THE LOCAL LANGLANDS CONJECTURE FOR GSp(4) II: THE CASE OF INNER FORMS

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Abstract. In an earlier paper of the first author and Shuichiro Takeda, the local Langlands correspondence for $\mathrm{GSp}_4(F)$ over a non-archimedean local field F of characteristic zero was established and several expected properties of the correspondence verified. In this paper, we extend the local Langlands correspondence to the inner form $\mathrm{GU}_2(D)$ of GSp_4.

1. Introduction

In an earlier paper [GT] of the first author with S. Takeda, the local Langlands conjecture for $\mathrm{GSp}_4(F)$ over a non-archimedean local field F of characteristic zero was established. More precisely, let $\Pi(\mathrm{GSp}_4)$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GSp}_4(F)$ and let $\Phi(\mathrm{GSp}_4)$ denote the set of isomorphism classes of L-parameters for $\mathrm{GSp}_4(F)$. Hence, an element of $\Phi(\mathrm{GSp}_4)$ is an admissible homomorphism

$$\phi : WD_F = W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

where W_F is the Weil group of F and $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ is the Weil-Deligne group. Given an L-parameter ϕ, set

$$A_\phi = \pi_0(Z_{\mathrm{GSp}_4(\mathbb{C})}(Im(\phi))),$$

i.e. the component group of the centralizer in $\mathrm{GSp}_4(\mathbb{C})$ of the image of ϕ. Then the main theorem of [GT] says that there is a surjective finite-to-one map

$$L : \Pi(\mathrm{GSp}_4) \longrightarrow \Phi(\mathrm{GSp}_4)$$

such that the fiber over $\phi \in \Phi(\mathrm{GSp}_4)$ is naturally indexed by the set $\widehat{A}_\phi$ of irreducible characters of the finite group A_ϕ. In other words, L induces a bijection

$$L : \Pi(\mathrm{GSp}_4) \longrightarrow \{(\phi, \eta) : \phi \in \Phi(\mathrm{GSp}_4),\ \eta \in \widehat{A}_\phi\}.$$

Moreover, it was shown in [GT] that the map L satisfies a number of expected properties, which characterize it uniquely. For example, it preserves certain natural invariants attached to both sides, namely certain L-factors and ϵ-factors.

In this paper, which is a sequel to [GT], we prove an analogous result for the inner form $\mathrm{GU}_2(D)$ of GSp_4. The inner form $\mathrm{GU}_2(D)$ is the similitude group of the unique 2-dimensional Hermitian vector space over the quaternion division F-algebra D. It is isomorphic as an algebraic group to $\mathrm{GSpin}_{4,1}$, the general spin group associated to the (unique up to scaling) non-split quadratic space of dimension 5 over F.

As above, we let $\Pi(\mathrm{GU}_2(D))$ denote the set of irreducible admissible representations of $\mathrm{GU}_2(D)$. On the other hand, the expected set $\Phi(\mathrm{GU}_2(D))$ of L-parameters for $\mathrm{GU}_2(D)$ is a certain subset of $\Phi(\mathrm{GSp}_4)$, as we now explain.

The unique 2-dimensional Hermitian space over D is isotropic, so that $\mathrm{GU}_2(D)$ has relative semisimple rank 1. In particular, $\mathrm{GU}_2(D)$ has a unique (up-to-conjugacy) minimal parabolic F-subgroup P whose Levi factor is

$$M = D^\times \times \mathrm{GL}_1$$

and whose unipotent radical is abelian. Hence, P is a form of the Siegel parabolic subgroup of GSp_4 and it determines a dual parabolic subgroup $P^\vee(\mathbb{C})$ in the dual group $\mathrm{GSp}_4(\mathbb{C})$ of $\mathrm{GU}_2(D)$ by [B, §3]. The parabolic subgroup $P^\vee(\mathbb{C})$ is the Heisenberg parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$ and its conjugacy class is said to be relevant for $\mathrm{GU}_2(D)$ while all other proper parabolic subgroups are irrelevant (see [B, §3]).

An L-parameter $\phi \in \Phi(\mathrm{GSp}_4)$ is relevant for $\mathrm{GU}_2(D)$ if ϕ does not factor through any irrelevant parabolic subgroups of $\mathrm{GSp}_4(\mathbb{C})$. The subset of relevant ϕ's is by definition the set $\Phi(\mathrm{GU}_2(D))$ of L-parameters for $\mathrm{GU}_2(D)$.

Further, for the purpose of stating the local Langlands correspondence for inner forms, it is more natural to consider a modified component group, following Vogan [V]. More precisely, for $\phi \in \Phi(\mathrm{GSp}_4)$, we set

$$B_\phi = \pi_0(Z_{\mathrm{Sp}_4(\mathbb{C})}(Im(\phi))),$$

i.e. the component group of the centralizer in $\mathrm{Sp}_4(\mathbb{C})$ of the image of ϕ. Then one has an exact sequence

$$Z_{\mathrm{Sp}_4(\mathbb{C})} = \langle \pm 1 \rangle \longrightarrow B_\phi \longrightarrow A_\phi \longrightarrow 0$$

so that there is an injection of the group of irreducible characters

$$\widehat{A}_\phi \hookrightarrow \widehat{B}_\phi$$

which identifies $\widehat{A}_\phi$ as the subgroup (of index at most 2) of characters trivial on the image of the center $Z_{\mathrm{Sp}_4(\mathbb{C})}$ of $\mathrm{Sp}_4(\mathbb{C})$. It is not difficult to check that $\widehat{B}_\phi \neq \widehat{A}_\phi$ if and only if ϕ is relevant for $\mathrm{GU}_2(D)$, i.e. $\phi \in \Phi(\mathrm{GU}_2(D))$.

With these notions introduced, the main result of this paper is:

Main Theorem

There is a natural surjective finite-to-one map

$$L : \Pi(\mathrm{GU}_2(D)) \longrightarrow \Phi(\mathrm{GU}_2(D))$$

satisfying the following properties:

(i) π is a (essentially) discrete series representation of $\mathrm{GU}_2(D)$ if and only if its L-parameter $\phi_\pi := L(\pi)$ does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.

(ii) For an L-parameter ϕ, its fiber L_ϕ can be naturally parametrized by the set $\widehat{B}_\phi \smallsetminus \widehat{A}_\phi$. This set has size either 1 or 2.

(iii) The similitude character $\mathrm{sim}(\phi_\pi)$ of ϕ_π is equal to the central character ω_π of π (via local class field theory). Here, $\mathrm{sim} : \mathrm{GSp}_4(\mathbb{C}) \longrightarrow \mathbb{C}^\times$ is the similitude character of $\mathrm{GSp}_4(\mathbb{C})$.

(iv) The L-parameter of $\pi \otimes (\chi \circ \lambda)$ is equal to $\phi_\pi \otimes \chi$. Here, $\lambda : \mathrm{GU}_2(D) \longrightarrow F^\times$ is the similitude character of $\mathrm{GU}_2(D)$, and we have regarded χ as both a character of $F^\times$ and a character of W_F by local class field theory.

(v) Suppose that π is a non-supercuspidal representation. Then for any irreducible representation σ of $\mathrm{GL}_r(F)$, with L-parameter ϕ_σ, we have:

$$\begin{cases} \gamma(s,\pi\times\sigma,\psi)=\gamma(s,\phi_\pi\otimes\phi_\sigma,\psi) \\ L(s,\pi\times\sigma)=L(s,\phi_\pi\otimes\phi_\sigma) \\ \epsilon(s,\pi\times\sigma,\psi)=\epsilon(s,\phi_\pi\otimes\phi_\sigma,\psi). \end{cases}$$

Here the functions on the RHS are the local factors of Artin type associated to the relevant representations of the Weil-Deligne group WD_F, whereas those on the LHS are defined more precisely in §8.

(vi) Suppose that π is a supercuspidal representation. For any irreducible supercuspidal representation σ of $\mathrm{GL}_r(F)$ with L-parameter ϕ_σ, let $\mu(s,\pi\boxtimes\sigma,\psi)$ denote the Plancherel measure associated to the family of induced representations $I_P(\pi\boxtimes\sigma,s)$ on $\mathrm{GSpin}_{r+4,r+1}$, where we have regarded $\pi\boxtimes\sigma$ as a representation of the Levi subgroup $\mathrm{GSpin}_{4,1}\times\mathrm{GL}_r\cong\mathrm{GU}_2(D)\times\mathrm{GL}_r$. Then $\mu(s,\pi\boxtimes\sigma)$ is equal to

$$\gamma(s,\phi_\pi^\vee\otimes\phi_\sigma,\psi)\cdot\gamma(-s,\phi_\pi\otimes\phi_\sigma^\vee,\overline{\psi})\cdot\gamma(2s,Sym^2\phi_\sigma\otimes\mathrm{sim}(\phi_\pi)^{-1},\psi)\cdot\gamma(-2s,Sym^2\phi_\sigma^\vee\otimes\mathrm{sim}(\phi_\pi),\overline{\psi}).$$

(vii) The map L is uniquely determined by the properties (i), (iii), (v) and (vi), with $r\le 4$ in (v) and (vi). □

In particular, combining the above result with the main theorem of [GT], we see that there is a natural bijection

$$\Pi(\mathrm{GSp}_4)\sqcup\Pi(\mathrm{GU}_2(D))\longleftrightarrow\{(\phi,\eta):\phi\in\Phi(\mathrm{GSp}_4),\ \eta\in\widehat{B}_\phi\},$$

satisfying a list of natural properties which characterize it uniquely.

The proof of the main theorem is parallel to that of the main theorem in [GT]. Recall that in [GT], one examines the theta correspondences given by the following diagram:

$$\begin{tikzcd}[arrows=-] & & \mathrm{GSO}_{3,3} \\ & \mathrm{GSp}_4 \arrow[ur] \arrow[dr] & \\ \mathrm{GSO}_{4,0} \arrow[ur] & & \mathrm{GSO}_{2,2} \end{tikzcd}$$

and exploits the accidental isomorphisms:

$$\begin{cases} \mathrm{GSO}_{2,2}\cong(\mathrm{GL}_2\times\mathrm{GL}_2)/\{(z,z^{-1}):z\in F^\times\} \\ \mathrm{GSO}_{4,0}\cong(D^\times\times D^\times)/\{(z,z^{-1}):z\in F^\times\} \\ \mathrm{GSO}_{3,3}\cong(\mathrm{GL}_4\times\mathrm{GL}_1)/\{(z,z^{-2}):z\in F^\times\}. \end{cases}$$

In particular, the local Langlands conjecture is known for each of these 3 groups. The proof in [GT] then consists of:

(a) showing that each element of $\Pi(\mathrm{GSp}_4)$ participates in theta correspondence with exactly one of $\mathrm{GSO}_{4,0}$ or $\mathrm{GSO}_{3,3}$. Moreover, the set of representations which participate in theta correspondence with $\mathrm{GSO}_{3,3}$ can further be partitioned into two disjoint sets, depending on whether they participate in theta correspondence with $\mathrm{GSO}_{2,2}$ or not. The existence of this partition of $\Pi(\mathrm{GSp}_4)$ into 3 disjoint subsets is largely provided by the results of Kudla-Rallis [KR];

(b) characterizing the representations of $\mathrm{GSO}_{3,3}$ which participate in theta correspondence with GSp_4 as those whose L-parameter $\phi : WD_F \longrightarrow \mathrm{GL}_4(\mathbb{C})$ factors through $\mathrm{GSp}_4(\mathbb{C})$. This is largely provided by the results of Muic-Savin [MS].

For the case of inner forms considered in this paper, one has to consider inner form versions of the above theta correspondences, namely theta correspondences associated to quaternionic hermitian groups. Indeed, given a Hermitian space V over D and a skew-Hermitian space W over D, one has a theta correspondence between $\mathrm{GU}(V) = \mathrm{GSp}_{m,n}$ (an inner form of a symplectic similitude group) and $\mathrm{GU}(W) = \mathrm{GO}^*_{p,q}$ (an inner form of an orthogonal similtiude group). We shall describe the quaternionic hermitian and skew-hermitian spaces in Section 2 and review the set-up of theta correspondences (for similitudes) in Section 3. More precisely, we shall consider the theta correspondence associated to:

$$\mathrm{GO}^*_{3,0} \text{ ——— } \mathrm{GSp}(1,1) \text{ ——— } \mathrm{GO}^*_{1,1}$$

and exploit the accidental isomorphisms:

$$\begin{cases} \mathrm{GO}^*_{1,1} \cong (D^\times \times \mathrm{GL}_2)/\{(z, z^{-1}) : z \in F^\times\} \\ \mathrm{GO}^*_{3,0} \cong (D_4^\times \times \mathrm{GL}_1)/\{(z, z^{-2}) : z \in F^\times\}, \end{cases}$$

where $D_4^\times$ denotes the multiplicative group of a degree 4 division algebra over F. There are two such division algebras (with invariants $1/4$ or $-1/4 \in \mathbb{Q}/\mathbb{Z}$), but they are opposite of each other and hence have canonically isomorphic multiplicative groups.

The main work of this paper then consists of showing the analogs of (a) and (b) above. Unfortunately, there is relatively little in the literature concerning the explicit theta correspondence associated to these quaternionic dual pairs over non-archimedean local fields (the general theory is of course covered in [MVW]). Hence, we need to extend the results of [KR] and [MS] to the setting of quaternionic dual pairs. These conjectural extensions are described in Section 4 and we verify what we need for the low rank groups encountered in this paper in Sections 5 and 6.

Since this paper was completed in 2010, the conjectural extension of the conservation conjecture of Kudla-Rallis has subsequently been shown by Minguez [M] for supercuspidal representations and by Sun-Zhu [SZ] in general. However, since the papers [M] and [SZ] refer to the conjectures stated here, it seems to us best to leave this part of our paper as it is.

In any case, we shall show:

(a) each element π of $\Pi(\mathrm{GU}_2(D))$ participates in theta correspondence with exactly one of $\mathrm{GO}^*_{1,1}$ or $\mathrm{GO}^*_{3,0}$; this is Thm. 5.8, which leads to a partition of the set $\Pi(\mathrm{GU}_2(D))$ into two disjoint subsets from the point of view of theta correspondence;

(b) a representation $\Pi \boxtimes \mu$ of $D_4^\times \times F^\times$ participates in theta correspondence with $\mathrm{GU}_2(D)$ if and only if the L-parameter of Π factors through $\mathrm{GSp}_4(\mathbb{C})$ with similitude character μ; this is Thm. 6.1.

These two statements allow us to give a definition of the map L in Section 7 and go a long way towards the proof of the main theorem. We verify the remaining properties of the map L in Section 8 and the characterization of L is shown in Section 9. Finally, in the appendix, we give the explicit determination of the local theta correspondences mentioned above.

As in the case of split groups treated in [GT], we do not address the harmonic analytic issues of stability and character relations for the L-packets we define here. Moreover, each $\phi \in \Phi(\mathrm{GU}_2(D))$ determines an L-packet L_ϕ for GSp_4 and an L-packet L'_ϕ for the inner form $\mathrm{GU}_2(D)$, and these L-packets are "Jacquet-Langlands transfers" of each other. In particular, one expects that the sum of the characters of representations in L_ϕ should be the negative of the analogous sum for L'_ϕ. These issues will not be discussed here, but will be addressed in a joint paper [CG] of the first author with Ping-Shun Chan, which is the third paper in this series.

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2. **Quaternionic Dual Pairs**

We begin by recalling the classification of quaternionic Hermitian and skew-Hermitian spaces over the quaternion F-algebra D. The reader can consult the classic text [S] of Scharlau, the papers of Tsukamoto [Ta] and Satake [Sa], or the book [MVW, Pg. 7-8] for this material.

