in terms of enhanced root theoretic data similar in spirit to (but simpler than) the BD data. We give a sketch in the case when $\mathbb{G}_{\kappa}$ is split. Then such extensions are classified by the category of pairs $(\mathcal{E}_{\kappa}, f_{\kappa})$ with

- $1 \to \mathbb{Z} \to \mathcal{E}_{\kappa} \to Y \to 1$ is an extension of free $\mathbb{Z}$-modules;
- $f_{\kappa} : Y^{sc} \longrightarrow \mathcal{E}_{\kappa}$ is a splitting of $\mathcal{E}_{\kappa}$ over Y^{sc}.

Moreover, the extension $\tilde{\mathbb{G}}_{\kappa}$ in question is split if and only if the map f_{κ} can be extended to a splitting $Y \longrightarrow \mathcal{E}_{\kappa}$, or equivalently, if and only if the extension
$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E}_{\kappa}/f_{\kappa}(Y^{sc}) \longrightarrow Y/Y^{sc} \longrightarrow 1$$
is split. This holds for example if Y/Y^{sc} is free. In particular, if $\mathbb{G} = \mathbb{G}^{sc}$ is simply connected, then $\tilde{\mathbb{G}}_{\kappa}$ is split and the splitting is unique.

Given an extension $\tilde{\mathbb{G}}_{\kappa}$, one obtains the above two data as follows. If $\mathbb{T}_{\kappa}$ is a maximal split torus of $\mathbb{G}_{\kappa}$, with preimage $\tilde{\mathbb{T}}_{\kappa}$, then the cocharacter lattice of $\tilde{\mathbb{T}}_{\kappa}$ gives the extension $\mathcal{E}_{\kappa}$. The pullback of $\tilde{\mathbb{G}}_{\kappa}$ to $\mathbb{G}_{\kappa}^{sc}$ is canonically split. On restricting this canonical splitting to the maximal split torus $\mathbb{T}_{\kappa}^{sc}$ (which is the pullback of $\mathbb{T}_{\kappa}$), one obtains the splitting f_{κ} on the level of cocharacter lattices.

4.4. **Splitting of $\overline{K}$.** Since the kernel of the reduction map is a pro-p group, the set of splittings of the topological extension $\overline{K} = \overline{\mathbb{G}}(\mathcal{O})$ is in bijection with those of the abstract extension $\tilde{G}_{\kappa}$. Further, a splitting of the residual extension $\tilde{\mathbb{G}}_{\kappa}$ gives rise to a splitting of $\tilde{G}_{\kappa}$ and thus of $\overline{\mathbb{G}}(\mathcal{O})$. We will investigate the existence of splittings for $\tilde{\mathbb{G}}_{\kappa}$: they give rise to splittings of $\overline{\mathbb{G}}(\mathcal{O})$ of "algebraic origin".

For example, when $\mathbb{G} = \mathbb{G}^{sc}$ is simply-connected, the unique splitting of the residual extension $\tilde{\mathbb{G}}_{\kappa}$ gives rise to a unique compatible system of splittings of $\mathbb{G}(\mathcal{O}')$ for all unramified extensions $\mathcal{O}'$ of $\mathcal{O}$. Indeed, one has a natural bijection
$$\{\text{splittings of residual extension } \tilde{\mathbb{G}}_{\kappa}\} \longleftrightarrow \{\text{compatible system of splittings of } \mathbb{G}(\mathcal{O}')\}.$$

4.5. **Determining $\tilde{\mathbb{G}}_{\kappa}$.** One can now figure out the residual extension $\tilde{\mathbb{G}}_{\kappa}$ obtained from a BD extension $\overline{\mathbb{G}}$ with associated BD data $(Q, \mathcal{E}, f)$.

Proposition 4.1. *If $\overline{\mathbb{G}}$ has BD data $(Q, \mathcal{E}, f)$, then the associated data $(\mathcal{E}_{\kappa}, f_{\kappa})$ for the residual extension $\tilde{\mathbb{G}}_{\kappa}$ is obtained as follows:*

- *$\mathcal{E}_{\kappa}$ is the pushout of $\mathcal{E}$ by the valuation map* $\mathrm{ord} : F^{\times} \longrightarrow \mathbb{Z}$*;*

- f_κ *is deduced from the associated map* $\mathrm{ord}_* \circ f : \mathcal{E}_{Q^{sc}} \to \mathcal{E} \to \mathcal{E}_\kappa$*:*

$$f_\kappa(\alpha^\vee) = \mathrm{ord}_*(f(s_{Q^{sc}}(\alpha^\vee))) \quad \textit{for } \alpha \in \Delta.$$

Proof. This question is systematically and more elegantly addressed in [W6], and we shall give a more ad hoc argument here. We assume that $\overline{\mathbb{G}}$ is incarnated by (D, η) for concreteness. As we discussed above, for any unramified extension $\mathcal{O}'$ of $\mathcal{O}$, there is a commutative diagram of extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \kappa'^\times & \longrightarrow & \overline{\mathbb{G}}(\mathcal{O}') & \longrightarrow & \mathbb{G}(\mathcal{O}') & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m(\kappa') & \longrightarrow & \tilde{\mathbb{G}}_\kappa(\kappa') & \longrightarrow & \mathbb{G}_\kappa(\kappa') & \longrightarrow & 1, \end{array}$$

and our goal is to determine the invariants $(\mathcal{E}_\kappa, f_\kappa)$ for the extension $\tilde{\mathbb{G}}_\kappa$.

Now we note:

(a) From the construction in [BD, §12.11], one has a commutative diagram of extensions and splittings (s_κ of $\tilde{\mathbb{G}}_\kappa^{sc}$) over κ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T}_\kappa^{sc} & \xrightarrow{s_\kappa} & \tilde{\mathbb{T}}_\kappa^{sc} & \longrightarrow & \tilde{\mathbb{T}}_\kappa & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_\kappa & \xrightarrow{s_\kappa} & \tilde{\mathbb{G}}_\kappa^{sc} & \longrightarrow & \tilde{\mathbb{G}}_\kappa & \longrightarrow & 1 \end{array}$$

pulling back to a compatible system

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T}^{sc}(\mathcal{O}') & \xrightarrow{s_{\mathcal{O}'}} & \overline{\mathbb{T}}^{sc}(\mathcal{O}') & \longrightarrow & \overline{\mathbb{T}}(\mathcal{O}') & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}^{sc}(\mathcal{O}') & \xrightarrow{s_{\mathcal{O}'}} & \overline{\mathbb{G}}^{sc}(\mathcal{O}') & \longrightarrow & \overline{\mathbb{G}}(\mathcal{O}') & \longrightarrow & 1. \end{array}$$

(b) Using the description of $\overline{\mathbb{T}}$ in terms of (D, η), one sees immediately that $\overline{\mathbb{T}}(\mathcal{O}') = \mathbb{T}(\mathcal{O}') \times \kappa'^\times$ (a direct product of groups), since the tame symbol is trivial on $\mathcal{O}'^\times \times \mathcal{O}'^\times$. Thus $\tilde{\mathbb{T}}_\kappa = \mathbb{T}_\kappa \times \mathbb{G}_m$. From this, one deduces that

$$\mathcal{E}_\kappa = Y \times \mathbb{Z} \quad \text{(as groups)}.$$

(c) The invariant f_κ is deduced from the composite map

$$\tilde{f}_\kappa : \mathbb{T}_\kappa^{sc} \xrightarrow{s_\kappa} \tilde{\mathbb{T}}_\kappa^{sc} \longrightarrow \tilde{\mathbb{T}}_\kappa = \mathbb{T}_\kappa \times \mathbb{G}_m$$

from (a). Suppose that

$$f_\kappa \circ \alpha^\vee : t \mapsto (\alpha^\vee(t), t^{m_\alpha}) \quad \text{with } m_\alpha \in \mathbb{Z} \text{ and for } t \in \mathbb{G}_m.$$

Then we need to show that

$$m_\alpha = \mathrm{ord}(\eta(\alpha^\vee)) \quad \text{for all } \alpha \in \Delta..$$

(d) Now the splitting $s_\kappa : \mathbb{G}_\kappa^{sc} \longrightarrow \tilde{\mathbb{G}}_\kappa^{sc}$ from (a) is uniquely determined by its restriction to the root subgroups $\mathrm{U}_{\alpha,\kappa}$ for $\alpha \in \Delta$. Hence s is determined by $s_\kappa \circ x_\alpha$, where $x_\alpha : \mathbb{G}_a \to \mathrm{U}_\alpha$ is part of the fixed épinglage. Since

$$\alpha^\vee(t) = n_\alpha(t) \cdot n_\alpha(-1) \in \mathbb{T}_\kappa^{sc} \subset \mathbb{T}_\kappa \quad \text{with} \quad n_\alpha(t) = x_\alpha(t) \cdot x_{-\alpha}(-t^{-1}) \cdot x_\alpha(t),$$

this implies that

$$(\alpha^\vee(t), t^{m_\alpha}) = f_\kappa \circ \alpha^\vee(t) = \text{image of } n_\alpha(t) \cdot n_\alpha(-1) \text{ in } \tilde{\mathbb{T}}_\kappa.$$

(e) Likewise, the induced system of splittings $s_{\mathcal{O}'} : \mathbb{G}^{sc}(\mathcal{O}') \to \overline{\mathbb{G}}^{sc}(\mathcal{O}')$ is determined by the unique splitting

$$\tilde{x}_\alpha : F' \longrightarrow \overline{G}_{F'}^{\text{tame}} \quad \text{for all } \alpha \in \Delta.$$

This implies that the composite

$$f_{\mathcal{O}'} : \mathbb{T}^{sc}(\mathcal{O}') \xrightarrow{s_{\mathcal{O}'}} \overline{\mathbb{T}}^{sc}(\mathcal{O}') \longrightarrow \overline{\mathbb{T}}(\mathcal{O}')$$

from (a) is given by

$$f_{\mathcal{O}'} \circ \alpha^\vee(\tilde{t}) = \text{image of } \tilde{n}_\alpha(\tilde{t}) \cdot \tilde{n}_\alpha(-1) \text{ in } \overline{T}(\mathcal{O}').$$

The RHS is nothing but the section

$$s_\alpha(\tilde{t}) = (\alpha^\vee(\tilde{t}), (\eta(\alpha^\vee), \tilde{t})_n) \in \mathbb{T}(\mathcal{O}')$$

for $\tilde{t} \in \mathcal{O}'^\times$, whose image under the reduction map is

$$(\alpha^\vee(t), t^{\mathrm{ord}(\eta(\alpha^\vee))}) \in \tilde{T}_\kappa(\kappa') = \mathbb{T}_\kappa(\kappa') \times \kappa'^\times.$$

By (a),

$$f_\kappa \circ \alpha^\vee(t) = \text{the image of } f_{\mathcal{O}'} \circ \alpha^\vee(\tilde{t}) \text{ under reduction map.}$$

Hence, it follows that

$$m_\alpha = \mathrm{ord}(\eta(\alpha^\vee))$$

for $\alpha \in \Delta$, as desired.

□

In terms of the bisector data (D, η), one has $\mathcal{E} = Y \times_D F^\times$ and $f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee))$ for $\alpha \in \Delta$. On pushing out by the valuation map, one has:

$$\mathcal{E}_\kappa = Y \times \mathbb{Z} \quad \text{(direct product of groups)}$$

and

$$f_\kappa(\alpha^\vee) = (\alpha^\vee, \mathrm{ord}(\eta(\alpha^\vee))) \quad \text{for } \alpha \in \Delta.$$

In particular, one has:

Theorem 4.2. *If Y/Y^{sc} is free or if η takes value in $\mathcal{O}_F^\times$, then the algebraic extension $\tilde{\mathbb{G}}_\kappa$ is split. Thus, the topological central extension $\overline{K}$ of K is also split.*

We have assumed that $n = \#\kappa^\times$ above. In general, when p does not divide n, the n-th Hilbert symbol map $\mathbb{K}_2(F) \longrightarrow \mu_n$ factors through $\mathbb{K}_2(F) \longrightarrow \kappa^\times \longrightarrow \mu_n$. So the degree n BD covering $\overline{\mathbb{G}}$ is obtained from the one of degree $\#\kappa^\times$ as a pushout. In particular, when the conditions of the above corollary holds, the degree n cover $\overline{K}$ is split as well. Indeed, whenever η takes value in $\mathcal{O}^\times \cdot F^{\times n}$, the cover $\overline{K}$ splits.

Note that we have merely given some simple sufficient conditions for $\overline{K}$ to be split. These conditions may not be necessary in a given case, but as we will see below, it is possible for $\overline{K}$ to be non-split when they fail. Moreover, note that the splitting of $\overline{K}$ is not necessarily unique (if it exists).

4.6. **Running example.** We illustrate the discussion in this section with our running example: $\mathbb{G} = \mathrm{PGL}_2$, $D = 0$ and $n = 2$. Then we have the BD extensions

$$\overline{\mathbb{G}}_\eta = (\mathrm{GL}_2 \times \mathbb{K}_2)/\{(z, \{\eta, z\}) : z \in \mathbb{G}_m\}.$$

The associated BD covering groups are:

$$\overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\}.$$

Let $\pi_\eta : \mathrm{GL}_2(F) \times \mu_2 \longrightarrow \overline{G}_\eta$ be the natural projection map, and let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in F^\times \right\} \subset \mathrm{GL}_2(F).$$

The projection map identifies A with a maximal split torus T of $\mathrm{PGL}_2(F)$ and π_η identifies $A \times \mu_2$ with $\overline{T}$ of $\mathrm{PGL}_2(F)$. In this case, $\overline{T}$ is abelian (since $D = 0$) and so $\overline{T} = \overline{T}_{Q,2}$ and $f : T_{Q,2} \longrightarrow T$ is the identity map.

Now consider the issue of whether the covering splits over $K = \mathrm{PGL}_2(\mathcal{O})$. We have already seen from general arguments that it does when $\eta \in \mathcal{O}^\times \cdot F^{\times 2}$. When $\eta = \varpi$ is a uniformizer, we shall show now that the covering $\overline{K}_\eta$ is not split.

If a splitting $K \longrightarrow \overline{K}_\eta$ exists, we would have a group homomorphism

$$\phi : \mathrm{GL}_2(\mathcal{O}) \longrightarrow (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\}$$

which is trivial on the center $Z(\mathcal{O})$ of $\mathrm{GL}_2(\mathcal{O})$. For $k \in \mathrm{GL}_2(\mathcal{O})$, we may write

$$\phi(k) = \text{the class of } (k, \mu(k))$$

for some $\mu(k) = \pm 1$. Now it is easy to check that $\mu : \mathrm{GL}_2(\mathcal{O}) \longrightarrow \mathcal{O}^\times/\mathcal{O}^{\times 2} = \{\pm 1\}$ is a group homomorphism and thus μ factors as

$$\mu : \mathrm{GL}_2(\mathcal{O}) \xrightarrow{\det} \mathcal{O}^\times \longrightarrow \kappa^\times \longrightarrow \pm 1.$$

If now $k = z \in Z(\mathcal{O})$ is a scalar matrix, then the fact that $\phi(z)$ is trivial means that

$$\mu(z) = (\varpi, z)_2.$$

Since μ factors through det, we see that $\mu(z) = 1$, but $(\varpi, z)_2$ is not 1 for some $z \in \mathcal{O}^\times$. With this contradiction, we see that the covering $\overline{K}_\eta$ is not split when $\eta = \varpi$ is a uniformizer.

5. Dual and L-Groups

In this section, we shall recall the definition of the L-group ${}^L\overline{G}$ of a BD covering $\overline{G}$ for a split $\mathbb{G}$ over a local field, following Weissman [W3, W4]. The construction in [W3] is quite involved, using a double twisting of the Hopf algebra of a candidate dual group. Moreover, it turns out to give the "correct" L-group only in the case (D,η) with η trivial. In a letter to Deligne [W4], Weissman gave a revision of his construction in [W3], using fully the BD data $(Q,\mathcal{E},f,n)$ or the bisector data (D,η). We shall follow this more streamlined treatment in [W4]. We note that the construction in [W4] is subsequently extended to the case of quasi-split $\mathbb{G}$, using the language of étale gerbes, and we again caution the reader that these constructions may continue to evolve as the subject develops.

Since we will simply be presenting the construction of these objects in this section, the definition of the dual group or L-group of $\overline{G}$ may seem rather unmotivated at the end of the section. Whether they are the right objects or not will largely depend on whether they give the right framework to describe the representation theory of $\overline{G}$. In the subsequent sections, we will address these concerns.

5.1. **Dual group à la Finkelberg-Lysenko-McNamara-Reich.** Let $\mathbb{G}$ be a split connected reductive group over F, with maximal split torus $\mathbb{T}$ and cocharacter lattice Y. Let $\Phi^\vee \subset Y$ be the set of coroots of $\mathbb{G}$ and let $Y^{sc} \subset Y$ be the sublattice generated by $\Phi^\vee$. Similarly, let $\Phi \subset X$ be the set of roots generating a sublattice X^{sc} in the character lattice X of $\mathbb{T}$.

Suppose that $\overline{\mathbb{G}}$ is a multiplicative $\mathbb{K}_2$-torsor with associated a BD data $(Q,\mathcal{E},\phi)$. The data $(Q,\mathcal{E},\phi,n)$ (with $|\mu_n(F)|=n$) then gives a central extension of locally compact groups

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

Using the data $(Y,\Phi^\vee,Q,n)$, we may define a modified root datum as follows:

- we have already set
$$Y_{Q,n} = Y \cap nY^*$$
where $Y^* \subset Y \otimes_{\mathbb{Z}} \mathbb{Q}$ is the dual lattice of Y relative to B_Q. Let $X_{Q,n} \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual lattice to $Y_{Q,n}$.
- for each $\alpha^\vee \in \Phi^\vee$, set
$$n_\alpha = \frac{n}{\gcd(n, Q(\alpha^\vee))},$$
and
$$\alpha^\vee_{Q,n} = n_\alpha \cdot \alpha^\vee.$$
Denote by $\Phi^\vee_{Q,n}$ the set of such $\alpha^\vee_{Q,n}$'s and observe that
$$\Phi^\vee_{Q,n} \subset Y_{Q,n}.$$
We let $Y^{sc}_{Q,n}$ denote the sublattice of $Y_{Q,n}$ generated by $\Phi^\vee_{Q,n}$.
- likewise, for $\alpha \in \Phi$, set
$$\alpha_{Q,n} = n_\alpha^{-1} \cdot \alpha$$
and denote by $\Phi_{Q,n}$ the set of such $\alpha_{Q,n}$'s, so that $\Phi_{Q,n} \subset X_{Q,n}$.

Then it was shown in [Mc2] and [W3] that the quadruple $(Y_{Q,n}, \Phi^\vee_{Q,n}, X_{Q,n}, \Phi_{Q,n})$ is a root datum, and hence determine a split connected reductive group $\overline{G}^\vee$ over $\mathbb{C}$. *The group $\overline{G}^\vee$ is by definition the dual group of the BD extension $\overline{G}$.* Observe that it only depends on (Q,n) and is independent of the third ingredient f of a BD data $(Q,\mathcal{E},f)$; equivalently, it only depends on (D,n) but not on η.

Let $Z(\overline{G}^\vee)$ be the center of $\overline{G}^\vee$. Then note that

$$Z(\overline{G}^\vee) = \mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times).$$

5.2. **L-group à la Weissman.** We can now describe Weissman's proposal for the L-group of $\overline{G}$ in [W4]. This is done by defining an extension

$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow E \longrightarrow F^\times/F^{\times n} \longrightarrow 1$$

followed by pushing out by the natural inclusion $Z(\overline{G}^\vee) \to \overline{G}^\vee$ and pulling back via the natural projection $W_F \to F^\times/F^{\times n}$. This results in an extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1,$$

which we call Weissman's L-group extension. Observe that by construction, the extension ${}^L\overline{G}$ is equipped with a canonical splitting over the finite-index subgroup

$$W_{F,n} = \mathrm{Ker}(W_F \longrightarrow F^\times/F^{\times n}).$$

The construction of E is as a Baer sum $E_1 + E_2$ of two extensions E_1 and E_2. These are defined as follows:

- E_1 is defined explicitly using the cocycle

  $$c_1 : F^\times/F^{\times n} \times F^\times/F^{\times n} \longrightarrow Z(\overline{G}^\vee) = \mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times)$$

  given by

  $$c_1(a,b)(y) = (a,b)_n^{Q(y)}.$$

  Since $2\cdot Q(y) = B_Q(y,y) \in n\mathbb{Z}$ for $y \in Y_{Q,n}$, we see that this cocycle is trivial when n is odd, and is valued in the 2-torsion subgroup $Z(\overline{G}^\vee)[2]$ when n is even. Note that E_1 depends only on (Q,n).

  Here is another description of E_1. Set

  $$m = \begin{cases} n \text{ if } n \text{ is odd;} \\ n/2 \text{ if } n \text{ is even.} \end{cases}$$

  Consider the extension

  $$1 \longrightarrow \mu_2 \longrightarrow E_0 \longrightarrow F^\times/F^{\times n} \longrightarrow 1$$

  defined by the cocycle

  $$c(a,b) = (a,b)_n^m.$$

  Let

  $$j : \mu_2 \longrightarrow Z(\overline{G}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times)$$

be the homomorphism defined by

$$j(-1)(y) = e^{2\pi i \cdot \frac{Q(y)}{n}} = (-1)^{\frac{2}{n}Q(y)} \in \mathbb{C}^\times.$$

This is a homomorphism because the map $y \mapsto \frac{Q(y)}{n}$ is a group homomorphsim $Y_{Q,n} \longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and $Q(y) \in n\mathbb{Z}$ for all $y \in Y_{Q,n}^{sc}$. Then E_1 is the pushout of E_0 by j.

- the construction of E_2 is slightly more involved and uses the full BD data $(Q, \mathcal{E}, f)$, where we recall that $\mathcal{E}$ is an extension

$$1 \longrightarrow F^\times \longrightarrow \mathcal{E} \longrightarrow Y \longrightarrow 1,$$

and

$$f : \mathcal{E}_{Q^{sc}} \longrightarrow q^*(\mathcal{E})$$

is an isomorphism, with $q : Y^{sc} \to Y$ the natural map.

Since we have the inclusion $Y_{Q,n}^{sc} \to Y_{Q,n} \to Y$, we may pullback the extensions $\mathcal{E}_{Q^{sc}}$ and $\mathcal{E}$ and pushout via $F^\times \longrightarrow F^\times/F^{\times n}$ to obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q^{sc},n} & \longrightarrow & Y_{Q,n}^{sc} & \longrightarrow & 1 \\ & & \| & & \downarrow f & & \downarrow & & \\ 1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q,n} & \longrightarrow & Y_{Q,n} & \longrightarrow & 1. \end{array}$$

Note that both $\mathcal{E}_{Q,n}$ and $\mathcal{E}_{Q^{sc},n}$ are abelian groups.

For each $\alpha^\vee \in \Phi^\vee \subset Y^{sc}$, we have defined before an element $s_{Q^{sc}}(\alpha^\vee) \in \mathcal{E}_{Q^{sc}}$ lying over $\alpha^\vee$. Indeed, $s_{Q^{sc}}(\alpha^\vee)$ is the image of the element $s_\alpha(\alpha^\vee(\tau)) \in \overline{\mathbb{T}}^{sc}(F(\!(\tau)\!))$ under pushout by the residue map $\mathrm{Res} : K_2(F(\!(\tau)\!)) \longrightarrow F^\times$. Analogously, we have the element $s_{Q^{sc}}(n_\alpha \cdot \alpha^\vee) \in \mathcal{E}_{Q^{sc}}$ which is the image of the element

$$s_\alpha((n_\alpha \cdot \alpha^\vee)(\tau)) = s_\alpha(\alpha^\vee(\tau^{n_\alpha})) \in \overline{\mathbb{T}}^{sc}(F(\!(\tau)\!)).$$

It lies over $\alpha_{Q,n}^\vee = n_\alpha \alpha^\vee \in Y_{Q,n}^{sc}$. Weissman showed that this induces a group homomorphism

$$s_{Q^{sc}} : Y_{Q,n}^{sc} \longrightarrow \mathcal{E}_{Q^{sc},n}.$$

Composing this with f, one obtains

$$s_f = f \circ s_{Q^{sc}} : Y_{Q,n}^{sc} \longrightarrow \mathcal{E}_{Q,n}.$$

Viewing $Y_{Q,n}^{sc}$ as a subgroup of $\mathcal{E}_{Q,n}$ by the splitting s_f, we inherit an extension

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \overline{\mathcal{E}}_{Q,n} = \mathcal{E}_{Q,n}/s_f(Y_{Q,n}^{sc}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{sc} \longrightarrow 1. \tag{5.1}$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ (which is exact, since $\mathbb{C}^\times$ is divisible and hence injective), we obtain the desired extension:

$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow E_2 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F^\times/F^{\times n}, \mathbb{C}^\times) \cong F^\times/F^{\times n} \longrightarrow 1 \tag{5.2}$$

where the last isomorphism is via the n-th Hilbert symbol: $a \in F^\times/F^{\times n}$ giving rise to the character $\chi_a : b \mapsto (b, a)_n$.

5.3. **Description using bisectors.** We may describe the construction of E_2 in terms of the bisector data (D,η). The bisector D allows us to realise the extension $\mathcal{E}$ as a set $Y \times F^\times$ with group law

$$(y_1, a)\cdot(y_2, b) = (y_1+y_2, ab\cdot(-1)^{D(y_1,y_2)}).$$

Pushing this out by $F^\times \to F^\times/F^{\times n}$ and pulling back to $Y_{Q,n}$ gives the extension $\mathcal{E}_{Q,n} = Y_{Q,n}\times F^\times/F^{\times n}$ with the same group law as above. In particular, if n is odd, $-1\in F^{\times n}$ so that the cocycle $(-1)^{D(y_1,y_2)}$ is trivial.

