The Local Langlands Correspondence

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Introduction

$\mathbb{Q}$ = the field of rational numbers.

Among other things, number theorists are interested in giving a useful and reasonable classification of the set of finite extensions of $\mathbb{Q}$:

$$\begin{array}{l} L \\ | \\ \mathbb{Q} \end{array} \quad \text{with } deg(L/\mathbb{Q}) < \infty$$

$$\updownarrow \text{ Galois}$$

$$\begin{array}{l} H \\ \cap \\ G_{\mathbb{Q}} \end{array} \quad \text{of finite index}$$

Here,

$$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \operatorname{proj\,lim}_{L} \mathrm{Gal}(L/\mathbb{Q})$$

is a profinite group, and so is a compact topological group.

To understand the latter set, one can try to linearize the problem by asking for a classification of

irreducible (finite dimensional) continuous representations over $\mathbb{C}$ of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)

An element of this set is:

$$\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$$

where V is an n-dimensional vector space over $\mathbb{C}$.

Thus, we want to understand the sets in the left hand column in the diagram. A simpler problem is to first understand the subsets depicted in the right hand column.

$\mathbb{Q} \subset L : deg(L/\mathbb{Q}) < \infty$	$\supset$	$L :$ Gal$(L/\mathbb{Q})$ abelian
$\updownarrow$		$\updownarrow$
$H \subset G_{\mathbb{Q}}$ of finite index	$\supset$	$H : G_{\mathbb{Q}}/H$ abelian
$\updownarrow$		$\updownarrow$
irreps of $G_{\mathbb{Q}}$	$\supset$	irreps of $G_{\mathbb{Q}}^{ab}$

It turns out the right hand column was understood a long time ago: this is Global Class Field Theory. But we won't go into this today.

Local-Global Principle

On the other hand, a time-honored principle in number theory is the so-called local-global principle or the **Hasse principle**.

Hasse Principle: to understand a problem over $\mathbb{Q}$, one can first understand the analgous problem over $\mathbb{R}$ and $\mathbb{Q}_p$ for each p, and then patch the answers together to obtain the solution over $\mathbb{Q}$.

The geometric analog of this is:

To understand a space M, one can first understand the local structure of M at each point $m \in M$, and then patch one's knowledge to obtain a global understanding of M.

Example of Hasse Principle:

Consider a quadratic form

$$Q(x_1, x_2, ..., x_n) = \sum_{i \le j} a_{ij} x_i x_j, \qquad a_{ij} \in \mathbb{Q}.$$

The equation

$$Q(x_1, ... x_n) = 0$$

has solutions in $\mathbb{Q}$ if and only if it has solutions in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p.

Local Fields

The fields $\mathbb{R}$ and $\mathbb{Q}_p$ are examples of local fields. They are obtained from $\mathbb{Q}$ as follows.

The usual absolute value $|x|$ defines a metric on $\mathbb{Q}$:

$$d_\infty(x, y) = |x - y|$$

with respect to which $\mathbb{Q}$ is not complete. Completing $\mathbb{Q}$ yields $\mathbb{R}$.

For each p, one has the p-adic absolute value:

$$|p^a \cdot \frac{m}{n}|_p = p^{-a}, \quad (m, p) = (n, p) = 1.$$

Completing $\mathbb{Q}$ with respect to the associated metric yields $\mathbb{Q}_p$.

The point of this is that in $\mathbb{Q}_p$, there is only 1 prime number, namely p!

Basic Structure of $\mathbb{Q}_p$

The completion of $\mathbb{Z}$ in $\mathbb{Q}_p$ is the subring $\mathbb{Z}_p$ of p-adic integers. The p-adic absolute value gives a group homomorphism

$$| - |_p : \mathbb{Q}_p^\times \longrightarrow \mathbb{R}_+^\times,$$

whose image is the subgroup $p^{\mathbb{Z}}$ and whose kernel is the group of units

$$U_p = \mathbb{Z}_p^\times.$$

So one has

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_p & \longrightarrow & \mathbb{Q}_p^\times & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & & & p & \longrightarrow & 1 & & \end{array}$$

Hasse Principle suggests that we should first study the local problem of classifying extensions of $\mathbb{Q}_p$.