2.1. **Quaternionic Hermitian spaces.** Given a positive integer k, there is a unique quaternionic Hermitian space $V_k = D^k$ which has dimension k over D. The space V_k is maximally split, in the sense that its maximal isotropic subspace has dimension $[k/2]$. Thus, V_1 is the only anisotropic quaternionic Hermitian space and $V_2 = De_1 \oplus De_2$ is the hyperbolic plane, with inner product given by

$$\langle e_1, e_2 \rangle = 1 \quad \text{and} \quad \langle e_i, e_i \rangle = 0.$$

This gives two Witt towers of quaternionic Hermitian spaces, characterized by the parity of their dimensions:

$$V_{2n} = V_2^{\oplus n} \quad \text{and} \quad V_{2n+1} = V_1 \oplus V_2^{\oplus n}.$$

The automorphism group $U(V_k)$ of V_k is an inner form of Sp_{2k} and the similitude group $\mathrm{GU}(V_k)$ is an inner form of GSp_{2k}. Depending on the context, we will denote the associated similitude group by any one of the following:

$$\begin{cases} \mathrm{GU}(V_{2n}) = \mathrm{GSp}_{n,n}; \\ \mathrm{GU}(V_{2n+1}) = \mathrm{GSp}_{n+1,n}, \end{cases}$$

and the isometry groups by

$$\begin{cases} \mathrm{U}(V_{2n}) = \mathrm{Sp}_{n,n}; \\ \mathrm{U}(V_{2n+1}) = \mathrm{Sp}_{n+1,n}\,. \end{cases}$$

The notation $\mathrm{GSp}_{m,n}$ is commonly used in the literature when one is working over $\mathbb{R}$ and should remind the reader that the group in question is an inner form of $\mathrm{GSp}_{2(m+n)}$.

2.2. **Quaternionic skew-Hermitian spaces.** On the other hand, for a given dimension k, the quaternionic skew-Hermitian spaces over D are classified by their discriminant, which is an element in $F^\times/F^{\times 2}$ and thus determined by a separable quadratic F-algebra K. We let W_k^K denote the unique skew-Hermitian space of dimension k and discriminant K, except when $k=1$ and K is split, in which case W_1^K does not exist.

In most of this paper, we shall only be concerned with the case of trivial discriminant, in which case we simply write W_k and suppress the mention of the discriminant. The space $W_2 = Df_1 \oplus Df_2$ is isotropic, with inner product

$$(f_1, f_2) = 1 \quad \text{and} \quad (f_i, f_i) = 0.$$

The space W_3 is anisotropic and can be described as follows (see [Ta] and [Sa]). Let K_1, K_2 and K_3 be 3 quadratic extensions which are contained in a biquadratic extension of F; equivalently, if K_i corresponds to $a_i \in F^\times/F^{\times 2}$, then

$$a_1 a_2 a_3 = 1 \in F^\times/F^{\times 2}.$$

Then

$$W_3 \cong W_1^{K_1} \oplus W_1^{K_2} \oplus W_1^{K_3},$$

where $W_1^{K_i} = D \cdot f$ with $(f, f) = \sqrt{a_i}$. Indeed, the RHS defines a skew-Hermitian space over D of dimension 3 over D and trivial discriminant, and so has no choice but to be isomorphic to W_3. Similarly, when the residue characteristic p of F is equal to 2, there is more than one biquadratic extension, but by the classification theorem of skew-Hermitian spaces of D [MVW, Pg. 7-8], the above construction gives the same skew-Hermitian spaces over D.

As in the Hermitian case, there are 2 Witt towers of skew-Hermitian spaces for a given discriminant. In the case of trivial discriminant, the two towers are:

$$W_{2m} = W_2^{\oplus m} \quad \text{and} \quad W_{2m+3} = W_3 \oplus W_2^{\oplus m}.$$

These two towers are again distinguished by the parity of dimensions. Depending on the context, we shall denote the similitude groups by

$$\mathrm{GU}(W_{2m}) = \mathrm{GO}^*_{m,m} \quad \text{and} \quad \mathrm{GU}(W_{2m+3}) = \mathrm{GO}^*_{m+3,m},$$

and the isometry groups by

$$\mathrm{U}(W_{2m}) = \mathrm{O}^*_{m,m} \quad \text{and} \quad \mathrm{U}(W_{2m+3}) = \mathrm{O}^*_{m+3,m}.$$

As before, this notation is motivated by the archimedean literature and serves to remind the reader that the group in question is an inner form of an orthogonal similitude group.

Now $\mathrm{GU}(W_m)$ and $\mathrm{U}(W_m)$ are disconnected as algebraic groups [MVW, Pg. 21] and we denote their identity components by $\mathrm{GSU}(W_m)$ and $\mathrm{SU}(W_m)$ respectively. However, unlike the orthogonal case, the non-identity connected component of $\mathrm{GU}(W_m)$ or $\mathrm{U}(W_m)$ does not contain any F-rational points [MVW, Pg. 21-22], so that

$$\mathrm{GU}(W_m)(F) = \mathrm{GSU}(W_m)(F) \quad \text{and} \quad \mathrm{U}(W_m)(F) = \mathrm{SU}(W_m)(F).$$

2.3. **Accidental isomorphisms.** Let us now describe the groups $\mathrm{GU}(W_2)$ and $\mathrm{GU}(W_3)$ more concretely. With $W_2 = D \cdot f_1 \oplus D \cdot f_2$, we have an action of $D^\times \times \mathrm{GL}_2(F)$ on W_2 with $\mathrm{GL}_2(F)$ acting as in its standard representation and $D^\times$ acting via:

$$a \cdot f_1 + b \cdot f_2 \mapsto a \cdot \alpha \cdot f_1 + b \cdot \alpha \cdot f_2$$

for $\alpha \in D^\times$. This gives an isomorphism

$$\mathrm{GU}(W_2)(F) = \mathrm{GSU}(W_2)(F) \cong (D^\times \times \mathrm{GL}_2(F))/\{(z, z^{-1}) : z \in F^\times\}. \tag{2.1}$$

On the other hand, the group of F-rational points of the similitude group of W_3 is:

$$\mathrm{GU}(W_3)(F) = \mathrm{GSU}(W_3)(F) \cong (D_4^\times \times F^\times)/\{(z, z^{-2}) : z \in F^\times\}, \tag{2.2}$$

where $D_4^\times$ is the multiplicative group of a degree 4 division algebra over F. This is not as easy to see, in the sense that it is not easy to write down an action of $D_4^\times$ on W_3. However, this was precisely what Satake did in [Sa, §2, Case 3]. In the following, we will give an alternative argument, using classification.

Since W_3 is an anisotropic skew-Hermitian space, the algebraic group $\mathrm{SU}(W_3)$ over F is an anisotroppic group which is isomorphic to $\mathrm{SO}_6 \cong \mathrm{SL}_4/\mu_2$ as an algebraic group over $\overline{F}$. Since one knows the classification of algebraic groups over F (see [T]), one sees that the anisotropic form of SL_4 is simply $\mathrm{SL}_1(D_4)$ for D_4 a degree 4 division algebra over F, so that $\mathrm{SU}(W_3) \cong \mathrm{SL}_1(D_4)/\mu_2$. Note that $\mathrm{SL}_1(D_4)/\mu_2$ has centre μ_2, since the center of $\mathrm{SL}_1(D_4)$ is μ_4.

Now

$$\mathrm{GSU}(W_3) = (\mathrm{SU}(W_3) \times \mathbb{G}_m)/\Delta\mu_2 \cong (\mathrm{SL}_1(D_4)/\mu_2 \times \mathbb{G}_m)/\Delta\mu_2 = (\mathrm{SL}_1(D_4) \times \mathbb{G}_m)/\{(z, z^{-2}) : z \in \mu_4\}$$

as algebraic groups over F, where $\mathbb{G}_m$ acts on W_3 by scaling. However, this description of $\mathrm{GSU}(W_3)$ is not so convenient, as it is not so easy to read off the group of F-points (since μ_4 is not cohomologically trivial). Instead, one verifies easily that the natural map

$$(\mathrm{SL}_1(D_4) \times \mathbb{G}_m)/\{(z, z^{-2}) : z \in \mu_4\} \longrightarrow (D_4^\times \times \mathbb{G}_m)/\{(z, z^{-2}) : z \in \mathbb{G}_m\}$$

is an isomorphism of algebraic groups. The description of $\mathrm{GSU}(W_3)$ on the RHS is what we shall use, because it is easy to describe the F-points of the RHS, since $H^1(F, \mathbb{G}_m) = 0$.

Finally, we note that the similitude character $\lambda_{W_3} : \mathrm{GU}(W_3) \longrightarrow \mathbb{G}_m$ is given by:

$$\lambda_{W_3}(g, t) = N(g) \cdot t^2$$

where N denotes the the reduced norm map on D_4.

2.4. **Dual pairs.** Now we come to the quaternionic dual pairs. If V_n is Hermitian and W_m is skew-Hermitian, then $V_n \otimes_D W_m$ has the natural structure of a symplectic vector space over F, with symplectic form given by:

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \frac{1}{2} \cdot Tr(\langle v_1, v_2 \rangle \cdot \overline{(w_1, w_2)}) \in F.$$

This gives a natural map

$$\mathrm{U}(V_n) \times \mathrm{U}(W_m) \longrightarrow \mathrm{Sp}(V_n \otimes W_m)$$

which realizes $(\mathrm{U}(V_n), \mathrm{U}(W_m))$ as a dual pair in $\mathrm{Sp}(V_n \times W_m)$. Similarly, one has a natural map

$$\mathrm{GU}(V_n) \times \mathrm{GU}(W_m) \longrightarrow \mathrm{GSp}(V_n \otimes W_m).$$

3. **Theta Correspondences for Similitudes**

In this section, we recall the Weil representation of $\mathrm{U}(V_n) \times \mathrm{U}(W_m)$ and its extension to the similitude group $\mathrm{GU}(V_n) \times \mathrm{GU}(W_m)$. In fact, for the purpose of this paper, we shall only consider the pair $\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)$ (the only exception is in §5.1). We fix a non-trivial additive character ψ of F.

Since V_{2n} is maximally split, we may fix a Witt decomposition

$$V_{2n} = X \oplus Y$$

where X and Y are maximal isotropic. Let

$$P(Y) = M(Y) \cdot N(Y)$$

be the maximal parabolic subgroup which stabilizes the space Y, so that

$$M(Y) = \mathrm{GL}(Y) \cong \mathrm{GL}_n(D) \quad \text{and} \quad N(Y) \cong \{\text{skew-hermitian forms on } X\} \subset Hom_D(X, Y).$$

Now the Weil representation $\omega_\psi^{2n,m}$ for $\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)$ can be realized on the space $S(X \otimes W_m)$ of Schwartz-Bruhat functions on $X \otimes W_m$. It is determined by the formulas (see [K]):

$$\begin{cases} \omega_\psi^{2n,m}(h)\phi(x) = \phi(h^{-1}x), \quad \text{for } h \in \mathrm{U}(W_m); \\ \omega_\psi(a)\phi(x) = |\det_Y(a)|^m \cdot \phi(a^{-1} \cdot x), \quad \text{for } a \in \mathrm{GL}(Y); \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x\rangle) \cdot \phi(x), \quad \text{for } b \in N(Y), \end{cases}$$

where $\langle -, - \rangle$ is the natural symplectic form on $V_{2n} \otimes W_m$. To describe the full action of $\mathrm{U}(V_{2n})$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $\mathrm{U}(V_{2n})$ (resp. $\mathrm{U}(W_m)$), the maximal π-isotypic quotient has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation of $\mathrm{U}(W_m)$ (resp. $\mathrm{U}(V_{2n})$). We call $\Theta_\psi(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible [MVW, chap. 4, IV, Théorème Principal (2), Pg. 69]. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$; we call it the small theta lift of π. Then it was a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is non-zero;
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger [W] when the residual characteristic p of F is not 2 and can be checked in many low-rank cases, regardless of the residual characteristic of F. Because we would like to include the case $p = 2$ in our discussion, we shall refrain from assuming Howe's conjecture in this paper. With this in mind, we take note of the following result [MVW, chap. 4, IV, Théorème Principal, Pg. 69] which holds for any residual characteristic p:

Proposition 3.1. *(i) If π is supercuspidal, $\Theta_\psi(\pi) = \theta_\psi(\pi)$ is irreducible or zero.*

(ii) If $\theta_\psi(\pi_1) = \theta_\psi(\pi_2) \neq 0$ for two supercuspidal representations π_1 and π_2, then $\pi_1 = \pi_2$.

One of the main purposes of this section is to extend this result to the case of similitude groups.

Let λ_V and λ_W be the similitude factors of $\mathrm{GU}(V_{2n})$ and $\mathrm{GU}(W_m)$ respectively. Note that both λ_V and λ_W are surjective onto $F^\times$ (since $\mathrm{disc}(W_m)$ is trivial). We shall consider the group

$$R = \mathrm{GU}(V_{2n}) \times \mathrm{GU}(W_m).$$

The group R contains the subgroup

$$R_0 = \{(g,h) \in R : \lambda_V(g) \cdot \lambda_W(h) = 1\},$$

which projects subjectively onto both $\mathrm{GU}(V_{2n})$ and $\mathrm{GU}(W_m)$. The Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(g,h)\phi = |\lambda_W(h)|^{-m}\omega(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{U}(V_{2n}).$$

Observe that the central elements $(t, t^{-1}) \in R_0$ act trivially.

Now consider the (compactly) induced representation

$$\Omega = ind_{R_0}^R \omega_\psi.$$

As a representation of R, Ω is independent of ψ. For any irreducible representation π of $\mathrm{GU}(V_{2n})$ (resp. $\mathrm{GU}(W_m)$), the maximal π-isotypic quotient of Ω has the form

$$\pi \otimes \Theta(\pi)$$

where $\Theta(\pi)$ is some smooth representation of $\mathrm{GU}(W_m)$ (resp. $\mathrm{GU}(V_{2n})$). Note that $\Theta(\pi)$ (if nonzero) has a central character which is equal to the central character of π. Further, we let $\theta(\pi)$ be the maximal semisimple quotient of $\Theta(\pi)$. The extended Howe conjecture for similitudes says that $\theta(\pi)$ is irreducible whenever $\Theta(\pi)$ is non-zero, and the map $\pi \mapsto \theta(\pi)$ is injective on its domain.

In [Ro], Roberts showed that the extended Howe conjecture for similitude groups would follow from the Howe conjecture for isometry groups (and thus would hold if $p \neq 2$) provided that restrictions of irreducible representations from similitude to isometry groups are known to be multiplicity-free. In the quaternionic case considered here, the multiplicity-freeness of restrictions may fail. However, we shall show below that Roberts' result actually holds without this multiplicity-freeness assumption; this will be very useful to know in practice. Note that though we are considering quaternionic dual pairs here, the argument applies to all dual pairs.

In the following, we shall use the following simple lemma on several occasions:

Lemma 3.2. *(i) If π is an irreducible representation of a similitude group $\mathrm{GU}(V)$, then the restriction of π to the isometry group $\mathrm{U}(V)$ is of the form $k \cdot \oplus_i \tau_i$ for some positive integer k and with τ_i irreducible representations of $\mathrm{U}(V)$.*

(ii) If π is an irreducible representation of a similitude group and $\sigma \subset \theta(\pi)$ is an irreducible summand, then $\pi \subset \theta(\sigma)$.

Now we have:

Proposition 3.3. *(i) If π is a representation of $\mathrm{GU}(V_{2n})$ (resp. $\mathrm{GU}(W_m)$) and the restriction of π to the relevant isometry group is $k \cdot \oplus_i \tau_i$, then as representations of $\mathrm{U}(W_m)$ (resp. $\mathrm{U}(V_{2n})$),*

$$\Theta(\pi) \cong k \cdot \bigoplus_i \Theta_\psi(\tau_i).$$

In particular, $\Theta(\pi)$ is of finite length and hence admissible. Moreover, $\Theta(\pi) = \theta(\pi)$ is semisimple if $\Theta_\psi(\tau_i) = \theta_\psi(\tau_i)$ is semisimple for all i.