The map $f:\mathcal{E}_{Q^{sc}} \longrightarrow \mathcal{E}$ is defined by

$$f(s_{Q^{sc}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)) \in Y\times F^\times, \quad \text{for } \alpha\in\Delta.$$

Then the splitting s_f is given by

$$s_f(\alpha^\vee_{Q,n}) = (\alpha^\vee_{Q,n}, \eta(\alpha^\vee_{Q,n})) \in Y_{Q,n}\times F^\times, \quad \text{for } \alpha\in\Delta.$$

It is instructive to note that the above constructions are functorial in nature. Given any isomorphism $\xi:(D,\eta)\longrightarrow(D',\eta')$, ξ carries the map s_f corresponding to (D,η) to the map $s_{f'}$ corresponding to (D',η').

5.4. **Running example.** Again, we illustrate the discussion in this section using our running example: $G=\mathrm{PGL}_2(F)$, $Q=0$ and $n=2$. In this case, $Y=\mathbb{Z}=Y_{Q,n}$ and $Y^{sc}=2\mathbb{Z}=Y^{sc}_{Q,n}$. So

$$Z(\overline{G}^\vee_\eta) = \mu_2 \subset \overline{G}^\vee_\eta = \mathrm{SL}_2(\mathbb{C}) \quad \text{for any } \eta.$$

Moreover, E_1^η is the split extension $Z(\overline{G}^\vee_\eta)\times F^\times/F^{\times 2}$ and $\mathcal{E}_\eta = Y\times F^\times/F^{\times 2}$ is split. Hence

$$\overline{\mathcal{E}}_\eta = (\mathbb{Z}\times F^\times/F^{\times 2})/\{(2y,\eta^y): y\in\mathbb{Z}\}$$

and

$$E_1^\eta + E_2^\eta = \mathrm{Hom}(\overline{\mathcal{E}}_\eta, \mathbb{C}^\times) = \{(t,a)\in\mathbb{C}^\times\times F^\times/F^{\times 2} : t^2 = (\eta,a)_2\}.$$

This contains $Z(\overline{G}^\vee_\eta)=\mu_2$ as the subgroup of elements $(\pm1,1)$, and the associated quotient is via the second projection to $F^\times/F^{\times 2}$.

Observe that when $F=\mathbb{R}$ and $\eta=-1\in\mathbb{R}^\times$, then $E_1^\eta+E_2^\eta$ is the cyclic group μ_4 and so the above extension is non-split! However, when we push out via the natural map $Z(\overline{G}^\vee_\eta)=\mu_2 \hookrightarrow \overline{T}^\vee_\eta = \mathbb{C}^\times$, then the pushout sequence is split. Indeed the sequence splits once we pushout by $\mu_2\hookrightarrow\mu_4$.

For general local field F, one sees that when one pushes $\overline{\mathcal{E}}_\eta$ out by $\mu_2\hookrightarrow \mathrm{SL}_2(\mathbb{C})$, one obtains the short exact sequence:

$$\mathrm{SL}_2(\mathbb{C}) \longrightarrow \{(g,a)\in \mathrm{GL}_2(\mathbb{C})\times F^\times/F^{\times 2} : \det(g) = (\eta,a)_2\} \longrightarrow F^\times/F^{\times 2}.$$

Pulling back to W_F, one obtains:

$${}^L\overline{G}_\eta = \{(g,w)\in\mathrm{GL}_2(\mathbb{C})\times W_F : \det(g)=\chi_\eta(w)\} \cong \mathrm{SL}_2(\mathbb{C})\rtimes_\eta W_F$$

where $w\in W_F$ acts on $\mathrm{SL}_2(\mathbb{C})$ by the conjugation action of the diagonal matrix $\mathrm{diag}(\chi_\eta(w),1)$. Thus, while the L-group extension is always split, ${}^L\overline{G}_\eta$ is not isomorphic to the direct product $\mathrm{SL}_2(\mathbb{C})\times W_F$ for general η.

5.5. **Functoriality for Levi subgroups.** Suppose that $\mathbb{M} \subset \mathbb{G}$ is a proper Levi subgroup, then a BD-covering $\overline{G}$ restricts to one on M. If the bisector data for $\overline{G}$ is (D, η, n), then that for $\overline{M}$ is $(D, \eta|_{Y^{sc}_{\mathbb{M}}}, n)$, where we have restricted η to the sublattice $Y^{sc}_{\mathbb{M}}$ generated by the simple coroots of M in Y. The above construction produces L-groups extensions ${}^L\overline{M}$ and ${}^L\overline{G}$. An examination of the construction shows:

Lemma 5.3. *This is a natural commutative diagram of short exact sequences:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_G & \longrightarrow & W_F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & Z(\overline{M}^\vee) & \longrightarrow & E_M & \longrightarrow & W_F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \overline{M}^\vee & \longrightarrow & {}^L\overline{M} & \longrightarrow & W_F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G} & \longrightarrow & W_F & \longrightarrow & 1
\end{array}$$

Proof. It suffices to exhibit a natural map from the extension E_G to E_M, which gives:

$$E_M \cong \frac{Z(\overline{M}^\vee) \times E_G}{\Delta Z(\overline{G}^\vee)}$$

and thus induces a map

$${}^L\overline{M} := \frac{\overline{M}^\vee \times E_M}{\Delta Z(\overline{M}^\vee)} \cong \frac{\overline{M}^\vee \times E_G}{\Delta Z(\overline{G}^\vee)} \longrightarrow \frac{\overline{G}^\vee \times E_G}{\Delta Z(\overline{G}^\vee)} =: {}^L\overline{G}$$

making the above diagram commute.

Write $E_G = E_1 + E_2$ and $E_M = E_{M,1} + E_{M,2}$ as Baer sums. Let $Y^{sc}_{M,Q,n} \subseteq Y^{sc}_{\mathbb{M}} \cap Y^{sc}_{Q,n}$ be the sublattice of $Y^{sc}_{\mathbb{M}}$ generated by $\alpha^\vee_{Q,n}$ for $\alpha^\vee \in Y^{sc}_{\mathbb{M}}$. The cocycle defining E_1 takes value in $\mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times)$, and the cocycle for $E_{M,1}$ takes the same formula and is valued in $\mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{M,Q,n}, \mathbb{C}^\times)$. Thus there is a natural map from E_1 to $E_{M,1}$, which in fact is canonically isomorphic to the push-out of E_1.

Consider E_2 and $E_{M,2}$. The pull-back of $\mathcal{E}_{Q^{sc},n}$ via $Y^{sc}_{M,Q,n} \subseteq Y^{sc}_{Q,n}$ gives an extension

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathcal{E}_{M,Q^{sc},n} \longrightarrow Y^{sc}_{M,Q,n} \longrightarrow 1$$

which has the splitting $s_{M,Q^{sc}}$ from the restriction of $s_{Q^{sc}}$ to $Y^{sc}_{M,Q,n}$. Let $s_{M,f} := f \circ s_{M,Q^{sc}}$ and $\overline{\mathcal{E}}_{M,Q,n} := \mathcal{E}_{M,Q^{sc},n}/s_{M,f}(Y^{sc}_{M,Q,n})$. Then one obtains the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \overline{\mathcal{E}}_{Q,n} & \longrightarrow & Y_{Q,n}/Y^{sc}_{Q,n} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \overline{\mathcal{E}}_{M,Q,n} & \longrightarrow & Y_{Q,n}/Y^{sc}_{M,Q,n} & \longrightarrow & 1.
\end{array}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ and followed by the identification $\mathrm{Hom}_{\mathbb{Z}}(F^\times/F^{\times n}, \mathbb{C}^\times) \simeq F^\times/F^{\times n}$ via the n-th Hilbert symbol, we obtain a natural map from E_2 to $E_{M,2}$ as in

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_2 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Z(\overline{M}^\vee) & \longrightarrow & E_{M,2} & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & 1. \end{array}$$

Combining the two extensions for the Baer sum, we see that there is a natural map from E_G to E_M. This gives a natural isomorphism

$$Z(\overline{M}^\vee) \times E_G/\Delta Z(\overline{G}^\vee) \longrightarrow E_M.$$

□

5.6. **Functoriality for z-extensions.** If

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1$$

is a z-extension, and $\overline{G}$ is a BD covering with bisector data (D, η), then we obtain a BD covering $\overline{H}$ with essentially the same bisector data, such that

$$1 \longrightarrow Z \stackrel{i}{\longrightarrow} \overline{H} \longrightarrow \overline{G} \longrightarrow 1.$$

From the construction of the L-group extension, one deduces:

Lemma 5.4. *There is a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G} & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{H}^\vee & \longrightarrow & {}^L\overline{H} & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & Z^\vee & = & Z^\vee. & & & & \end{array}$$

6. **Distinguished Splittings of L-Groups**

In this section, we study the L-group extension proposed in the previous section. More precisely, we will investigate whether this extension actually splits. Since the L-group extension is defined from the extension $E_1 + E_2$, we have seen that it has a canonical splitting over the finite index subgroup $W_{F,n} = \mathrm{Ker}(W_F \longrightarrow F^\times/F^{\times n})$ and it is natural to ask if this canonical splitting extends to W_F. This amounts to asking whether $E_1 + E_2$ splits: this is the question addressed in this section. We have seen in the last section that it does not in general, but we would like to understand where the obstruction lies. We shall assume that $\overline{\mathbb{G}}$ is defined by a pair (D, η) where D is a *fair* bisector.

6.1. **Splittings of $E_1 + E_2$.** What does it mean to give a splitting of $E_1 + E_2$? We first observe that when $Z(\overline{G}^\vee)$ is a finite group with order relatively prime to that of $F^\times/F^{\times n}$, then the abelian extensions E_1 and E_2 split uniquely over $F^\times/F^{\times n}$. Therefore, in this case $E_1 + E_2$ splits and one has ${}^L\overline{G} \simeq \overline{G}^\vee \times W_F$.

In general, since E_1 is given by an explicit cocycle c_1 (valued in $Z(\overline{G}^\vee)[2]$), the issue of equipping $E_1 + E_2$ with a splitting is equivalent to finding a set theoretic section of E_2 whose associated cocycle is equal to c_1. In addressing this question, we shall work explicitly using a bisector data (D, η).

In the construction of E_2 above, the extension (5.1) is equipped with a cocycle inherited from the fair bisector D. However, in taking $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ to obtain (5.2), we have partially lost the data of a cocycle. To obtain a cocycle, we need to choose a set theoretic section of (5.2). Hence, for each $a \in F^\times/F^{\times n}$, we need to extend the character

$$\chi_a \in \mathrm{Hom}_\mathbb{Z}(F^\times/F^{\times n}, \mathbb{C}^\times)$$

to a character $\tilde{\chi}_a$ of $\mathcal{E}_{Q,n}$. Another way of expressing this is to consider the pushout of $\mathcal{E}_{Q,n}$ by χ_a:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q,n} & \longrightarrow & Y_{Q,n} & \longrightarrow & 1 \\ & & \big\downarrow \chi_a & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \tilde{\mathcal{E}}_{Q,n} & \longrightarrow & Y_{Q,n} & \longrightarrow & 1. \end{array}$$

Then extending χ_a amounts to finding a genuine character of $\tilde{\mathcal{E}}_{Q,n}$ which is trivial when precomposed with $f \circ s$.

Note that $\tilde{\mathcal{E}}_{Q,n} = Y_{Q,n} \times \mu_n$ has group law

$$(y_1, 1) \cdot (y_2, 1) = (y_1 + y_2, \chi_a(-1)^{D(y_1,y_2)}) = (y_1 + y_2, (a, a)_n^{D(y_1,y_2)}).$$

Thus to give a genuine character $\tilde{\chi}_a$ of $\tilde{\mathcal{E}}_{Q,n}$ is to give a function

$$\tilde{\chi}_a : Y_{Q,n} \longrightarrow \mathbb{C}^\times$$

satisfying

$$\tilde{\chi}_a(y_1) \cdot \tilde{\chi}_a(y_2) = \tilde{\chi}_a(y_1 + y_2) \cdot (a, a)_n^{D(y_1,y_2)}.$$

For the composite $\tilde{\chi}_a \circ f \circ s$ to be trivial, one needs to require that

$$\tilde{\chi}_a(\alpha_{Q,n}^\vee) = (a, \eta(\alpha_{Q,n}^\vee))_n \quad \text{for all } \alpha \in \Delta.$$

Let us fix such a genuine character $\tilde{\chi}_a$ for each $a \in F^\times/F^{\times n}$. Using this as a set theoretic section for (5.2), we may write

$$E_2 = F^\times/F^{\times n} \times \mathrm{Hom}_\mathbb{Z}(Y_{Q,n}, \mathbb{C}^\times)$$

with cocycle

$$c_2(a, b)(y) = \tilde{\chi}_a(y)\tilde{\chi}_b(y)\tilde{\chi}_{ab}(y)^{-1} \quad \text{for all } y \in Y_{Q,n}.$$

To impose the condition that $c_2 = c_1$ means:

$$\tilde{\chi}_a(y)\tilde{\chi}_b(y) = \tilde{\chi}_{ab}(y) \cdot (a, b)_n^{Q(y)}$$

for all $y \in Y_{Q,n}$.

To summarise, we have shown:

Lemma 6.1. *Giving a splitting of* $E_1 + E_2$ *is equivalent to giving a function*

$$\chi : F^\times \times Y_{Q,n} \longrightarrow \mathbb{C}^\times$$

such that

(a)
$$\chi(ab^n, y) = \chi(a, y) \quad \text{for } a, b \in F^\times \text{ and } y \in Y_{Q,n}.$$

(b)
$$\chi(a, y_1) \cdot \chi(a, y_2) = \chi(a, y_1 + y_2) \cdot (a, a)_n^{D(y_1, y_2)}$$

(c)
$$\chi(a, y) \cdot \chi(b, y) = \chi(ab, y) \cdot (a, b)_n^{Q(y)}.$$

(d)
$$\chi(a, \alpha_{Q,n}^\vee) = (a, \eta(\alpha_{Q,n}^\vee))_n \quad \text{for } a \in F^\times \text{ and } \alpha \in \Delta.$$

In §3.3 and §3.4, we have described the covering torus $\overline{T}_{Q,n}$ by generators and relations, using the bisector D. It follows immediately from these that to give a splitting of $E_1 + E_2$ is equivalent to giving a genuine character χ of $\overline{T}_{Q,n}$ (by properties (b) and (c)) satisfying some properties (dictated by properties (a) and (d)). More precisely, properties (a) and (d) can be rephrased as:

(a') the inclusion $nY_{Q,n} \hookrightarrow Y_{Q,n}$ gives the n-power isogeny $n : T_{Q,n} \to T_{Q,n}$ and this lifts to a group homomoprhism

$$i_n : T_{Q,n} \longrightarrow \overline{T}_{Q,n}$$

given by

$$i_n(y(a)) = (ny)(a) = y(a^n) \quad y \in Y_{Q,n}.$$

Then property (a) says that $\chi \circ i_n$ is trivial.

(d') the inclusion $Y_{Q,n}^{sc} \hookrightarrow Y_{Q,n}$ induces an isogeny $T_{Q,n}^{sc} \to T_{Q,n}$. and the map

$$\alpha_{Q,n}^\vee(a) \mapsto \alpha_{Q,n}^\vee(a) \cdot (\eta(\alpha_{Q,n}^\vee), a)_n \in \overline{T}_{Q,n}, \quad \text{for all } \alpha \in \Delta,$$

defines a splitting of this isogeny to give

$$s_\eta : T_{Q,n}^{sc} \longrightarrow \overline{T}_{Q,n}.$$

Then property (d) says that $\chi \circ s_\eta$ is trivial. Because D is fair, we have in fact, for any $y \in Y_{Q,n}^{sc}$,

$$s_\eta(y(a)) = y(a) \cdot (\eta(y), a)_n \in \overline{T}_{Q,n}. \tag{6.2}$$

6.2. **Obstruction.** The question is thus: does there exist genuine characters of $\overline{T}_{Q,n}$ such that (a') and (d') are satisfied? We shall see that there will be some obstructions. More precisely, suppose that $y(a) \in T^{sc}_{Q,n}$ ($y \in Y^{sc}_{Q,n}$) belongs to

$$\mathrm{Ker}(T^{sc}_{Q,n} \to T_{Q,n}) = \mathrm{Tor}_1(Y_{Q,n}/Y^{sc}_{Q,n}, F^\times).$$

Note that this kernel is generated by such pure tensors $y(a)$. Then (d') requires χ to be trivial on the element $(\eta(y), a)_n \in \mu_n(F)$. But χ is supposed to be a genuine character. So we have our first obstruction:

Obstruction 1: A genuine character χ satisfying (d') exists if and only if

$$(\eta(y), a)_n = 1 \quad \text{whenever } y \otimes a = 0 \text{ in } Y_{Q,n} \otimes F^\times \text{ with } y \in Y^{sc}_{Q,n}. \tag{6.3}$$

This condition does not hold automatically. It does hold, however, if $\eta_n|_{Y^{sc}_{Q,n}}$ can be extended to a homomorphism $Y_{Q,n} \longrightarrow F^\times/F^{\times n}$.

Another obstruction is the following. Suppose that $y \in nY_{Q,n} \cap Y^{sc}_{Q,n}$. Then properties (a') and (d') require that

$$\chi(y(a)) = 1 = \chi(y(a)) \cdot (\eta(y), a)_n \quad \text{for any } a \in F^\times.$$

Thus we have our second obstruction:

Obstruction 2: A genuine character χ satisfying (a') and (d') exists if and only if (6.3) holds and

$$(a, \eta(y))_n = 1 \quad \text{for any } y \in nY_{Q,n} \cap Y^{sc}_{Q,n} \text{ and any } a \in F^\times. \tag{6.4}$$

Again, this condition does not hold automatically, but it does hold if $\eta_n|_{Y^{sc}_{Q,n}}$ is extendable to $Y_{Q,n}$.

6.3. **Existence of splitting.** To summarise, we have shown:

Proposition 6.5. *(i) To give a splitting of E_1+E_2 is equivalent to giving a genuine character of $\overline{T}_{Q,n}$ satisfying conditions (a') and (d') above. Any such genuine character is of finite order.*

(ii) If $\eta_n|_{Y^{sc}_{Q,n}}$ is extendable to a homomorphism $Y_{Q,n} \to F^\times/F^{\times n}$, then such genuine characters as in (i) exist, so that the sequence $E_1 + E_2$ is split.

(iii) Under the hypothesis in (ii), the set of such splittings is a torsor under the group

$$\mathrm{Hom}(F^\times/F^{\times n}, Z(\overline{G}^\vee)) = \mathrm{Hom}(W_F, Z(\overline{G}^\vee)[n]).$$

Proof. We have already shown (i) . For (ii), the existence of such genuine characters follows by Pontrjagin duality, noting that $\overline{T}_{Q,n}$ is abelian. Finally (iii) is clear. ☐

We shall see later that these obstructions do occur in our running example.

6.4. **Distinguished splitting,** The definition of $E_1 + E_2$ uses essentially only the data $(n, Y_{Q,n}, \Phi^\vee_{Q,n}, Q|_{Y_{Q,n}}, \mathcal{E}_{Q,n})$. In particular, it does not make use of the full information available in (Y, Q). Recall that there is a short exact sequence

$$1 \longrightarrow \mathrm{Ker}(h) \longrightarrow \overline{T}_{Q,n} \longrightarrow Z(\overline{T}) \longrightarrow 1.$$

Since we are after all defining the L-group of $\overline{G}$, it is natural to consider those splittings of $E_1 + E_2$ which corresponds to those genuine characters of $\overline{T}_{Q,n}$ which are trivial on $\mathrm{Ker}(h)$ and thus factors to give characters of $Z(\overline{T})$. For this, it follows by Lemma 3.1 that it suffices to consider those genuine characters which are trivial on the image of $\tilde{g} : T \to \overline{T}_{Q,n}$, with

$$\tilde{g} : y(a) \mapsto (ny)(a) \in \overline{T}_{Q,n}.$$

Then we have:

(a") a genuine character χ of $\overline{T}_{Q,n}$ factors to $Z(\overline{T})$ if $\chi \circ \tilde{g} = 1$.

Note that $\tilde{g}$ agrees with i_n when pulled back to $T_{Q,n}$, so that this requirement subsumes condition (a'). It is now natural to make the following definition:

Definition: We call a genuine character of $\overline{T}_{Q,n}$ which satisfies (a") and (d') a *distinguished genuine character*. We call the corresponding splitting of $E_1 + E_2$ a *distinguished splitting.*

Assume that we have a genuine character satisfying (a') and (d') already. As before, we see that there is an obstruction to (a"). Namely,

Obstruction 3: A genuine character satisfying (a") and (d') exists if and only if (6.3) holds and

$$(a, \eta(y))_n = 1 \quad \text{for any } y \in nY \cap Y^{sc}_{Q,n} \text{ and any } a \in F^\times.$$

This condition is not automatic, but is satisfied when $\eta_n : Y^{sc} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y.

We have thus shown:

Theorem 6.6. *Assume that $\eta_n : Y^{sc} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y, and let $J = nY + Y^{sc}_{Q,n}$.*

(i) The set of distinguished genuine characters of $\overline{T}_{Q,n}$ is nonempty.

(ii) Consider the subgroup

$$Z^\heartsuit(\overline{G}^\vee) := \mathrm{Hom}(Y_{Q,n}/J, \mathbb{C}^\times) \subset Z(\overline{G}^\vee).$$

Then the set of distinguished splittings of $E_1 + E_2$ is a torsor under

$$\mathrm{Hom}(F^\times/F^{\times n}, Z^\heartsuit(\overline{G}^\vee)) = \mathrm{Hom}(W_F, Z^\heartsuit(\overline{G}^\vee)[n]).$$

(iii) Each distinguished splitting of $E_1 + E_2$ gives rise to a splitting $s : W_F \longrightarrow {}^L\overline{G}$ which agrees with the canonical splitting over $W_{F,n}$ and induces an isomorphism

$$\overline{G}^\vee \times W_F \cong {}^L\overline{G}.$$

Henceforth, when $\eta_n : Y^{sc} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y, we shall consider the L-group of a degree n BD covering $\overline{G}$ as the extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1$$

equipped with the set of distinguished splittings. In particular, when $\mathbb{G} = \mathbb{T}$ is a split torus, then the set of distinguished splittings is nonempty since $Y^{sc} = 0$.

6.5. **Weyl invariance.** Under the hypothesis of Theorem 6.6, we have distinguished genuine characters of $\overline{T}_{Q,n}$. These distinguished characters possess another desirable property. Namely, the action of $N(\mathbb{T})$ on $\overline{\mathbb{T}}$ gives rise to an action of $N(\mathbb{T})$ on $\overline{T}$ which preserves the center $Z(\overline{T})$. Moreover, the action on $Z(\overline{T})$ factors through the Weyl group $W = N(\mathbb{T})/\mathbb{T}$. Since $\overline{T}_{Q,n}$ is the pullback of $Z(\overline{T})$ using the map $T_{Q,n} \longrightarrow T$, it inherits an action of W as well. Hence it makes sense to ask if a genuine character of $Z(\overline{T})$ or $\overline{T}_{Q,n}$ is W-invariant.

For $\alpha \in \Phi$, let w_α denote the element of W corresponding to the element $q(n_\alpha(1))$. Consider $\mathrm{Int}(w_\alpha)(\tilde{t}) := w_\alpha \cdot \tilde{t} \cdot w_\alpha^{-1}$. Then we have seen in (2.3) that for $\tilde{t} \in Z(\overline{T})$,

$$\mathrm{Int}(w_\alpha)(\tilde{t}) = \tilde{t} \cdot q(s_\alpha(\alpha^\vee(\alpha(t)^{-1}))).$$

One has an analogous formula for the W-action on $\overline{T}_{Q,n}$. More precisely, if $t = y(a) \in T_{Q,n}$, with $y \in Y_{Q,n}$, then

$$\mathrm{Int}(w_\alpha)(\tilde{t}) = \tilde{t} \cdot s_\eta(-\langle \alpha, y\rangle \cdot \alpha^\vee(a)). \tag{6.7}$$

The following proposition shows that $\langle \alpha, y\rangle \cdot \alpha^\vee \in Y_{Q,n}^{sc}$ so that the right hand side of the above formula is well-defined.

Theorem 6.8. *For $y \in Y_{Q,n}$, n_α divides $\langle \alpha, y\rangle$ for each $\alpha \in \Phi$. Hence, χ is W-invariant if χ satisfies (d'). In particular, distinguished genuine characters of $\overline{T}_{Q,n}$ gives rise to W-invariant genuine characters of $Z(\overline{T})$.*

Proof. Since Q is W-invariant, we have

$$Q(y) = Q(w_\alpha y) = Q(y - \langle y, \alpha\rangle\alpha^\vee) = Q(y) - \langle y, \alpha\rangle \cdot B(y, \alpha^\vee) + \langle y, \alpha\rangle^2 Q(\alpha^\vee).$$

This implies that

$$\langle \alpha, y\rangle = 0 \quad \text{or} \quad B(y, \alpha^\vee) = \langle \alpha, y\rangle \cdot Q(\alpha^\vee).$$

In the latter case, note that $B(y, \alpha^\vee)$ is divisible by n if $y \in Y_{Q,n}$, in which case n_α divides $\langle \alpha, y\rangle$ as desired.

∎

6.6. **Splitting of ${}^L\overline{G}$.** We have so far considered splittings of the fundamental extension $E := E_1 + E_2$ of $F^\times/F^{\times n}$ by $Z(\overline{G}^\vee)$ with good properties. Since the L-group of $\overline{G}$ is obtained from this fundamental extension by a combination of pushout and pullback, one may consider splittings of the extensions derived from these operations. Of course, these extensions will possess more splittings than the fundamental extension from which they are derived.