$\boxed{\mathbb{Q}_p \subset L : deg(L/\mathbb{Q}_p) < \infty}$	$\supset$	$\boxed{L : \text{Gal}(L/\mathbb{Q}_p) \text{ abelian}}$
$\updownarrow$		$\updownarrow$
$\boxed{H \subset G_{\mathbb{Q}_p} \text{ of finite index}}$	$\supset$	$\boxed{H : G_{\mathbb{Q}_p}/H \text{ abelian}}$
$\updownarrow$		$\updownarrow$
$\boxed{\text{irreps of } G_{\mathbb{Q}_p}}$	$\supset$	$\boxed{\text{irreps of } G^{ab}_{\mathbb{Q}_p}}$

As in the global case, the right hand column is understood: this is **Local Class Field Theory**, to which we shall turn shortly.

The Weil Group $W_{\mathbb{Q}_p}$

Local CFT gives a description of $G_{\mathbb{Q}_p}^{ab}$.

The group $G_{\mathbb{Q}_p}$ sits in a short exact sequence

$$1 \longrightarrow I_p \longrightarrow G_{\mathbb{Q}_p} \longrightarrow \widehat{\mathbb{Z}} \longrightarrow 1$$

where

$$\widehat{\mathbb{Z}} = \operatorname{proj}\lim_N \mathbb{Z}/N\mathbb{Z}$$

is the profinite completion of $\mathbb{Z}$. Consider:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_p & \longrightarrow & G_{\mathbb{Q}_p} & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & I_p & \longrightarrow & W_{\mathbb{Q}_p} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

The group $W_{\mathbb{Q}_p}$ is called the **Weil group** of $\mathbb{Q}_p$. It is a dense subgroup of $G_{\mathbb{Q}_p}$. So

$$\{\text{irreps of } G_{\mathbb{Q}_p}\} \hookrightarrow \{\text{irreps of } W_{\mathbb{Q}_p}\}.$$

An element of $W_{\mathbb{Q}_p}$ which maps to $1 \in \mathbb{Z}$ is called a **Frobenius element**.

Local CLass Field Theory

There is a natural isomorphism of topological groups

$$art : \mathbb{Q}_p^\times \longrightarrow W_{\mathbb{Q}_p}^{ab}$$

which induces an isomorphism

$$art : \widehat{\mathbb{Q}_p^\times} \longrightarrow G_{\mathbb{Q}_p}^{ab}$$

where $\widehat{\mathbb{Q}_p^\times}$ is the profinite completion of $\mathbb{Q}_p^\times$. Indeed one has:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_p & \longrightarrow & \mathbb{Q}_p^\times & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & I_p^{ab} & \longrightarrow & W_{\mathbb{Q}_p}^{ab} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

Corollary:

The isomorphism art induces a natural bijection:

$$\boxed{\text{irreducible representations of } \mathbb{Q}_p^\times}$$

$$\updownarrow$$

$$\boxed{\text{1-dimensional representations of } W_{\mathbb{Q}_p}}$$

The **local Langlands correspondence** extends this to general n-dimensional representations of $W_{\mathbb{Q}_p}$.

How is this done?

Corollary:

The isomorphism art induces a natural bijection:

$$\boxed{\text{irreducible representations of } \mathbb{Q}_p^\times}$$

$$\updownarrow$$

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The **local Langlands correspondence** extends this to general n-dimensional representations of $W_{\mathbb{Q}_p}$.

How is this done?

Answer: Replace 1 by n.