(ii) Suppose that

$$\mathrm{Hom}_R(\Omega, \pi \boxtimes \pi') \neq 0.$$

Suppose further that for each constituent τ in the restriction of π to $\mathrm{U}(V_{2n})$, $\theta_\psi(\tau)$ is irreducible and the map $\tau \mapsto \theta_\psi(\tau)$ is injective on the set of irreducible constituents of $\pi|_{\mathrm{U}(V_{2n})}$. Then there is a uniquely determined bijection

$$\begin{array}{c} \{\text{isomorphism classes of irreducible summands of } \pi|_{\mathrm{U}(V_{2n})}\} \\ \Big\downarrow f \\ \{\text{isomorphism classes of irreducible summands of } \pi'|_{\mathrm{U}(W_m)}\} \end{array}$$

such that for any irreducible summand τ or τ' in the restriction of π or π' to the relevant isometry group,

$$\tau' = f(\tau) \Longleftrightarrow \mathrm{Hom}_{\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)}(\omega_\psi, \tau \boxtimes \tau') \neq 0.$$

One has the analogous statement with the roles of $\mathrm{U}(V_{2n})$ *and* $\mathrm{U}(W_m)$ *exchanged.*

(iii) Suppose that π *is an irreducible representation of a similitude group considered here and* τ *is a constituent of the restriction of* π *to the isometry group. Then* $\theta_\psi(\tau) \neq 0$ *if and only if* $\theta(\pi) \neq 0$.

(iv) Assume that the Howe conjecture holds for the dual pair $\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)$ *of isometry groups. Then the extended Howe conjecture holds for the similitude dual pair* $\mathrm{GU}(V_{2n}) \times \mathrm{GU}(W_m)$.

Proof. (i) Without loss of generality, suppose that π is a representation of $\mathrm{GU}(V_{2n})$. Then by Frobenius reciprocity,

$$\begin{aligned}
\Theta(\pi)^* &\cong \mathrm{Hom}_{\mathrm{GU}(V_{2n})}(\Omega, \pi) \\
&\cong \mathrm{Hom}_{\mathrm{GU}(V_{2n})}(\pi^\vee, \Omega^\vee) \\
&\cong \mathrm{Hom}_{\mathrm{GU}(V_{2n})}(\pi^\vee, \mathrm{Ind}_{\mathrm{U}(V_{2n})}^{\mathrm{GU}(V_{2n})} \omega_\psi^\vee) \\
&\cong \mathrm{Hom}_{\mathrm{U}(V_{2n})}(\pi^\vee|_{\mathrm{U}(V_{2n})}, \omega_\psi^\vee) \\
&\cong \mathrm{Hom}_{\mathrm{U}(V_{2n})}((\pi|_{\mathrm{U}(V_{2n})})^\vee, \omega_\psi^\vee) \\
&\cong \mathrm{Hom}_{\mathrm{U}(V_{2n})}(\omega_\psi, \pi|_{\mathrm{U}(V_{2n})}) \\
&\cong \mathrm{Hom}_{\mathrm{U}(V_{2n})}(\omega_\psi, k \cdot \bigoplus_i \tau_i) \\
&\cong k \cdot \bigoplus_i \Theta_\psi(\tau_i)^*.
\end{aligned}$$

Here, the first three isomorphisms are $\mathrm{GU}(W_m)$-equivariant and the remaining ones are $\mathrm{U}(W_m)$-equivariant. Moreover, in the 5th isomorphism, we have used the fact that since π has a central character and $Z_{\mathrm{GU}(V_{2n})} \cdot \mathrm{U}(V_{2n})$ is of finite index and hence open in $\mathrm{GU}(V_{2n})$, the space $(\pi|_{U(V_{2n})})^\vee$ of $\mathrm{U}(V_{2n})$-smooth vectors in π^* is the space $\pi^\vee$ of $\mathrm{GU}(V_{2n})$-smooth vectors in π^*.

Thus we have a $\mathrm{U}(W_m)$-equivariant isomorphism of $\mathrm{U}(W_m)$-smooth vectors:

$$\Theta(\pi)^\vee \cong k \cdot \bigoplus_i \Theta_\psi(\tau_i)^\vee.$$

Again, since $\Theta(\pi)$ has a central character and $Z_{\mathrm{GU}(W_m)} \cdot \mathrm{U}(W_m)$ is open in $\mathrm{GU}(W_m)$, the subspace of $\mathrm{U}(W_m)$-smooth vectors in $\Theta(\pi)^*$ is the same as the subspace of $\mathrm{GU}(W_m)$-smooth vectors. Finally, since the $\Theta_\psi(\tau_i)$'s are admissible, so is $\Theta(\pi)^\vee$ and the desired result follows by taking contragredient.

(ii) This is similar to the proof of [Ro, Lemma 4.2]. Suppose that

$$\pi|_{\mathrm{U}(V_{2n})} = k \cdot \bigoplus_i \tau_i \quad \text{and} \quad \pi'|_{\mathrm{U}(W_m)} = \ell \cdot \bigoplus_j \tau_j'.$$

Since $\mathrm{Hom}_R(\Omega, \pi \boxtimes \pi') \neq 0$, one sees by Frobenius reciprocity that

$$\mathrm{Hom}_{R_0}(\omega_\psi, \pi \boxtimes \pi') \neq 0.$$

Hence, there are two irreducible constituents, say τ_1 and τ_1', such that

$$\mathrm{Hom}_{\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)}(\omega_\psi, \tau_1 \boxtimes \tau_1') \neq 0.$$

Now recall that the group R_0 normalizes $\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)$, so that R_0 acts on the set of irreducible representations of the latter, and the Weil representation ω_ψ extends to R_0. If $r \in R_0$ and L is a non-zero element of $\mathrm{Hom}_{\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)}(\omega_\psi, \tau_1 \boxtimes \tau_1')$, then the map $v \mapsto L(r \cdot v)$ defines a non-zero element of $\mathrm{Hom}_{\mathrm{U}(V_{2n})\times \mathrm{U}(W_m)}(\omega_\psi, r(\tau_1 \boxtimes \tau_1'))$, where $r(\tau_1 \boxtimes \tau_1')$ denotes the representation of $\mathrm{U}(V_{2n}) \times \mathrm{U}(W_m)$ obtained from $\tau_1 \boxtimes \tau_1'$ under the action of $r \in R_0$.

Now the group R_0 acts transitively on the set of isomorphism classes of irreducible constituents of $\pi|_{\mathrm{U}(V_{2n})}$, as well as on the set of those of $\pi'|_{\mathrm{U}(W_m)}$, since the projections of R_0 to $\mathrm{GU}(V_{2n})$ and $\mathrm{GU}(W_m)$ are surjective. Thus, for each τ_i, there is a τ_i' such that

$$\mathrm{Hom}_{\mathrm{U}(V_{2n})\times \mathrm{U}(W_m)}(\omega_\psi, \tau_i \boxtimes \tau_i') \neq 0,$$

and vice versa. Moreover, by our assumptions, the equivalence classes of τ_i and τ_i' determine each other. This gives the desired bijection f.

(iii) This follows immediately from (i) and (ii).

(iv) We are assuming the Howe conjecture for isometry groups, so that in the context of (i), $\sigma_i := \theta_\psi(\tau_i)$ is irreducible for each i and $\theta_\psi(\sigma_i) = \tau_i$. By (ii), the maximal semisimple quotient of $\Theta(\pi)|_{\mathrm{U}(W_m)}$ is isomorphic to $k \cdot \bigoplus_i \theta_\psi(\tau_i)$. Since $\theta(\pi)|_{\mathrm{U}(W_m)}$ is a semisimple quotient of $\Theta(\pi)|_{\mathrm{U}(W_m)}$, we conclude that the isomorphism of (ii) induces

$$\theta(\pi)|_{\mathrm{U}(W_m)} \subset k \cdot \bigoplus_i \theta_\psi(\tau_i) = k \cdot \bigoplus_i \sigma_i.$$

Thus, if an irreducible representation Σ of $\mathrm{GU}(W_m)$ is a summand of $\theta(\pi)$, we have

$$\Sigma|_{\mathrm{U}(W_m)} = \ell \cdot \bigoplus_i \theta_\psi(\tau_i) = \ell \cdot \bigoplus_i \sigma_i \quad \text{with } \ell \leq k.$$

But we claim that the equality $k = \ell$ must hold. This is because the above argument applies with the roles of $\mathrm{U}(V_{2n})$ and $\mathrm{U}(W_m)$ exchanged, so that any irreducible summand π' of $\theta(\Sigma)$ satisfies

$$\pi'|_{\mathrm{U}(V_{2n})} = k' \cdot \bigoplus_i \theta_\psi(\sigma_i) = k' \cdot \bigoplus_i \tau_i \quad \text{with } k' \leq \ell.$$

But we could take $\pi' = \pi$ and deduce the reverse inequality $k \leq \ell$.

Thus, we deduce that $\theta(\pi)$ must in fact be irreducible and (iv) is proved. □

Of course, since we are not assuming Howe's conjecture for isometry group, we will have no occasion to use part (iv) of the above proposition. Instead, we have the following proposition which extends Proposition 3.1 to the situation of similitude groups; its proof is along the lines of that of Prop. 3.3(iv).

Proposition 3.4. *Suppose that π is a supercuspidal representation of $\mathrm{GU}(V_{2n})$ (resp. $\mathrm{GU}(W_m)$). Then we have:*

(i) $\Theta(\pi)$ is either zero or is an irreducible representation of $\mathrm{GU}(W_m)$ (resp. $\mathrm{GU}(V_{2n})$).

(ii) If π' is another superucspidal representation such that $\Theta(\pi') = \Theta(\pi) \neq 0$, then $\pi' = \pi$.

Proof. (i) Without loss of generality, we suppose that π is a representation of $\mathrm{GU}(V_{2n})$ such that

$$\pi|_{\mathrm{U}(V_{2n})} = k \cdot \bigoplus_i \tau_i.$$

By Prop. 3.1(i),

$$\sigma_i := \Theta_\psi(\tau_i) = \theta_\psi(\tau_i)$$

is irreducible. By Prop. 3.3(i), $\Theta(\pi) = \theta(\pi)$ is semisimple with

$$\Theta(\pi)|_{\mathrm{U}(W_m)} = k \cdot \bigoplus_i \sigma_i.$$

If $\Theta(\pi)$ is nonzero, we then need to show that it is irreducible.

Suppose that Σ is an irreducible summand of $\Theta(\pi)$. Then Prop. 3.3(i) implies that

$$\Sigma|_{\mathrm{U}(W_m)} = \ell \cdot \bigoplus_i \sigma_i \quad \text{with } \ell \leq k.$$

We will be done if we could show that $k = \ell$. Applying Prop. 3.3(i) to Σ, we also have

$$\Theta(\Sigma)|_{\mathrm{U}(V_{2n})} = \ell \cdot \bigoplus_i \Theta_\psi(\sigma_i).$$

Considering the maximal $\mathrm{U}(V_{2n})$-semisimple quotient, we see as in the proof of Prop. 3.3(iv) that

$$\theta(\Sigma)|_{\mathrm{U}(V_{2n})} \subset \ell \cdot \bigoplus_i \theta_\psi(\sigma_i).$$

Since $\pi \subset \theta(\Sigma)$, we see that

$$\pi|_{\mathrm{U}(V_{2n})} \subset \ell \cdot \bigoplus_i \theta_\psi(\sigma_i).$$

But since τ_i is supercuspidal, it follows by Prop. 3.1(i) that τ_i occurs with multiplicity one in $\theta_\psi(\sigma_i)$ and does not occur in $\theta_\psi(\sigma_j)$ for $j \neq i$. Indeed, if τ_i occurs with multiplicity > 1 in $\theta_\psi(\sigma_i)$, then σ_i would occur with multiplicity > 1 in $\theta_\psi(\tau_i) = \sigma_i$; similarly, if τ_i occurs in $\theta_\psi(\sigma_j)$ for $j \neq i$, then σ_j would occur in $\theta_\psi(\tau_i) = \sigma_i$. Thus, the multiplicity of τ_i on the RHS is precisely ℓ, whereas its multiplicity on the LHS is k. This implies the reverse inequality $k \leq \ell$ and proves (i).

(ii) Suppose that $\Sigma = \Theta(\pi) = \Theta(\pi') \neq 0$ and

$$\pi|_{\mathrm{U}(V_{2n})} = k \cdot \bigoplus_{i=1}^{a} \tau_i \quad \text{and} \quad \pi'|_{\mathrm{U}(V_{2n})} = k' \cdot \bigoplus_{i=1}^{a'} \tau_i'.$$

Then by Prop. 3.3(i), we see that

$$k \cdot \bigoplus_{i=1}^{a} \theta_\psi(\tau_i) = \Sigma|_{U(W_m)} = k' \cdot \bigoplus_{i=1}^{a'} \theta_\psi(\tau_i'),$$

so that after rearrangement of indices, we have $k = k'$, $a = a'$ and

$$\sigma_i := \theta_\psi(\tau_i) = \theta_\psi(\tau_i').$$

By Prop. 3.1(ii), we deduce that $\tau_i = \tau_i'$, so that

$$\pi|_{\mathrm{U}(V_{2n})} = \pi'|_{\mathrm{U}(V_{2n})}.$$

Now we can argue as in the proof of (i). Namely, applying Prop. 3.3(i) to Σ, we have

$$\Theta(\Sigma)|_{\mathrm{U}(V_{2n})} = k \cdot \bigoplus_i \Theta_\psi(\sigma_i),$$

and considering the maximal $\mathrm{U}(V_{2n})$-semisimple quotient, we see that

$$\theta(\Sigma)|_{\mathrm{U}(V_{2n})} \subset k \cdot \bigoplus_i \theta_\psi(\sigma_i).$$

If $\pi \neq \pi'$, then we would have

$$(\pi \oplus \pi')|_{\mathrm{U}(V_{2n})} \subset \theta(\Sigma)|_{\mathrm{U}(V_{2n})} \subset k \cdot \bigoplus_i \theta_\psi(\sigma_i),$$

so that τ_i occurs with multiplicity at least $2k$ on the RHS. However, as in (i), Prop. 3.1 implies that τ_i occurs with multiplicity precisely k on the RHS. Thus we have the desired contradiction and (ii) is proved. $\square$

4. **Conservation Relation**

In this section, we shall formulate a conjecture about the first occurrence index of a representation in a tower of theta liftings. For the symplectic/orthogonal and unitary dual pairs, such conjectures were first made by Kudla and Rallis and are usually referred to as *conservation relations.*

To formulate the conjecture, let us fix the quaternionic Hermitian space V_n and an irreducible representation π of $\mathrm{U}(V_n)$. As we mentioned in Section 2, there are two towers of quaternionic skew-Hermitian spaces with trivial discriminant:

$$W_{2m} = W_2^{\oplus m} \quad \text{and} \quad W_{2m+1} = W_3 \oplus W_2^{\oplus(m-1)},$$

which are characterized by the parity of their dimensions. We may consider the theta lift of π to $\mathrm{U}(W_m)$, denoting this theta lift by $\theta_\psi^{n,m}(\pi)$. Let

$$m^+(\pi) = \text{the smallest } 2m \text{ such that } \theta_\psi^{n,2m}(\pi) \neq 0$$

and

$$m^-(\pi) = \text{the smallest } 2m+1 \text{ such that } \theta_\psi^{n,2m+1}(\pi) \neq 0.$$

These are the first occurrence indices of π in the two towers. Here is the conjecture:

Conjecture 1: For each irreducible representation π of $\mathrm{U}(V_n)$,

$$m^+(\pi) + m^-(\pi) = 2n + 3.$$

By Prop. 3.3(iii), this conjecture applies to the case of similitude groups as well. In particular, we have the following consequence of Conjecture 1 if we specialize to the case $n = 2$: each irreducible representation of $\mathrm{GSp}_{1,1}$ must participate in theta correspondence with exactly one of $\mathrm{GO}^*_{1,1}$ or $\mathrm{GO}^*_{3,0}$. We shall prove this consequence in the following section.

One also has an analogous conjecture with the role of V_n and W_m reversed. More precisely, one may fix an irreducible representation π of $\mathrm{U}(W_m)$ and consider its theta lifts to the two towers of quaternionic Hermitian spaces:

$$V_{2n} = V_2^{\oplus n} \quad \text{and} \quad V_{2n+1} = V_1 \oplus V_2^{\oplus n}.$$

As before, let $n^+(\pi)$ and $n^-(\pi)$ be the first occurrence indices of π in the two respective towers. Then one has:

Conjecture 2: For each irreducible representation π of $\mathrm{U}(W_m)$,

$$n^+(\pi) + n^-(\pi) = 2m + 1.$$

Specializing to the case $m = 2$, we see that as a consequence of Conjecture 2, each irreducible representation of $\mathrm{GO}^*_{1,1}$ must participate in theta correspondence with exactly one of $\mathrm{GSp}_{1,0}$ or $\mathrm{GSp}_{1,1}$. We shall prove this in the next section.