For example, one may pull back the fundamental extension to obtain

$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow \tilde{E} \longrightarrow W_F \longrightarrow 1$$

and one may ask for distinguished splittings of this central extension. This amounts to dropping condition (a') above, so that one is finding genuine characters of $\overline{T}_{Q,n}$ which satisfy only (d'). To ensure that this character factors through to $Z(\overline{T})$, it would not be reasonable to require the condition (a") (since (a') is not assumed); one would simply require the character to be trivial on $\mathrm{Ker}(h)$.

Now recall from §5.5 that the L-group construction is functorial with respect to inclusion of Levi subgroups, so that one has an embedding ${}^L\overline{T} \hookrightarrow {}^L\overline{G}$. Since we know that the extension ${}^L\overline{T}$ splits over W_F, we deduce:

Proposition 6.9. *The extension ${}^L\overline{G}$ splits over W_F, so that ${}^L\overline{G}$ is abstractly a semi-direct product.*

As we have seen in our running example, it is not a direct product in general. Rather, it is the type of split extensions which one typically encounters in the usual theory of endoscopy (as we explained in the introduction).

Further, suppose we fix a fair bisector D and consider all BD-extensions with BD invariants (D, η). All these BD covering groups $\overline{G}_\eta$ have isomorphic covering tori $\overline{T} = T \times_D \mu_n$. When $\eta_n = 1$ is trivial, we have seen that the set of distinguished splittings is nonempty. If we fixed a distinguished splitting of E for $\eta_n = 1$, we would have fixed a splitting of ${}^L\overline{T}$ and hence for ${}^L\overline{G}_\eta$ for all η. If χ is the associated genuine character of $Z(\overline{T})$, note that χ is Weyl-invariant for the Weyl action associated to $\eta_n = 1$. But this χ need not be Weyl-invariant for general η, since the Weyl action on $Z(\overline{T})$ depends on η.

6.7. **Running example.** We consider our running example to illustrate the discussion of this section. Recall that we have:

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\} \longrightarrow \mathrm{PGL}_2(F) \longrightarrow 1$$

and

$$E_1^\eta + E_2^\eta = \{(t, a) \in \mathbb{C}^\times \times F^\times/F^{\times 2} : t^2 = (\eta, a)_2\}.$$

Moreover, $Y_{Q,2} = Y \supset Y_{Q,n}^{sc} = Y^{sc} = 2Y$. In this case, Obstruction 1 says that

$$(\eta, -1)_2 = 1.$$

Clearly, this may fail if $-1 \notin F^{\times 2}$. However, it does hold if $\eta \in F^{\times 2}$, or equivalently, if η_2 is trivial. Obstruction 2 says that

$$(\eta, a)_2 = 1 \quad \text{for all } a \in F^\times.$$

This is clearly a stronger condition than the one above, and it holds if and only if η_2 is trivial. Thus, we see that if $\eta \notin F^{\times 2}$, then the sequence $E_1^\eta + E_2^\eta$ does not split.

When η_2 is trivial, however, the above two obstructions, as well as Obstruction 3, are all absent and so a distinguished character of $\overline{T}_\eta = Z(\overline{T}_\eta)$ exists. If we identify $\overline{T}_\eta$ with $T \times \mu_2$, so that genuine characters of $\overline{T}_\eta$ is in natural bijection with the characters of T, then the distinguished characters correspond naturally to quadratic characters of T.

The Weyl group action on $\overline{T}_\eta$ is given by

$$\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, 1\right) \mapsto \left(\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}, (\eta, a)_2\right),$$

so that the Weyl action on genuine characters is given by

$$\chi \mapsto \chi^{-1} \cdot (\eta, -)_2.$$

In particular, we see that χ is W-invariant if and only if

$$\chi^2 = (\eta, -)_2,$$

i.e. if χ is a square root of the quadratic character $(\eta, -)_2$. Such a square root exists if and only if

$$(\eta, -1)_2 = 1,$$

i.e. if and only if Obstruction 1 is absent.

7. Construction of Distinguished Genuine Characters

It is useful in practice to have an explicit construction of the distinguished genuine characters (when they exist). Note that once one constructs one such distinguished character, the others are obtained by twisting it by a character of the finite group $T_{Q,n}/T_J$ (see §7.1 below for the definition of T_J). We shall see that a distinguished genuine character can be constructed using the Hilbert symbol $(-,-)_n$ and the Weil index γ_ψ (cf. [Ra]) associated to a nontrivial additive character ψ of F. Here,

$$\gamma_\psi : F^\times \longrightarrow \mu_4 \subset \mathbb{C}^\times$$

satisfies

$$\gamma_\psi(a) \cdot \gamma_\psi(b) = \gamma_\psi(ab) \cdot (a, b)_2$$

for all $a, b \in F^\times$. Moreover, $\gamma_{\psi_{a^2}} = \gamma_\psi$. Such Weil indices play an important role in the classical theory of the metaplectic groups $\mathrm{Mp}_{2n}(F)$.

7.1. **Reduction to rank 1 case.** By the elementary divisor theorem, we may pick a basis $\{y_i\}$ of $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for the lattice $J = nY + Y^{sc}_{Q,n}$ for some $k_i \in \mathbb{Z}$. Let $\mathbb{T}_J$ be the split torus associated to J and let T_J be its F-rational points. This gives a decomposition

$$J = k_1 Y_1 \oplus \ldots \oplus k_r Y_r \subset Y_1 \oplus \ldots \oplus Y_r = Y$$

and a map

$$T_J = \prod_i (k_i Y_i) \otimes_{\mathbb{Z}} F^\times \longrightarrow \prod_i Y_i \otimes_{\mathbb{Z}} F^\times$$

which is the k_i-power map on the i-th coordinate. Write $T_{Q,n,i}$ for the 1-dimensional torus corresponding to Y_i and $T_{J,i}$ for that corresponding to $k_i Y_i$. Now, because $\overline{T}_{Q,n}$ is abelian, we see that

$$\overline{T}_{Q,n} \cong \overline{T}_1 \times \ldots\ldots \times \overline{T}_r / Z$$

where

$$Z = \{(\epsilon_i) \in \prod_{i=1}^r \mu_n : \prod_i \epsilon_i = 1\}.$$

Moreover, the group law on $\overline{T}_{Q,n,i} = (Y_i \otimes F^\times) \times \mu_n$ is given by

$$y_i(a) \cdot y_i(b) = y_i(ab) \cdot (a,b)_n^{Q(y_i)}.$$

It follows that the map $T_{J,i} \to T_{Q,n,i}$ splits naturally to give

$$T_{J,i} \longrightarrow \overline{T}_{Q,n,i}.$$

Thus, to construct a distinguished genuine character on $\overline{T}_{Q,n}$, it suffices to construct a genuine character of $\overline{T}_{Q,n,i}$ which is trivial on the image of $T_{J,i}$; the product of these characters will then be a distinguished genuine character of $\overline{T}_{Q,n}$. We are thus reduced to constructing genuine characters of 1-dimensional covering tori.

7.2. **The definition.** We set

$$\chi((y_i(a)) = \gamma_\psi(a)^{f_i}$$

for some $f_i \in \mathbb{Z}$ to be determined. Now we need to check various requirements:

- we first need to check the relation for $\overline{T}_{Q,n,i}$:

  $$\chi(y_i(a)) \cdot \chi(y_i(b)) = \chi(y_i(ab)) \cdot (a,b)_n^{Q(y_i)}.$$

  This amounts to the requirement that

  $$f_i \equiv A_i := \frac{2}{n} \cdot Q(y_i) \mod 2.$$

- next we need to ensure that χ is trivial on the image of $T_{J,i}$, i.e. trivial on $y_i(a^{k_i})$. But a short computation gives:

  $$\chi(y_i(a^{k_i})) = \gamma_\psi(a)^{k_i f_i + k_i(k_i-1)\cdot A_i}.$$

  Thus we need

  $$k_i \cdot (f_i + (k_i - 1)A_i) \equiv 0 \mod 4.$$

  To ensure this, we shall simply take

  $$f_i := -(k_i - 1)A_i.$$

  Then when k_i is even, it is automatic that $f_i \equiv A_i \mod 2$. We need to ensure that this continues to hold when k_i is odd.

  For this, we need to show that $A_i = 0 \mod 2$ when k_i is odd; equivalently, we need to show that $Q(y_i) \equiv 0 \mod n$. Since we already know that $Q(y_i)$ is divisible by $n/2$, it remains to show that if 2^e divides n, then 2^e divides $Q(y_i)$. As $k_i y_i \in J = Y_{Q,n}^{sc} + nY$, we know that $Q(k_i y_i) \equiv 0 \mod n$. So 2^e divides $k_i^2 \cdot Q(y_i)$. Since k_i is odd, we see that 2^e does divide $Q(y_i)$, as desired.

We have completed the construction of a distinguished genuine character χ_ψ. A formula for χ_ψ can be given as follows. For $y = \sum_i n_i y_i \in Y_{Q,n}$ and $a \in F^\times$,

$$\chi_\psi(y(a)) = \prod_i \gamma_\psi(a^{n_i})^{f_i} \cdot (a,a)_n^{\sum_{i<j} n_i n_j D(y_i,y_j)},$$

with

$$f_i = -(k_i - 1) \cdot \frac{2}{n} \cdot Q(y_i).$$

Though explicit, a slightly unsatisfactory aspect of this formula is that one first needs to find compatible bases for the lattice chain $J \subset Y_{Q,n}$.

8. LLC for Covering Tori

We are going to specialise the investigation of §6 to several examples. In this section, we assume that $\mathbb{G} = \mathbb{T}$ is a split torus.

8.1. **LLC for $\overline{T}_{Q,n}$.** When $\mathbb{G} = \mathbb{T}$, one has $Y^{sc} = 0$, so that $\eta = 1$, and the extension $E_1 + E_2$ is

$$1 \longrightarrow \overline{T}^{\vee}_{Q,n} \longrightarrow E_1 + E_2 \longrightarrow F^\times / F^{\times n} \longrightarrow 1.$$

In the previous section, we have seen that to give a splitting of this sequence is equivalent to giving a genuine character on $\overline{T}_{Q,n}$ satisfying conditions (a') and (d'). Since $Y^{sc} = 0$, (d') is vacuous, and since $\eta = 1$, (a') holds. So we obtain a bijection between the set of splittings of $E_1 + E_2$ and the set of genuine characters of $\overline{T}_{Q,n}$ satisfying (a').

On the other hand, to obtain the L-group of $\overline{T}$, we pull $E_1 + E_2$ back by $W_F \longrightarrow F^\times \longrightarrow F^\times / F^{\times n}$. The same considerations show that to give a splitting of ${}^L\overline{T}$ is equivalent to giving a genuine character of $\overline{T}_{Q,n}$, where we don't insist on condition (a') anymore. Hence, we have obtained a natural bijection

$$\{\text{Splittings of } {}^L\overline{T}\} \longleftrightarrow \{\text{genuine characters of } \overline{T}_{Q,n}\}.$$

This is a classification of the genuine characters of the abelian group $\overline{T}_{Q,n}$.

We can explicate the bijection above by tracing through the discussion in §6. Let $\chi \in \mathrm{Hom}(\overline{T}_{Q,n}, \mathbb{C}^\times)$ be a genuine character of $\overline{T}_{Q,n}$ and write ρ_χ for the splitting of ${}^L\overline{T}$ given by the LLC. Note that splittings of ${}^L\overline{T}$ over W_F are in bijection with splittings of the $E_1 + E_2$ extension (pulled-back to be) over $F^\times$. Recall that we have

$$E_1 = \overline{T}^{\vee} \times_{c_1} F^\times \quad \text{defined using the cocycle } c_1,$$

and

$$E_2 = \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbb{C}^\times)$$

with

$$\mathcal{E}_{Q,n} = Y_{Q,n} \times_D F^\times \quad \text{defined using a cocycle determined by the bisector } D.$$

With this identification, one has

$$\rho_\chi(a) = (1, a) +_{\text{Baer}} \tilde{\chi}_a \in E_1 + E_2 \tag{8.1}$$

for $a \in F^\times$, where

- $(1, a) \in E_1 = \overline{T}^{\vee} \times_{c_1} F^\times$;
- $\tilde{\chi}_a \in E_2 = \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbb{C}^\times)$ is the character of $\mathcal{E}_{Q,n}$ given by

  $$\tilde{\chi}_a(y, b) = (b, a)_n \cdot \chi(y(a)) \quad \text{for } (y, b) \in Y_{Q,n} \times_D F^\times,$$

  noting that $y(a) \in \overline{T}_{Q,n}$.

8.2. **Construction for $\overline{T}$.** The covering torus $\overline{T}$ is a Heisenberg type group (cf. [KP1, Mc2, W1]), and one has a natural bijection

$$\{\text{genuine characters of } Z(\overline{T})\} \longleftrightarrow \{\text{irreducible genuine representations of } \overline{T}\}.$$

This bijection is defined as follows. Choose and fix a maximal abelian subgroup H of $\overline{T}$ containing $Z(\overline{T})$. Given a genuine character χ of $Z(\overline{T})$, extend χ arbitrarily to a character χ_H of H. Then the induced representation

$$i(\chi) = \mathrm{ind}_H^{\overline{T}} \chi_H$$

is irreducible and independent of the choice of (H, χ_H). By the analog of the Stone-von-Neumann theorem, it is characterized as the unique irreducible genuine representation of $\overline{T}$ which has central character χ.

8.3. **LLC for $\overline{T}$.** Combining these, we obtain the following result which is the LLC for covering (split) tori:

Theorem 8.2. *There is a natural injective map*

$$\mathcal{L}_{\overline{T}} : \{\textit{irreducible genuine representations of } \overline{T}\} \hookrightarrow \{\textit{Splittings of } {}^L\overline{T}\}.$$

The image of this injection can be described as follows. Tensoring the short exact sequence

$$0 \longrightarrow X \longrightarrow X_{Q,n} \longrightarrow X_{Q,n}/X \longrightarrow 0$$

with $\mathbb{C}^\times$, the associated long exact sequence gives:

$$0 \longrightarrow \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times) \longrightarrow X \otimes \mathbb{C}^\times = T^\vee \longrightarrow X_{Q,n} \otimes \mathbb{C}^\times = \overline{T}^\vee \longrightarrow 0.$$

This is a short exact sequence of W_F-modules and gives an exact sequence

$$H^1(W_F, T^\vee) \xrightarrow{f_*} H^1(W_F, \overline{T}^\vee) \xrightarrow{\delta} H^2(W_F, \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)).$$

If we fix a distinguished splitting s_0 of ${}^L\overline{T}$, which corresponds to a genuine character χ_0 of $Z(\overline{T})$, then all other splittings of ${}^L\overline{T}$ are of the form $s = s_0 \cdot \rho$ with $\rho \in H^1(W_F, \overline{T}^\vee)$. This gives an identification

$$H^1(W_F, \overline{T}^\vee) = \{\text{Splittings of } {}^L\overline{T}\}.$$

Then one sees that a splitting s of ${}^L\overline{T}$ is in the image of the map $\mathcal{L}_{\overline{T}}$ if and only if s lies in the image of f_*, or equivalently

$$\delta(s) = 0 \in H^2(W_F, \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)).$$

Note that if $X_{Q,n}/X \cong \prod_i \mathbb{Z}/n_i\mathbb{Z}$, then

$$\mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)) \cong \prod_i \mu_{n_i}(\mathbb{C}).$$

9. LLC for Unramified Representations

We consider the tame case in this section, so that p does not divide n. We assume that there is a splitting

$$s : K = \mathbb{G}(\mathcal{O}) \longrightarrow \overline{G}.$$

Then one may consider the s-unramified genuine representations of $\overline{G}$, i.e, those with nonzero $s(K)$-fixed vectors. We would like to obtain an LLC for such s-unramified genuine representations.

9.1. **Torus case.**

We first consider the case when $\mathbb{G} = \mathbb{T}$ is a split torus. Suppose that $\overline{\mathbb{T}}$ has bisector data D, so that

$$\overline{T} = T \times_D \mu_n$$

In the tame case, the trivial section $y(a) \mapsto (y(a), 1) \in T \times_D \mu_n$ is a splitting over $\mathbb{T}(\mathcal{O})$ and any splitting $s : \mathbb{T}(\mathcal{O}) \longrightarrow \overline{T}$ is given by

$$s(t) = (t, \mu_s(t)) \quad \text{for } t \in \mathbb{T}(\mathcal{O})$$

where $\mu_s : \mathbb{T}(\mathcal{O}) \longrightarrow \mu_n$ is a group homomorphism. Such an s will give rise to a splitting of $\mathbb{T}_{Q,n}(\mathcal{O})$, denoted by s as well, via pulling back.

We call a genuine representation $i(\chi)$ (see §8.2) of $\overline{T}$ s-unramified if $i(\chi)$ has a nonzero vector fixed by $s(\mathbb{T}(\mathcal{O}))$. In this tame case, one can check that $H = Z(\overline{T}) \cdot s(\mathbb{T}(\mathcal{O}))$ is a maximal abelian subgroup of $\overline{T}$. From this, one deduces:

Lemma 9.1. *A representation $i(\chi)$ is s-unramified if and only if χ is trivial when restricted to $Z(\overline{T}) \cap s(\mathbb{T}(\mathcal{O}))$. In this case, the space of $s(\mathbb{T}(\mathcal{O}))$-fixed vectors is 1-dimensional.*

We say that a genuine character of $Z(\overline{T})$ or $\overline{T}_{Q,n}$ is s-unramfied if it is trivial on the image of s. Thus, the above lemma says that $i(\chi)$ is s-unramified if and only if χ is s-unramified. Such a χ will pullback to an s-unramified character of $\overline{T}_{Q,n}$. Conversely, observe that an s-unramified genuine character of $\overline{T}_{Q,n}$ automatically factors through to a genuine character of $Z(\overline{T})$ since $\mathrm{Ker}(h) \subset s(\mathbb{T}_{Q,n}(\mathcal{O}))$. Under the LLC for $\overline{T}$ defined in the last section, the s-unramified genuine characters correspond to a subset of splittings of ${}^L\overline{T}$. We would like to explicate this subset.

Lemma 9.2. *The L-parameters $\rho_\chi : W_F \longrightarrow {}^L\overline{T}$ of the s-unramified characters χ have the same restriction ρ_s to the inertia group I_F.*

Proof. We need to show that, as maps from $W_F^{ab} = F^\times$ to ${}^L\overline{T}$, all these ρ_χ's have the same restriction ρ_s on $\mathcal{O}^\times$. This follows from an examination of the construction of the LLC for $\overline{T}$. In particular, this restriction ρ_s can be described explicitly as follows.

Suppose an s-unramified χ is given. By (8.1), we see that for $a \in \mathcal{O}^\times$, $\rho_\chi(a) \in {}^L\overline{T}$ is determined by the character $\tilde{\chi}_a : \mathcal{E}_{Q,n} \longrightarrow \mathbb{C}^\times$ described there. Using the notations in (8.1), the s-unramified condition says that for $(y, b) \in \mathcal{E}_{Q,n}$,

$$\tilde{\chi}_a(y, b) = (b, a)_n \cdot \mu_s(y(a))^{-1},$$

which is independent of χ. □

We shall call a splitting ρ of ${}^L\overline{T}$ *s-unramified* if $\rho|_{I_F} = \rho_s$. Dividing ${}^L\overline{T}$ by $\rho_s(I_F)$, we obtain a short exact sequence

$$1 \longrightarrow \overline{T}^\vee \longrightarrow {}^L\overline{T}_s \stackrel{p}{\longrightarrow} \mathbb{Z}\cdot \mathrm{Frob}_s \longrightarrow 1$$

To summarize, we have shown:

Proposition 9.3. *Under the LLC for covering tori, one has a bijection*

$$\{\textit{irreducible } s\textit{-unramified genuine representations of } \overline{T}\}$$
$$\updownarrow$$
$$\{\overline{T}^\vee\textit{-conjugacy classes of } s\textit{-unramified splittings } \rho \textit{ of } {}^L\overline{T}\ \}$$
$$\updownarrow$$
$$\{\overline{T}^\vee\textit{-orbits in } p^{-1}(\mathrm{Frob}_s).\}$$

9.2. **Satake isomorphism.** We now consider the case of general $\mathbb{G}$. The key step in understanding the s-unramified representation of $\overline{G}$ is the Satake isomorphism. More precisely, let $\mathcal{H}(\overline{G}, s)$ be the $\mathbb{C}$-algebra of anti-genuine locally constant, compactly supported functions on $\overline{G}$ which are bi-invariant under $s(\mathbb{G}(\mathcal{O}))$. Let $\mathcal{H}(\overline{T}, s)$ denote the analogous $\mathbb{C}$-algebra for $\overline{T}$. One can check that for an element $f \in \mathcal{H}(\overline{T}, s)$, the support of f is contained in $Z(\overline{T})\cdot s(\mathbb{T}(\mathcal{O}))$, and thus f is completely determined by its restriction to $Z(\overline{T})$. Moreover, the Weyl group W acts naturally on $Z(\overline{T})\cdot s(\mathbb{T}(\mathcal{O}))$ and thus on $\mathcal{H}(\overline{T})$.

One has an explicit $\mathbb{C}$-algebra morphism

$$\mathcal{S}: \mathcal{H}(\overline{G}, s) \longrightarrow \mathcal{H}(\overline{T}, s)$$

given by

$$\mathcal{S}(f)(t) = \delta(t)^{1/2}\int_U f(tu)du \text{ for all } f\in \mathcal{H}(\overline{G}, s).$$

The following is shown in [Mc2, L2, W7]:

Theorem 9.4. *The Satake map $\mathcal{S}$ induces an isomorphism of $\mathbb{C}$-algebras:*

$$\mathcal{H}(\overline{G}, s) \cong \mathcal{H}(\overline{T}, s)^W.$$

As a consequence of this, one deduces a bijection

$$\{\text{irreducible } s\text{-unramified genuine representations of } \overline{G}\}$$
$$\updownarrow$$
$$\{\text{irreducible modules of } \mathcal{H}(\overline{G}, s)\}$$
$$\updownarrow$$
$$\{W\text{-orbits of } s\text{-unramified genuine characters of } Z(\overline{T})\}$$
$$\updownarrow$$
$$\{W\text{-orbits of } s\text{-unramified genuine characters of } \overline{T}_{Q,n}\}.$$

Explicitly, the bijection [Mc2] is constructed as in the linear case. Namely, given a s-unramified genuine character χ of $\overline{T}_{Q,n}$, we saw in §9.1 and Lemma 9.1 that χ descends to a genuine character of $Z(\overline{T})$ and gives an irreducible unramified genuine representation $i(\chi)$ of $\overline{T}$. Now we set

$$I(\chi) = \mathrm{Ind}_{\overline{B}}^{\overline{G}} i(\chi),$$

where the induction is normalized. Then $I(\chi)$ has a 1-dimensional space of $s(K)$-fixed vectors and thus a unique irreducible constituent which is unramified. The action of $\mathcal{H}(\overline{G}, s)$ on this 1-dimensional space of $s(K)$-fixed vectors is easily calculated to be via the character:

$$f \mapsto \int_{\overline{T}} \chi(t) \cdot \mathcal{S}(f)(t)\, dt.$$

Note that this character of $\mathcal{H}(\overline{G}, s)$ depends only on the W-orbit of χ since $\mathcal{S}(f)$ is W-invariant. This shows that the $s(K)$-unramified representation associated to χ is the unique $s(K)$-unramified constituent of $I(\chi)$.

9.3. **W-equivariance of LLC.** We would like to give an interpretation of spherical Hecke algebra in terms of the L-group ${}^L\overline{G}$. In view of the natural inclusion ${}^L\overline{T} \hookrightarrow {}^L\overline{G}$, it is natural to apply the LLC for $\overline{T}$ (or $\overline{T}_{Q,n}$) at this point. However, we first need to check the following proposition:

Theorem 9.5. *The local Langlands correspondence for $\overline{T}$ is equivariant with respect to the action of the Weyl group W on both sides.*

Proof. Let χ be a genuine character of $\overline{T}_{Q,n}$, and let ρ_χ be the associated splitting of the L-group given by LLC. It suffices to show that for any simple reflection $w = w_\alpha$, with $\alpha \in \Delta$,

$${}^w(\rho_\chi) = \rho_{{}^w\chi}.$$

The action of $N(\mathbb{T})$ on $\overline{\mathbb{T}}$ is given by formula (2.3). It induces actions of the Weyl group W on the topological cover $\overline{T}_{Q,n}$ (given by (6.7)), as well as the central extension $\mathcal{E}_{Q,n}$. The latter gives rise to an inherited action on E_2, and thus on ${}^L\overline{T}$. Note that the Weyl group acts trivially on E_1.

Fix $a \in F^\times$, by the explicit form of ρ_χ given in (8.1), it is enough show

$${}^w\tilde{\chi}_a = (\widetilde{{}^w\chi})_a \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbb{C}^\times).$$

Using the notations in (8.1), we need to verify that

$$(9.6) \qquad {}^w\tilde{\chi}_a(y, b) = (\widetilde{{}^w\chi})_a(y, b)$$

for all $(y, b) \in \mathcal{E}_{Q,n}$.