Recall:

irreducible representations of $\mathbb{Q}_p^\times = \mathsf{GL}_1(\mathbb{Q}_p)$

$$\updownarrow$$

1-dimensional representations of $W_{\mathbb{Q}_p}$

So we guess:

Conjecture (1st form) There is a natural bijection:

irreducible representations of $\mathsf{GL}_n(\mathbb{Q}_p)$

$$\updownarrow$$

n-dimensional representations of $W_{\mathbb{Q}_p}$

Remarks:

(i) (**Smooth Reps**)

One needs to say more precisely what representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ one means.

We shall consider **smooth $\mathbb{C}$-representations** of $\mathrm{GL}_n(\mathbb{Q}_p)$. This is a pair (π, V) where

- V is a $\mathbb{C}$-vector space, possibly infinite-dimensional, and

- $$\pi : \mathrm{GL}_n(\mathbb{Q}_p) \longrightarrow \mathrm{GL}(V)$$

  with the continuity assumption:

  (*) the stabilizer of each $v \in V$ is an open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Equivalently, for each $v \in V$, the orbit map

$$\mathrm{GL}_n(\mathbb{Q}_p) \longrightarrow V$$

$$g \mapsto \pi(g)(v)$$

is continuous if one gives V the discrete topology.

In particular, the topology of $\mathbb{C}$ is irrelevant: one could replace $\mathbb{C}$ by any uncountable algebraically closed field of characteristic 0.

Almost all irreducible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ are infinite-dimensional. The only finite dimensional ones are of the form:

$$g \mapsto \chi(\det g)$$

where

$$\det : \mathrm{GL}_n(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$$

and

$$\chi : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times.$$

(ii) (**Langlands parameters**) We also need to explicate what we mean by n-dimensional representations of $W_{\mathbb{Q}_p}$. This is a pair (ϕ, V) with

$$\phi : W_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}(V) \quad \dim_{\mathbb{C}} V = n$$

which is *continuous* and *semisimple*. Such a representation of $W_{\mathbb{Q}_p}$ is called a **Langlands parameter** or **L-parameter** for GL_n.

Finally, observe that we did not require the representation (ϕ, V) to be irreducible. This may seem strange, for one expects that one understands the reducible (ϕ, V) by induction on $\dim V$, so that the most fundamental part must be the irreducible (ϕ, V).

Now let's do some experiment to see if the conjecture is reasonable as stated.

Given a number of representations

$$\phi_i : W_{\mathbb{Q}_p} \longrightarrow \mathsf{GL}(V_i) \qquad \dim V_i = n_i$$

one can form the direct sum

$$\phi = \bigoplus_i \phi_i : W_{\mathbb{Q}_p} \longrightarrow \mathsf{GL}(V)$$

with

$$V = \bigoplus V_i, \qquad n = \dim V = \sum_i n_i.$$

Is there a similar operation on the other side? In other words, given irreducible representations π_i of $\mathsf{GL}_{n_i}(\mathbb{Q}_p)$, is there a way of producing an irreducible representation π of $\mathsf{GL}_n(\mathbb{Q}_p)$ from the π_i's?

Parabolic Induction

A parabolic subgroup P of GL_n is a subgroup of block upper triangular matrices. Given any partition

$$n = n_1 + ...n_r$$

one has an associated parabolic subgroup, whose diagonal entries are blocks of sizes $n_1, ..., n_r$. The parabolic subgroup P is a semi-direct product

$$P = M \ltimes N$$

where M is the subgroup of block-diagonal matrices. So

$$M = \mathsf{GL}_{n_1}(\mathbb{Q}_p) \times .... \times \mathsf{GL}_{n_r}(\mathbb{Q}_p)$$

and N consists of those elements of P whose diagonal block are identity matrices.

Example: Take the partition

$$6 = 3 + 2 + 1.$$

Then elements of P look like:

$$\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & c \end{pmatrix}$$

where A is 3×3, B is 2×2 and c is a scalar.

Elements of M are matrices above with all

$$* = 0.$$

Elements in N are matrices above with A, B and c equal to the identity matrices of the relevant sizes.