We remark that since this paper was first written, Conjectures 1 and 2 have been proven. Firstly, in a paper of Minguez [M] , Conjectures 1 and 2 were proved for supercuspidal π. Then in a recent paper of Sun-Zhu [SZ], they were proved in general. However, since these papers refer to the conjectures made here, we have allowed this paper to stand as it was.

5. **Theta Lifting and Theta Dichotomy**

In this section, we shall determine the theta correspondence in some low rank cases, and verify the consequences of Conjectures 1 and 2 highlighted in the previous section. In particular, we shall examine the dual pairs

$$\begin{cases} \mathrm{GU}(V_1)\times\mathrm{GU}(W_2)=\mathrm{GSp}_{1,0}\times\mathrm{GO}^*_{1,1};\\ \mathrm{GU}(V_2)\times\mathrm{GU}(W_2)=\mathrm{GSp}_{1,1}\times\mathrm{GO}^*_{1,1};\\ \mathrm{GU}(V_2)\times\mathrm{GU}(W_3)=\mathrm{GSp}_{1,1}\times\mathrm{GO}^*_{3,0}\,. \end{cases}$$

5.1. **Theta lifts from** $\mathrm{GSp}_{1,0}$ **to** $\mathrm{GO}^*_{1,1}$**.** We first consider the dual pair $\mathrm{GU}(V_1)\times\mathrm{GU}(W_2)$. Note that

$$\mathrm{GU}(V_1)\cong D^\times$$

and

$$\mathrm{GU}(W_2)\cong(D^\times\times\mathrm{GL}_2(F))/\{(t,t^{-1}):t\in F^\times\}.$$

Now we have the following proposition:

Proposition 5.1. *Let ρ be an irreducible representation of $\mathrm{GU}(V_1)\cong D^\times$. Then*

$$\Theta(\rho)\cong\rho\boxtimes JL(\rho)$$

where $JL(\rho)$ is the Jacquet-Langlands lift of ρ to $\mathrm{GL}_2(F)$.

Proof. We shall prove this by global means. More precisely, let $\mathbb{F}$ be a number field and $\mathbb{D}$ a quaternion $\mathbb{F}$-algebra such that for some place v_0 of $\mathbb{F}$, we have

$$\mathbb{F}_{v_0}\cong F\quad\text{and}\quad\mathbb{D}_{v_0}\cong D.$$

Let $\mathbb{V}_1$ denote the rank one Hermitian space over $\mathbb{D}$ with hermitian form

$$\langle x,y\rangle=x\cdot\overline{y}$$

and let $\mathbb{W}_2$ denote the split rank 2 skew-Hermitian space over $\mathbb{D}$, so that

$$\mathbb{V}_{1,v_0}\cong V_1\quad\text{and}\quad\mathbb{W}_{2,v_0}\cong W_2.$$

Then one has the dual pair $\mathrm{GU}(\mathbb{V}_1)\times\mathrm{GU}(\mathbb{W}_2)$ over $\mathbb{F}$ and one may consider the global theta lift from

$$\mathrm{GU}(\mathbb{V}_1)\cong\mathbb{D}^\times$$

to

$$\mathrm{GU}(\mathbb{W}_2)\cong(\mathbb{D}^\times\times\mathrm{GL}_2(\mathbb{F}))/\{(t,t^{-1}):t\in\mathbb{F}^\times\}.$$

Let Σ be a cuspidal infinite-dimensional representation of $\mathrm{GU}(\mathbb{V}_1)$ whose local component at v_0 is ρ, and consider the theta lift $\Theta(\Sigma)$ of Σ to $\mathrm{GU}(\mathbb{W}_2)$. This global theta lift is nonzero cuspidal because we are in the so-called stable range. Moreover, at the places where $\mathbb{D}$ is unramified, one knows that the local theta lift of Σ_v is the representation $\Sigma_v\boxtimes\Sigma_v$ of $\mathrm{GU}(W_2)(\mathbb{F}_v)$. By the strong multiplicity one theorem for $\mathbb{D}^\times$ and GL_2, we conclude that

$$\Theta(\Sigma)\cong\Sigma\boxtimes JL(\Sigma).$$

By the local-global compatibility of theta correspondence (see [G, §2.11 and Prop. 2.12]), we have: $\Theta(\Sigma)_v = \theta_v(\Sigma_v)$ for each place v. Hence, we obtain the proposition by extracting the component at v_0. □

Corollary 5.2. *The only irreducible representations of* $\mathrm{GU}(W_2)$ *which participate in theta correspondence with* $\mathrm{GU}(V_1)$ *are precisely those of the form* $\rho \boxtimes JL(\rho)$*, where* ρ *is an irreducible representation of* $D^\times$*.*

5.2. **Principal Series Representations of** $\mathrm{GU}(W_2)$**.** Before coming to the dual pair $\mathrm{GU}(W_2) \times \mathrm{GU}(V_2)$, we need to establish some notations for the principal series representations of these two groups. Recall that $W_2 = Df_1 \oplus Df_2$ and we have an isomorphism

$$\mathrm{GU}(W_2) \cong (D^\times \times \mathrm{GL}_2(F))/\{(t, t^{-1}) : t \in F^\times\}.$$

We shall need to describe this isomorphism more explicitly. Let $Q = LU$ be the parabolic subgroup which stabilizes the isotropic line Df_1, so that

$$L = \mathrm{GL}(Df_1) \times \mathrm{GL}_1(F) = D^\times \times F^\times,$$

where $\lambda \in \mathrm{GL}_1(F)$ acts by

$$\begin{cases} f_1 \mapsto f_1 \\ f_2 \mapsto \lambda \cdot f_2. \end{cases}$$

Hence the projection onto $\mathrm{GL}_1(F)$ gives the similitude character on L. We fix the above isomorphism so that it sends the parabolic Q to the corresponding parabolic subgroup determined by the upper triangular Borel subgroup of GL_2. In particular, its restriction to L has the form:

$$(a, t) \mapsto \left(a, \begin{pmatrix} 1 & \\ & t \cdot N(a)^{-1} \end{pmatrix}\right),$$

for $(a, t) \in L$. The reverse isomorphism is such that

$$\left(a, \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right) \mapsto (at_1, N(a) \cdot t_1 t_2).$$

We can now describe the principal series representations of $\mathrm{GU}(W_2)$. For a representation $\rho \boxtimes \chi$ of $D^\times \times F^\times$, we let $I_Q(\rho, \chi)$ denote the normalized parabolically induced representation of $\mathrm{GU}(W_2)$. Under the above isomorphism, this principal series representation corresponds to the representation

$$(\rho \otimes \chi) \boxtimes \pi(\chi\omega_\rho, \chi) \quad \text{of } D^\times \times \mathrm{GL}_2(F),$$

where $\pi(\chi_1, \chi_2)$ stands for the principal series representation for $\mathrm{GL}_2(F)$ parabolically induced from the character $\chi_1 \boxtimes \chi_2$ of the diagonal torus. From this, we see that $I_Q(\rho, \chi)$ is irreducible unless

$$\omega_\rho = |-|^{\pm 1}.$$

5.3. **Principal Series Representations of** $\mathrm{GU}(V_2)$**.** We now consider the principal series representation of $\mathrm{GU}(V_2)$ associated to the parabolic subgroup $P = P(Y) = M(Y) \cdot N(Y)$, with

$$M(Y) = \mathrm{GL}(Y) \times \mathrm{GL}_1(F) \cong D^\times \times F^\times.$$

For a representation $\rho \boxtimes \chi$ of $M(Y)$, we let $I_P(\rho, \chi)$ denote the normalized parabolically induced representation. The reducibility points of this family of induced representations are known (cf. [H] and [Y1]):

Proposition 5.3. *(i) If* $\dim \rho > 1$*, then* $I_P(\rho, \chi)$ *reduces iff* $\rho = \rho_0|-|^{\pm 1/2}$ *where* ρ_0 *has trivial central character. Moreover, one has the short exact sequence:*

$$0 \longrightarrow St(\rho_0, \chi_0) \longrightarrow I_P(\rho_0|-|^{1/2}, \chi_0|-|^{-1/2}) \longrightarrow Sp(\rho_0, \chi_0) \longrightarrow 0,$$

where $St(\rho_0, \chi_0)$ *is a discrete series representation and* $Sp(\rho_0, \chi_0)$ *is a non-tempered Langlands quotient.*

(ii) If $\dim \rho = 1$*, then* $I_P(\rho, \chi)$ *reduces iff one of the following holds:*

(a) $\rho = \rho_0|-|^{\pm 1/2}$ *with* ρ_0 *a non-trivial quadratic character, in which case*

$$0 \longrightarrow St(\rho_0, \chi_0) \longrightarrow I_P(\rho_0|-|^{1/2}, \chi_0|-|^{-1/2}) \longrightarrow Sp(\rho_0, \chi_0) \longrightarrow 0,$$

where $St(\rho_0, \chi_0)$ *is a discrete series representation and* $Sp(\rho_0, \chi_0)$ *is a non-tempered Langlands quotient.*

(b) $\rho = |-|^{\pm 3/2}$*, in which case*

$$0 \longrightarrow St_{\mathrm{GU}(V_2)} \otimes \chi \longrightarrow I_P(|-|^{3/2}, \chi \cdot |-|^{-3/2}) \longrightarrow \chi \longrightarrow 0,$$

where $St_{\mathrm{GU}(V_2)}$ *is the Steinberg representation of* $\mathrm{GU}(V_2)$*.*

5.4. **Theta Lifts from $\mathrm{GO}^*_{1,1}$ to $\mathrm{GSp}_{1,1}$.** Now we are ready to compute the theta lift from $\mathrm{GU}(W_2)$ to $\mathrm{GU}(V_2)$. The result is:

Proposition 5.4. *(i) If* $\pi = J_Q(\rho, \chi)$ *is the Langlands quotient of* $I_Q(\rho, \chi) = (\rho \otimes \chi) \boxtimes \pi(\chi\omega_\rho, \chi)$*, then*

$$\theta(\pi) = J_P(\rho, \chi).$$

In fact, $\Theta(\pi) = \theta(\pi)$ *except when* $\rho = |-|^{3/2}$*, in which case* $\Theta(\pi) = I_P(|-|^{3/2}, \chi)$ *and* $\theta(\pi)$ *is its unique irreducible quotient* $\chi|-|^{3/2}$*.*

(ii) If $\pi = \rho \boxtimes JL(\rho)$*, then*

$$\Theta(\pi) = 0.$$

(iii) If $\pi = \rho \boxtimes st_\chi$ *with* $\rho \neq \chi$*, then*

$$\Theta(\pi) = \theta(\pi) = St(\rho\chi^{-1}, \chi).$$

(iv) If $\pi = \rho \boxtimes \tau$ *for* τ *a supercuspidal representation* $\neq JL(\rho)$*, then* $\Theta(\pi)$ *is a nonzero, irreducible supercuspidal representation.*

Proof. The proof of this proposition is given in the appendix. □

Corollary 5.5. *Each irreducible representation of* $\mathrm{GO}^*_{1,1}$ *participates in the theta correspondence with exactly one of* $\mathrm{GSp}_{1,0}$ *or* $\mathrm{GSp}_{1,1}$*.*

Corollary 5.6. *Let* π *be a non-supercuspidal representation of* $\mathrm{GSp}_{1,1}$ *such that* π *is not a twisted Steinberg representation. Then* π *has a non-zero theta lift to* $\mathrm{GO}^*_{1,1}$*. If* π *is a twisted Steinberg representation, then its theta lift to* $\mathrm{GO}^*_{1,1}$ *is zero.*

5.5. **Theta Lifts from** $\mathrm{GSp}_{1,1}$ **to** $\mathrm{GO}^*_{3,0}$**.** We now come to the dual pair $\mathrm{GU}(V_2) \times \mathrm{GU}(W_3)$. The following proposition describes the theta lifts of non-supercuspidal representations of $\mathrm{GU}(V_2)$.

Proposition 5.7. *Let π be a non-supercuspidal representation of $\mathrm{GSp}_{1,1}$ which is not a twisted Steinberg representation. Then the theta lift of π to $\mathrm{GO}^*_{3,0}$ is zero. If $\pi = St_\chi$ is a twisted Steinberg representation, then*

$$\theta(St_\chi) = \chi \circ \lambda_{W_3}.$$

Proof. The proof of this proposition is given in the appendix. □

5.6. **Dichotomy.** Here is the main result of this section, which confirms the consequence of Conjecture 1 highlighted in the previous section. Of course, it follows from the recent result of [SZ] and [M], but we shall give a sketch of a direct proof.

Theorem 5.8. *Let π be an irreducible representation of $\mathrm{GU}(V_2) = \mathrm{GSp}_{1,1}$. Then π participates in the theta correspondence with exactly one of $\mathrm{GU}(W_2) = \mathrm{GO}^*_{1,1}$ or $\mathrm{GU}(W_3) = \mathrm{GO}^*_{3,0}$.*

Proof. Cor. 5.6 and Prop. 5.7 together imply the theorem for non-supercuspidal representations. Hence, we assume that π is supercuspidal in the rest of the proof.

Let τ be an irreducible constituent of $\pi|_{\mathrm{U}(V_2)}$. One knows that $\Theta^{2,r}(\pi) \neq 0$ iff $\Theta^{2,r}_\psi(\tau) \neq 0$. To see if the latter holds, we consider the doubling method (cf. [LR] and [Y2]) associated to the situation

$$V_2 \oplus (-V_2) \cong V_4.$$

Let X be the maximal isotropic subspace ΔV_2 of V_4 and let P denote the parabolic subgroup of $\mathrm{U}(V_4)$ stabilizing X. Then one has the normalized degenerate principal series representation $I_P(s)$ of $\mathrm{U}(V_4)$ associated to the character $|\det_X|^s$ of P.

By the local seesaw identity of the doubling method, one sees that the nonvanishing of $\Theta^{2,r}_\psi(\tau)$ is in turn equivalent to

$$\mathrm{Hom}_{\mathrm{U}(V_2)\times\mathrm{U}(V_2)}(\Theta^{r,4}_\psi(1), \tau \boxtimes \tau^\vee) \neq 0,$$

where $\Theta^{r,4}_\psi(1)$ denotes the big theta lift of the trivial representation. We are interested in the cases $r = 2$ and $r = 3$. In these cases, the results of Yamana [Y1, Thm. 1.4] shows that

$$0 \longrightarrow \Theta^{3,4}_\psi(1) \longrightarrow I_P(1/2) \longrightarrow \Theta^{2,4}_\psi(1) \longrightarrow 0$$

with both $\Theta^{2,4}_\psi(1)$ and $\Theta^{3,4}_\psi(1)$ irreducible. Now, by a Mackey-type argument as given in [KR, §1], one knows that

$$\dim \mathrm{Hom}_{\mathrm{U}(V_2)\times\mathrm{U}(V_2)}(I_P(s), \tau \boxtimes \tau^\vee) = 1$$

for any $s \in \mathbb{C}$. Since τ is supercuspidal, we have

$$1 = \dim \mathrm{Hom}_{\mathrm{U}(V_2)\times\mathrm{U}(V_2)}(\Theta^{2,4}_\psi(1), \tau \boxtimes \tau^\vee) + \dim \mathrm{Hom}_{\mathrm{U}(V_2)\times\mathrm{U}(V_2)}(\Theta^{3,4}_\psi(1), \tau \boxtimes \tau^\vee).$$

Hence, we conclude that

$$\dim \mathrm{Hom}_{\mathrm{U}(V_2)\times\mathrm{U}(V_2)}(\Theta^{r,4}_\psi(1), \tau \boxtimes \tau^\vee) = 1$$

for $r = 2$ or 3 but not both. This proves the theorem. □

6. **Doubling Zeta Integrals and Gamma Factors**

In this section, we consider the question of characterizing those representations of $\mathrm{GO}^*_{3,0}$ which participate in the theta correspondence with $\mathrm{GSp}_{1,1}$. The main tool used will be the theory of doubling zeta integrals of Piatetski-Shapiro and Rallis, as refined by Lapid-Rallis [LR] and Yamana [Y2].