For ease of notation, set

$$y_\alpha = -\langle \alpha, y\rangle \cdot \alpha^\vee \in Y_{Q,n}^{sc} \quad \text{since } n_\alpha \text{ divides } \langle \alpha, y\rangle$$

and write y for $(y, 1) \in \mathcal{E}_{Q,n}$. The left hand side of the desired identity (9.6) is

$$\tilde{\chi}_a\big((y, b) \cdot s_{Q^{sc}}(y_\alpha)\big) = (b, a)_n \cdot \chi(y(a)) \cdot \tilde{\chi}_a\big(s_{Q^{sc}}(y_\alpha)\big).$$

On the other hand, the right hand side of the desired identity (9.6) is

$$(b, a)_n \cdot ({}^w\chi)\big(y(a)\big) = (b, a)_n \cdot \chi\big(y(a) \cdot s_\eta(y_\alpha(a))\big),$$

where $s_\eta : T_{Q,n}^{sc} \longrightarrow \overline{T}_{Q,n}$ was introduced in (6.2).

Thus, to obtain the equality (9.6), it now suffices to show

$$\tilde{\chi}_a\big(\underbrace{s_{Q^{sc}}(y_\alpha)}_{\in \mathcal{E}_{Q,n}}\big) = \chi\big(\underbrace{s_\eta(y_\alpha(a))}_{\in \overline{T}_{Q,n}}\big). \tag{9.7}$$

However, on the LHS of (9.7),

$$s_{Q^{sc}}(y_\alpha) = (y_\alpha, \eta(y_\alpha)) \in \mathcal{E}_{Q,n},$$

whereas on the RHS of (9.7),

$$s_\eta(y_\alpha(a)) = \big(\eta(y_\alpha), a\big)_n \cdot y_\alpha(a) \in \overline{T}_{Q,n}.$$

Thus, both sides of the desired identity (9.7) are equal to

$$\big(\eta(y_\alpha), a\big)_n \cdot \chi\left(y_\alpha(a)\right).$$

The proof of the theorem is thus completed. □

9.4. **Passing to dual side.** Consequently, there is a bijection between

$$\{W\text{-orbits of } s\text{-unramified genuine characters of } \overline{T}_{Q,n}\}$$
$$\updownarrow$$
$$\{W\text{-orbits of } s\text{-unramified splittings of } {}^L\overline{T}\}.$$

We know from Lemma 9.2 that there is a splitting $\rho_s : I_F \hookrightarrow {}^L\overline{T}$ such that all s-unramified splittings of ${}^L\overline{T}$ restricts to ρ_s on I_F. Dividing out by $\rho_s(I_F)$, one has a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{T}^\vee & \longrightarrow & {}^L\overline{T}_s & \xrightarrow{p} & \mathbb{Z}\cdot \mathrm{Frob}_s & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_s & \xrightarrow{p} & \mathbb{Z}\cdot \mathrm{Frob}_s & \longrightarrow & 1. \end{array}$$

Hence, we have bijections

$$\{W\text{-orbits of } s\text{-unramified splittings of } {}^L\overline{T}\}$$
$$\updownarrow$$
$$\{W\text{-orbits of splittings of } {}^L\overline{T}_s\ \}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\text{-orbits of splittings of } {}^L\overline{G}_s\ \}.$$

Here the last bijection can be seen by the arguments in [Bo, §6.3-§6.7].

9.5. **Representation ring.** Finally, we note that the bijections above are induced by algebra isomorphisms

$$\mathcal{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W \simeq \mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s),$$

where $\mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)$ denotes the algebra of functions on the preimage of Frob_s in ${}^L\overline{T}_s$ obtained as the restriction of the representation ring of ${}^L\overline{T}_s$ and $\mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s)$ is analogously defined. Here the second isomorphism can be shown following [Bo, §6.3- §6.7] and the first can be reduced to the linear algebraic case as follows.

(1) We claim that there exists a W-invariant s-unramified genuine character χ of $\overline{T}_{Q,n}$, i.e. $\chi \circ s$ is trivial and $\chi\big(s_\eta((\langle \alpha, y\rangle \cdot \alpha^\vee)(a))\big) = 1$ for all $a \in F^\times, y \in Y_{Q,n}$ and $\alpha \in \Delta$. However, for $a \in \mathcal{O}_F^\times$, one has

$$s\big((\langle \alpha, y\rangle \cdot \alpha^\vee)(a)\big) = s_\eta\big((\langle \alpha, y\rangle \cdot \alpha^\vee)(a)\big), \quad \text{with } (\langle \alpha, y\rangle \cdot \alpha^\vee)(a) \in T_{Q,n}^{sc}(\mathcal{O}),$$

since s_η is by definition given by the unique splitting on unipotent subgroups on which it agrees with s. Thus the two conditions are compatible, and by Pontryagin duality, there exists such unramified Weyl-invariant genuine character.

(2) Fix a χ as in (1). Dividing by χ gives a Weyl-equivariant algebra isomorphism

$$\mathcal{H}(\overline{T}, s) \cong \mathbb{C}[Y_{Q,n}].$$

The unramified character χ gives rise to an element $t_\chi \in p^{-1}(\mathrm{Frob}_s)$ under the LLC for $\overline{T}_{Q,n}$. By Theorem 9.5, t_χ is Weyl-invariant. Therefore, it gives a Weyl-equivariant algebra isomorphism

$$\mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s) \cong \mathrm{Rep}(\overline{T}^\vee).$$

It follows that there is a natural isomorphism

$$\mathcal{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W,$$

which can be checked to be independent of the choice of χ.

To summarize, we have shown:

Theorem 9.8. *The Satake isomorphism gives isomorphisms*

$$\mathcal{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W \simeq \mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s),$$

which induces bijections

$$\{\textit{irreducible } s\textit{-unramified genuine representations of } \overline{G}\}$$
$$\updownarrow$$
$$\{W\textit{-orbits of } s\textit{-unramified splittings of } {}^L\overline{T}\}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\textit{-orbits of } s\textit{-unramified splittings of } {}^L\overline{G}\}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\textit{-orbits of semisimple elements in } p^{-1}(\mathrm{Frob}_s)\}.$$

10. L-Groups: Second Take

After the discussion of the previous sections and a study of our running example, we may draw the following conclusions:

(i) For a fixed *fair* bisector D, and among all BD covering groups (of degree n) with bisector data (D, η), those with $\eta_n = 1$ are most nicely behaved. For example, their maximal covering tori $\overline{T}$ have certain distinguished Weyl-invariant genuine representations and $\overline{G}$ splits over the hyperspecial maximal compact subgroup $G(\mathcal{O})$. Moreover, their L-groups are isomorphic to a direct product $\overline{G}^{\vee} \times W_F$.

(ii) The BD covering groups $\overline{G}_\eta$ for a fixed bisector data are closely related, and it may be useful to consider them together, both structurally as well as from the point of view of representation theory. For example, they all have the same dual group $\overline{G}_Q^{\vee}$.

In this section, we would like to suggest a slightly different take on the L-group extension, so as to treat the closely related groups $\overline{G}_\eta$ together.

10.1. **The case $Q = 0$.** To guide our efforts, we shall consider the genuine representation theory of the covering groups in the case when $Q = 0$. This expands upon our running example and will provide a clue about the modifications needed.

When $Q = 0 = D$, the objects in $\mathbf{Bis}_{\mathbb{G},Q}$ are simply homomorphisms $\eta : Y^{sc} \to F^{\times}$. Choose a z-extension

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1,$$

so that $Y_{\mathbb{H}}^{sc} = Y^{sc}$ and $Y_{\mathbb{H}}/Y^{sc}$ is free. For any η, we have a corresponding short exact sequence

$$1 \longrightarrow Z = Z_\eta \longrightarrow \overline{\mathbb{H}}_\eta \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

Since all $\eta : Y_{\mathbb{H}}^{sc} = Y^{sc} \to F^{\times}$ are equivalent to the trivial homomorphism 1 as objects of $\mathbf{Bis}_{\mathbb{H},Q}$, we may choose an isomorphism

$$\xi : \overline{\mathbb{H}}_1 = \mathbb{H} \times \mathbb{K}_2 \longrightarrow \overline{\mathbb{H}}_\eta.$$

After taking F-points, and noting that $H^1_{Zar}(F, Z) = 0$, we then have

$$\overline{H}_1 = H \times \mu_n \xrightarrow{\ \xi\ } \overline{H}_\eta \longrightarrow \overline{G}_\eta,$$

and the kernel of this map is the subgroup

$$\xi^{-1}(Z_\eta) = \{(z, \chi_{\eta,\xi}(z)^{-1}) : z \in Z\} \subset H \times \mu_n,$$

where $\chi_{\eta,\xi}$ is the map

$$Z = Y_Z \otimes F^{\times} \xrightarrow{\ \xi\ } F^{\times} \otimes F^{\times} \xrightarrow{(-,-)_n} \mu_n.$$

Hence, the set of genuine representations of $\overline{G}_\eta$ can be identified (by pulling back) with a subset of the genuine representations of the split extension $H \times \mu_n$ whose restriction to the central subgroup $Z \subset H$ is the character $\chi_{\eta,\xi}$.

Now the L-group of $\overline{H} = H \times \mu_n$ is a short exact sequence

$$H^{\vee} \longrightarrow {}^L\overline{H} \longrightarrow W_F$$

which is equipped with a finite set of distinguished splittings. For example one may take the distinguished splitting s_0 which corresponds to the trivial character of the maximal torus T_H of H. Then we may identify the set of all splittings (modulo conjugacy by $H^\vee$) with the set of L-parameters

$$W_F \longrightarrow H^\vee$$

of H. Thus, if the LLC holds for the linear group H, there is a finite-to-one map

$$\bigcup_\eta \mathrm{Irr}_{gen}(\overline{G}_\eta) \longrightarrow \{\text{L-parameters for } H\}$$

which may be construed as a (weak) LLC for the family of covering groups $\overline{G}_\eta$ (as η varies). Moreover, the image of $\mathrm{Irr}_{gen}(\overline{G}_\eta)$ for a particular η can be described as follows. By Lemma 5.4, there is a natural short exact sequence:

$$1 \longrightarrow G^\vee \longrightarrow H^\vee \xrightarrow{\rho} Z^\vee = \mathrm{Hom}(Y_Z, \mathbb{C}^\times) \longrightarrow 1.$$

Then $\mathrm{Irr}_{gen}(\overline{G}_\eta)$ is the set of L-parameters ϕ of H such that $\rho \circ \phi : W_F \longrightarrow Z^\vee$ is the L-parameter of the character $\chi_{\eta,\xi}$. Observe that the genuine representations of $\overline{G}_1 = G \times \mu_n$ is then parametrized by L-parameters of H which factors through $G^\vee$.

There is an obvious notion of a z-extension H dominating another H', and one can easily check that the above classification of the genuine representations of $\overline{G}_\eta$ behave functorially with respect to dominance. This suggests that it is possible (and certainly desirable) to formulate the LLC for $\overline{G}_\eta$ without reference to the z-extension H. Such use of z-extensions is similar to the use of z-extensions in the usual theory of endoscopy for linear groups.

10.2. **Modification of L-group.** Motivated by the above discussion, we can revisit the L-group construction in the general setting. Fix the quadratic form Q on Y. The crucial E_2 construction starts with

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathcal{E}_{Q,n} \longrightarrow Y_{Q,n} \longrightarrow 1$$

and then use the section

$$s_\eta : Y_{Q,n}^{sc} \longrightarrow \mathcal{E}_{Q,n}$$

to form the quotient

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathcal{E}_{Q,n}/s_\eta(Y_{Q,n}^{sc}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{sc} \longrightarrow 1,$$

before applying $\mathrm{Hom}(-, \mathbb{C}^\times)$. To incorporate all η's together, we observe that the section s_η is independent of η when restricted to the sublattice $nY_{Q,n}^{sc}$. Then one has the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q,n}/s_\eta(nY_{Q,n}^{sc}) & \longrightarrow & Y_{Q,n}/nY_{Q,n}^{sc} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q,n}/s_1(Y_{Q,n}^{sc}) & \longrightarrow & Y_{Q,n}/Y_{Q,n}^{sc} & \longrightarrow & 1.
\end{array}$$

Taking $\mathrm{Hom}(-,\mathbb{C}^\times)$, one obtains the commutative diagram with exact rows and columns, which defines the modification $\tilde{E}_2$ of E_2:

$$\begin{array}{ccccc} Z(\overline{G}_1^\vee) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbb{C}^\times) & \longrightarrow & E_2 & \longrightarrow & F^\times/F^{\times n} \\ \downarrow & & \downarrow & & \| \\ \mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{sc}, \mathbb{C}^\times) & \longrightarrow & \tilde{E}_2 & \longrightarrow & F^\times/F^{\times n} \\ \downarrow & & \downarrow & & \downarrow \\ (T_{Q,n}^{sc})^\vee[n] = \mathrm{Hom}(Y_{Q,n}^{sc}/nY_{Q,n}^{sc}, \mathbb{C}^\times) & = & (T_{Q,n}^{sc})^\vee[n] & \longrightarrow & 1 \end{array}$$

The cocycle defining E_1 also defines an extension

$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{sc}, \mathbb{C}^\times) \longrightarrow \tilde{E}_1 \longrightarrow F^\times/F^{\times n} \longrightarrow 1.$$

Then we can form the Baer sum and obtain

$$\begin{array}{ccccc} Z(\overline{G}_1^\vee) & \longrightarrow & E = E_1 + E_2 & \longrightarrow & F^\times/F^{\times n} \\ \downarrow & & \downarrow & & \| \\ \mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{sc}, \mathbb{C}^\times) & \longrightarrow & \tilde{E} = \tilde{E}_1 + \tilde{E}_2 & \longrightarrow & F^\times/F^{\times n} \\ \downarrow & & \downarrow & & \downarrow \\ (T_{Q,n}^{sc})^\vee[n] & = & (T_{Q,n}^{sc})^\vee[n] & \longrightarrow & 1 \end{array}$$

From this, we infer the short exact sequence:

$$1 \longrightarrow Z(\overline{G}_1^\vee) \longrightarrow \tilde{E} \longrightarrow F^\times/F^{\times n} \times (T_{Q,n}^{sc})^\vee[n] \longrightarrow 1.$$

This is our enlarged fundamental extension. Pushing this out by $Z(\overline{G}_1^\vee) \hookrightarrow \overline{G}^\vee$ and pulling back to W_F, one obtains:

$$1 \longrightarrow \overline{G}_Q^\vee \longrightarrow {}^L\overline{G}_Q^\# \longrightarrow W_F \times (T_{Q,n}^{sc})^\vee[n] \longrightarrow 1$$

which is our enlarged L-group extension for the family of BD covers with fixed BD-invariant Q. Here we also use the notation $\overline{G}_Q^\vee$ for $\overline{G}^\vee$.

Note that $(T_{Q,n}^{sc})^\vee$ is a maximal torus in the adjoint quotient $(\overline{G}^\vee)_{ad}$ of $\overline{G}^\vee$, so that $(T_{Q,n}^{sc})^\vee[n]$ is its n-torsion subgroup.

10.3. **Relation with ${}^L\overline{G}_\eta$.** How can one recover the L-group of $\overline{G}_\eta$, as previously defined, from the enlarged L-group defined here? Given an $\eta : Y_{Q,n}^{sc} \longrightarrow F^\times$, one obtains a natural map

$$\varphi_\eta : W_F \longrightarrow W_F^{ab} = F^\times \longrightarrow (T_{Q,n}^{sc})^\vee[n] = \mathrm{Hom}(Y_{Q,n}^{sc}, \mu_n)$$

given by

$$\varphi_\eta(a)(y) = (\eta(y), a)_n \quad \text{for all } a \in F^\times \text{ and } y \in Y_{Q,n}^{sc}.$$

Pulling back the enlarged L-group extension by the diagonal map

$$\mathrm{id} \times \varphi_\eta : W_F \longrightarrow W_F \times (T_{Q,n}^{sc})^\vee[n],$$

one obtains the L-group extension ${}^L\overline{G}_\eta$.

10.4. **The modified dual group.** Consider the kernel $\overline{G}^{\#}_Q \subseteq {}^L\overline{G}^{\#}_Q$ of the following composition of surjections:

$$^L\overline{G}^{\#}_Q \longrightarrow W_F \times (T^{sc}_{Q,n})^\vee[n] \longrightarrow W_F,$$

where the second map is the projection on the first component. By the definition of ${}^L\overline{G}^{\#}_Q$, the group $\overline{G}^{\#}_Q$ lies in the exact sequence

$$1 \longrightarrow \overline{G}^\vee_Q \longrightarrow \overline{G}^{\#}_Q \longrightarrow (T^{sc}_{Q,n})^\vee[n] \longrightarrow 1, \tag{10.1}$$

which is the push out of

$$1 \longrightarrow Z(\overline{G}^\vee_Q) \longrightarrow \mathrm{Hom}(Y_{Q,n}/nY^{sc}_{Q,n}, \mathbb{C}^\times) \longrightarrow (T^{sc}_{Q,n})^\vee[n] \longrightarrow 1. \tag{10.2}$$

Since the adjoint group $(\overline{G}^\vee)_{ad}$ acts naturally on $\overline{G}^\vee$, preserving $Z(\overline{G}^\vee)$, there is a canonical splitting of $\overline{G}^{\#}_Q$ over $(T^{sc}_{Q,n})^\vee[n]$, which gives a canonical isomorphism

$$\overline{G}^{\#}_Q \simeq \overline{G}^\vee_Q \rtimes (T^{sc}_{Q,n})^\vee[n].$$

Moreover, the action of $(T^{sc}_{Q,n})^\vee[n]$ on $\overline{G}^\vee$ is identity on its maximal torus $\overline{T}^\vee = X_{Q,n} \otimes \mathbb{C}^\times$ and preserves the maximal unipotent subgroup corresponding to the set of simple roots $\Delta^\vee_{Q,n}$. This shows that every irreducible representation is invariant under the action of $(T^{sc}_{Q,n})^\vee[n]$, and thus extends (non-canonically) to $\overline{G}^{\#}_Q$. In other words, the representation theory of the disconnected group $\overline{G}^{\#}_Q$ is not more complicated than that of $\overline{G}^\vee$.

10.5. **Running example.** Consider the case $\mathbb{G} = \mathrm{PGL}_2$ and $D = 0$, $n = 2$. The dual group is $\overline{G}^\vee_Q = \mathrm{SL}_2(\mathbb{C})$ and the exact sequence (10.2) is

$$1 \longrightarrow \mu_2 \longrightarrow \mu_4 \longrightarrow \mu_2 \longrightarrow 1.$$

We obtain $\overline{G}^{\#}_Q \simeq \mathrm{SL}_2(\mathbb{C}) \rtimes \mu_2$. The action of the nontrivial element $\epsilon \in \mu_2$ on $\mathrm{SL}_2(\mathbb{C})$ is given by

$$\epsilon : g \mapsto \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix} g \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix}^{-1},$$

where $\xi \in \mu_4$ is any square root of ϵ. As there is no splitting of $\overline{G}^{\#}_Q$ over μ_2 valued in the center $Z(\overline{G}^\vee_Q)$, the group $\overline{G}^{\#}_Q$ is not isomorphic to the direct product $\mathrm{SL}_2(\mathbb{C}) \times \mu_2$.

This example shows that in general $\overline{G}^{\#}_Q \simeq \overline{G}^\vee_Q \rtimes (T^{sc}_{Q,n})^\vee[n]$ is not a direct product of the two groups.

11. **The LLC**

After the discussion in the previous sections, we can now formulate the LLC for BD covering groups.

11.1. **L-parameters.** After introducing the L-group extension, one now has the following notions:

- An L-parameter for the covering group $\overline{G}_\eta$ is a splitting $\phi : W_F \longrightarrow {}^L\overline{G}_\eta$ of the extension ${}^L\overline{G}_\eta$, taken up to conjugacy by $\overline{G}^\vee$. Equivalently, it is a splitting $\phi : W_F \longrightarrow {}^L\overline{G}^\#_Q$ of the enlarged L-group extension such that
$$p \circ \phi = \varphi_\eta,$$
where $p : {}^L\overline{G}^\#_Q \longrightarrow (T^{sc}_{Q,n})^\vee[n]$ is the natural projection.
- we have demonstrated the existence of a finite set of distinguished splittings for ${}^L\overline{G}_1$ and thus for ${}^L\overline{G}^\#_Q$. If we fix one such splitting ϕ_0, then all splittings of ${}^L\overline{G}^\#_Q$ are of the form $\phi_0 \cdot \phi$ where
$$\phi : W_F \longrightarrow \overline{G}^\#_Q = \overline{G}^\vee_Q \rtimes (T^{sc}_{Q,n})^\vee[n].$$
We call such ϕ's the L-parameters relative to the distinguished splitting ϕ_0.

11.2. **Local L-factors.** Given a representation
$$R : {}^L\overline{G}_\eta \longrightarrow \mathrm{GL}(V)$$
where V is a complex finite-dimensional vector space over $\mathbb{C}$, and a splitting ϕ of ${}^L\overline{G}_\eta$, one obtains a complex representation $R \circ \phi$ of W_F and hence an Artin L-factor $L(s, \phi, R)$. Alternatively, if $\phi : W_F \longrightarrow \overline{G}^\#$ is an L-parameter relative to a distinguished splitting ϕ_0 of ${}^L\overline{G}^\#$ over W_F, and $R : \overline{G}^\# \longrightarrow \mathrm{GL}(V)$ is a representation, then one has an associated L-factor $L_{\phi_0}(s, \phi, R)$. As we noted before, the irreducible representations of $\overline{G}^\#$ are simply extensions of those of $\overline{G}^\vee$. We shall give a more detailed treatment of this in the next section, where we introduce automorphic L-functions.

11.3. **The LLC.** In view of the unramified LLC discussed in Section 9, one is tempted to conjecture the existence of a finite-to-one map giving rise to a commutative diagram:
$$\begin{array}{ccc}
\mathcal{L}_\eta : \mathrm{Irr}\overline{G}_\eta & \xrightarrow{\mathcal{L}_\eta} & \{\text{splittings of } {}^L\overline{G}_\eta\} \\
\downarrow & & \downarrow \\
\bigcup_\eta \mathrm{Irr}\overline{G}_\eta & \xrightarrow{\mathcal{L}} & \{\text{splittings of } {}^L\overline{G}^\#_Q\}.
\end{array}$$
This is a weak LLC. As shown in the case when $\mathbb{G} = \mathbb{T}$ is a split torus, one should not expect this map to be surjective. Thus, one would also like to have a conjectural parametrization of the fibers of this map. This would be a strong LLC.

11.4. **Reduction to $\eta = 1$.** We shall show that this weak LLC for general $\overline{G}_\eta$ can be reduced to the case of trivial η. This is similar to the discussion at the beginning of the last section and relies on the consideration of z-extensions.

More precisely, we choose a z-extension $1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1$ giving rise to
$$1 \longrightarrow Z \longrightarrow \overline{\mathbb{H}}_\eta \xrightarrow{p} \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

Choose an isomorphism $\xi : \overline{\mathbb{H}}_1 \longrightarrow \overline{\mathbb{H}}_\eta$, which realises

$$\overline{\mathbb{G}}_\eta \cong \overline{\mathbb{H}}_1/\xi^{-1}(Z).$$

Thus one has an injection

$$\xi^* \circ p^* : \mathrm{Irr}\overline{G}_\eta \hookrightarrow \mathrm{Irr}\overline{H}_1$$

whose image consists of those irreducible genuine representations of $\overline{H}_1$ whose restriction to Z is a prescribed character χ_ξ. If the LLC holds for the case of trivial η, then one would have a map

$$\mathcal{L} \circ \xi^* \circ p^* : \mathrm{Irr}\overline{G}_\eta \longrightarrow \{\text{splittings of } {}^L\overline{H}_1\}.$$

Now recall that by Lemma 5.4, there is a natural map

$$p : {}^L\overline{H}_1 \longrightarrow Z^\vee.$$

If one assumes the (weak) LLC for $\overline{H}_1$ satisfies the natural property that the restriction of the central character of $\pi \in \mathrm{Irr}(\overline{H}_1)$ to Z corresponds to the parameter $p \circ \mathcal{L}(\pi)$ under the usual LLC for the (linear) torus Z, then one sees that

$$\mathrm{Irr}\overline{G}_\eta \longrightarrow \{\text{splittings } \phi \text{ of } {}^L\overline{H}_1\text{: } p \circ \phi \text{ corresponds to } \chi_\xi\} = \{\text{splittings of } {}^L\overline{G}_\eta\}.$$

The relation with the enlarged L-group ${}^L\overline{G}^\#$ is as follows. One has a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{G}^\# & \longrightarrow & {}^L\overline{G}^\# & \longrightarrow & W_F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \overline{H}_1^\vee & \longrightarrow & {}^L\overline{H}_1 & \longrightarrow & W_F & \longrightarrow & 1,
\end{array}$$

and the L-parameters of $\overline{H}_1$ which intervene in the description of the LLC for $\overline{G}_\eta$ (as η varies) are those which factors through ${}^L\overline{G}^\#$.

11.5. **Reduction to discrete series.** The existence of the (weak) LLC map $\mathcal{L}$ can be reduced to the case of (quasi)-discrete series representation, much like the case of linear reductive groups. More precisely,

- Ban and Jantzen [BJ] have established the analog of the Langlands classification for general covering groups over p-adic fields; the case of covers of real groups is also known [BW]. This says that every irreducible representation is uniquely expressed as the unique irreducible quotient of a standard module. As in the linear case, this reduces the definition of $\mathcal{L}$ to (quasi)-tempered representations.
- As shown in the work of W.-W. Li [L4], any tempered representation is contained in a representation parabolically induced from (quasi-)discrete series representations; moreover the decomposition of this induced representation is governed by an R-group [L4]. This reduces the definition of $\mathcal{L}$ to the case of (quasi-)discrete series representations.