In any case, given π_i's as above, we have the representation

$$\pi_1 \boxtimes ... \boxtimes \pi_r \quad \text{of } M.$$

We may pull this back to P and then induce to $\mathsf{GL}_n(\mathbb{Q}_p)$. Thus we set

$$\pi = Ind_P^{\mathsf{GL}_n(\mathbb{Q}_p)} \pi_1 \boxtimes ... \boxtimes \pi_r.$$

It is comforting to note that:

If one permutes the π_i's, then the new induced representation is isomorphic to the original one, ar least after semi-simplification.

Reduction to Basic Case

The compatibility of the direct-sum construction and parabolic induction reduces the Conjecture (first form) to:

Conjecture (Basic Case)

There is a natural bijection

irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ not obtainable via parabolic induction

$$\updownarrow$$

n-dimensional irreducible representations of $W_{\mathbb{Q}_p}$

The representations in the former set are called **supercuspidal representations**.

It turns out that Conjecture (Basic Case) is true, but Conjecture (first form) is wrong.

Why?

There is a little problem with the reduction step:

Problem: the parabolically induced representation

$$\pi = Ind_P^{\mathsf{GL}_n(\mathbb{Q}_p)} \pi_1 \boxtimes ... \boxtimes \pi_r$$

is irreducible most of the time but not always.

This means: there are more elements in

$$\boxed{\text{irreducible representations of } \mathsf{GL}_n(\mathbb{Q}_p)}$$

than

$$\boxed{\text{n-dim representations of } WD_{\mathbb{Q}_p}}$$

To restore the bijectivity, we enlarge the latter set, following Deligne.

The Weil-Deligne Group

Deligne's idea: consider continuous ℓ-adic representations of $W_{\mathbb{Q}_p}$ instead of $\mathbb{C}$-representations.

But he showed that it is equivalent to consider the $\mathbb{C}$-representations of

$$WD_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}).$$

This is the Weil-Deligne group. So the right conjecture is:

Local Langlands Conjecture for $\mathbf{GL}_n$:

There is a natural bijection

$$\boxed{\text{irreducible representations of } \mathrm{GL}_n(\mathbb{Q}_p)}$$

$$\updownarrow$$

$$\boxed{\text{n-dimensional representations of } WD_{\mathbb{Q}_p}}$$

What does the word "**natural**" in the conjecture refers to?

Theorem of Harris-Taylor and Henniart

The Local Langlands Conjecture (LLC) is true. More precisely, there is a **unique** system of bijections

$$\boxed{\text{irreducible representations of } \mathrm{GL}_n(\mathbb{Q}_p)}$$
$$\Big\downarrow rec_n$$
$$\boxed{\text{n-dimensional representations of } WD_{\mathbb{Q}_p}}$$

satisfying:

- rec_1 is the bijection induced by local class field theory.

- rec_n is compatible with twisting by 1-dimensional characters of $\mathrm{GL}_1(\mathbb{Q}_p)$ and $W_{\mathbb{Q}_p}$:

$$rec_n(\pi \otimes \chi \circ \det) = rec_n(\pi) \otimes rec_1(\chi).$$

- the central character of π corresponds to the determinant character of $rec_n(\pi)$:

$$rec_1(\omega_\pi) = \det rec_n(\pi).$$

- rec_n respects the formation of dual representations:

$$rec_n(\pi^\vee) = rec_n(\pi)^\vee$$

- rec_n respects certain natural invariants attached to both sides, namely L-factors and ϵ-factors of pairs.

This last condition is the most important one.

L- and ϵ-factors of pairs

More precisely, given a pair of representations

$$\begin{cases} \pi_n & \text{of } \mathsf{GL}_n(\mathbb{Q}_p) \\ \pi_m & \text{of } \mathsf{GL}_m(\mathbb{Q}_p), \end{cases}$$

Jacquet, Piatetski-Shapiro and Shalika defined two invariants

$$L(s, \pi_m \times \pi_N) \quad \text{and} \quad \epsilon(s, \pi_m \times \pi_n, \psi),$$

where ψ is an additive character of $\mathbb{Q}_p$.