6.1. **L-parameters for $\mathrm{GO}^*_{3,0}$.** Recall that

$$\mathrm{GO}^*_{3,0} \cong (D_4^\times \times F^\times)/\{(t, t^{-2}) : t \in F^\times\},$$

and so a representation of $\mathrm{GO}^*_{3,0}$ is of the form $\Pi \boxtimes \mu$ where Π is a representation of $D_4^\times$ and μ is a character of $F^\times$ such that $\mu^2 = \omega_\Pi$. By the local Jacquet-Langlands correspondence due to Deligne-Kazhdan-Vigneras [DKV], one has a bijection

$$\{\text{irreducible representations of } D_4^\times\} \longleftrightarrow \{\text{irreducible discrete series representations of } \mathrm{GL}_4(F)\}.$$

Thus, one can attach an L-parameter to Π by using the Jacquet-Langlands correspondence and the local Langlands correspondence for $\mathrm{GL}_4(F)$. Namely, if the Jacquet-Langlands transfer of Π to $\mathrm{GL}_4(F)$ is the discrete series representation $JL(\Pi)$, we set

$$\phi_\Pi := \phi_{JL(\Pi)} : WD_F \longrightarrow \mathrm{GL}_4(\mathbb{C}).$$

This is the local Lanlgands correspondence for $D_4^\times$. Given an irreducible representation $\Pi \boxtimes \mu$ of $\mathrm{GO}^*_{3,0}$, we can then associate an L-parameter $(\phi_\Pi, \mu) \in \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$.

For example, if $\Pi \boxtimes \mu = (\chi \circ N_{D_4}) \boxtimes \chi^2 = \chi \circ \lambda_{W_3}$ is a character of $\mathrm{GU}(W_3)$, then its Jacquet-Langlands lift is the twisted Steinberg representation $(St \otimes \chi) \boxtimes \chi^2$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$, and so its L-parameter is $(\chi \otimes S_4, \chi^2)$, where S_4 is the 4-dimensional representation of $\mathrm{SL}_2(C)$.

6.2. **Main result.** Now the main result of this section is:

Theorem 6.1. *Let $\Pi \boxtimes \mu$ be an irreducible representation of $\mathrm{GO}^*_{3,0}$. Then the following are equivalent:*

(a) $\Pi \boxtimes \mu$ participates in theta correspondence with $\mathrm{GSp}_{1,1}$;

(b) $\gamma(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1}, \psi)$ has a pole at $s = 1$;

(c) the L-parameter ϕ_Π factors through $\mathrm{GSp}_4(\mathbb{C})$ with similitude character μ.

The rest of this section is devoted to the proof of the theorem. In [GT], the analogous result was supplied by a theorem of Muić-Savin [MS] which was shown using Shahidi's theory of L-functions. For the case of inner forms, Shahidi's theory is not available. The main tool used in the proof of Theorem 6.1 is the local zeta integral arising from the doubling method, which can be used to define the standard gamma factors for representations of all classical groups. This was due to Piatetski-Shapiro and Rallis [PSR] but the definitive account is the relatively recent paper [LR] of Lapid-Rallis. Unfortunately, [LR] only treats the symplectic, orthogonal and unitary cases and does not treat the quaternionic Hermitian or skew-Hermitian cases. Thankfully, the recent paper [Y2] of Yamana gives a complete treatment including the quaternionic cases we need here.

6.3. **The local zeta integral.** We first recall the definition of the local zeta integral. Consider the quaternionic skew-Hermitian space

$$W_{2n} \cong W_n \oplus (-W_n)$$

and the corresponding embedding

$$U(W_n) \times U(W_n) \hookrightarrow U(W_{2n}).$$

The subspace $Y = \Delta W_n \subset W_{2n}$ is maximal isotropic and so determines a maximal parabolic subgroup $P = P(Y)$. Let $I_P(s)$ be the normalized degenerate principal series representation associated to the character $|\det_Y|^s$ of $P(Y)$. Suppose that π is an irreducible representation of $U(W_n)$ with contragredient $\pi^\vee$. Given a standard section F_s of $I_P(s)$ and vectors $f \in \pi$ and $f^\vee \in \pi^\vee$, the local zeta integral is defined by

$$Z(s, F, f, f^\vee) = \int_{U(W_n)} F_s(h, 1) \cdot \langle \pi(h) f, f^\vee \rangle \, dh,$$

for a fixed Haar measure on $U(W_n)$. It converges when $Re(s)$ is sufficiently large and admits a meromorphic continuation to all of $\mathbb{C}$. If π is supercuspidal, then the integral converges for all $s \in \mathbb{C}$ and hence defines an entire function.

6.4. **Local functional equation.** The function $Z(s, F, f, f^\vee)$ satisfies a functional equation which can be described as follows. There is a standard intertwining operator

$$M(s) : I_P(s) \longrightarrow I_P(-s)$$

defined by the usual integral formula which extends to a meromorphic function in s. One normalizes this operator as in [Y2, Appendix B] (using a local Fourier coefficient) and denotes this normalized operator by $M_\psi^\dagger(s)$. Observe that this normalized operator depends on the choice of a non-trivial additive character ψ of F. In any case, the local functional equation is:

$$Z(-s, M_\psi^\dagger(s) F, f, f^\vee) = \gamma^{PSR}(s + \frac{1}{2}, \pi, \psi) \cdot \pi(-1) \cdot Z(s, F, f, f^\vee),$$

where the constant of proportionality $\gamma^{PSR}(s, \pi, \psi)$ is defined to be the standard γ-factor of π of Piatetski-Shapiro and Rallis.

6.5. **The standard gamma factor.** Henceforth, we specialize to the case of interest $n = 3$. In this case, π is an irreducible constituent of the restriction of a representation $\Pi \boxtimes \mu$ of $D_4^\times \times F^\times$ to $\mathrm{O}_{3,0}^*$. Observe that since $\mathrm{O}_{3,0}^*$ is compact, the zeta integral is entire. Now we have the following crucial result:

Proposition 6.2. *If $\pi \subset (\Pi \boxtimes \mu)|_{\mathrm{O}_{3,0}^*}$, then*

$$\gamma^{PSR}(s, \pi, \psi) = \alpha \cdot \gamma(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1}, \psi)$$

for some root of unity α. In particular, $\gamma^{PSR}(s, \pi, \psi)$ has a pole of order at most 1 at $s = 1$.

Proof. The proof of the proposition is global. We first find a number field $\mathbb{F}$ which has 2 places v_1 and v_2 such that $\mathbb{F}_{v_i} = F$. Now let $\mathbb{D}$ be the quaternion division algebra which is ramified precisely at v_1 and v_2. Also, let $\mathbb{D}_4$ be the degree 4 division algebra over $\mathbb{F}$ whose invariants at v_1 and v_2 are $1/4$ and $-1/4$ respectively, and which is split for all $v \neq v_i$. Then the algebraic group

$$\mathrm{GO}_{3,0}^* = (\mathbb{D}_4^\times \times \mathbb{G}_m)/\{(t, t^{-2}) : t \in \mathbb{G}_m\}$$

is the similitude group of a rank 3 skew-Hermitian vector space over $\mathbb{D}$.

Given the representation $\Pi \boxtimes \mu$ of $\mathrm{GO}_{3,0}^*(F)$, we find a cuspidal representation $\Sigma \boxtimes \Xi$ of $\mathrm{GO}_{3,0}^*(\mathbb{A})$ such that its local components at v_1 and v_2 are both equal to $\Pi \ \boxtimes \mu$. We can also ensure that the global Jacquet-Langlands lift [Ba] of $\Sigma \boxtimes \Xi$ to $\mathrm{GL}_4 \times \mathrm{GL}_1$ is a cuspidal representation $JL(\Sigma) \boxtimes \Xi$ (for example, by requiring that Σ_v is supercuspidal for some $v \neq v_i$). In particular, this implies that the local components of Σ at $v \neq v_i$ are all generic.

Now by results of Shahidi, we have the global functional equation for the twisted exterior square L-function of $JL(\Sigma) \boxtimes \Xi$:

$$L^S(s, \Sigma, \bigwedge^2 \otimes \Xi^{-1}) = \prod_{v \in S} \gamma_v^{Sh}(s, \Sigma_v, \bigwedge^2 \otimes \Xi_v^{-1}, \psi_v) \cdot L^S(1-s, JL(\Sigma)^\vee, \bigwedge^2 \otimes \Xi),$$

where the γ-factors which intervene are those of Shahidi's. On the other hand, we may take an irreducible constituent Δ of $\Sigma \boxtimes \Xi|_{\mathrm{O}_{3,0}^*}$ whose local component at v_i is π and obtain the global functional equation for the standard L-function of Δ provided by the doubling method:

$$L^S(s, \Delta, std) = \prod_{v \in S} \gamma_v^{PSR}(s, \Delta_v, \psi_v) \cdot L^S(1-s, \Delta^\vee, std).$$

But one knows by [LR, Thm. 4(3) and §10] and [Y2, Prop. 5.3] that

$$L^S(s, \Sigma, \bigwedge^2 \otimes \Xi^{-1}) = L^S(s, \Delta, std) \quad \text{and} \quad L^S(1-s, JL(\Sigma)^\vee, \bigwedge^2 \otimes \Xi) = L^S(1-s, \Delta^\vee, std).$$

Moreover, if $v \in S$ but $v \neq v_i$, then the group $\mathrm{GO}_{3,0}^*(F_v)$ is split and it follows by [LR, Thm. 4 and Pg. 351] and [Sh1, Thm. 3.5] that

$$\gamma_v^{Sh}(s, \Sigma_v, \bigwedge^2 \otimes \Xi_v^{-1}, \psi_v) = \gamma_v^{PSR}(s, \Delta_v, \psi_v).$$

Taking all these into account, we deduce an identity of the two types of γ-factors at the places v_1 and v_2, and thus

$$\gamma^{Sh}(s, \Pi, \bigwedge^2 \otimes \mu^{-1}, \psi)^2 = \gamma^{PSR}(s, \pi, \psi)^2.$$

By a result of Henniart [He2], the local Langlands correspondence for GL_n respects the (Galois theoretic and Shahidi's versions of) twisted exterior square γ-factors up to a root of unity. Hence, we deduce that there is a root of unity α such that

$$\alpha \cdot \gamma(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1}, \psi) = \gamma^{PSR}(s, \pi, \psi).$$

The proposition is proved. □

6.6. **Degenerate principal series $I_P(1/2)$ on $\mathrm{O}_{3,3}^*$.** Before coming to the proof of the theorem, we need another ingredient: the structure of the degenerate principal series $I_P(1/2)$. This is a very special case of a general result of S. Yamana [Y1], who determines the structure of such degenerate principal series representations on $\mathrm{O}_{n,n}^*$. It was shown in [Y1] that $I_P(1/2)$ is non-semisimple and has length 2. Moreover, its irreducible subquotients can be related to theta correspondence as follows. Consider the dual pairs

$$\mathrm{Sp}_{1,1} \times \mathrm{O}_{3,3}^* \quad \text{and} \quad \mathrm{Sp}_{2,1} \times \mathrm{O}_{3,3}^*$$

and in particular, the theta lifts of the trivial representations of $\mathrm{Sp}_{1,1}$ and $\mathrm{Sp}_{2,1}$ to $\mathrm{O}_{3,3}^*$. By [Y1, Thm. 1.3], we have

Lemma 6.3. *The representation $I_P(1/2)$ has length 2 and sits in a short exact sequence*

$$0 \longrightarrow \Theta_\psi^{3,6}(1) \longrightarrow I_P(1/2) \longrightarrow \Theta_\psi^{2,6}(1) \longrightarrow 0.$$

6.7. **The Normalized Intertwining Operator.** We also need to understand the behaviour of the normalized intertwining operator

$$M_\psi^\dagger(s) : I_P(s) \longrightarrow I_P(-1/2)$$

at $s = 1/2$. The following follows from [Y1, Cor. 4.5] and [Y2, Appendix B].

Lemma 6.4. *The normalized intertwining operator $M_\psi^\dagger(s)$ has a pole of order* 1 *at $s = 1/2$.*

6.8. **Proof of Theorem 6.1.** We are finally ready to prove Theorem 6.1. Let $\Pi \boxtimes \mu$ be the given representation of $\mathrm{GO}^*_{3,0}$. Since ϕ_Π is an irreducible 4-dimensional representation of the Weil-Deligne group WD_F, it is easy to see that (b) is equivalent to the L-function $L(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1})$ having a pole at $s = 0$. This is in turn equivalent to (c), thus showing the equivalence of (b) and (c).

Now suppose that (a) holds. Let π be an irreducible constituent of the restriction of $\Pi \boxtimes \mu$ to $\mathrm{O}^*_{3,0}$. Then

$$\mathrm{Hom}_{\mathrm{O}^*_{3,0} \times \mathrm{O}^*_{3,0}}(\Theta^{2,6}_\psi(1), \pi \boxtimes \pi^\vee) \neq 0.$$

Since $\Theta^{2,6}_\psi(1)$ is the irreducible submodule of $I_P(-1/2)$, one can find a vector F in this submodule such that

$$Z(-1/2, F, f, f^\vee) \neq 0 \quad \text{for some } f \in \pi \text{ and } f^\vee \in \pi^\vee.$$

By Lemma 6.4, if we set

$$M^*_\psi(s) = (s - \frac{1}{2}) \cdot M^\dagger_\psi(s)$$

then $M^*_\psi(s)$ is holomorphic at $s = 1/2$ and $M^*_\psi(1/2)$ is a non-zero operator with image equal to the submodule $\Theta^{2,6}_\psi(1)$ of $I_P(-1/2)$. We can thus find an element $\tilde{F} \in I_P(1/2)$ such that $M^*(1/2)\tilde{F} = F$.

Now let us examine the local functional equation of the zeta integral:

$$(6.5) \qquad (s - \frac{1}{2})^{-1} \cdot Z(-s, M^*(s)\tilde{F}, f, f^\vee) = \gamma^{PSR}(s + 1/2, \pi, \psi) \cdot \pi(-1) \cdot Z(s, \tilde{F}, f, f^\vee).$$

By our choice of $\tilde{F}$, f and $f^\vee$, we see that the LHS of the functional equation has a pole of order 1 at $s = 1/2$. Hence, so must the RHS. Since the local zeta integral is entire in this case, we conclude that $\gamma^{PSR}(s + 1/2, \pi, \psi)$ must have a pole at $s = 1/2$, as desired.

Conversely, suppose that $\Pi \boxtimes \mu$ satisfies (b). By Prop. 6.2, $\gamma^{PSR}(s, \pi, \psi)$ has a pole at $s = 1$. Then we need to show that

$$Z(-1/2, M^*(1/2)\tilde{F}, f, f^\vee) \neq 0 \qquad \text{for some } \tilde{F},\ f \text{ and } f^\vee.$$

But if not, then the LHS of the functional equation (6.5) is holomorphic at $s = 1/2$ for any choice of data. However, the RHS has a pole of order 1 for some choice of data. With this contradiction, the theorem is proved.

Remark: We remark that the technique of the proof of Theorem 6.1 has been used in the paper [GI] of the first author with A. Ichino to prove a theorem relating local theta correspondence and poles of standard gamma factors (arising from the doubling method) for general dual pairs, which significantly extends the results of [MS]. See [GI, Prop. 11.2 and Thm. 11.5].

7. Definition of the Map L

We are now ready to define the map

$$L : \Pi(\mathrm{GSp}_{1,1}) \longrightarrow \Phi(\mathrm{GSp}_{1,1})$$

of our main theorem and verify some of its properties.

7.1. **Definition of L.** Let π be an irreducible representation of $\mathrm{GSp}_{1,1}$. By Thm. 5.8, π participates in theta correspondence with exactly one of $\mathrm{GO}^*_{1,1}$ or $\mathrm{GO}^*_{3,0}$. Thus we have the following two cases:

Case 1: suppose that π participates in theta correspondence with $\mathrm{GO}^*_{1,1}$, say

$$\theta(\pi) = \rho \boxtimes \tau,$$

with $\omega_\pi = \omega_\rho = \omega_\tau$. By Prop. 5.4, we have $JL(\rho) \neq \tau$. If $\phi_{JL(\rho)}$ and ϕ_τ are the L-parameters of $JL(\rho)$ and τ respectively, we set

$$L(\pi) = \phi_{JL(\rho)} \oplus \phi_\tau.$$

This factors through $\mathrm{GSp}_4(\mathbb{C})$ with similitude character $\omega_\rho = \omega_\tau$. Moreover, $L(\pi)$ is relevant for $\mathrm{GSp}_{1,1}$ because $JL(\rho) \neq \tau$.