11.6. **The example of** Mp_{2n}**.** The only nontrivial case where one has rather complete results for the LLC is the classical case of the metaplectic double covering group $\overline{G} = \mathrm{Mp}_{2n}(F)$ of $\mathrm{Sp}_{2n}(F)$. In this case, the L-group ${}^L\overline{G}$ is isomorphic (non canonically) to the direct product $\mathrm{Sp}_{2n}(\mathbb{C}) \times W_F$, and such an isomorphism is given by the choice of a distinguished genuine character of the covering torus $\overline{T} \subset \mathrm{Mp}_{2n}$. In §16.1.2, we show that such a distinguished genuine character of $\overline{T}$ is simply a genuine character χ_ψ defined using the Weil index associated to a nontrivial additive character ψ of F. Thus the choice of ψ gives a bijection

$$\mathcal{S}_\psi : \{\text{splittings of } {}^L\overline{G} \text{ over } WD_F\} \longleftrightarrow \{\text{homomorphisms } WD_F \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})\}$$

On the other hand, using the theory of theta correspondence, it was shown in [AB] (for archimedean F) and [GS] (for nonarchimedean F) that there is a bijection

$$\Theta_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longleftrightarrow \bigcup_{V_n} \mathrm{Irr}(\mathrm{SO}(V_n))$$

where V_n runs over all isomorphism classes of quadratic spaces of dimension $2n+1$ and trivial discriminant. Combining this with the LLC for odd special orthogonal groups (due to Arthur [Ar] and Moeglin [M]), one obtains an LLC map

$$\mathcal{L}_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\text{homomorphisms } WD_F \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})\}.$$

We thus have a surjective map

$$\mathcal{S}_\psi^{-1} \circ \mathcal{L}_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\text{L-parameters for } \mathrm{Mp}_{2n}\}.$$

The main observation we want to make is:

Theorem 11.1. *The composite $\mathcal{S}_\psi^{-1} \circ \mathcal{L}_\psi$ is independent of the choice of ψ. In particular, assuming the LLC for odd special orthogonal groups, one has an LLC for $\mathrm{Mp}_{2n}(F)$ in terms of the L-group considered in this paper.*

Proof. All nontrivial characters of F are of the form $\psi_a(x) = \psi(ax)$ for some $a \in F^\times$. The corresponding distinguished characters of $\overline{T}$ are related by

$$\chi_{\psi_a} = \chi_\psi \cdot \chi_a$$

where $\chi_a = (a, -)_2$. Thus, the distinguished splittings s_ψ and s_{ψ_a} differ by the quadratic character χ_a (regarded as a map $W_F \longrightarrow \mu_2 \subset \overline{T}^\vee$), and the bijections $\mathcal{S}_\psi$ and $\mathcal{S}_{\psi_a}$ differ by twisting by χ_a. On the other hand, it was shown in [AB, GS] that $\mathcal{L}_{\psi_a}$ and $\mathcal{L}_\psi$ are also related by twisting by χ_a. It follows that the two dependence on a cancel and $\mathcal{S}_\psi \circ \mathcal{L}_\psi$ is independent of the choice of ψ. □

12. Desiderata and Anomalies

In this section, we shall explain how the L-group formalism developed thus far allows one to explain (at least conjecturally) certain anomalies which have been empirically observed in the genuine representation theory of covering groups.

12.1. **Central characters.** In the LLC for Mp_{2n} discussed at the end of the previous section, it is known that the central character is not an invariant of an L-packet. This is in contrast to the case of linear groups, where all representations in a given L-packet have the same central character. Let us see how this anomaly is explained by the L-group formalism.

For a BD covering group $\overline{G}$, we have attached an L-group ${}^L\overline{G}$, which is abstractly isomorphic to a semi-direct product. Let us consider the case when ${}^L\overline{G}$ has a distinguished splitting, so that ${}^L\overline{G} \cong \overline{G}^\vee \times W_F$. Let $\mathbb{G}_{Q,n}$ be the split linear algebraic group over F, with dual group $\overline{G}^\vee$ (this $\mathbb{G}_{Q,n}$ should be the principal endoscopic group of $\overline{G}$, as we discuss later on). Then by construction $G_{Q,n} := \mathbb{G}_{Q,n}(F)$ contains $T_{Q,n}$ as maximal split torus. Now an L-parameter ϕ of $\overline{G}$ can be regarded as one for $G_{Q,n}$ (relative to the choice of a distinguished splitting). Such an L-parameter certainly encodes the central characters of representations of $G_{Q,n}$ in the associated L-packet.

Let us compare the centres of $G_{Q,n}$ and $\overline{G}$, which are contained in $T_{Q,n}$ and $\overline{T}$ respectively. One has

$$Z(G_{Q,n}) \cong \mathrm{Hom}(X_{Q,n}/X_{Q,n}^{sc}, F^\times) \subset T_{Q,n}$$

and

$$p(Z(\overline{G})) = Z(G) \cap p(\overline{T}) \subset \mathrm{Hom}(X/X^{sc}, F^\times) \subset T,$$

where $p : \overline{G} \longrightarrow G$ is the natural projection. Moreover, recall that there is an isogeny

$$i : T_{Q,n} \longrightarrow T,$$

associated to the natural embedding $Y_{Q,n} \hookrightarrow Y$ and such that

$$i(T_{Q,n}) = p(Z(\overline{T})).$$

It is easy to check that

$$i(Z(G_{Q,n})) \subset p(Z(\overline{G})).$$

This leads us to the following:

Speculation: under the LLC for $\overline{G}$, for an L-parameter ϕ for $\overline{G}$ with associated L-packet Π_ϕ, all representations in Π_ϕ transform by the same character (determined by ϕ) when restricted to the preimage in $\overline{G}$ of the central subgroup $i(Z(G_{Q,n}))$. In particular, this leaves open the possibility for the representations in Π_ϕ to transform by different characters under the whole center $Z(\overline{G})$.

If we apply this to the case of $\overline{G} = \mathrm{Mp}_{2n}$, then $\mathbb{G}_{Q,n} = \mathrm{SO}_{2n+1}$ so that $Z(G_{Q,n})$ is trivial, whereas $p(Z(\overline{G}))$ is $\mu_2(F)$. Thus, both genuine central characters are allowed in a given L-packet, as has been observed.

12.2. **Twisting by characters.** Suppose that π is an irreducible genuine representation of $\overline{G}$ with L-parameter $\phi : W_F \longrightarrow {}^L\overline{G}$. Let $\chi : G \longrightarrow \mathbb{C}^\times$ be a 1-dimensional character. Then we may consider the irreducible genuine representation $\pi \otimes \chi$. What should be the L-parameter of $\pi \otimes \chi$?

Let $\phi_\chi : W_F \longrightarrow G^\vee$ be the L-parameter of χ, with $G^\vee$ the Langlands dual group of G. One would like to twist the L-parameter ϕ of π by ϕ_χ, but the two parameters take value in different groups. However, one knows that ϕ_χ factors through the center $Z(G^\vee)$ of $G^\vee$:

$$\phi_\chi : W_F \longrightarrow Z(G^\vee) \longrightarrow G^\vee.$$

On the other hand, one has a natural map

$$\delta : Z(G^\vee) = \mathrm{Hom}(Y/Y^{sc}, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}), \mathbb{C}^\times)$$

given by restricting from Y to $Y_{Q,n}$. Then one has the composite

$$\delta \circ \phi_\chi : W_F \longrightarrow Z(G^\vee) \longrightarrow Z(\overline{G}^\vee) \subset \overline{G}^\vee.$$

Now it is natural to have the following expectation:

Speculation: If ϕ is the L-parameter of the irreducible genuine representation π of $\overline{G}$, then the L-parameter of $\pi \otimes \chi$ is given by $\phi \otimes (\delta \circ \phi_\chi)$.

12.3. **Restriction to derived subgroups.** Another anomaly concerns the restriction of representations of $\overline{G}$ to, for example, $\overline{G}^{der}$ or $\overline{G}^{sc}$. For example, in the linear case, all irreducible summands in the restriction of an irreducible representations of $\mathrm{GL}_2(F)$ to $\mathrm{SL}_2(F)$ belongs to the same L-packet. However, if one takes the degree 2 Kazhdan-Patterson cover $\overline{\mathrm{GL}}_2$ of $\mathrm{GL}_2(F)$, then the restriction of an irreducible genuine representation σ of $\overline{\mathrm{GL}}_2$ to $\overline{\mathrm{SL}}_2 = \mathrm{Mp}_2$ may contain constituents belonging to different L-packets of Mp_2. More precisely, one has

$$\sigma|_{\mathrm{Mp}_2} \cong \bigoplus_\chi \pi_\chi$$

where the sum runs over quadratic characters χ of $F^\times$. Moreover, if the L-parameter of π_1 is ϕ, then that of π_χ is $\phi \otimes \chi$. How can this be explained by the L-group formalism?

Let $\mathbb{G}$ be a linear reductive group over F and

$$\mathbb{G}^{sc} \longrightarrow \mathbb{G}^{der} \longrightarrow \mathbb{G}$$

the natural sequence of maps. In forming the L-group of $\overline{G}$, we consider the lattices

$$\begin{array}{ccc} Y^{sc}_{Q,n} & \longrightarrow & Y^{sc} \\ \downarrow & & \downarrow \\ Y_{Q,n} & \longrightarrow & Y \end{array}$$

Here the arrows are inclusions, and the first column is used to define the root datum for the dual group of $\overline{G}$. Now we may pullback the BD cover to G^{sc} and construct the L-group of $\overline{G}^{sc}$; this gives the diagram, writing $Z = Y^{sc}$ for readability:

$$\begin{array}{ccc} Z^{sc}_{Q,n} & \longrightarrow & Z \\ \downarrow & & \| \\ Z_{Q,n} & \longrightarrow & Z. \end{array}$$

Here the first column is used to form the dual group of $\overline{G}^{sc}$.

Let's examine how these two diagrams interact. One has:

$$\begin{array}{ccccc} Y_{Q,n} & \longleftarrow & Y_{Q,n} \cap Z & \longleftarrow & Y^{sc}_{Q,n} \\ & & \| & & \| \\ (Y^{sc})_{Q,n} = Z_{Q,n} & \longleftarrow & Y_{Q,n} \cap Z & \longleftarrow & Z^{sc}_{Q,n} \end{array}$$

and the point is that there is an inclusion

$$Z_{Q,n} \supset Y_{Q,n} \cap Z = Y_{Q,n} \cap Y^{sc}$$

which is not necessarily an equality (it would be if $n = 1$).

This means that one has the following diagram of dual groups:

$$\overline{G}^{\vee} \longrightarrow H^{\vee} \longleftarrow (\overline{G}^{sc})^{\vee}$$

where $H^{\vee}$ is the connected reductive group with root datum

$$(Y_{Q,n} \cap Z, \Delta^{\vee}_{Q,n}, \text{dual lattice of } Y_{Q,n} \cap Z, \Delta_{Q,n}),$$

and the second arrow is an isogeny. Now suppose one is given an L-parameter for $\overline{G}$, i.e.

$$\phi : W_F \longrightarrow \overline{G}^{\vee}.$$

Let

$$\overline{\phi} : W_F \longrightarrow \overline{G}^{\vee} \longrightarrow H^{\vee}.$$

This leads to the following speculation:

Speculation: If one takes a representation π in the associated L-packet Π_{ϕ}, then the pullback of π to $\overline{G}^{sc}$ will decompose into irreducible summands. The L-parameters of these summands are given by those

$$\phi' : W_F \longrightarrow (\overline{G}^{sc})^{\vee}$$

such that

$$\overline{\phi} = \overline{\phi}' \quad \text{i.e. equality when both } \phi \text{ and } \phi' \text{ are projected to } H^{\vee}.$$

Since the projection from $(\overline{G}^{sc})^{\vee}$ to $H^{\vee}$ is not an isomorphism in general, it is possible for several ϕ' to arise. This explains the above phenomenon in the examples mentioned above.

13. **Automorphic L-functions**

While we have considered only the case of local fields for most of this paper, we shall now consider the global setting, so that k is a number field with ring of adeles $\mathbb{A}$. We shall briefly explain how the construction of the L-group extension extends to the global situation, referring to [W7] for the details. The goal of the section is to give a definition of the notion of *automorphic L-functions*.

13.1. **Adelic BD covering.** Starting with a BD extension $\overline{\mathbb{G}}$ over $\mathrm{Spec}(k)$ and a positive integer n such that $|\mu_n(k)| = n$, Brylinski and Deligne showed using results of Moore [Mo] that one inherits the following data:

- for each place v of k, a local BD covering group $\overline{G}_v$ of degree n;
- for almost all v, a splitting $s_v : \mathbb{G}(\mathcal{O}_v) \longrightarrow \overline{G}_v$;
- a restricted direct product $\prod'_v \overline{G}_v$ with respect to the family of subgroups $s_v(\mathbb{G}(\mathcal{O}_v))$, from which one can define:
$$\overline{G}(\mathbb{A}) := \prod_v{}' \, \overline{G}_v / \{(\zeta_v) \in \oplus_v \mu_n(k_v) : \prod_v \zeta_v = 1\},$$
which gives a topological central extension
$$1 \longrightarrow \mu_n(k) \longrightarrow \overline{G}(\mathbb{A}) \longrightarrow \mathbb{G}(\mathbb{A}) \longrightarrow 1,$$
called the adelic or global BD covering group;
- a natural inclusion
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n(k_v) & \longrightarrow & \overline{G}_v & \longrightarrow & \mathbb{G}(k_v) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n(k) & \longrightarrow & \overline{G}(\mathbb{A}) & \longrightarrow & \mathbb{G}(\mathbb{A}) & \longrightarrow & 1 \end{array}$$
for each place v of k;
- a natural splitting
$$i : \mathbb{G}(k) \longrightarrow \overline{G}(\mathbb{A}),$$
which allows one to consider the space of automorphic forms on $\overline{G}(\mathbb{A})$.

13.2. **Global L-group extension.** One may define the L-group extension for the adelic BD cover $\overline{G}(\mathbb{A})$ following the same procedure as in the local setting. We briefly summarise the process, highlighting the differences. Suppose that $\overline{\mathbb{G}}$ has BD invariant $(Q, \mathcal{E}, f)$ or bisector data (D, η). Then one has:

- The dual group of $\overline{G}(\mathbb{A})$ is defined in exactly the same way as in the local setting. Namely, one may define the lattice $Y_{Q,n}$ and the modified coroot lattice $Y_{Q,n}^{sc}$ in the same way. This gives the dual group $\overline{G}^\vee$. Indeed, since $\mathbb{G}$ is split, the definition of these objects works over any k or k_v and gives the same complex dual group $\overline{G}^\vee$.
- The role of $F^\times/F^{\times n}$ in the local setting is replaced by $\mathbb{A}^\times/k^\times\mathbb{A}^{\times n}$. More precisely, with $(-,-)_n$ denoting the global n-th Hilbert symbol, the 2-cocycle
$$c_1(a,b)(y) = (a,b)_n^{Q(y)}$$
for $a, b \in \mathbb{A}^\times/\mathbb{A}^{\times n}$ defines an extension
$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbb{C}^\times) \longrightarrow E_1' \longrightarrow \mathbb{A}^\times/\mathbb{A}^{\times n} \longrightarrow 1.$$
Since c_1 is trivial on $k^\times \times k^\times$, this sequence splits canonically over the image of $k^\times$ in $\mathbb{A}^\times/\mathbb{A}^{\times n}$. Dividing out by the image of $k^\times$ under the splitting gives the extension

E_1:

$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbb{C}^\times) \longrightarrow E_1 \longrightarrow \mathbb{A}^\times/k^\times\mathbb{A}^{\times n} \longrightarrow 1.$$

- The extension E_2 is defined in the same way, applying $\mathrm{Hom}(-, \mathbb{C}^\times)$ to the sequence

$$1 \longrightarrow k^\times/k^{\times n} \longrightarrow \mathcal{E}/s_f(Y_{Q,n}^{sc}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{sc} \longrightarrow 1,$$

and pulling back by

$$\mathbb{A}^\times/k^\times\mathbb{A}^{\times n} \longrightarrow \mathrm{Hom}(k^\times/k^{\times n}, \mathbb{C}^\times).$$

Forming Baer sum with E_1 gives the global fundamental sequence:

$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow E \longrightarrow \mathbb{A}^\times/k^\times\mathbb{A}^{\times n} \longrightarrow 1.$$

Pulling back to the global Weil group W_k and pushing out to $\overline{G}^\vee$ gives the global L-group extension, which fits into a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_v & \longrightarrow & W_{k_v} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \end{array}$$

for each place v of k.

- This global L-group extension satisfies the same functoriality with respect to Levi subgroups and z-extensions as in the local case.

13.3. **Distinguished splittings.** One may examine the question of splitting of the global L-group extension. By construction, it has a canonical splitting over the subgroup

$$W_{k,n} = \mathrm{Ker}(W_k \longrightarrow k^\times\mathbb{A}^{\times n}\backslash\mathbb{A}^\times).$$

As in the local case, the problem of extending this canonical splitting to one over W_k is equivalent to finding a splitting of the global fundamental sequence. One can define the notion of distinguished splitting of ${}^L\overline{G}_{\mathbb{A}}$ analogously as in the local case, and this amounts to finding a genuine automorphic character

$$\chi : \mathbb{T}_{Q,n}(k) \cdot \mathbb{T}_J(\mathbb{A})\backslash\overline{T}_{Q,n}(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

where $J = Y_{Q,n}^{sc} + nY \subset Y_{Q,n}$. Such a character exists when $\eta_n = 1$ is trivial, and thus a distinguished splitting of the fundamental sequence E exists in this case. If we fix such an automorphic character $\chi = \prod_v \chi_v$, then each χ_v corresponds to a distinguished splitting of the local fundamental sequence. Moreover, χ is invariant under the Weyl group $W(k) = N(\mathbb{T})(k)/\mathbb{T}(k)$. The explicit construction of a distinguished genuine character given in §7 produces an automorphic character.

We deduce from the above discussion that the L-group extension is always split, and thus is abstractly a semi-direct product, but it may not be a direct product in general.

13.4. **Automorphic L-functions.** We now have all the ingredients to define the notion of (partial) automorphic L-functions. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\overline{G}_{\mathbb{A}}$. For almost all v, π_v is s_v-unramified. By the unramified LLC, π_v gives rise to an s_v-unramified splitting

$$\rho_{\pi,v} : W_{k_v} \longrightarrow {}^L\overline{G}_v \subset {}^L\overline{G}_{\mathbb{A}}.$$

Let $R : {}^L\overline{G}_{\mathbb{A}} \longrightarrow \mathrm{GL}(V)$ be a continuous finite dimensional representation which is trivial on the subgroup

$$W_{k,n} = \mathrm{Ker}(W_k \longrightarrow k^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times) \subset {}^L\overline{G}_{\mathbb{A}}.$$

Recall here that the subgroup $W_{k,n} \subset W_k$ admits a canonical splitting into ${}^L\overline{G}_{\mathbb{A}}$ and observe that $W_k/W_{k,n}$ has exponent n. We may form the local Artin L-factor for the representation $R \circ \rho_{\pi,v} : W_{k_v} \longrightarrow \mathrm{GL}(V)$:

$$L(s, \pi_v, R) = \frac{1}{\det(1 - \rho_{\pi,v}(\mathrm{Frob}_v) \cdot q_v^{-s} | V^{I_v})}$$

where q_v denotes the size of the residue field of k_v. Then we may form the partial global L-function relative to R:

$$L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R)$$

for a sufficiently large finite set of places of k including all archimedean ones.

Theorem 13.1. *For a cuspidal representation π, the Euler product*

$$L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R)$$

converges uniformly when $\mathrm{Re}(s)$ *is sufficiently large.*

Proof. The proof is essentially the same as that of Langlands' in [La] and we shall give a sketch, following Borel's exposition in [Bo, §13] closely. Fixing a splitting $\phi : W_k \longrightarrow {}^L\overline{T}_{\mathbb{A}} \subset {}^L\overline{G}_{\mathbb{A}}$ which agrees with the canonical splitting on $W_{k,n}$, we may write ${}^L\overline{G}_{\mathbb{A}}$ as a semidirect product $\overline{G}^\vee \rtimes W_k$. Moreover, the conjugation action of W_k fixes $\overline{T}^\vee$ pointwise and normalizes the Borel subgroup $\overline{B}^\vee$. For each place v, the restriction of ϕ to W_{k_v} corresponds under LLC to a genuine character χ_v of $\overline{T}_{Q,n,v}$ which has finite order by Proposition 6.5(i), so that the positive-valued character $|\chi_v|$ is trivial.

Now the representation R is thus pulled back from a representation (still denoted by R) of $\overline{G}^\vee \rtimes W_k/W_{k,n}$. We need to show that there exists $A > 0$ such that almost all places v, any eigenvalue α of $R(\rho_{\pi,v}(\mathrm{Frob}_v))$ satisfies

$$|\alpha| \leq q_v^A.$$

Let us write:

$$\rho_{\pi,v}(\mathrm{Frob}_v) = (t_v, \mathrm{Frob}_v) \in \overline{T}^\vee \rtimes W_k/W_{k,n}.$$

Since $W_k/W_{k,n}$ has exponent n and t_v commutes with Frob_v, one has:

$$(t_v, \mathrm{Frob}_v)^n = (t_v, 1)^n.$$

It suffices to show that there exists $A > 0$ such that for almost all places v, any eigenvalue α of $R(t_v) \in \mathrm{GL}(V)$ satisfies

$$|\alpha| \leq q_v^A.$$

Since π is cuspidal, we may assume without loss of generality that π has unitary central character and hence is unitary. For almost all v, π_v is a unitary s_v-unramified representation, which is associated to an unramified genuine character χ_v of $\overline{T}_{Q,n,v}$. The character

$$\alpha(\chi_v) : \overline{T}_{Q,n,v} \xrightarrow{\chi_v} \mathbb{C}^\times \xrightarrow{|-|} \mathbb{R}^\times \xrightarrow{\log_{q_v}} \mathbb{R}$$

factors through $T_{Q,n,v}/T_{Q,n,v} \cap K_v \cong Y_{Q,n}$ and thus can be identified with an element of $X(\mathbb{T}_{Q,n}) \otimes \mathbb{R} = X(\mathbb{T}) \otimes \mathbb{R}$ (it is called the real logarithm of χ_v in [Bo, §13]). We may assume that $\alpha(\chi_v)$ lies in the closure of the positive Weyl chamber associated to a Borel subgroup $\mathbb{B}$. Then the principal series representation $I(\chi)$ is a direct sum of standard modules, exactly one of which is s_v-unramfied. Then π_v is the unique Langlands quotient of this s_v-unramfied standard module. This follows from the Gindikin-Karpelevich formula shown in [Ga], which shows that the standard intertwining operator associated to this standard module is nonzero on the spherical vector.

Now the spherical matrix coefficient f_{π_v} of π_v is a bounded genuine function on $\overline{G}_v$. On the other hand, the asymptotic of f_{π_v} is governed by the Jacquet module of π_v along the Borel subgroup $\overline{B}_v$, i.e. by the central exponents of π_v; this is a result of Casselman for linear algebraic groups but has been extended to the covering case by Ban-Jantzen; see [BJ, §3.2], especially the proof of [BJ, Theorem 3.4]. By the discussion of the previous paragraph, the normalized Jacquet module of π_v contains the character χ_v^{-1} as a submodule. From this, one deduces as in the linear case that for f_{π_v} to be bounded, $\alpha(\chi_v) \in X(\mathbb{T}) \otimes \mathbb{R}$ must satisfy the following. With

$$\rho = \frac{1}{2} \cdot \sum_{\alpha \in \Phi} \alpha,$$

one has

$$\langle \alpha^\vee, \alpha(\chi_v) \rangle \leq \langle \alpha^\vee, \rho \rangle = 1 \quad \text{for all } \alpha^\vee \in \Delta^\vee.$$

For any weight λ of $\overline{T}^\vee$ in the representation R, one may write λ as a $\mathbb{Q}$-linear combination of $\alpha^\vee \in \Delta^\vee$ and elements in the character group $X(Z(\overline{G}^\vee))$ of $Z(\overline{G}^\vee)$:

$$\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \cdot \alpha^\vee \mod X(Z(\overline{G}^\vee)) \otimes \mathbb{Q}.$$

Setting

$$\lambda^{abs} = \sum_{\alpha \in \Delta} |\lambda_\alpha| \cdot \alpha^\vee \mod X(Z(\overline{G}^\vee)) \otimes \mathbb{Q},$$

we then infer from the above inequality that

$$|\lambda(t_v)| = q_v^{\langle \lambda, \alpha(\chi_v) \rangle} \leq q_v^{\langle \lambda^{abs}, \rho \rangle}.$$

Since there are only finitely many such λ's and hence λ^{abs}'s (as $\dim V$ is finite), we deduce the desired upper bound on the eigenvalues of $R(t_v)$ for almost all v. □

We want to highlight some instances where one can write down these automorphic L-functions.

(i) (L-functions relative to a distinguished splitting) If ${}^L\overline{G}_{\mathbb{A}}$ possesses a distinguished splitting ρ_0 (e.g. if $\eta_n = 1$), then ρ_0 is s_v-unramified for almost all v, and so we have an unramified homomorphism

$$\rho_{\pi,v}/\rho_{0,v} : W_{k_v} \longrightarrow k_v^\times \longrightarrow \mathbb{Z} \longrightarrow \overline{G}^\vee.$$

In other words, for almost all v, we have a Satake parameter $s_{\pi_v} \in \overline{G}^\vee$, well-defined up to conjugacy, and depending on $\rho_{0,v}$.

In this setting, if one has a representation $R : \overline{G}^\vee \longrightarrow \mathrm{GL}(V)$, one can form the partial L-function

$$L^S(s, \pi, R, \rho_0) := \prod_{v \notin S} \frac{1}{\det(1 - s_{\pi_v} \cdot q_v^{-s} | V)}.$$

We call this the (R, ρ_0) L-function of π.