Case 2: suppose that π participates in theta correspondence with $\mathrm{GO}^*_{3,0}$, say

$$\theta(\pi) = \Pi \boxtimes \mu.$$

Then by Thm. 6.1, the L-parameter of $JL(\Pi)$ factors through $\mathrm{GSp}_4(\mathbb{C})$ with similitude character $\mu = \omega_\pi$. Thus, we set

$$L(\pi) = \phi_\Pi : WD_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

with similitude character μ. Since ϕ_Π is irreducible as a 4-dimensional representation, $L(\pi)$ is relevant for $\mathrm{GSp}_{1,1}$.

This completes the definition of the map L. We can already read off some simple properties of L:

Proposition 7.1. *(i) The similitude character of $L(\pi)$ is equal to the central character of π, i.e. property (iii) of the main theorem holds.*

(ii) The L-parameter of $\pi \otimes (\chi \circ \lambda_{V_2})$ is equal to $L(\pi) \otimes \chi$, i.e. property (iv) of the main theorem holds.

(iii) The map L defines a bijection between the non-discrete series representations and the non-discrete series parameters of $\mathrm{GSp}_{1,1}$.

(iv) If $\pi = St(\rho_0, \chi_0)$ is the non-supercuspidal discrete series representation introduced in Prop. 5.3(i) and (ii)(a), then its L-parameter is

$$L(\pi) = \chi_0 \cdot (\phi_{\rho_0} \oplus S_2)$$

with similitude character χ_0^2, where S_2 is the two-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

(v) If $\pi = St_{\mathrm{GU}(V_2)} \otimes \chi$ is a twisted Steinberg representation (as in Prop. 5.3(ii)(b)), then its L-parameter is

$$L(\pi) = \chi \boxtimes S_4.$$

Proof. The statements (i) and (ii) follow from the definition of L and the following two facts. The first is that the theta correspondences used in the definition of L and the Jacquet-Langlands correspondence preserve central character and are compatible with twisting. The second is that the local Langlands correspondence for GL_n is compatible with twisting and identifies the central character of a representation with the determinant character of its L-parameter.

To prove (iii), note that every non-discrete series representation of $\mathrm{GSp}_{1,1}$ is of the form $\pi = J_P(\rho, \chi)$, i.e. the Langlands quotient of a standard module. We have shown in Prop. 5.4 that such a π has nonzero theta lift to $\mathrm{GO}^*_{1,1}$ and

$$\theta(\pi) = J_Q(\rho, \chi) \twoheadleftarrow (\rho \otimes \chi) \boxtimes \pi(\chi\omega_\rho, \chi).$$

So we have

$$L(\pi) = \chi \cdot (\phi_\rho \oplus \omega_\rho \oplus 1)$$

with similitude character $\omega_\rho \chi^2$. It is easy to see that these are all the non-discrete series parameters of $\mathrm{GSp}_{1,1}$. Further, for any non-discrete series parameter ϕ of $\mathrm{GSp}_{1,1}$, the set $\widehat{B}_\phi \smallsetminus \widehat{A}_\phi$ is a singleton, since one has

$$B_\phi = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad A_\phi = \{1\}.$$

Finally, (iv) and (v) follow immediately from the definition of L and the determination of the relevant theta correspondences given in Prop. 5.4 and Prop. 5.7. □

7.2. **Surjectivity and Fibers.** We now show that L is surjective and determine its fibers. By (iii) of the above proposition, it remains to treat the discrete series representations and parameters. Hence, let ϕ be a discrete series parameter for $\mathrm{GSp}_{1,1}$. There are then two cases:

Case 1: Suppose that as a 4-dimensional representation of WD_F,

$$\phi = \phi_1 \oplus \phi_2$$

with ϕ_i irreducible of dimension 2 and $\phi_1 \neq \phi_2$ but $\det \phi_1 = \det \phi_2$. Then one has

$$A_\phi = Z_{\mathrm{Sp}_4} = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Delta} B_\phi = Z_{\mathrm{SL}_2}(\phi_1) \times Z_{\mathrm{SL}_2}(\phi_2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Thus, the set $\widehat{B}_\phi \smallsetminus \widehat{A}_\phi$ has size 2 and consists of the two characters η_{+-} and η_{-+} of B_ϕ, where η_{+-} is trivial on $Z_{\mathrm{SL}_2}(\phi_1)$ and nontrivial on $Z_{\mathrm{SL}_2}(\phi_2)$, and similarly for η_{-+}.

In this case, let τ_i be the irreducible discrete series representation of $\mathrm{GL}_2(F)$ associated to ϕ_i. One can then form 2 irreducible representations of $D^\times \times \mathrm{GL}_2(F)$, namely

$$JL(\tau_1) \boxtimes \tau_2 \quad \text{and} \quad JL(\tau_2) \boxtimes \tau_1.$$

By Prop. 5.4, both these representations have nonzero theta lifts to $\mathrm{GSp}_{1,1}$, yielding the representations

$$\pi^{+-} = \theta(JL(\tau_2) \boxtimes \tau_1) \quad \text{and} \quad \pi^{-+} = \theta(JL(\tau_1) \boxtimes \tau_2).$$

It is easy to see that the fiber of L over ϕ is equal to $\{\pi^{+-}, \pi^{-+}\}$, with π^{+-} (resp. π^{-+}) indexed by the character η_{+-} (resp. η_{-+}).

Case 2: Suppose that ϕ is irreducible as a representation of WD_F, so that A_ϕ is trivial and $B_\phi = Z_{Sp_4} = \mathbb{Z}/2\mathbb{Z}$. Hence $\widehat{B}_\phi \smallsetminus \widehat{A}_\phi$ is a singleton set consisting of the nontrivial characetr of B_ϕ. In this case, ϕ corresponds to a discrete series representation of $\mathrm{GL}_4(F)$ and hence to a representation Π of $D_4^\times$ via the Jacquet-Langlands correspondence. Moreover, the central character of Π is equal to μ^2 where $\mu = \mathrm{sim}(\phi)$. Hence, $\Pi \boxtimes \mu$ is a representation of $\mathrm{GO}^*_{3,0}$. Since $\gamma(s, \bigwedge^2 \phi \otimes \mu^{-1}, \psi)$ has a pole at $s = 1$, it follows by Thm. 6.1 that $\Pi \boxtimes \mu$ participates in theta correspondence with $\mathrm{GSp}_{1,1}$. If π is the representation of $\mathrm{GSp}_{1,1}$ such that $\theta(\pi) = \Pi \boxtimes \mu$, then it is easy to see that the fiber of L over ϕ contains only the representation π, indexed by the nontrivial character of $B_\phi = \mathbb{Z}/2\mathbb{Z}$.

With this, we have shown the surjectivity of L and also verified properties (i)-(iv) of the main theorem.

8. Preservation of Plancherel Measures and Local Factors

In this section, we shall verify properties (v) and (vi) of our main theorem. For that, we first need to explain how we attach L-factors and ϵ-factors to non-supercuspidal representations of $\mathrm{GSp}_{1,1}$.

Suppose that π is a non-tempered representation of $\mathrm{GSp}_{1,1}$. Then π is the quotient of a unique standard module $I_P(\tau, \chi)$, which is induced from a representation $\tau \boxtimes \chi$ of $M = D^\times \times F^\times$. If σ is an irreducible representation of GL_r, then the local factors for $\pi \boxtimes \sigma$ are defined using multiplicativity (cf. [Sh2]) and the Jacquet-Langlands correspondence, so that

$$\begin{cases} L(s, \pi \times \sigma) = L(s, (JL(\tau) \otimes \chi) \times \sigma) \cdot L(s, \chi \times \sigma) \cdot L(s, \chi\omega_\tau \cdot \sigma), \\ \epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, (JL(\tau) \otimes \chi) \times \sigma, \psi) \cdot \epsilon(s, \chi \times \sigma, \psi) \cdot \epsilon(s, \chi\omega_\tau \times \sigma, \psi), \\ \gamma(s, \pi \times \sigma, \psi) = \gamma(s, (JL(\tau) \otimes \chi \times \sigma, \psi) \cdot \gamma(s, \chi \times \sigma, \psi) \cdot \gamma(s, \chi\omega_\tau \times \sigma, \psi). \end{cases}$$

Note that the factors on the RHS are local factors attached to $\mathrm{GL}_2 \times \mathrm{GL}_r$ and $\mathrm{GL}_1 \times \mathrm{GL}_r$ and thus have all been defined.

On the other hand, if π is tempered but non-supercuspidal, then we still have $\pi \hookrightarrow I_P(\tau, \chi)$ for some representation $\tau \boxtimes \chi$ of $M = D^\times \times F^\times$. In this case, if σ is an irreducible representation of GL_r, one still defines the gamma factor by multiplicativity as above:

$$\gamma(s, \pi \times \sigma, \psi) = \gamma(s, JL(\tau) \cdot \chi \times \sigma, \psi) \cdot \gamma(s, \chi \times \sigma, \psi) \cdot \gamma(s, \chi\omega_\tau \times \sigma, \psi).$$

Then, as in [Sh1], we define the L-factor $L(s, \pi \times \sigma)$ such that $L(s, \pi \times \sigma)^{-1}$ is the polynomial in q^{-s} with constant term 1 which is the numerator of $\gamma(s, \pi \times \sigma, \psi)$, and we define the epsilon factor by the relation

$$\gamma(s, \pi \times \sigma, \psi) = \epsilon(s, \pi \times \sigma, \psi) \cdot \frac{L(1-s, \pi^\vee \times \sigma^\vee)}{L(s, \pi \times \sigma)}.$$

Now the above definition of the local factors may not be very satisfactory; it would be more satisfactory to have a general theory of L-functions (such as that developed by Tate in his thesis for GL_1 or by Shahidi [Sh1] for generic representations of quasi-split groups), with the above identities as consequences. However, any such general theory of L-functions can only be considered reasonable if the above identities in the non-supercuspidal case do hold, so that the definition of local factors adopted above is the only possible one.

In any case, with the above definition of local factors and the explicit determination of the L-parameters of non-supercuspidal representations given in Prop. 7.1 and its proof, it is immediate that property (v) of the main theorem holds. Hence, it remains to verify the property (vi).

Let π be a supercuspidal representation of $\mathrm{GSp}_{1,1}$ and σ a supercuspidal representation of $\mathrm{GL}_r(F)$, so that $\pi \boxtimes \sigma$ is a supercuspidal representation of the Levi subgroup $M_r = \mathrm{GSpin}_{4,1} \times \mathrm{GL}_r$ of a maximal parabolic P_r of $\mathrm{GSpin}_{r+4,r+1}$ (see [GT, §9]). We want to compute the Plancherel measure attached to the family of principal series representions $I_{P_r}(\pi \boxtimes \sigma, s)$ in terms of the γ-factors of the L-parameters ϕ_π and ϕ_σ.

Now recall that the representation π has a nonzero supercuspidal theta lift $\tau = \theta(\pi)$ on $\mathrm{GU}(W_k)$ for $k = 2$ or 3 (but not both). In either case, one may consider the Jacquet-Langlands transfer $JL(\tau)$ of τ, which is a generic representation of $\mathrm{GSO}_{k,k}$. Using the identification of $\mathrm{GSO}_{k,k}$ with quotients of $\mathrm{GL}_2 \times \mathrm{GL}_2$ or $\mathrm{GL}_4 \times \mathrm{GL}_1$, we then obtain a representation $\sigma(\tau) \boxtimes \mu$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$ whose L-parameter is essentially the L-parameter of π (composed with the 4-dimensional representation and the similitude character of $\mathrm{GSp}_4(\mathbb{C})$). Given this, the key is then to show:

Proposition 8.1. *If $\tau = \theta(\pi)$ is a representation of $\mathrm{GU}(W_k)$ for $k = 2$ or 3, so that $JL(\tau)$ is a representation of $\mathrm{GSO}_{k,k}$ giving rise to $\sigma(\tau) \boxtimes \mu$ on $\mathrm{GL}_4 \times \mathrm{GL}_1$, then $\mu(s, \pi \times \sigma)$ is equal to*

$$\gamma(s, \sigma(\tau)^\vee \times \sigma, \psi) \cdot \gamma(-s, \sigma(\tau) \times \sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \sigma, Sym^2 \times \mu^{-1}, \psi) \cdot \gamma(-2s, \sigma^\vee, Sym^2 \times \mu, \overline{\psi}),$$

where the γ-factors here are those of Shahidi's.

The rest of the section is devoted to the proof of the proposition, which is a global argument. To save space, we shall write $\nu(s, \tau \times \sigma)$ to denote the product of γ-factors in the proposition, so that

$$\nu(s, \tau \times \sigma) := \gamma(s, \sigma(\tau)^\vee \times \sigma, \psi) \cdot \gamma(-s, \sigma(\tau) \times \sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \sigma, Sym^2 \times \mu^{-1}, \psi) \cdot \gamma(-2s, \sigma^\vee, Sym^2 \times \mu, \overline{\psi}).$$

Thus, what we need to show is the equality

$$\mu(s, \pi \times \sigma) = \nu(s, \tau \times \sigma).$$

To begin, let $\mathbb{F}$ be a totally complex number field such that at two places v_1 and v_2, we have $\mathbb{F}_{v_i} = F$. Let $\mathbb{D}$ be the quaternion division algebra over $\mathbb{F}$ which is ramified at precisely v_1 and v_2. Consider the split quaternionic Hermitian space $\mathbb{V}_2$ of rank 2 over $\mathbb{D}$, so that its similitude group $\mathrm{GU}(\mathbb{V}_2)$ is isomorphic to $\mathrm{GSp}_{1,1}$ over F_{v_i} ($i = 1$ or 2) and is split at all other places. In particular, $\mathrm{GU}(\mathbb{V}_2)$ is an inner form of the split group GSp_4 over $\mathbb{F}$. Similarly, let $\mathbb{W}_k$ be a quaternionic skew-Hermitian space over $\mathbb{D}$ such that its similitude group $\mathrm{GU}(\mathbb{W}_k)$ is split for all $v \neq v_i$ and is isomorphic to $\mathrm{GU}(W_k)$ at v_1 and v_2.

Next we globalize the local representations involved, in such a way that the global representations are global theta lifts of each other. To do this, note that if N is the unipotent radical N of the Siegel parabolic P of $\mathrm{GU}(V_2)$, then there exists a nontrivial character χ of N such that

$$\pi_{N,\chi} \neq 0.$$

By a standard computation (see [PT, §6, Cor. 6.6 and Thm. 8]), one has

$$\pi_{N,\chi} \subset \Theta(\tau)_{N,\chi} \cong \mathrm{Hom}_H(\tau, \mathbb{C}) \tag{8.2}$$

for some subgroup $H \subset \mathrm{GU}(W_k)$.

Now one can find a subgroup $\mathbb{H} \subset \mathrm{GU}(\mathbb{W}_k)$ such that $\mathbb{H}_{v_1} = \mathbb{H}_{v_2} = H$. Then using a result of Prasad-Schulze-Pillot [PSP, Thm. 4.1], one can find a cuspidal representation $\mathfrak{T}$ of $\mathrm{GU}(\mathbb{W}_k)$ such that

- $\mathfrak{T}_v \cong \tau$ if $v = v_1$ or v_2;
- $\mathfrak{T}_v$ is unramified for all finite $v \neq v_1$ or v_2;
- $\mathfrak{T}$ has a nonzero period integral over $\mathbb{H}$.

By the global analog of (8.2) [PT, Thm. 11], the last condition implies that $\mathfrak{T}$ has a nonzero global theta lift

$$\Pi := \Theta(\mathfrak{T}) \quad \text{on } \mathrm{GU}(\mathbb{V}_2),$$

and the first two imply that $\Pi_v \cong \pi$ for $v = v_1$ or v_2, and Π_v is unramified for all other finite v.