More generally, if one fixes a distinguish splitting ρ_0 of the enlarged L-group ${}^L\overline{G}^{\#}$ (which always exists), one has the notion of unramified L-parameters relative to ρ_0:

$$\rho_{\pi,v} : W_{k_v} \to k_v^\times \to \mathbb{Z} \to \overline{G}^{\#},$$

which gives rise to a Satake parameter $s_{\pi_v} \in \overline{G}^{\#}$. If one extends R above to the disconnected group $\overline{G}^{\#}$, one can define the partial L-function $L^S(s, \pi, R, \rho_0)$ as above.

(ii) (Adjoint type L-functions) If $\overline{G}^\vee_{ad} := \overline{G}^\vee / Z(\overline{G}^\vee)$ denotes the adjoint quotient of $\overline{G}^\vee$, there is a natural commutative diagram of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow p & & \downarrow & & \\
1 & \longrightarrow & \overline{G}^\vee_{ad} & \longrightarrow & \overline{G}^\vee_{ad} \times W_k & \longrightarrow & W_k & \longrightarrow & 1.
\end{array}$$

Thus, if $R : \overline{G}^\vee_{ad} \longrightarrow \mathrm{GL}(V)$ is any representation, we may pull it back to ${}^L\overline{G}_{\mathbb{A}}$ and obtain a partial L-function $L^S(s, \pi, R)$.

(iii) (Langlands-Shahidi L-functions) More generally, suppose that $\mathbb{P} = \mathbb{M}\mathbb{N} \subset \mathbb{G}$ is a parabolic subgroup, and π is an automorphic representation of the BD covering $\overline{M}_{\mathbb{A}}$. By functoriality of the L-group construction, one has inclusions

$$E_{\overline{G}} \hookrightarrow {}^L\overline{M}_{\mathbb{A}} \hookrightarrow {}^L\overline{G}_{\mathbb{A}}$$

where $E_{\overline{G}}$ is the fundamental sequence for $\overline{G}_{\mathbb{A}}$. As in (ii) above, one has a natural commutative diagram of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{M}^\vee & \longrightarrow & {}^L\overline{M}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \overline{M}^\vee / Z(\overline{G}^\vee) & \longrightarrow & \overline{M}^\vee / Z(\overline{G}^\vee) \times W_k & \longrightarrow & W_k & \longrightarrow & 1,
\end{array}$$

i.e. a canonically split extension. Then any representation of $\overline{M}^\vee$ which is trivial on $Z(\overline{G}^\vee)$ pulls back to a representation of ${}^L\overline{M}_{\mathbb{A}}$.

A source of such representations is the adjoint action of ${}^L\overline{M}_{\mathbb{A}}$ on $\mathrm{Lie}(N^\vee)$. Let R be an irreducible summand, so that R is trivial on $Z(\overline{G}^\vee)$. Then we obtain a partial automorphic L-function $L^S(s,\pi,R)$. As shown in the PhD thesis [Ga] of the second author, these Langlands-Shahidi type L-functions appear in the constant term of the Eisenstein series on $\overline{G}_{\mathbb{A}}$, as in the case of linear groups.

A basic open question is whether such automorphic L-functions associated to automorphic representations of BD covering groups have the usual nice properties such as meromorphic continuation and functional equations. In [Ga], the second author has shown that the Langlands-Shahidi L-functions for BD covers have meromorphic continuation. A related question is whether such an automorphic L-function agrees with one for a linear reductive group. We shall examine this question in §15.

14. **Langlands Functoriality: Base Change**

Besides giving the definition of automorphic L-functions, the L-group formalism allows one to define the notion of "Langlands functoriality". In this section, we return to the local setting and examine an instance of Langlands functoriality, namely the notion of base change. Hence, F is again a local field in this section.

14.1. **Base change.** For linear groups, the notion of base change can be directly defined in terms of character identities (in the theory of twisted endoscopy) or defined on the L-group side as the restriction of L-parameters from WD_F to WD_K for a field extension K/F. We adopt the second approach. Thus, given a BD extension $\overline{\mathbb{G}}$ over F, a positive integer n such that $|\mu_n(F)| = n$ and a Galois extension K of F, we have the topological degree n covering groups $\overline{G}_F$ and $\overline{G}_K$ and their associated L-groups ${}^L\overline{G}_F$ and ${}^L\overline{G}_K$. Observe that the dual groups of $\overline{G}_F$ and $\overline{G}_K$ are identical by definition; we shall simply denote this dual group by $\overline{G}^\vee$. Then we would like to define a natural commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_K & \longrightarrow & W_K & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta} & & \downarrow & & \\
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_F & \xrightarrow{\pi} & W_F & \longrightarrow & 1.
\end{array}$$

Such a diagram would induce an isomorphism

$${}^L\overline{G}_K \cong \pi^{-1}(W_K) \subset {}^L\overline{G}_F.$$

Then any splitting of ${}^L\overline{G}_F$ (i.e. any L-parameter of $\overline{G}$) will give by restriction to W_K a splitting of ${}^L\overline{G}_K$ (i.e. an L-parameter for $\overline{G}_K$), thus defining the notion of base change on the L-group side. Moreover, by the construction of the L-group extension, it suffices to construct a natural commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E(\overline{G}_K) & \longrightarrow & K^\times & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E(\overline{G}_F) & \longrightarrow & F^\times & \longrightarrow & 1. \end{array}$$

14.2. **An example.** Before going on, let us consider an example: the case of Mp_2 or even more pertinently, the covering split torus $\overline{T}$ in Mp_2. Let's convince ourselves that there is a canonical base change in this case.

We have $\overline{T}_F = T(F) \times \mu_2$ with the group law given by the quadratic Hilbert symbol of F:

$$(t_1, \epsilon_1) \cdot (t_2, \epsilon_2) = (t_1 t_2, \epsilon_1 \epsilon_2 \cdot (t_1, t_2)_F).$$

For each nontrivial additive character ψ of F, we have a genuine character χ_ψ determined by the Weil index. Having fixed ψ, all other genuine characters are of the form $\chi_\psi \chi$ for χ a character of $F^\times$. Now suppose K/F is a Galois extension and consider the covering torus $\overline{T}_K$ defined analogously by the quadratic Hilbert symbol of K. The character ψ of F gives rise to an additive character $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$ of K which is $\mathrm{Gal}(K/F)$-invariant and hence a genuine character χ_{ψ_K}. One would imagine that "base change" should be the map

$$\chi_\psi \cdot \chi \mapsto \chi_{\psi_K} \cdot (\chi \circ \mathrm{N}_{K/F}).$$

In particular, it carries χ_ψ to χ_{ψ_K}. Let us call this the ψ-base change map for the moment.

Now let's observe that the ψ-base change map is independent of the choice of ψ. Any other additive character is of the form ψ_a for $a \in F^\times$ and we have

$$\chi_\psi \cdot \chi = \chi_{\psi_a} \cdot (a, -)_F \cdot \chi \quad \text{as characters of } \overline{T}_F.$$

Then the ψ_a-base change of the RHS is, by definition,

$$\chi_{\psi_{a,K}} \cdot ((a, -)_F \circ \mathrm{N}_{K/F}) \cdot (\chi \circ \mathrm{N}_{K/F}).$$

Since

$$\chi_{\psi_{a,K}} = \chi_{\psi_K} \cdot (a, -)_K \quad \text{and} \quad (a, -)_F \circ \mathrm{N}_{K/F} = (a, -)_K,$$

we see that the ψ_a-base change of RHS is the same as the ψ-base change of the LHS. Thus we see that there should be a canonical base change map for $\overline{T}$.

14.3. **Base change map.** The next question is whether it is given by a map of the L-group-extensions. Recall that the L-group of $\overline{T}_F$ is defined as the Baer sum $E_1 + E_2$ of two extensions E_i of $F^\times / F^{\times n}$ by $\overline{T}^\vee$ (pulled back via $W_F \to F^\times \to F^\times / F^{\times n}$). It is easy to see that the construction of E_2 is functorial with respect to base change, in the sense that there is a natural map

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{T}^\vee \cong \mathbb{C}^\times & \longrightarrow & E_{2,K} & \longrightarrow & K^\times & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & \overline{T}^\vee \cong \mathbb{C}^\times & \longrightarrow & E_{2,F} & \longrightarrow & F^\times & \longrightarrow & 1. \end{array}$$

One would like an analogous diagram for E_1, but the construction of E_1 does not immediately lead to such a diagram.

To be more precise, one would like to define a *natural* map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E_{1,K} & \longrightarrow & K^\times & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_f} & & \downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E_{1,F} & \longrightarrow & F^\times & \longrightarrow & 1 \end{array}$$

where the top row is defined by the cocycle $(-,-)_K$ whereas the bottom is defined by $(-,-)_F$. Writing N for $\mathrm{N}_{K/F}$ for simplicity, the map η_f has the form

$$\eta_f(a,z) = (\mathrm{N}(a), z\cdot f(a)) \quad \text{with } z\in\mathbb{C}^\times \text{ and } a\in K^\times,$$

for some $f : K^\times \longrightarrow \mathbb{C}^\times$ satisfying:

$$f(ab)f(a)^{-1}f(b)^{-1} = (a,b)_K \cdot (\mathrm{N}(a),\mathrm{N}(b))_F \quad \text{for all } a,b\in K^\times.$$

Does such an f exist? The following theorem was proved by Bender [B]:

Theorem 14.1. *Define $f : K^\times \longrightarrow \{\pm 1\} \subset \mathbb{C}^\times$ by:*

$$f(a) = \frac{\textit{(Hasse-Witt invariant of the quadratic form } Tr_{K/F}(ax^2))}{\textit{(Hasse-Witt -invariant of the quadratic form } Tr_{K/F}(x^2))}.$$

Then

$$f(ab)f(a)^{-1}f(b)^{-1} = (a,b)_K \cdot (N(a),N(b))_F \quad \textit{for all } a,b\in K^\times.$$

The Hasse-Witt invariant of the trace quadratic forms in the theorem has been studied by Serre [Se]. Thus we have a completely natural function (valued in μ_2) which depends only on the arithmetic of K/F and which induces a natural homomorphism ${}^L\overline{T}_K \longrightarrow {}^L\overline{T}_F$. This induces a map from the set of L-parameters (i.e. splitting of L-group) of $\overline{T}_F$ to $\overline{T}_K$.

Explicitly, this map of L-parameters is given as follows. Since the L-group is the Baer sum of two extensions E_1 and E_2, an L-parameter is a section of E_2 whose associated 2-cocycle is equal to that of E_1. Suppose ϕ_F is an L-parameter for $\overline{T}_F$, thought of as a section of E_1 (i.e. as a function $\phi_F : F^\times \longrightarrow \mathbb{C}^\times$) such that

$$\phi_F(ab) = \phi_F(a)\phi_F(b)\cdot(a,b)_F, \quad \text{with } a,b\in F^\times \cong W_F^{ab}.$$

Then ϕ_F is mapped to the L-parameter ϕ_K of $\overline{T}_K$, thought of as a section of E_1 (i.e. a function $K^\times \longrightarrow \mathbb{C}^\times$), given by:

$$\phi_K(a) = \phi_F(\mathrm{N}(a))\cdot f(a).$$

Via the LLC for covering tori, we can work out the base change of genuine characters of $\overline{T}_F$ to $\overline{T}_K$. The map

$$\mathcal{N}_{\overline{T}} : \overline{T}_K \longrightarrow \overline{T}_F$$

given by

$$\mathcal{N}_{\overline{T}}(t,\epsilon) = (\mathrm{N}_{K/F}(t), \epsilon\cdot f(t))$$

is easily checked to be a group homomorphism; we call it the norm map for the covering torus $\overline{T}$ as it lifts the usual norm map of the linear torus T so that one has a commutative diagram:

$$\begin{array}{ccc} \overline{T}_K & \xrightarrow{\mathcal{N}_{\overline{T}}} & \overline{T}_F \\ \downarrow & & \downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathcal{N}_T} & \mathbb{T}(F). \end{array}$$

Then the base change of genuine characters corresponding to the base change of L-parameters is simply the pullback by the norm map:

$$\mathrm{BC}(\chi)(t,\epsilon) = \chi(\mathcal{N}_{\overline{T}}(t,\epsilon))$$

for any genuine character χ of $\overline{T}_F$.

14.4. **Consistency.** Of course, the question is whether the functoriality implied by this homomorphim of L-groups for $\overline{T}$ agrees with the base change constructed above. In particular, if ϕ_F corresponds to the genuine character χ_ψ, we would like to see that ϕ_K corresponds to the character $\chi_{\psi_K} = \chi_{\psi\circ\mathrm{Tr}}$. This means that we need to check:

Proposition 14.2.

$$\chi_{\psi_K}(a) = \chi_\psi(N(a)) \cdot f(a) \quad \textit{for } a \in K^\times.$$

Proof. We shall verify this proposition by a computation. Since both sides of the identity are roots of unity, it suffices to verify the identity in $\mathbb{C}^\times/\mathbb{R}^\times_{>0}$. Recall that

$$\chi_{\psi_K}(a) = \frac{\gamma_{\psi_K}}{\gamma_{\psi_{K,a}}}$$

where the factors on the RHS are the Weil indices defined by the equation (of distributions):

$$(*) \qquad \int_K \psi_K(ax^2)\cdot\psi_K(xy)\,dx = \gamma(\psi_{K,a})\cdot|a|_K^{-1/2}\cdot\psi_K(a^{-1}y^2).$$

Now we may compute the LHS as follows. Replacing x by x/a, we get

$$\mathrm{LHS} = |a|_K^{-1}\cdot\int_K \psi(\mathrm{Tr}(a^{-1}x^2))\cdot\psi(\mathrm{Tr}(a^{-1}xy))\,dx.$$

Let $q_{a^{-1}}$ denote the quadratic form $x \mapsto \mathrm{Tr}(a^{-1}x^2)$: it is a quadratic form on the F-vector space K. We may find an F-basis $\{\alpha_i\}$ of K such that

$$q_{a^{-1}}(x) = \sum_i a_i x_i^2 \quad (\text{with } x = \sum_i x_i\alpha_i).$$

Then the integral over K factories into $[K:F]$ integrals over F to give:

$$\begin{aligned}\mathrm{LHS} &= \prod_i \int_F \psi(a_i x_i^2)\cdot\psi(a_i x_i y_i)\,dx_i \mod \mathbb{R}^\times_{>0} \\ &= \prod_i \gamma(\psi_{a_i})\cdot\psi(a_i y^2) \mod \mathbb{R}^\times_{>0}.\end{aligned}$$

Comparing this with the RHS of (*), we deduce that

$$\gamma_{\psi_{K,a}} = \prod_i \gamma_{\psi_{a_i}} = \gamma_\psi^{-[K:F]}\cdot\prod_i \chi_\psi(a_i) = \gamma_\psi^{-[K:F]}\cdot\chi_\psi(\mathrm{disc}(q_{a^{-1}}))\cdot\mathrm{HW}(q_{a^{-1}}).$$

Thus,

$$\chi_{\psi_K}(a) = \frac{\chi_\psi(\mathrm{disc}q_1)}{\chi_\psi(\mathrm{disc}q_a)} \cdot \frac{\mathrm{HW}(q_1)}{\mathrm{HW}(q_a)}.$$

(there is no harm replacing in $q_{a^{-1}}$ by q_a here). On the other hand, we have (see [Se, Pg. 668])

$$\mathrm{disc}(q_a) = \mathrm{N}(a) \cdot \mathrm{disc}(K/F) \in F^\times/F^{\times 2},$$

so that

$$\mathrm{disc}(q_a) = \mathrm{disc}(q_1) \cdot \mathrm{N}(a).$$

From this, we deduce that

$$\chi_{\psi_K}(a) = \chi_\psi(\mathrm{N}(a)) \cdot \frac{\mathrm{HW}(q_1)}{\mathrm{HW}(q_a)},$$

as desired. Here, note that we have used $(\mathrm{N}(a), \mathrm{disc}(q_1))_F = (\mathrm{N}(a), \mathrm{disc}(K/F))_F = 1$. The proposition is proved. □

14.5. **General case.** We have focused exclusively on a very simple covering torus $\overline{T}$ above, but the discussion in fact applies generally. The point is that, for any BD cover $\overline{G}$, the miraculous function f allows us to define a natural map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_1(\overline{G}_K) & \longrightarrow & K^\times & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_f} & & \downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_1(\overline{G}_F) & \longrightarrow & F^\times & \longrightarrow & 1 \end{array}$$

by the same formula and using the natural map $j : \mu_2 \longrightarrow Z(\overline{G}^\vee)$ used in the definition of E_1 in §5.2 (recall that E_1 is a pushout of an extension E_0 of $F^\times$ by μ_2 under the map j). Since the construction of E_2 is functorial with respect to base change, one has an analogous diagram for E_2 and taking the Baer sum (together with pushout and pullback), we obtain a natural diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_K & \longrightarrow & W_K & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_f} & & \downarrow & & \\ 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_F & \xrightarrow{\pi} & W_F & \longrightarrow & 1. \end{array}$$

It is this diagram that allows one to define the notion of base change for $\overline{G}$ on the side of the L-groups.

14.6. **Example of** Mp_{2n}**.** Since the LLC is known for Mp_{2n}, one does have base change of L-packets from $\mathrm{Mp}_{2n}(F)$ to $\mathrm{Mp}_{2n}(K)$. Recall that the LLC for Mp_{2n} is defined via theta correspondence with SO_{2n+1}. Thus, Proposition 14.2 allows us to describe this base change more concretely, via the theta correspondence and base change for odd special orthogonal groups.

More precisely, given an L-packet Π_{ϕ_F} of $\mathrm{Mp}_{2n}(F)$, the theta correspondence gives the corresponding L-packet $\Pi_{\phi'_F,\psi}$ on $\mathrm{SO}_{2n+1}(F)$ (which depends on an additive character ψ) and its base change $\Pi_{\phi'_K,\psi}$ on $\mathrm{SO}_{2n+1}(K)$; here ϕ'_K is simply the restriction of ϕ'_F from WD_F to WD_K. Theta correspondence then gives a packet $\Theta_\psi(\Pi_{\phi'_K,\psi})$ on $\mathrm{Mp}_{2n}(K)$. Similar to our treatment of base change for the covering torus $\overline{T}$, it is easy to see that this packet on $\mathrm{Mp}_{2n}(K)$ is independent of K and its L-parameter is that obtained from ϕ_F by the base change commutative diagram constructed above (using the map f provided by Bender).

14.7. **Example of covering tori.** Another case we should check is the case of general covering (split) tori $\overline{T}$. We may assume that $\overline{T}$ is defined by a bisector D, so that it is described by generators and relations as in §3.3. Then the covering torus $\overline{T}_{Q,n}$ is presented analogously as we explained in §3.4. For ease of reference, let us recall this presentation here. The covering torus $\overline{T}_{Q,n,F}$ is generated by $\epsilon \in \mu_n$ and $y(a)$ with $y \in Y_{Q,n}$ and $a \in F^\times$, subject to:

- $\overline{T}_{Q,n}$ is abelian;
- $y_1(a) \cdot y_2(a) = (y_1 + y_2)(a) \cdot (-1, a)_{F,n}^{D(y_1,y_2)}$ for $y_1, y_2 \in Y_{Q,n}$ and $a \in F^\times$, and where $(-,-)_{n,F}$ denotes the n-th Hilbert symbol for F;
- $y(a) \cdot y(b) = y(ab) \cdot (a,b)_{F,n}^{Q(y)}$ for $y \in Y_{Q,n}$ and $a, b \in F^\times$.

One has the analogous presentation for $\overline{T}_{Q,n,K}$. We also recall that for $y \in Y_{Q,n}$, $\frac{2}{n} \cdot Q(y) \in \mathbb{Z}$.

Now the base change map of L-parameters actually implies a base change of genuine characters of $\overline{T}_{Q,n,F}$ to those of $\overline{T}_{Q,n,K}$ and this is given by the following theorem:

Theorem 14.3. *(i) The map $\mathcal{N}_{\overline{T}} : \overline{T}_{Q,n,K} \longrightarrow \overline{T}_{Q,n,F}$ given by*

$$\mathcal{N}_{\overline{T}}(y(a)) = y(N_{K/F}(a)) \cdot f(a)^{\frac{2}{n} \cdot Q(y)} \quad \text{for } y \in Y_{Q,n} \text{ and } a \in K^\times,$$

and

$$\mathcal{N}_{\overline{T}}(\epsilon) = \epsilon \quad \text{for } \epsilon \in \mu_n,$$

is a group homomorphism, and there is a commutative diagram

$$\begin{array}{ccc} \overline{T}_{Q,n,K} & \xrightarrow{\mathcal{N}_{\overline{T}}} & \overline{T}_{Q,n,F} \\ \big\downarrow & & \big\downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathcal{N}_T} & \mathbb{T}(F). \end{array}$$

Moreover, the map $\mathcal{N}_{\overline{T}}$ descends to give a homomorphism

$$\mathcal{N}_{\overline{T}} : Z(\overline{T}_K) \longrightarrow Z(\overline{T}_F).$$

(ii) The base change map

$$\mathrm{BC} : \mathrm{Irr}\, \overline{T}_{Q,n,F} \longrightarrow \mathrm{Irr}\, \overline{T}_{Q,n,K}$$

given by the base change morphism of L-groups is given by the pullback of genuine characters defined by $\mathcal{N}_{\overline{T}}$. Moreover, BC *restricts to give a map*

$$\mathrm{BC} : \mathrm{Irr}\, \overline{T}_F \longrightarrow \mathrm{Irr}\, \overline{T}_K$$

Proof. (i) To show that $\mathcal{N}_{\overline{T}}$ is a group homomorphism, we need to verify that it respects the defining relations of the covering tori. The first relation to check is that

$$\mathcal{N}_{\overline{T}}(y_1(a)) \cdot \mathcal{N}_{\overline{T}}(y_2(a)) = \mathcal{N}_{\overline{T}}\big((y_1 + y_2)(a) \cdot (-1, a)_{K,n}^{D(y_1,y_2)}\big).$$

One the left, one has by definition:

$$y_1(\mathrm{N}_{K/F}(a)) \cdot y_2(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}\cdot(Q(y_1)+Q(y_2))},$$

whereas on the right, one has

$$(y_1+y_2)(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}\cdot Q(y_1+y_2)} \cdot (-1,a)_{K,n}^{D(y_1,y_2)}.$$

Since

$$(y_1+y_2)(\mathrm{N}_{K/F}(a)) = y_1(\mathrm{N}_{K/F}(a)) \cdot y_2(\mathrm{N}_{K/F}(a)) \cdot (-1,\mathrm{N}_{K/F}(a))_{F,n}^{D(y_1,y_2)},$$

the first relation then follows from

$$(-1,a)_{K,n} = (-1,\mathrm{N}_{K/F}(a))_{F,n}$$

and

$$f(a)^{\frac{2}{n}\cdot(Q(y_1+y_2)-Q(y_1)-Q(y_2))} = f(a)^{\frac{2}{n}\cdot B_Q(y_1,y_2)} = 1$$

since $B_Q(y_1,y_2) \equiv 0 \mod n$ for $y_1 \in Y_{Q,n}$.

The second relation to verify is the identity

$$\mathcal{N}_{\overline{T}}(y(a)) \cdot \mathcal{N}_{\overline{T}}(y(a)) = \mathcal{N}_{\overline{T}}(y(ab)) \cdot (a,b)_{K,n}^{Q(y)}.$$

On the left, one has

$$\begin{aligned} &y(\mathrm{N}_{K/F}(a)) \cdot y(\mathrm{N}_{K/F}(b)) \cdot f(a)^{\frac{2}{n}\cdot Q(y)} \cdot f(b)^{\frac{2}{n}\cdot Q(y)} \\ =&y(\mathrm{N}_{K/F}(ab)) \cdot (\mathrm{N}_{K/F}(a),\mathrm{N}_{K/F}(b))_{F,n} \cdot f(a)^{\frac{2}{n}\cdot Q(y)} \cdot f(b)^{\frac{2}{n}\cdot Q(y)}, \end{aligned}$$

whereas on the right, one has

$$y(\mathrm{N}_{K/F}(ab)) \cdot (a,b)_{K,n}^{Q(y)} \cdot f(ab)^{\frac{2}{n}\cdot Q(y)}.$$

The desired identity follows by noting that

$$(a,b)_{K,n}^{Q(y)} = (a,b)_{K,2}^{\frac{2}{n}\cdot Q(y)} \quad \text{and} \quad (\mathrm{N}_{K/F}(a),\mathrm{N}_{K/F}(b))_{F,n}^{Q(y)} = (\mathrm{N}_{K/F}(a),\mathrm{N}_{K/F}(b))_{F,2}^{\frac{2}{n}\cdot Q(y)}$$

and applying Theorem 14.1.

To prove the last assertion of (i), it suffices to show that $\mathcal{N}_{\overline{T}}$ sends $\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K)$ to $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$. By Lemma 3.1, one has

$$\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K) = \tilde{g}(T_K[n]))$$

where $\tilde{g}(y(a)) = (ny)(a) \in \overline{T}_{Q,n,K}$. From the proof of Lemma 3.1, we see that this kernel is generated by certain elements of the form $(ny)(\zeta)$, with $\zeta \in \mu_n(K) = \mu_n(F)$. Then

$$\mathcal{N}_{\overline{T}}((ny)(\zeta)) = (ny)(\zeta) \cdot f(\zeta)^{\frac{2}{n}\cdot Q(ny)} = (ny)(\zeta),$$

which lies in $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$ by Lemma 3.1 again.

(ii) If $\chi \in \mathrm{Irr}(\overline{T}_{Q,n,F})$ has L-parameter ϕ, then by Lemma 6.1, ϕ is a map

$$\phi : F^\times \times Y_{Q,n} \longrightarrow \mathbb{C}^\times$$

satisfying the conditions (b) and (c) in Lemma 6.1. As we remarked after Lemma 6.1, the presentation of $\overline{T}_{Q,n,F}$ shows that this map ϕ defines a genuine character of $\overline{T}_{Q,n,F}$, and this was how the LLC for $\overline{T}$ was shown.