Consider the Jacquet-Langlands transfer [Ba] $JL(\mathfrak{T})$ of $\mathfrak{T}$ on the split group $\mathrm{GSO}_{k,k}$, which gives rise to an automorphic representation $\Sigma(\mathfrak{T}) \boxtimes \daleth$ of $\mathrm{GL}_4 \times \mathrm{GL}_1$ whose local components at v_1 and v_2 are $\sigma(\tau) \boxtimes \mu$ and are unramified at all other finite v. Finally, we let Σ be a cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$ such that $\Sigma_v = \sigma$ for $v = v_1$ or v_2, and Σ_v is unramified for all other finite v.

Now one considers the global induced representation

$$I_{P_r}(\Pi \boxtimes \Sigma, s) \quad \text{of } \mathrm{GSpin}_{r+4,r+1}.$$

There is a global standard intertwining operator

$$A(s, \Pi \boxtimes \Sigma, N_r, \bar{N}_r) : I_{P_r}(\Pi \boxtimes \Sigma, s) \longrightarrow I_{\bar{P}_r}(\Pi \boxtimes \Sigma, s),$$

where $\bar{P}_r = M_r \cdot \bar{N}_r$ is the maximal parabolic subgroup opposite to P_r. This satisfies the functional equation

$$\mu(s, \Pi \boxtimes \Sigma) := A(s, \Pi \boxtimes \Sigma, \bar{N}_r, N_r) \circ A(s, \Pi \boxtimes \Sigma, N_r, \bar{N}_r) = 1.$$

On the other hand, one has the global functional equations

$$\gamma(s, \Sigma(\mathfrak{T}) \boxtimes \Sigma^\vee) := \epsilon(s, \Sigma(\mathfrak{T}) \times \Sigma^\vee) \cdot \frac{L(1-s, \Sigma(\mathfrak{T})^\vee \times \Sigma)}{L(s, \Sigma(\mathfrak{T}) \times \Sigma^\vee)} = 1$$

and

$$\gamma(s, \Sigma, Sym^2 \times \daleth^{-1}) := \epsilon(s, \Sigma^\vee, Sym^2 \times \daleth^{-1}) \cdot \frac{L(1-s, \Sigma^\vee, Sym^2 \times \daleth)}{L(s, \Sigma, Sym^2 \times \daleth^{-1}))} = 1.$$

In particular, if we define $\nu(s, \mathfrak{T} \boxtimes \Sigma)$ as the product of the above global γ-factors as in the proposition:

$$\nu(s, \mathfrak{T} \boxtimes \Sigma) = \gamma(s, \Sigma(\mathfrak{T})^\vee \times \Sigma) \cdot \gamma(-s, \Sigma(\mathfrak{T}) \times \Sigma^\vee) \cdot \gamma(2s, \Sigma, Sym^2 \times \daleth^{-1}) \cdot \gamma(-2s, \Sigma^\vee, Sym^2 \times \daleth, \overline{\psi}),$$

then the global functional equations imply that

$$\mu(s, \Pi \boxtimes \Sigma) = 1 = \nu(s, \mathfrak{T} \boxtimes \Sigma). \tag{8.3}$$

Finally, for all $v \neq v_1$ or v_2, one knows that

$$\mu(s, \Pi_v \times \Sigma_v) = \nu(s, \mathfrak{T}_v \boxtimes \Sigma_v),$$

because one understands the theta correspondence of unramified representations [GT2] or the theta correspondence for complex groups [AB] completely. For finite $v \neq v_i$, one may also appeal to [GT, Prop. 9.2 and Thm. 9.3], which gives the above identity in the case of GSp_4. Together with (8.3), this implies that

$$\mu(s, \pi \boxtimes \sigma)^2 = \nu(s, \tau \boxtimes \sigma)^2 \quad \text{for all } s \in \mathbb{C}.$$

However, the Plancherel measure on the LHS is ≥ 0 when s is purely imaginary [Si, §5]. Similarly, by [Sh1, (7.8.1)], when s is purely imaginary, one has

$$\overline{\gamma(s, \sigma(\tau)^\vee \times \sigma, \psi)} = \gamma(-s, \sigma(\tau) \times \sigma^\vee, \overline{\psi})$$

and

$$\overline{\gamma(2s, \sigma, Sym^2 \times \mu^{-1}, \psi)} = \gamma(-2s, \sigma^\vee, Sym^2 \times \mu, \overline{\psi}),$$

so that

$$\nu(s, \tau \boxtimes \sigma) = |\gamma(s, \sigma(\tau)^\vee \times \sigma, \psi)|^2 \cdot |\gamma(2s, \sigma, Sym^2 \times \mu^{-1}, \psi)|^2 \geq 0.$$

Hence we have the equality:

$$\mu(s, \pi \boxtimes \sigma) = \nu(s, \tau \boxtimes \sigma),$$

as desired. Property (vi) of the main theorem then follows by the fact that the local Langlands correspondence for GL_n respects γ-factors of pairs and the twisted symmetric square γ-factors [He2].

9. Characterization of the Map L

The goal of this section is to show the following result:

Theorem 9.1. *There is at most one map*

$$L : \Pi(\mathrm{GU}_2(D)) \longrightarrow \Phi(\mathrm{GU}_2(D))$$

satisfying:

(a) the central character ω_π of π corresponds to the similitude character $sim(\phi_\pi)$ of $\phi_\pi := L(\pi)$ under local class field theory;

(b) π is an essentially discrete series representation if and only if ϕ_π does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$;

(c) if π is not a supercuspidal representation, then for any irreducible representation σ of $\mathrm{GL}_r(F)$ with $r \leq 2$,

$$\begin{cases} L(s, \pi \times \sigma) = L(s, \phi_\pi \times \phi_\sigma), \\ \epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\pi \times \phi_\sigma, \psi). \end{cases}$$

(d) if π is a supercuspidal representation, then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$ with $r \leq 4$, the Plancherel measure $\mu(s, \pi \times \sigma)$ is equal to

$$\gamma(s, \phi_\pi^\vee \times \phi_\sigma, \psi) \cdot \gamma(-s, \phi_\pi \times \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \sigma, Sym^2 \otimes \omega_\pi^{-1}, \psi) \cdot \gamma(-2s, \sigma^\vee, Sym^2 \otimes \omega_\pi^{-1}, \overline{\psi}).$$

We note that in [GT, Thm. 10.1(d)], the analogous condition as Theorem 9.1(d) is stated with $r \leq 2$, rather than $r \leq 4$. The reason why we have a weaker result here is that (c) does not apply to supercuspidal representations, so that one is not in the position to apply the local converse theorem of J. P. Chen [C].

The rest of the section is devoted to the proof of this theorem. If L and L' are two maps satisfying the conditions of the theorem, we first show that they must agree on any non-supercuspidal representation π. If we set

$$\phi_1 := L(\pi) \quad \text{and} \quad \phi_2 := L'(\pi),$$

then as in the proof of [GT, Thm. 10.1], it suffices to show that $\phi_1 \cong \phi_2$ as 4-dimensional representations of the Weil-Deligne group WD_F. The condition (c) implies that for any irreducible representation of GL_r with $r \leq 2$,

$$\begin{cases} L(s, \phi_1 \otimes \phi_\sigma) = L(s, \phi_2 \times \phi_\sigma), \\ \epsilon(s, \phi_1 \otimes \phi_\sigma, \psi) = \epsilon(s, \phi_2 \otimes \phi_\sigma, \psi). \end{cases}$$

But as shown in [GT, proof of Thm. 10.1, Case 1] (following the arguments in [He1]), these conditions imply that $\phi_1 \cong \phi_2$, as desired.

It remains to show that L and L' must agree on any supercuspidal representation π. By conditions (a) and (b), $L(\pi)$ and $L'(\pi)$ are both 4-dimensional semisimple representations of WD_F which are multiplicity-free sum of irreducible symplectic summands with a fixed similitude character $\mu = \omega_\pi$. In view of condition (d), we can now appeal to the following general lemma:

Lemma 9.2. *Suppose that ϕ_1 and ϕ_2 are $2n$-dimensional semisimple representations of WD_F, each of which is a multiplicity-free sum of irreducible symplectic summands with a fixed similitude character μ. If*

$$\gamma(s, \phi_1^\vee \otimes \phi_\rho, \psi) \cdot \gamma(-s, \phi_1 \otimes \phi_\rho^\vee, \overline{\psi}) = \gamma(s, \phi_2^\vee \otimes \phi_\rho, \psi) \cdot \gamma(-s, \phi_2 \otimes \phi_\rho^\vee, \overline{\psi}) \tag{9.3}$$

for every irreducible representation ϕ_ρ of W_F of dimension $\leq 2n$, then

$$\phi_1 \cong \phi_2$$

as representations of WD_F.

Proof. We shall proceed by induction on $\dim \phi_i = 2n$. Suppose that ϕ_0 is an irreducible representation of W_k such that $\phi_0 \boxtimes S_r$ is contained in ϕ_1 for some $r \geq 1$. Here S_r is the r-dimensional irreducible repersentation of $\mathrm{SL}_2(\mathbb{C})$. Let r_0 be the smallest such r. Taking $\phi_\rho = \phi_0$ and evaluating at $s = (r_0 - 1)/2 \geq 0$, one sees from the minimality of r_0 that the LHS of (9.3) has a zero at $s = (r_0 - 1)/2$, and hence so must the RHS. This implies that $L(-s, \phi_2 \otimes \phi_0^\vee) \cdot L(s, \phi_2^\vee \otimes \phi_0)$ must have a pole at $s = (r_0 - 1)/2$. It is not difficult to see that this can only happen if ϕ_2 contains $\phi_0 \boxtimes S_{r_0}$ as well. Thus, we may cancel $\phi_0 \boxtimes S_r$ from both ϕ_1 and ϕ_2, and still have the analog of (9.3). The lemma then follows by induction. □

10. Appendix: Explicit Determination of Theta Lifts

In this appendix, we give the proofs of Props. 5.4 and 5.7.

10.1. **Theta Lifts from $\mathrm{GO}^*_{1,1}$ to $\mathrm{GSp}_{1,1}$.** We begin with Prop. 5.4, which determines the theta lift from $\mathrm{GU}(W_2) = \mathrm{GO}^*_{1,1}$ to $\mathrm{GU}(V_2) = \mathrm{GSp}_{1,1}$. Recall that according to Prop. 5.1, the representations of $\mathrm{GO}^*_{1,1} \cong D^\times \times \mathrm{GL}_2 / \{((t, t^{-1}) : t \in \mathrm{GL}_1\}$ which participate in theta correspondence with $\mathrm{GU}(V_1) = \mathrm{GSp}_{1,0} \cong D^\times$ are precisely those of the form $\rho \boxtimes JL(\rho)$ for some irrreducible representation ρ of $D^\times$. One of the things we need to show is that such representations of $\mathrm{GO}^*_{1,1}$ do not participate in theta correspondence with $\mathrm{GSp}_{1,1}$. We consider the following three cases separately: supercuspidal case, non-discrete-series case and discrete series case.

10.2. **Supercuspidal Case.** We first consider the theta lifts of supercuspidal representations $\rho \boxtimes \tau$ of $\mathrm{GO}^*_{1,1}$, following the arguments of Thm. 5.8. Let σ be an irreducible constituent of $(\rho \boxtimes \tau)|_{U(W_2)}$. Then $\Theta^{2,r}(\rho \boxtimes \tau) \neq 0$ iff $\Theta^{2,r}_\psi(\sigma) \neq 0$. By the seesaw identity of the doubling method, this is in turn equivalent to

$$\mathrm{Hom}_{U(W_2)\times U(W_2)}(\Theta^{r,4}_\psi(1), \sigma \boxtimes \sigma^\vee) \neq 0.$$

We are interested in the cases $r = 1$ and $r = 2$. In these cases, we have:

$$\Theta^{r,4}_\psi(1) = \theta^{r,4}_\psi(1)$$

since we are in the stable range. By a result of Yamana [Y1, Thm. 1.3], one has

$$0 \longrightarrow \theta^{2,4}_\psi(1) \longrightarrow I_P(1/2) \longrightarrow \theta^{1,4}_\psi(1) \longrightarrow 0.$$

Moreover, by a Mackey type argument similar to [KR, §1], one has

$$\dim \mathrm{Hom}_{U(W_2)\times U(W_2)}(I_P(s), \sigma \boxtimes \sigma^\vee) = 1$$

for any $s \in \mathbb{C}$. Since σ is supercuspidal, we deduce that

$$\dim \mathrm{Hom}_{U(W_2)\times U(W_2)}(\Theta^{1,4}_\psi(1), \sigma \boxtimes \sigma^\vee) + \dim \mathrm{Hom}_{U(W_2)\times U(W_2)}(\Theta^{2,4}_\psi(1), \sigma \boxtimes \sigma^\vee) = 1.$$

Hence, we conclude that

$$\dim \mathrm{Hom}_{U(W_2)\times U(W_2)}(\Theta^{r,4}_\psi(1), \sigma \boxtimes \sigma^\vee) = 1$$

for $r = 1$ or 2 but not both. In particular, one sees by Prop. 5.1 that when τ is supercuspidal,

$$\Theta^{2,2}(\rho \boxtimes \tau) = 0 \Longleftrightarrow \tau = JL(\rho).$$

10.3. **Jacquet modules.** Before coming to the non-discrete-series case, we need to compute the normalized Jacquet modules of the induced Weil representation Ω for $\mathrm{GU}(W_2) \times \mathrm{GU}(V_2)$ with respect to the parabolic subgroups $Q = L \cdot U$ of $\mathrm{GU}(W_2)$ and $P = M \cdot N$ of $\mathrm{GU}(V_2)$. This is a standard calculation (see [GT2] for the details in the split cases) and the results are given by the following two lemmas and their corollaries.