By the base change homomorphism of L-group, the L-parameter ϕ gives rise to an L-parameter ϕ_K for $\overline{T}_{Q,n,K}$. Regarding ϕ_K as a map from $K^\times \times Y_{Q,n} \longrightarrow \mathbb{C}^\times$, the map ϕ_K is given by

$$\phi_K(a,y) = \phi(\mathrm{N}_{K/F}(a), y) \cdot f(a)^{\frac{2}{n}\cdot Q(y)}.$$

From this and the construction of the LLC, we deduce that the genuine character defined by ϕ_K is the pullback of χ by $\mathcal{N}_{\overline{T}}$. If χ is trivial on $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$, then the last part of (i) implies that $\chi \circ \mathcal{N}_{\overline{T}}$ is trivial on $\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K)$; this proves the last assertion of (ii). □

14.8. **Principal series.** Having described the base change for covering (split) tori, we can obtain the base change of principal series (on the level of L-packets) for a BD covering group $\overline{G}$ (attached to the invariant (D,η)). Let $T \subset G$ be a maximal split torus. In this case, the Weyl group W acts naturally on $\overline{T}_{Q,n,F}$ and $\overline{T}_{Q,n,K}$. We note the following lemma:

Lemma 14.4. *The norm map* $\mathcal{N}_{\overline{T}} : \overline{T}_{Q,n,K} \longrightarrow \overline{T}_{Q,n,F}$ *is* W*-equivariant.*

Proof. The W-action on $\overline{T}_{Q,n,F}$ is given by (6.7): for a root α and with $w_\alpha \in W$ the image of $q(n_\alpha(1))$, one has:

$$\mathrm{Int}(w_\alpha)(y(a)) = y(a) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^\vee(a)) \in \overline{T}_{Q,n,F},$$

where for $z \in Y_{Q,n}^{sc}$ (such as $z = -\langle \alpha, y\rangle \cdot \alpha^\vee$),

$$s_{\eta,F}(z(a)) = z(a) \cdot (\eta(z), a)_{F,n}.$$

One has the analogous formula for the action of w_α on $\overline{T}_{Q,n,K}$.

Now for $y \in Y_{Q,n}$ and $a \in K^\times$, we have

$$\begin{aligned}
&\mathcal{N}_{\overline{T}}(\mathrm{Int}(w_\alpha)(y(a))) \\
=&\mathcal{N}_{\overline{T}}(y(a)) \cdot \mathcal{N}_{\overline{T}}(s_{\eta,K}(-\langle \alpha, y\rangle \cdot \alpha^\vee(a))) \\
=&y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))) \cdot f(a)^{\frac{2}{n}Q(y)} \cdot f(a)^{\frac{2}{n}\cdot \langle \alpha, y\rangle^2 \cdot Q(\alpha^\vee)} \\
=&y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))) \cdot f(a)^{\frac{2}{n}Q(y)}.
\end{aligned}$$

Here the second equality holds because

$$(\eta(z), a)_{K,n} = (\eta(z), \mathrm{N}_{K/F}(a))_{F,n} \quad \text{for } z \in Y_{Q,n}^{sc} \text{ and } a \in K^\times,$$

as $\eta(z) \in F^\times$. Further, the last equality holds since

$$\frac{2}{n} \cdot \langle \alpha, y\rangle^2 \cdot Q(\alpha^\vee) = 0 \mod 2$$

by the proof of Theorem 6.8 (which shows that either $\langle \alpha, y\rangle = 0$ or $B(y, \alpha^\vee) = \langle \alpha, y\rangle \cdot Q(\alpha^\vee)$).

On the other hand,

$$\begin{aligned}
&\mathrm{Int}(w_\alpha)\left(\mathcal{N}_{\overline{T}}(y(a))\right) \\
=&\mathrm{Int}(w_\alpha)\left(y(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}Q(y)}\right) \\
=&y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}\left(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))\right) \cdot f(a)^{\frac{2}{n}Q(y)}.
\end{aligned}$$

This proves the desired Weyl-equivariance. □

Corollary 14.5. *(i) The base change of a W-invariant distinguished character of $\overline{T}_{Q,n,F}$ with respect to K/F is a W-invariant character of $\overline{T}_{Q,n,K}$.*

(ii) The base change of a distinguished character of $\overline{T}_{Q,n,F}$ with respect to K/F is a distinguished character of $\overline{T}_{Q,n,K}$.

(iii) The base change of a tempered principal series representation $I_F(\chi)$ of $\overline{G}_F$ with respect to K/F is the tempered principal series representation $I_K(BC(\chi))$ of $\overline{G}_K$.

15. **Endoscopy**

Another instance of Langlands functoriality is the theory of endoscopy. The L-group formalism should lead naturally to the notions of "endoscopic groups" or "endoscopic datum" $(H, \mathcal{H}, s, \xi)$ for the BD covering groups. Since the LLC can be reduced to the case of trivial η, we shall assume $\eta = 1$ for the rest of this section.

15.1. **Endoscopic groups.** Given a BD covering group $\overline{G}$, we have associated with it a dual group $\overline{G}^\vee$ which is defined using a based root datum

$$(Y_{Q,n}, \Delta^\vee_{Q,n}, X_{Q,n}, \Delta_{Q,n}),$$

and an L-group extension ${}^L\overline{G}$ which we have shown to be a split extension. Since $\eta = 1$ by hypothesis, the L-group extension has a distinguished splitting. Any such distinguished splitting gives an isomorphism ${}^L\overline{G} \cong \overline{G}^\vee \times W_F$. Let $\mathbb{G}_{Q,n}$ be the split linear algebraic group over F whose dual group is isomorphic to $\overline{G}^\vee$, which comes equipped with a maximally split torus $\mathbb{T}_{Q,n}$. Recall that one has a natural isogeny $i : T_{Q,n} \longrightarrow T \subset G$.

It is natural to regard $\mathbb{G}_{Q,n}$ as the principal endoscopy group of $\overline{G}$ and the elliptic endoscopic groups $\mathbb{H}_{Q,n}$ of $\mathbb{G}_{Q,n}$ as the elliptic endoscopic groups of $\overline{G}$ relative to a distinguished splitting. Indeed, it follows by definition that there is a natural map of dual groups

$$H^\vee_{Q,n} \longrightarrow \overline{G}^\vee.$$

The choice of a distinguished splitting of ${}^L\overline{G}$ then gives rise to a map of L-groups

$${}^L H_{Q,n} \longrightarrow {}^L\overline{G}.$$

15.2. **Speculations.** We do not have anything substantive to say beyond this, but content ourselves with a few highly speculative remarks.

- For any such endoscopic group $\mathbb{H}_{Q,n}$, the L-group formalism leads to a matching of stable conjugacy classes of regular semisimple elements for $H_{Q,n}$ and G. This matching is obtained by a twist of the isogeny $i : T_{Q,n} \longrightarrow T$ as explained by Langlands-Shelstad [LS]. In particular, the stable conjugacy classes of G which occur in the orbit-matching are all "good", in the sense that they can support genuine invariant distributions.

- The notion of isomorphisms of endoscopic data in the covering case should be slightly different from that in the linear case. This is already evident from the example of Mp_{2n}, where a theory of endoscopy was developed by W.-W. Li [L1]. Here the endoscopic groups of Mp_{2n} (relative to the choice of a distinguished splitting) are $\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}$ where (a, b) are ordered pairs of non-negative integers such that $a+b = n$. These are also the endoscopic groups of SO_{2n+1} except that the pairs (a, b) are unordered.

  We suspect that the notion of equivalence of endoscopic data $(H, \mathcal{H}, s, \xi)$ should be modified as follows. In the linear algebraic case, the semisimple element s in the dual group is taken modulo the center $Z(G^\vee)$ of the dual group $G^\vee$. In the covering case, we suspect that it should be taken modulo a smaller group. Namely, the isogeny $i : T_{Q,n} \longrightarrow T$ gives on the dual side a map
$$i^* : T^\vee \longrightarrow T_{Q,n}^\vee.$$
  The center $Z(G^\vee)$ of $G^\vee$ is a subgroup of $T^\vee$ and its image $i^*(Z(G^\vee))$ in $T_{Q,n}^\vee$ lies in the center $Z(\overline{G}^\vee) = Z(G_{Q,n}^\vee)$ of $\overline{G}^\vee$. We speculate that in the definition of endoscopic datum $(H, \mathcal{H}, s, \xi)$ of a BD covering group, the semisimple element s should be taken modulo the group $i^*(Z(G^\vee)) \subset Z(G^\vee)$. This is a wild speculation at the moment, but it is related to the anomaly discussed in §12 about central characters being non-constant in an L-packet in the covering case.
- After the orbit-matching, one needs to define the transfer factors for endoscopic transfers. This is a function $\Delta_{H_{Q,n},\overline{G}}$ which is supported only on matching pairs of elements in $H_{Q,n} \times \overline{G}$ and which satisfies certain properties.
- These transfer factors should allow one to transfer orbital integrals on $\overline{G}$ to stable orbital integrals on $H_{Q,n}$. One would imagine that such a transfer can be proved as a consequence of the companion fundamental lemma. One would further imagine that the fundamental lemma itself could be proved by using Harish-Chandra descent to reduce it to a fundamental lemma on the Lie algebra. We believe that on the level of Lie algebra, this fundamental lemma should be a consequence of the so-called nonstandard fundamental lemma formulated by Waldspurger [Wa] and established by Ngo [N]. In other words, we believe that once the definitions of the transfer factors are set up correctly, the ultimate ingredients for proving it should already be available by the work of Waldspurger [Wa] and Ngo [N].

15.3. **Principal series.** The isogeny $i : T_{Q,n} \longrightarrow T$ allows one to transfer principal series representations of $G_{Q,n}$ to those on $\overline{G}$, subject to picking a distinguished splitting of ${}^L\overline{G}$, or equivalently a Weyl-invariant genuine character χ_0 of $\overline{T}_{Q,n}$ which factors to $Z(\overline{T})$. Given any character χ of $T_{Q,n}$ which factors through i, one then obtains a genuine character $\chi_0 \cdot \chi$ of $Z(\overline{T})$, from which one obtains an irreducible genuine character $i(\chi_0\chi)$ of $\overline{T}$ and a principal series representation $I_{\overline{G}}(\chi_0\chi)$ of $\overline{G}$ by parabolic induction. Since the association $\chi \mapsto \chi_0\chi$ is W-equivariant (because χ_0 is Weyl-invariant), this gives a well-defined lifting
$$I_{G_{Q,n}}(\chi) \mapsto I_{\overline{G}}(\chi_0\chi)$$

which depends on the choice of χ_0 and for those principal series representations $I_{G_{Q,n}}(\chi)$ for χ trivial on $\mathrm{Ker}(i)$.

15.4. **Iwahori-Hecke algebra isomorphisms.** When F is p-adic and $\mathbb{G}$ is simply-connected, Savin has studied the Iwahori-Hecke algebra of covers of G and established Iwahori-Hecke algebra isomorphisms with those of linear reductive groups. One can show the same results in the generality of this paper.

For this subsection, we resume the notations of §4. In particular, we have $\gcd(p, n) = 1$ and $K = \mathbb{G}(\mathcal{O})$ for a smooth reductive group scheme $\mathbb{G}$ over $\mathcal{O}$. Consider the natural reduction map $\mathbb{G}(\mathcal{O}) \to \mathbb{G}_\kappa(\kappa)$, and let I be the standard Iwahori subgroup, defined to the inverse image of $\mathbb{B}_\kappa(\kappa) \subset \mathbb{G}_\kappa(\kappa)$ with respect to the reduction map. We fix a splitting of K into $\overline{G}$ which gives a splitting of I. One may consider the Iwahori Hecke algebra $\mathcal{H}_\epsilon(\overline{G}, I)$; it is the algebra of anti-genuine I-biinvariant locally constant and compactly supported functions on $\overline{G}$.

In this subsection, we only consider those $\overline{G}$ for which $Z(\overline{T})$ possesses distinguished unramified genuine characters. In this case, ${}^L\overline{G}$ is isomorphic to $\overline{G}^\vee \times W_F$ (relative to the choice of a distinguished character) and thus we have the split "principal endoscopic group" $G_{Q,n}$ of $\overline{G}$ and its Iwahori-Hecke algebra $\mathcal{H}(G_{Q,n}, I_{Q,n})$.

Theorem 15.1 (Savin)**.** *Consider a BD covering group $\overline{G}$, for which there exist distinguished unramified genuine characters of $Z(\overline{T})$ (for example when $\eta = 1$). Then, its Iwahori-Hecke algebra $\mathcal{H}_\epsilon(\overline{G}, I)$ has the following description:*

$$\mathcal{H}_\epsilon(\overline{G}, I) = \left\langle T_y, E_{w_\alpha} : \ y \in Y_{Q,n}, \alpha^\vee \in \Delta^\vee \right\rangle$$

with relations given by

- $(E_{w_\alpha} - q)(E_{w_\alpha} + 1) = 0.$
- $(E_{w_\alpha} E_{w_\beta})^r = (E_{w_\beta} E_{w_\alpha})^r$ *if $w_\alpha w_\beta$ is of order $2r$.*
- $(E_{w_\alpha} E_{w_\beta})^r E_{w_\alpha} = (E_{w_\beta} E_{w_\alpha})^r E_{w_\beta}$ *if $w_\alpha w_\beta$ is of order $2r+1$.*
- $T_y \cdot T_{y'} = T_{y+y'}.$
- *Write $\langle y, \alpha \rangle = m n_\alpha$. Then*

$$E_{w_\alpha} \cdot T_y = \begin{cases} T_{y^{w_\alpha}} \cdot E_{w_\alpha} + (q-1) \sum_{k=0}^{m-1} T_{y - k n_\alpha \alpha^\vee} & \text{if } m > 0, \\ T_y \cdot E_{w_\alpha} & \text{if } m = 0, \\ T_{y^{w_\alpha}} \cdot E_{w_\alpha} - (q-1) \sum_{k=0}^{1-m} T_{y + k n_\alpha \alpha^\vee} & \text{if } m < 0. \end{cases}$$

The Iwahori-Hecke algebra $\mathcal{H}(G_{Q,n}, I_{Q,n})$ has the same description by generators and relations. Consequently, one has an isomorphism $\mathcal{H}_\epsilon(\overline{G}, I) \cong \mathcal{H}(G_{Q,n}, I_{Q,n})$ depending on the choice of the distinguished genuine character of $Z(\overline{T})$.

Proof. The proof can be taken almost verbatim from [Sa], by noting the following. First, the argument in [Sa] relies crucially on the existence of a Weyl-invariant unramified character of $Z(\overline{T})$ (c.f. [Sa, Lemma 4.5]). Our assumption on $\overline{G}$ yields such existence as discussed in §6.5. Second, to obtain the above explicit relations between the generators of $\mathcal{H}(\overline{G}, I)$, one makes use of the property (d') for any distinguished character χ, namely that

$$\chi \circ s_\eta\big(\alpha_{Q,n}^\vee(a)\big) = 1, \quad \text{for } \alpha \in \Delta \text{ and } a \in F^\times.$$

This is the result generalizing [Sa, §4], used in the proof of [Sa, Proposition 7.2] there. Besides these, Savin's argument could be carried out in our setting word for word. □

16. **Examples**

In this section, we give a number of examples to illustrate some of the topics treated in this paper. These examples are the ones which have been studied in the literature. As these groups arise as the cover $\overline{\mathbb{G}}$ of $\mathbb{G}$ which has simply-connected derived group, we may assume that $\overline{\mathbb{G}}$ is incarnated by a fair $(D, 1)$ without loss of generalities. For fixed $n \in \mathbb{N}$, we have the associated degree n cover $\overline{G}$.

We have seen that there always exist distinguished splittings of ${}^L\overline{G}$ for such $\overline{G}$, with respect to which ${}^L\overline{G} \simeq \overline{G}^\vee \times W_F$. In this section, we use χ_ψ to denote a distinguished character constructed in section 7. It will be shown explicitly that our construction in the simply-connected simply-laced case agrees with the one given by Savin [Sa]. It is also compatible with the one for the classical double cover $\overline{\mathrm{Sp}}_{2r}$, as in [K, Ra].

The computation of the bilinear form B_Q below uses crucially the identity

$$B_Q(\alpha^\vee, y) = Q(\alpha^\vee) \cdot \langle \alpha, y \rangle,$$

where $\alpha \in \Phi^\vee$ is any coroot and $y \in Y$.

16.1. **Simply-connected case.**

Consider a simply-connected simple group $\mathbb{G}$ of arbitrary type. There is up to unique isomorphism a $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ associated to a Weyl-invariant quadratic form on $Y^{sc} = Y$. Consider $\overline{\mathbb{G}}$ incarnated by (D, η). As indicated above, there is no loss of generality in assuming D fair and $\eta = 1$, and we will do so in the following.

For simplicity we assume $n = 2$ except for the case of the exceptional G_2 where the computation is very simple for general n. We also assume that Q is the unique Weyl-invariant quadratic form which takes value 1 on the short coroots of $\mathbb{G}$. The general case of n and Q follows from similar computations.

Note that whenever we have assumed $n = 2$, we will write $Y_{Q,2}$ and $Y_{Q,2}^{sc}$ for the lattices $Y_{Q,n}$ and $Y_{Q,n}^{sc}$ which are of interest. We also have $J = 2Y + Y_{Q,2}^{sc} = Y_{Q,2}^{sc}$ since $Y = Y^{sc}$.

16.1.1. **The simply-laced case A_r, D_r, E_6, E_7, E_8 and compatibility.**

Now let $\mathbb{G}$ be a simply-laced simply-connected group of type A_r for $r \geq 1$, D_r for $r \geq 3$, and E_6, E_7, E_8. Let $\Delta = \{\alpha_1, ..., \alpha_r\}$ be a fixed set of simple roots of $\mathbb{G}$. Let $\overline{\mathbb{G}}$ be the extension of $\mathbb{G}$ determined by the quadratic form Q with $Q(\alpha_i^\vee) = 1$ for all coroots $\alpha_i^\vee$. We obtain the two-fold cover $\overline{G}$ of G.

Clearly we have $n_\alpha = 2$ for all $\alpha \in \Phi$ in this case. Let $\alpha_i^\vee \in \Delta^\vee$ for $i = 1, ..., r$ be the simple coroots of $\mathbb{G}$. It is easy to compute the bilinear form B_Q associated with Q:

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} -1 & \text{if } \alpha_i \text{ and } \alpha_j \text{ connected in the Dynkin diagram,} \\ 0 & \text{otherwise.} \end{cases} \tag{16.1}$$

In order to show compatibility with Savin, we may further assume that $\overline{G}$ is incarnated by the following fair bisector D associated with B_Q as given in [Sa],

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 0 & \text{if } i < j, \\ Q(\alpha_i^\vee) & \text{if } i = j, \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases} \tag{16.2}$$

The following lemma is in [Sa] and reproduced here for convenience. The stated result can also be checked by straightforward computation.

Lemma 16.3. *Let Ω be a subset of the vertices in the Dynkin diagram of $\mathbb{G}$ satisfying:*

(i) no two vertices in Ω are adjacent;
(ii) every vertex not in Ω is adjacent to an even number of vertices in Ω.

Then the map given by $\Omega \longmapsto e_\Omega$ with $e_\Omega := \sum_{\alpha_i \in \Omega} \alpha_i^\vee$ gives a well-defined correspondence between such sets Ω and the cosets of $Y_{Q,2}/J$. In particular, the empty set corresponds to the trivial coset J.

By properties of B_Q and (i) of Ω above, it follows that

$$Q(e_\Omega) = |\Omega|.$$

We now give a brief case by case discussion.

The A_r case. There are two situations according to the parity of r.

Case 1: r is even. As an illustration, we first do the straightforward computation. Let $\sum_i k_i \alpha_i^\vee \in Y_{Q,2}$ for proper $k_i \in \mathbb{Z}$. Then $B_Q(\sum_i k_i \alpha_i^\vee, \alpha_j^\vee) \in 2\mathbb{Z}$ for all $1 \le j \le r$ by the definition of $Y_{Q,2}$. In view of (16.1), it is equivalent to

$$\begin{cases} 2k_1 + (-1)k_2 & \in 2\mathbb{Z}, \\ (-1)k_1 + 2k_2 + (-1)k_3 & \in 2\mathbb{Z}, \\ (-1)k_2 + 2k_3 + (-1)k_4 & \in 2\mathbb{Z}, \\ \vdots & \\ (-1)k_{r-2} + 2k_{r-1} + (-1)k_r & \in 2\mathbb{Z}, \\ (-1)k_{r-1} + 2k_r & \in 2\mathbb{Z}. \end{cases} \tag{16.4}$$

It follows that k_2 is even and so are the successive $k_4, ..., k_r$ (we have assumed r to be even). Similarly, k_{r-1} is even and therefore all $k_{r-3}, ..., k_1$ are also even. This gives $Y_{Q,2} = J$.

Note that we could simply apply the lemma to get $Y_{Q,2} = J$, which corresponds to the fact that only the empty set satisfies properties (i) and (ii). There is nothing to check in this case, and the character χ such that $\chi \circ s_\eta$ is trivial will be a distinguished character.

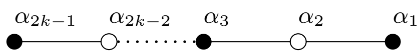
Case 2: $r = 2k - 1$ is odd. Recall the notation

$$\alpha_{i,Q,2}^\vee := n_{\alpha_i} \cdot \alpha_i^\vee.$$

The consideration as in (16.4) works. However, for convenience we will apply the lemma to get $[Y_{Q,2}, J] = 2$ with a basis of $Y_{Q,2}$ given by

$$\{\alpha^\vee_{r,Q,2}, \alpha^\vee_{r-1,Q,2}, ..., \alpha^\vee_{2,Q,2}, e_\Omega = \sum_{m=1}^{k} \alpha^\vee_{2m-1}\}.$$

The nontrivial coset corresponds to the set Ω indicated by alternating bold circles below



A basis for $J = 2Y^{sc}$ is given by

$$\{\alpha^\vee_{r,Q,2}, \alpha^\vee_{r-1,Q,2}, ..., \alpha^\vee_{2,Q,2}, 2e_\Omega\}.$$

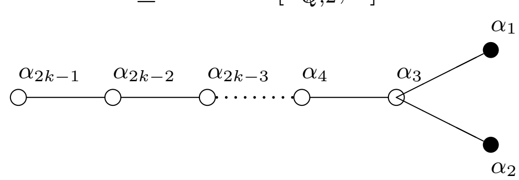
Now the distinguished character χ_ψ we constructed in §7 is determined by

$$\begin{cases} \chi_\psi(\alpha^\vee_{i,Q,2}(a)) = 1, \ 2 \le i \le r, \\ \chi_\psi(e_\Omega(a)) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}. \end{cases} \tag{16.5}$$

This agrees with the formula in [Sa] when we substitute $a = \varpi$ in $\gamma_\psi(a)$ for ψ of conductor $\mathcal{O}_F$. See [Sa, pg 118].

The D_r case. We also have two cases.

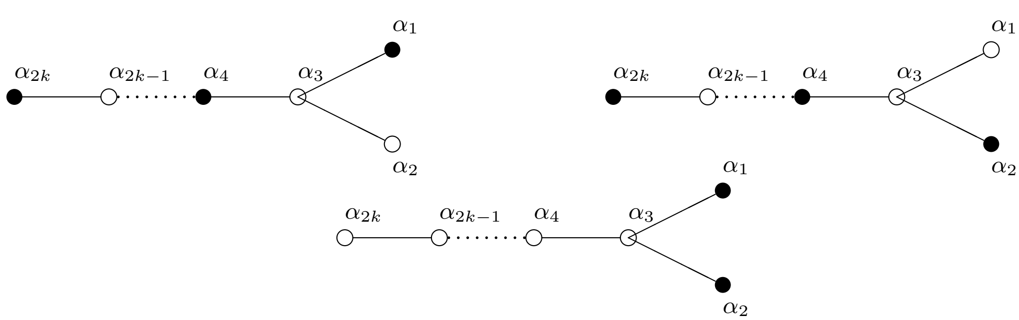
<u>Case 1:</u> $r = 2k-1$ is odd with $k \ge 2$. Then $[Y_{Q,2}, J] = 2$ with the nontrivial $\Omega = \{\alpha_1, \alpha_2\}$:



Consider the basis $\{\alpha^\vee_{i,Q,2} : 2 \le i \le r\} \cup \{e_\Omega\}$ of $Y_{Q,2}$, then the construction of distinguished χ_ψ in previous section is determined by

$$\chi_\psi(e_\Omega(a)) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}.$$

<u>Case 2:</u> $r = 2k$ is even. Then $[Y_{Q,2} : J] = 4$. There are three nontrivial sets Ω_i for $i = 1, 2, 3$ as indicated by the bold circles below.



That is, $\Omega_1 = \{\alpha_1\} \cup \{\alpha_{2m} : 2 \le m \le k\}$, $\Omega_2 = \{\alpha_2\} \cup \{\alpha_{2m} : 2 \le m \le k\}$ and $\Omega_3 = \{\alpha_1, \alpha_2\}$. Note $|\Omega_1| = |\Omega_2| = k$ and $|\Omega_2| = 2$.