Lemma 10.1. *The normalized Jacquet module $R_Q(\Omega)$ of Ω with respect to the parabolic $Q = L \cdot U$ of $\mathrm{GU}(W_2)$ sits in a short exact sequence of $(L \times \mathrm{GU}(V_2))$-modules*

$$0 \longrightarrow A \longrightarrow R_Q(\Omega) \longrightarrow B \longrightarrow 0.$$

Here,

$$A \cong I_P(S(F^\times) \otimes S(D^\times))$$

where $(l, \alpha) \in L \cong D^\times \times \mathrm{GL}_1$ acts on $S(F^\times \times D^\times)$ by

$$((l, \alpha) \cdot f)(t, h) = |N_D(l)|^{3/2} \cdot |\alpha|^{-3/2} \cdot f(\alpha t, hl),$$

and $(m, \beta) \in M \cong D^\times \times \mathrm{GL}_1$ acts by

$$((m, \beta) \cdot f)(t, h) = |N_D(m)|^{-3/2} \cdot |\beta|^{3/2} \cdot f(\beta t, \beta \cdot m^{-1} h).$$

Moreover,

$$B \cong S(F^\times)$$

where $((l, \alpha), g) \in L \times \mathrm{GU}(V_2) = D^\times \times F^\times \times \mathrm{GU}(V_2)$ acts by

$$((l, \alpha, g) \cdot f)(t) = |N_D(l)|^{3/2} \cdot |\alpha|^{-3/2} \cdot f(t\alpha\lambda_{V_2}(g)).$$

Corollary 10.2. *For a representation $\rho \boxtimes \chi$ of L, we have:*

$$\mathrm{Hom}_L(A, \rho \boxtimes \chi) \cong I_P(\rho^\vee, \chi \cdot \omega_\rho)^* \quad \textit{(full linear dual)}$$

as representations of $\mathrm{GU}(V_2)$. *Moreover,*

$$\mathrm{Hom}_L(B, \rho \boxtimes \chi) \neq 0 \Longleftrightarrow \rho = |N_D|^{3/2}.$$

In particular, if $\rho \neq |N_D|^{3/2}$, then

$$\mathrm{Hom}_{\mathrm{GU}(W_2)}(\Omega, I_Q(\rho, \chi)) \cong I_P(\rho^\vee, \chi\omega_\rho)^*.$$

Hence, if π is an irreducible quotient of $I_P(\rho^\vee, \chi\omega_\rho)$, then there is a nonzero (but not necessarily surjective) equivariant map

$$\Omega \longrightarrow I_Q(\rho, \chi) \boxtimes \pi.$$

Lemma 10.3. *The normalized Jacquet module $R_P(\Omega)$ of Ω with respect to the parabolic $P = M \cdot N$ of* $\mathrm{GU}(V_2)$ *sits in a short exact sequence of* $(\mathrm{GU}(W_2) \times M)$*-modules:*

$$0 \longrightarrow A' \longrightarrow R_P(\Omega) \longrightarrow B' \longrightarrow 0.$$

Here,

$$A' \cong I_Q(S(F^\times) \otimes S(D^\times))$$

where $(l, \alpha) \in L \cong D^\times \times \mathrm{GL}_1$ acts by

$$((l, \alpha) \cdot f)(t, d) = |N_D(l)|^{-1/2} \cdot |\alpha|^{-3/2} \cdot f(\alpha t, l^{-1} d),$$

and $(m, \beta) \in M = D^\times \times \mathrm{GL}_1$ acts by

$$((m, \beta) \cdot f)(t, d) = |N_D(m)|^{1/2} \cdot |\beta|^{-5/2} \cdot f(\beta t, d \cdot m \cdot \beta^{-1}).$$

Moreover,

$$B' \cong S(F^\times),$$

where $(h, (m, \beta)) \in \mathrm{GU}(W_2) \times M = \mathrm{GU}(W_2) \times D^\times \times F^\times$ acts by

$$((h, m, \beta) \cdot f)(t) = \lambda_{W_2}(h)^{-2} \cdot |N_D(m)|^{1/2} \cdot \beta^{-5/2} \cdot f(t \cdot \beta \cdot \lambda_{W_2}(h)).$$

Corollary 10.4. *For a representation $\rho \boxtimes \chi$ of M, we have*

$$\mathrm{Hom}_M(A', \rho \boxtimes \chi) \cong I_Q(\rho^\vee, \chi \cdot \omega_\rho)^* \quad \textit{(full linear dual)}$$

as representations of $\mathrm{GU}(W_2)$. *Moreover,*

$$\mathrm{Hom}_M(B', \rho \boxtimes \chi) \neq 0 \Longleftrightarrow \rho = |N_D|^{1/2}.$$

In particular, if $\rho \neq |N_D|^{1/2}$, then

$$\mathrm{Hom}_{\mathrm{GU}(V_2)}(\Omega, I_P(\rho, \chi)) \cong I_Q(\rho^\vee, \chi\omega_\rho)^*.$$

Hence, if π is an irreducible quotient of $I_Q(\rho^\vee, \chi\omega_\rho)$, then there is a nonzero (but not necessarily surjective) equivariant map

$$\Omega \longrightarrow I_P(\rho, \chi) \boxtimes \pi.$$

10.4. **Non-discrete-series case.** Suppose now that $\pi = J_P(\rho,\chi)$ is a non-discrete-series representation of $\mathrm{GU}(W_2)$, so that

$$\pi \hookrightarrow I_Q(\rho^\vee, \chi\omega_\rho) = (\rho \otimes \chi) \boxtimes \pi(\chi, \chi\omega_\rho),$$

and $|\omega_\rho| = |-|^s$ with $s \geq 0$. In particular, $\rho^\vee \neq |N_D|^{3/2}$. Now by Cor. 10.2, we deduce that

$$\Theta(\pi)^* \hookrightarrow \mathrm{Hom}_{\mathrm{GU}(W_2)}(\Omega, I_Q(\rho^\vee, \chi\omega_\rho)) = I_P(\rho,\chi)^*.$$

Hence we have

$$I_P(\rho,\chi) \twoheadrightarrow \Theta(\pi)$$

so that

$$\theta(\pi) \subset J_P(\rho,\chi).$$

In fact, if $I_Q(\rho,\chi)$ is irreducible, then equality holds so that

$$\Theta(\pi) = I_P(\rho,\chi) \quad \text{and} \quad \theta(\pi) = J_P(\rho,\chi).$$

On the other hand, suppose henceforth that $I_Q(\rho,\chi)$ is reducible. Then we must have $\rho = \rho_0|-|^{1/2}$ for a quadratic character ρ_0 (possibly trivial). Setting $\chi = \chi_0|-|^{-1/2}$, we have

$$\pi = \rho_0\chi_0 \boxtimes \chi_0,$$

and

$$J_P(\rho,\chi) = \begin{cases} Sp(\rho_0,\chi_0), & \text{if } \rho_0 \text{ is nontrivial;} \\ I_P(\rho,\chi), & \text{if } \rho_0 \text{ is trivial.} \end{cases}$$

In this case, what we need to show is

$$\Theta(\pi) = J_P(\rho,\chi).$$

For this, we first apply Cor. 10.4 to see that

$$\mathrm{Hom}_M(\Omega, I_P(\rho^\vee, \chi\omega_\rho)) = I_Q(\rho,\chi)^*,$$

so that there is a nonzero equivariant map

$$\Omega \longrightarrow \pi \boxtimes I_P(\rho^\vee, \chi\omega_\rho).$$

This shows that $\Theta(\pi) \neq 0$ and $\theta(\pi) = J_P(\rho,\chi)$ as desired. Further, if $\rho = |-|^{1/2}$, i.e. if ρ_0 is trivial, then since $I_P(\rho,\chi)$ is irreducible, we have $\Theta(\pi) = \theta(\pi) = I_P(\rho,\chi) = J_P(\rho,\chi)$. On the other hand, if ρ_0 is a nontrivial quadratic character, then suppose for the sake of contradiction that $\Theta(\pi) \neq \theta(\pi)$. In that case, $\Theta(\pi) = I_P(\rho,\chi)$, so that by Cor. 10.4 again, one has

$$\pi^* \hookrightarrow \mathrm{Hom}_M(\Omega, I_P(\rho,\chi)) = I_Q(\rho^\vee, \chi\omega_\rho)^*.$$

Hence, we would deduce that π is a quotient of $I_Q(\rho^\vee, \chi\omega_\rho)$. This is a contradiction, which completes the proof of Prop. 5.4(i).

10.5. **Bessel models.** There is in fact another way to see that $\theta(\pi) \neq 0$ (and thus equals $J_P(\rho,\chi)$) when $I_Q(\rho,\chi)$ is reducible, namely by computing a local Bessel model of $\Theta(\pi)$ [PT]. As shown in [PT, Cor. 7.2], with $\pi = \rho' \boxtimes \tau'$, $\Theta(\pi)$ has a nonzero local Bessel model if and only if for some quadratic field extension E/F and some character μ of $E^\times$,

$$\mathrm{Hom}_{E^\times}(\rho', \mu) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{E^\times}(\tau', \mu) \neq 0.$$

For the case at hand, where $\rho' \boxtimes \tau' = \rho_0\chi_0 \boxtimes \chi_0$ with ρ_0 quadratic, we can simply take E/F to be the quadratic field extension determined by ρ_0 and $\mu = \chi_0 \circ N_{E/F}$. This shows that $\Theta(\pi) \neq 0$ so that $\theta(\pi) = J_P(\rho,\chi)$ whenever $\pi = J_Q(\rho,\chi)$. However, this alternative argument does not seem to give the stronger conclusion that $\Theta(\pi) = \theta(\pi)$. Nonetheless, this Bessel model argument will be useful when we study the discrete series case.

10.6. **Discrete Series Case.** Finally we consider the case when π is nonsupercuspidal discrete series. In this case,

$$\pi \hookrightarrow I_Q(\rho,\chi) = (\rho\otimes\chi)\boxtimes\pi(\chi\omega_\rho,\chi)$$

with

$$\omega_\rho = |-|.$$

Let us write

$$\rho = \rho_0|-|^{1/2} \quad \text{and} \quad \chi = \chi_0|-|^{-1/2}$$

so that ρ_0 has trivial central character and

$$\pi = \rho_0\chi_0 \boxtimes st_{\chi_0}.$$

Now Cor. 10.2 implies that

$$I_P(\rho^\vee,\chi\omega_\rho) = I_P(\rho_0^\vee|-|^{-1/2},\chi_0|-|^{1/2}) \twoheadrightarrow \Theta(\pi).$$

From Prop. 5.3, we see that

$$\theta(\pi) = St(\rho_0,\chi_0) \quad \text{or} \quad 0.$$

What we need to show in this case is that

$$\Theta(\pi) = \begin{cases} St(\rho_0,\chi_0), & \text{if } \rho_0 \text{ is nontrivial;} \\ 0, & \text{if } \rho_0 \text{ is the trivial representation.} \end{cases}$$

To show the desired result when ρ_0 is nontrivial, we can argue in a similar way as in the non-discrete-series case above. More precisely, Cor. 10.4 implies that

$$\mathrm{Hom}_M(\Omega, I_P(\rho,\chi)) = I_Q(\rho^\vee,\chi\omega_\rho)^*$$

if $\rho \neq |-|^{1/2}$, i.e. if ρ_0 is not trivial. Thus, if ρ_0 is not trivial, then there is a nonzero equivariant map

$$\Omega \longrightarrow \pi \boxtimes I_P(\rho,\chi).$$

In particular,

$$\Theta(\pi) \neq 0 \quad \text{and} \quad \theta(\pi) = St(\rho_0,\chi_0).$$

To see that $\Theta(\pi) = \theta(\pi)$, we suppose for the sake of contradiction that $\Theta(\pi) \neq \theta(\pi)$. Then $\Theta(\pi) = I_P(\rho^\vee,\chi\omega_\rho)$. But Cor. 10.4 implies that

$$\pi^* \hookrightarrow \mathrm{Hom}_M(\Omega, I_P(\rho^\vee,\chi\omega_\rho)) = I_Q(\rho,\chi)^*$$

so that π is a quotient of $I_Q(\rho,\chi)$. This is a contradiction.

We are thus left with the case when ρ_0 is trivial. In this case, $\pi = \chi_0 \boxtimes st_{\chi_0}$ and we want to show that $\Theta(\pi) = 0$. For this, we shall resort to a Bessel model argument.

More precisely, if $\Theta(\pi) \neq 0$, so that $\Theta(\pi) = \theta(\pi) = I_P(|-|^{-1/2},\chi_0|-|^{1/2})$, then $\Theta(\pi)$ must have a nonzero local Bessel model. As we shall see below, this is not the case. Indeed, the consideration of Bessel models gives an alternative proof of the nonvanishing of $\theta(\pi)$ when ρ_0 is nontrival.

10.7. **Bessel models again.** We consider general ρ_0 here (i.e. ρ_0 need not be trivial). By [PT, Cor. 7.2], $\theta(\pi)$ has some nonzero local Bessel model iff for some quadratic field extension E/F and some character μ of $E^\times$,

$$\mathrm{Hom}_{E^\times}(\rho_0\chi_0, \mu) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{E^\times}(st_{\chi_0}, \mu) \neq 0.$$

Take any E and write $\mu = \mu_0 \cdot (\chi_0 \circ N_{E/F})$. Then the above Hom spaces are nonzero iff

$$\mathrm{Hom}_{E^\times}(\rho_0, \mu_0) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{E^\times}(st, \mu_0) \neq 0.$$

But we know that

$$\mathrm{Hom}_{E^\times}(st, \mu_0) \neq 0 \Longleftrightarrow \mu_0 \text{ is nontrivial.}$$

Hence, it remains to see if there is a pair (E, μ_0) with $\mu_0 \neq 1$ such that

$$\mathrm{Hom}_{E^\times}(\rho_0, \mu_0) \neq 0.$$

We consider the following cases:

(a) If $\dim \rho_0 > 1$, then $\rho_0|_{E^\times}$ necessarily contains a nontrivial character of $E^\times$. So $\theta(\pi) \neq 0$ in this case.

(b) If ρ_0 is 1-dimensional but nontrivial, then it is a quadratic character (since ρ_0 has trivial central character). Now one can take E to be a quadratic extension distinct from the one determined by ρ_0 and $\mu_0 = \rho_0|_{E^\times} \neq 1$. So $\theta(\pi) \neq 0$ in this case.

(c) If $\rho_0 = 1$, then clearly
$$\mathrm{Hom}_{E^\times}(\rho_0, \mu_0) = 0$$
for any nontrivial character μ_0 of $E^\times$, so that $\theta(\pi) = 0$, as desired.

We have thus completed the proof of Prop. 5.4.

10.8. **Theta Lifts from** $\mathrm{GSp}_{1,1}$ **to** $\mathrm{GO}^*_{3,0}$**.** Now we come to Prop. 5.7, which determines the theta lift of nonsupercuspidal representations of $\mathrm{GSp}_{1,1}$ to $\mathrm{GO}^*_{3,0}$. The following standard lemma computes the Jacquet module of the relevant Weil representation Ω with respect to the parabolic subgroup $P(Y)$ of $\mathrm{GSp}_{1,1}$.

Lemma 10.5. *The normalized Jacquet module $R_{P(Y)}(\Omega)$ of the Weil representation Ω with respect to $P(Y)$ is given by*
$$R_{P(Y)}(\Omega) \cong S(F^\times) \otimes |N_D|^{3/2} \otimes |\lambda_{V_2}|^{-9/2} \otimes |\lambda_{W_3}|^{-3}$$
as a representation of $\mathrm{GU}(W_3) \times M(Y) \cong \mathrm{GU}(W_3) \times (D^\times \times \mathrm{GL}_1)$. Here the action of $\mathrm{GU}(W_3) \times D^\times \times \mathrm{GL}_1$ on $S(F^\times)$ is given by
$$((h, a, \lambda) \cdot \phi)(t) = \phi(t \cdot \lambda \cdot \lambda_{W_3}(h)).$$

Now if π is a nonsupercuspidal representation of $\mathrm{GSp}_{1,1}$, then
$$\pi \hookrightarrow I_{P(Y)}(\rho, \chi|-|^{-3/2})$$
for some representation $\rho \boxtimes \chi|-|^{-3/2}$ of $M(Y)$ and we have
$$\Theta(\pi)^* \hookrightarrow \mathrm{Hom}_{\mathrm{GSp}_{1,1}}(\Omega, I_{P(Y)}(\rho, \chi|-|^{-3/2})) = \mathrm{Hom}_{M(Y)}(R_{P(Y)}(\Omega), \rho \boxtimes \chi|-|^{-3/2})$$
as representations of $\mathrm{GU}(W_3) = \mathrm{GO}^*_{3,0}$. The lemma implies that the latter Hom space is zero unless $\rho = |N_D|^{3/2}$. Assume henceforth that $\rho = |N_D|^{3/2}$. By Prop. 5.3(ii)(b), one has
$$0 \longrightarrow St_{\mathrm{GU}(V_2)} \otimes \chi \longrightarrow I_P(|-|^{3/2}, \chi \cdot |-|^{-3/2}) \longrightarrow \chi \longrightarrow 0,$$

so that π is isomorphic to the twisted Steinberg representation $St_{\mathrm{GU}(V_2)} \otimes \chi$. This shows that $\Theta(\pi) = 0$ if π is nonsupercuspidal and not equal to a twisted Steinberg representation. Moreover, one has

$$\mathrm{Hom}_{M(Y)}(R_{P(Y)}(\Omega), |-|^{3/2} \boxtimes \chi|-|^{-3/2}) \cong (\chi \circ \lambda_{W_3})^*$$

as representations of $\mathrm{GU}(W_3)$. Hence, we see that

$$\Theta(St_{\mathrm{GU}(V_2)} \otimes \chi) = 0 \quad \text{or} \quad \chi \circ \lambda_{W_3},$$

and there is a nonzero equivariant map

$$\Omega \longrightarrow I_{P(Y)}(|-|^{3/2}, \chi|-|^{-3/2}) \boxtimes (\chi \circ \lambda_{W_3})).$$

Since we have shown that $\Theta(\chi \circ \lambda_{V_2}) = 0$, the image of this equivariant map must be $(St_{\mathrm{GU}(V_2)} \otimes \chi) \boxtimes (\chi \circ \lambda_{W_3})$. Thus we have shown that

$$\Theta(St_{\mathrm{GU}(V_2)} \otimes \chi) = \chi \circ \lambda_{W_3}$$

and the proof of Prop. 5.7 is complete.

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