A basis of $Y_{Q,2}$ is given by

$$\{\alpha_{i,Q,2}^\vee : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1}, e_{\Omega_2}, e_{\Omega_3}\}.$$

However, the construction of distinguished characters utilizes the elementary divisor theorem. Thus we have to provide bases for $Y_{Q,2}$ and J aligned in a proper way. To achieve this, consider the alternative basis of $Y_{Q,2}$ given by

$$\{\alpha_{i,Q,2}^\vee : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1}+e_{\Omega_2}+e_{\Omega_3}, e_{\Omega_2}+e_{\Omega_3}, e_{\Omega_3}\}.$$

Then it is easy to check that the set

$$\{\alpha_{i,Q,2}^\vee : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1}+e_{\Omega_2}+e_{\Omega_3}, 2(e_{\Omega_2}+e_{\Omega_3}), 2e_{\Omega_3}\}$$

is a basis for J. Note

$$Q(e_{\Omega_2}+e_{\Omega_3}) = |\Omega_2| + Q(2\alpha^\vee) = |\Omega_2| + 4.$$

Thus a distinguished character could be determined by

$$\begin{cases} \chi_\psi(\alpha_{i,Q,2}^\vee(a)) = 1, \ 3 \le i \le 2k-1, \\ \chi_\psi((e_{\Omega_1}+e_{\Omega_2}+e_{\Omega_3})(a)) = 1, \\ \chi_\psi((e_{\Omega_2}+e_{\Omega_3})(a)) = \gamma_\psi(a)^{Q(e_{\Omega_2}+e_{\Omega_3})} = \gamma_\psi(a)^{|\Omega_2|}, \\ \chi_\psi(e_{\Omega_3}(a)) = \gamma_\psi(a)^{|\Omega_3|}. \end{cases}$$

However, since we have assumed that D takes the special form given by (16.2), we have

$$D(e_{\Omega_1}, e_{\Omega_2}+e_{\Omega_3}) = |\Omega_1|$$
$$D(e_{\Omega_2}, e_{\Omega_3}) = Q(\alpha_i^\vee) = 1.$$

Thus

$$\chi_\psi(e_{\Omega_1}(a)) \cdot \chi_\psi((e_{\Omega_2}+e_{\Omega_3})(a)) = (a,a)_2^{|\Omega_1|} \cdot \chi_\psi((e_{\Omega_1}+e_{\Omega_2}+e_{\Omega_3})(a))$$
$$\chi_\psi((e_{\Omega_2}(a)) \cdot \chi_\psi(e_{\Omega_3}(a)) = (a,a)_2 \cdot \chi_\psi((e_{\Omega_2}+e_{\Omega_3})(a))$$

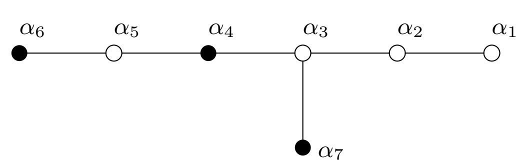
Recall $\gamma_\psi(a)^2 = (a,a)_2$. This combined with the above results gives

$$\begin{cases} \chi_\psi(e_{\Omega_1}(a)) = \gamma_\psi(a)^{|\Omega_1|}, \\ \chi_\psi(e_{\Omega_2}(a)) = \gamma_\psi(a)^{|\Omega_2|}, \\ \chi_\psi(e_{\Omega_3}(a)) = \gamma_\psi(a)^{|\Omega_3|}. \end{cases}$$

It agrees with the genuine character given by Savin.

The E_6, E_7, E_8 case.

For E_6 and E_8, $Y_{Q,2} = J$ and so the situation is trivial. Consider E_7, then $[Y_{Q,n} : J] = 2$. The nontrivial Ω is given by $\Omega = \{\alpha_4, \alpha_6, \alpha_7\}$.



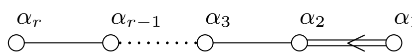
The set $\{\alpha_{i,Q,2}^\vee : 1 \le i \le 6\} \cup \{e_\Omega\}$ is a basis of $Y_{Q,2}$, while $\{\alpha_{i,Q,2}^\vee : 1 \le i \le 6\} \cup \{2e_\Omega\}$ a basis for J.

Our distinguished character is determined by

$$\chi_\psi(e_\Omega(a)) = \gamma_\psi(a)^{|\Omega|}.$$

This agrees with Savin also.

16.1.2. **The case C_r.** Let Sp_{2r} be the simply-connected simple group with Dynkin diagram:



Let $\{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee\}$ be the set of simple coroots with $\alpha_1^\vee$ the short one. Let $n = 2$. Here $\overline{\mathrm{Sp}}_{2r}$ is determined by the unique Weyl-invariant quadratic form Q on Y with $Q(\alpha_1^\vee) = 1$.

It follows $n_{\alpha_1} = 2$. Also $Q(\alpha_i^\vee) = 2$ and $n_{\alpha_i} = 1$ for $2 \le i \le r$. Moreover, a basis of $Y_{Q,2} = Y^{sc}$ is given by

$$\{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee\},$$

while a basis for $Y_{Q,2}^{sc}$ is

$$\{2\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee\}.$$

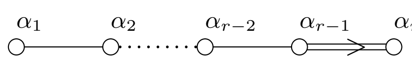
Since $J = Y_{Q,2}^{sc}$, by the construction of distinguished character χ_ψ, it is determined by

$$\begin{cases} \chi_\psi(\alpha_i^\vee(a)) = 1, \text{ if } i = 2, 3, ..., r; \\ \chi_\psi(\alpha_1^\vee(a)) = \gamma_\psi(a)^{(2-1)Q(\alpha_1^\vee)} = \gamma_\psi(a). \end{cases} \tag{16.6}$$

This uniquely determined a genuine character of $\overline{T}$ which is abelian. It can be checked that this agrees with the classical one (cf. [K, Ra] for example).

16.1.3. **The B_r, F_4 and G_2 case.** For completeness, we also give the explicit form of the distinguished character constructed in previous section for the double cover $\overline{G}$ of the simply connected group G of type B_r, F_4 and G_2. Recall that when $n = 2$ we have $J = Y_{Q,2}^{sc}$.

The B_r case. Consider the Dynkin diagram of B_r:



Let Q be the unique Weyl-invariant quadratic form with $Q(\alpha_i^\vee) = 1$ for $1 \le i \le r-1$. It gives $Q(\alpha_r^\vee) = 2$. We have also assumed that the double cover $\overline{G}$ is incarnated by a fair bisector D. The discussion now will be split into two cases according to the parity of r.

Case 1: r is odd. Direct computation gives $Y_{Q,2}^{sc} = Y_{Q,n}$ and therefore this case is trivial.

Case 2: r is even. It is not difficult to compute the index $[Y_{Q,2} : Y_{Q,2}^{sc}] = 2$. In fact, a basis of $Y_{Q,2}$ is given by

$$\{\alpha_1^\vee + \alpha_3^\vee + ... + \alpha_{r-1}^\vee\} \cup \{2\alpha_i^\vee : 2 \le i \le r-1\} \cup \{\alpha_r^\vee\}.$$

This gives a basis of $J = Y_{Q,2}^{sc}$:

$$\{2(\alpha_1^\vee + \alpha_3^\vee + ... + \alpha_{r-1}^\vee)\} \cup \{2\alpha_i^\vee : 2 \le i \le r-1\} \cup \{\alpha_r^\vee\}.$$

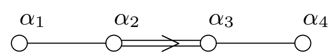
We have

$$Q(\alpha_1^\vee + \alpha_3^\vee + ... + \alpha_{r-1}^\vee) = r/2.$$

By the construction of distinguished character χ_ψ, it is determined by

$$\begin{cases} \chi_\psi\big((\alpha_1^\vee + \alpha_3^\vee + ... + \alpha_{r-1}^\vee)(a)\big) = \gamma_\psi(a)^{r/2}; \\ \chi_\psi\big((2\alpha_i^\vee)(a)\big) = 1, \text{ for } 2 \le i \le r-1; \\ \chi_\psi\big(\alpha_r^\vee(a)\big) = 1. \end{cases} \tag{16.7}$$

The F_4 case. Consider the Dynkim diagram of F_4:



Let Q be such that $Q(\alpha_i^\vee) = 1$ for $i = 1, 2$. It implies $Q(\alpha_i^\vee) = 2$ for $i = 3, 4$. Clearly $n_{\alpha_i} = 2$ for $i = 1, 2$ and $n_{\alpha_i} = 1$ for $i = 3, 4$. We can compute

$$B_Q(\alpha_1^\vee, \alpha_2^\vee) = -1, \quad B_Q(\alpha_2^\vee, \alpha_3^\vee) = -2, \quad B_Q(\alpha_3^\vee, \alpha_4^\vee) = -2.$$

Also $B_Q(\alpha_i^\vee, \alpha_j^\vee) = 0$ if α_i and α_j are not adjacent in the Dynkin diagram.

Moreover, any $\sum_i k_i\alpha_i^\vee \in Y^{sc}$ with certain $k_i \in \mathbb{Z}$ belongs to $Y_{Q,2}$ if and only if

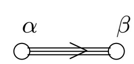
$$B_Q(\sum_i k_i\alpha_i^\vee, \alpha_j^\vee) \in 2\mathbb{Z} \text{ for all } 1 \le j \le 4,$$

which explicitly is given by

$$\begin{cases} 2k_1 + (-1)k_2 \in 2\mathbb{Z}, \\ (-1)k_1 + 2k_2 + (-2)k_3 \in 2\mathbb{Z}, \\ (-2)k_2 + 4k_3 + (-2)k_4 \in 2\mathbb{Z}, \\ (-2)k_3 + 4k_4 \in 2\mathbb{Z}. \end{cases}$$

Equivalently, $k_1, k_2 \in 2\mathbb{Z}$. This shows $Y_{Q,2} = Y_{Q,2}^{sc}$, and thus the situation is trivial.

The G_2 case. Consider the Dynkin diagram of G_2:



Let Q be such that $Q(\alpha^\vee) = 1$. This determines $Q(\beta^\vee) = 3$. Note $B_Q(\alpha^\vee, \beta^\vee) = -Q(\alpha^\vee) = -3$.

Since the computation is straightforward, we may assume $n \in \mathbb{N}_{\ge 1}$ is general instead of 2. It follows $n_\alpha = n$ and $n_\beta = n/\gcd(n, 3)$. Then $k_1\alpha^\vee + k_2\beta^\vee$ lies in $Y_{Q,n}$ if and only if

$$\begin{cases} 2k_1 - 3k_2 \in n\mathbb{Z}, \\ -3k_1 + 6k_2 \in n\mathbb{Z}. \end{cases}$$

Equivalently, $k_1 \in n\mathbb{Z}$ and k_2 divisible by $n/\gcd(n, 3)$. This exactly shows $Y_{Q,n} = Y_{Q,n}^{sc}$ for arbitrary n. Also in this case, it is trivial to define the distinguished character for the fair D.

16.2. **Kazhdan-Patterson coverings** $\overline{\mathrm{GL}}_r$ [KP1, KP2]**.** We consider the group GL_r with root data $(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee)$. Let $\{e_1, e_2, ..., e_r\}$ be a basis for the cocharacter lattice Y of GL_r. Let $\Delta^\vee = \{\alpha_i^\vee := e_i - e_{i+1} : 1 \le i \le r\}$ denote a set of simple coroots of GL_r.

Consider the Weyl-invariant bilinear form on Y determined by

$$B(e_i, e_j) = \begin{cases} 2c & \text{if } i = j, \\ 2c+1 & \text{otherwise,} \end{cases}$$

where $c \in \mathbb{Z}$ is an integer. It follows that $Q(\alpha_i^\vee) = -1$ for any $\alpha_i^\vee$. The covering groups $\overline{\mathrm{GL}}_r$ arising are exactly those studied by Kazhdan-Patterson, and c is the twisting parameter in [KP1].

Write $c_{n,r} := n/\gcd(2cr + r - 1, n)$. It follows

$$Y_{Q,n} = \left\{ \sum_{i=1}^r m_i e_i : m_i \equiv m_j \bmod n, \text{ and } c_{n,r} | m_i \text{ for all } i, j \right\}.$$

In particular, a basis for $Y_{Q,n}$ is given by

$$\{ne_i : 1 \le i \le r-1\} \cup \left\{ c_{n,r} \cdot \left(\sum_{i=1}^r e_i \right) \right\}.$$

On the other hand, $Y_{Q,n}^{sc}$ is spanned by $\{n \cdot \alpha_i^\vee : 1 \le i \le r-1\}$. It follows that $J = Y_{Q,n}^{sc} + nY$ has a basis given by

$$\{ne_i : 1 \le i \le r-1\} \cup \left\{ n \cdot \left(\sum_{i=1}^r e_i \right) \right\}.$$

A distinguished character χ_ψ is thus determined by

$$\begin{cases} \chi_\psi\big(ne_i(a)\big) = 1 \text{ for all } 1 \le i \le r-1, \\ \chi_\psi\big((\sum_{i=1}^r c_{n,r} e_i)(a)\big) = \gamma_\psi(a)^{-\frac{r(2cr+r-1)c_{n,r}}{n}(n - c_{n,r})}. \end{cases} \tag{16.8}$$

In fact, this distinguished character is basically the genuine character of $Z(\overline{T})$ given in [CO, Lemma 2], with associated parameter $s = 0$ in the notation of loc. cit.. More precisely, the first equality in (16.8) corresponds to $s = 0$ in [CO, Lemma 2], and the second equality in (16.8) corresponds to the equality (3.5) in the paper of Chinta and Offen.

An examination of the root datum shows that

$$\overline{G}^\vee \cong \{(g, \lambda) \in \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^{\gcd(2cr+r-1,n)}\} \subset \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Thus, in general, the dual group $\overline{G}^\vee$ may not be GL_r. However, if $\gcd(2cr + r - 1, n) = 1$, then $\overline{G}^\vee = \mathrm{GL}_r$ in this case. In particular, for $r = 2$ and $c = 0$, the untwisted n-fold covering groups of GL_2 studied by Flicker in [F] belong to this class.

For more examples, we consider the embedding of $\mathrm{GL}_r \hookrightarrow \mathrm{SL}_{r+1}$ given by $g \mapsto (g, \det(g)^{-1})$. Let $\overline{\mathrm{SL}}_{r+1}$ be the degree n covering associated with (n, Q) where $Q(\alpha_i^\vee) = -1$ for all i. Then the pull-back covering of GL_r will be the covering associated to the quadratic form above with $c = -1$. Consider $n = 2$. If r is even, then

$$\overline{\mathrm{GL}}_r^\vee = \mathrm{GL}_r.$$

If r is odd, then $\overline{\mathrm{GL}}_r^\vee \subseteq \mathrm{GL}_r \times \mathrm{GL}_1$ and is given by

$$\overline{\mathrm{GL}}_r^\vee = \{(g,a) \in \mathrm{GL}_r \times \mathrm{GL}_1 : \det(g) = a^2\}.$$

When $r = 1$ or $r = 3$ there is an isomorphism $\overline{\mathrm{GL}}_r^\vee \simeq \mathrm{GL}_r$ given by $(g,a) \mapsto ga^{-1}$. However, for odd $r \geq 5$, there exists an isogeny $\overline{\mathrm{GL}}_r^\vee \to \mathrm{GL}_r$ of degree two given by $(g,a) \mapsto g$.

16.3. **The cover $\overline{\mathrm{GSp}}_{2r}$.** Let $\mathbb{G}$ be the group GSp_{2r} of similitudes of symplectic type, and let $(X, \Delta, Y, \Delta^\vee)$ be its root data given as follows. The character group $X \simeq \mathbb{Z}^{r+1}$ has a standard basis $\{e_i^* : 1 \leq i \leq r\} \cup \{e_0^*\}$, and the roots are given by

$$\Delta = \{e_i^* - e_{i+1}^* : 1 \leq i \leq r-1\} \cup \{2e_r^* - e_0^*\}.$$

The cocharacter group $Y \simeq \mathbb{Z}^{r+1}$ is given with a basis $\{e_i : 1 \leq i \leq r\} \cup \{e_0\}$. The coroots are

$$\Delta^\vee = \{e_i - e_{i+1} : 1 \leq i \leq r-1\} \cup \{e_r\}.$$

Write $\alpha_i = e_i^* - e_{i+1}^*$, $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \leq i \leq r-1$, and also $\alpha_r = 2e_r^* - e_0^*$, $\alpha_r^\vee = e_r$. Consider a covering $\overline{\mathbb{G}}$ incarnated by $(D, 1)$. We are interested in those $\overline{\mathbb{G}}$ whose restricted to Sp_{2r} is the one with $Q(\alpha_r^\vee) = 1$. That is, we assume

$$Q(\alpha_i^\vee) = 2 \text{ for } 1 \leq i \leq r-1, \quad Q(\alpha_r^\vee) = 1.$$

Since $\Delta^\vee \cup \{e_0\}$ gives a basis for Y, to determine Q it suffices to specify $Q(e_0)$. Let $n = 2$, and we obtain a double cover $\overline{\mathrm{GSp}}_{2r}$ which restricts to the classical metaplectic double cover $\overline{\mathrm{Sp}}_{2r}$. Note also the number $Q(e_0) \in \mathbb{Z}/2\mathbb{Z}$ determines whether the similitude factor $F^\times$ corresponding to the cocharacter e_0 splits into $\overline{\mathrm{GSp}}_{2r}$ or not. To recover the classical double cover of GSp_{2r}, we should take $Q(e_0)$ to be even.

Back to the case of $n = 2$ and general $Q(e_0)$. We compute the root data for the complex dual group $\overline{\mathrm{GSp}}_{2r}^\vee$. We have

$$Y_{Q,2} = \left\{ \sum_{i=1}^r k_i \alpha_i^\vee + k e_0 \in Y : k_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r-1, k_r, k \in 2\mathbb{Z} \right\}$$

and the sublattice $Y_{Q,2}^{sc}$ is spanned by $\{\alpha_{i,Q,2}^\vee\}_{1 \leq i \leq r}$, i.e.

$$\{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_{r-1}^\vee, 2\alpha_r^\vee\}.$$

The lattice $J = Y_{Q,2}^{sc} + 2Y$ is thus equal to $Y_{Q,2}$. In this case, distinguished character χ_ψ will be determined by the condition (since we have assumed $(D,1)$ to be fair)

$$\chi_\psi\big(y(a)\big) = 1 \text{ for all } y \in Y_{Q,2}, \ a \in F^\times.$$

An examination of the root datum gives:

$$\overline{\mathrm{GSp}}_{2r}^\vee = \begin{cases} \mathrm{GSp}_{2r}(\mathbb{C}), \text{ if } r \text{ is odd;} \\ \mathrm{PGSp}_{2r}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}), \text{ if } r \text{ is even.} \end{cases}$$

This explains the difference in the representation theory of $\overline{\mathrm{GSp}}_{2r}$ for even and odd r observed in the work of Szpruch [Sz3].

17. **Problems and Questions**

In this final section, we highlight a few problems and questions which we feel are important to carry the program forward:

(a) (Real Groups) When $F = \mathbb{R}$, Harish-Chandra's classification of discrete series representation works equally well for BD covering groups. Might one be able to formulate this classification in terms of L-parameters in the spirit of this paper, analogous to what Langlands accomplished for linear real reductive groups? Moreover, can the results of the recent papers [AH, ABPTV] be formulated in the framework of this paper? In the paper [W7], Weissman has shown that the classification of discrete series of $\overline{G}$ (given by Harish-Chandra) can be formulated in terms of L-parameters; this gives strong supporting evidence for the notion of L-groups introduced in [W7] and described here.

(b) (Hecke algebra isomorphisms) In Theorem 15.1, we saw that there is an isomorphism of Iwahori-Hecke algebras for the covering group $\overline{G}$ and a linear algebraic group $G_{Q,n}$ in the tame case. There should perhaps be such Hecke algebra isomorphisms for other Bernstein components. One also expects some Hecke algebra isomorphisms outside the tame case, i.e. when the residue characteristic p divides the degree n of the covering. For Mp_{2n}, such results were obtained in [Wo] and [TW].

(c) (Supercuspidal representations) Depth zero supercuspidal representations of BD covering groups have been studied by Howard-Weissman [HW]. One would expect that the construction of supercuspidal representations of J.K. Yu [Yu] and S. Stevens [St] for linear reductive groups can be extended to BD covering groups. This is an ongoing investigation of the first author with J.L.Kim. One expects that Kim's proof of the exhaustion of Yu's construction [Yu] can be extended to give exhaustion in the covering case. For this, a better understanding of the Bruhat-Tits theory of BD covering groups is probably needed; a preliminary study has been conducted by Weissman [W2].

(d) (Harmonic Analysis) For covering groups, any conjugacy-invariant function or distribution is necessarily supported on the subset of "good" or "relevant" elements. These are elements $g \in \overline{G}$ such that g is not conjugate to $g \cdot \epsilon$ for any $\epsilon \in \mu_n$ (for a degree n cover). It is clear that to have a better understanding of invariant harmonic analysis, a better understanding of such "good" or "relevant" elements is necessary. One might ask if the BD structure theory is robust enough to give one a classification of such elements.

W.-W. Li has extended many foundational results in harmonic analysis to the covering case in [L4], such as the Plancherel theorem, the basic properties of invariant distributions and the properties of the standard intertwining operators. There are many further questions to investigate in invariant harmonic analysis for covering groups. One is to have a better understanding of the theory of R-groups for covering groups initiated in [L4]. Another is to extend foundational results such as the Howe conjecture (proved by Clozel [C] in the linear case).

(e) (Automorphic L-functions) We have defined the global partial automoprhic L-function associated to an automorphic representation of a BD covering group. One would like to have a definition of the local L-factors at the remaining set of places so as to obtain a complete L-function. One would also like to show that the partial L-function has a meromorphic continuation, and the complete L-function satisfies a standard functional equation relative to $s \leftrightarrow 1-s$. For this, one may ask if Langlands-Shahidi theory can be extended to the covering case, but this is not clear since uniqueness of Whittaker models is false in general for covering groups. The only such success is the thesis work [Sz1, Sz2] of D. Szpruch where the Langlands-Shahidi theory was extended to the group Mp_{2n}. One might also ask if the myriad of Rankin-Selberg integrals for various L-functions of linear groups have counterparts in the covering case. The recent preprint [CFGK] of Cai-Friedberg-Ginzburg-Kaplan is a very exciting and promising work in this direction.

(f) (Functoriality) More generally, one would like to show that this class of automorphic L-functions from BD covering groups belong to the class of automorphic L-functions of linear reductive group. In the context of (b), one might expect that there is a functorial transfer of automorphic representations from $\overline{G}$ to $G_{Q,n}$ which respects (partial) automorphic L-functions. One might imagine comparing the trace formula for these two groups. For this, we note that the work of W.-W. Li [L2, L4, L5, L6] has carried the theory of the trace formula for covering groups to the point where one has the invariant trace formula. The earlier works of Flicker [F] and Flicker-Kazhdan [FK] undertook such a comparison of trace formula for the Kazhdan-Patterson coverings.

(g) (Endoscopy) The next step in the theory of the trace formula for covering groups is undoubtedly the stable trace formula. For this, one needs to develop the theory of endoscopy for covering groups. This includes the definition of stable conjugation, the definition of endoscopic groups, the definition of correspondence of stable classes between a covering group and its endoscopic groups and the definition of the transfer factors. Since the theory of endoscopy for linear reductive groups is essentially of arithmetic origin and content, one might expect that a nice theory exists for the BD covering groups since these are of algebraic origin. The only covering group for which a theory of endoscopy exists is the group Mp_{2n}, where the theory is due to Adams [A1], Renard [R1, R2] and W.-W. Li [L1, L3]. The recent preprint [L7] of W.-W. Li has taken the first step towards the stabilisation of the trace formula for Mp_{2n}.

(h) (Automorphic Discrete Spectrum) Naturally, one hopes to have an analog of the Arthur's conjecture for BD covering groups, including an analog of the Arthur multiplicity formula. This will very much depend on the shape of the theory of endoscopy. The only BD covering group for which a precise conjecture exists is Mp_{2n}, beyond the work of Flicker [F] and Flicker-Kazhdan [FK].

(i) (General Covering Tori) The various questions highlighted above are already highly non-trivial and interesting when $\mathbb{G} = \mathbb{T}$ is a (not necessarily split) torus. The ongoing work of Weissman [W5, W7] and Hiraga-Ikeda aim to understand this case completely but many mysteries remain.

(j) (Applications) The impetus for a program naturally depends on its potential applications. The motivation for our investigations is simply in the naive hope of including the representation theory and automorphic forms of BD covering groups in the framework of the Langlands philosophy. This is reasonable enough (if naive) for the point of view of representation theory. But what about form the point of view of number theory? Automorphic forms of covering groups have traditionally found applications in analytic number theory, such as in the work of Bump-Friedberg-Hoffstein [BFH]. It is reasonable to demand concrete arithmetic applications of this potential theory. The only thought we have to offer is perhaps in various branching or period problems, such as the arithmetic information contained in Fourier coefficients or in the analogs of the Gross-Prasad conjecture.

(k) (Geometric Counterpart) The definition of the dual group of a BD cover first appeared in the context of the Geometric Langlands Program, through the work of Finkelberg-Lysenko [FL]. One might expect the geometric theory to offer more evidence for this program. From the geometric side, quantum groups seem to play an important role in the theory. This is also reflected to some extent in the work of Brubaker-Bump-Freidberg (c.f. [BBF, BBFH, BBCFG]) on the Whittaker-Fourier coefficients of metaplectic Eisenstein series and the work of Chinta-Offen [CO] and McNamara [Mc1, Mc3] the metaplectic Casselman-Shalika formula. However, quantum groups are conspicuously missing from the framework developed in this article, and one may wonder if and how they should be incorporated.

(l) (Function Fields) On the other hand, one can consider classical function fields (of curves over finite fields) and ask whether V. Lafforgue's recent construction [Laf] of the global Langlands correspondence for arbitrary linear reductive groups could be extended to the case of BD covering groups: this is a very tantalizing problem whose resolution should shed much light on the Langlands-Weissman program and [Laf] has suggested that this should follow from the methods there.

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