On the other hand, given representations ϕ_m and ϕ_n of $WD_{\mathbb{Q}_p}$, Artin, Deligne and Langlands defined

$$L(s, \phi_m \otimes \phi_n) \quad \text{and} \quad \epsilon(s, \phi_m \otimes \phi_n, \psi).$$

The last requirement in the theorem is:

$$\begin{cases} L(s, \pi_m \times \pi_n) = L(s, \phi_m \otimes \phi_n) \\ \epsilon(s, \pi_m \times \pi_n, \psi) = \epsilon(s, \phi_m \otimes \phi_n, \psi). \end{cases}$$

Very short sketch of the idea of proof

As we mention, we first reduce to the basic case. Then we find the desired bijection in nature.

More precisely, there is a "space" X which has an action of $W_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$. Then the coholomogy $H_c^*(X)$ of X becomes a representation for $W_{\mathbb{Q}_p} \times \mathrm{GL}_n(\mathbb{Q}_p)$. It decomposes as:

$$H_c^*(X) = \bigoplus_{\pi} \pi \boxtimes V_\pi$$

where the sum runs over the irreducible supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, and V_π is the multiplicity space, which naturally has an action of $W_{\mathbb{Q}_p}$. It turns out that V_π is an isotypic sum of an irreducible n-dimensional representation of $W_{\mathbb{Q}_p}$:

$$V_\pi = m_\pi \cdot \phi_\pi.$$

What next?

For a representation theorist, $\mathrm{GL}_n(\mathbb{Q}_p)$ is just one example of p-adic Lie groups.

If we view the LLC for GL_n as a classification of irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ in terms of Galois theoretic data, one can ask if there is a similar classification for any (connected split reductive) G.

So we want to generalize the bijection

$$\boxed{\text{irreducible representations of } \mathrm{GL}_n(\mathbb{Q}_p)}$$

$$\downarrow$$

$$\boxed{\text{n-dimensional representations of } WD_{\mathbb{Q}_p}}$$

$$\|$$

$$\boxed{\phi : WD_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{C})}$$

To do this, we replace GL_n by G................we get:

$$\boxed{\text{irreducible representations of } G(\mathbb{Q}_p)}$$
$$\downarrow$$
$$\boxed{\rho : WD_{\mathbb{Q}_p} \to G(\mathbb{C})}$$

This is almost right, but not quite. One needs the following corrections, which require two fundamental ideas (both due to Langlands):

(i) (**Langlands Dual Group**)

Need to replace $G(\mathbb{C})$ by the **Langlands dual group** $G^{\vee}(\mathbb{C})$ of G. This is a complex Lie group obtained from G by

"switching the long and short roots of the root system attached to G".

G	GL_n	SL_n	Sp_{2n}	SO_{2n}	GSp_{2n}
$G^{\vee}$	GL_n	PGL_n	SO_{2n+1}	SO_{2n}	GSpin_{2n+1}

The introduction of this dual group is one of the most important contributions of Langlands.

(ii) (L-**indistinguishability**)

For various reasons, one only expects to have a **surjective, finite-to-one** map

$$\boxed{\text{irreducible representations of } G(\mathbb{Q}_p)}$$
$$\downarrow$$
$$\boxed{\rho : WD_{\mathbb{Q}_p} \to G^\vee(\mathbb{C})}$$

The fibers of this map are called **L-packets**, and representations in an L-packet are said to be L-indistinguishable.

Moreover, one expects the fiber over a point ϕ to be described as follows. Given ϕ, consider the centralizer $Z_{G^\vee}(\phi)$ of ϕ in $G^\vee$. Now we have the finite group

$$A_\phi = \pi_0(Z_G(\phi)/Z_{G^\vee}).$$

Then one expects that the fibers of ϕ are in bijection with

$$\widehat{A}_\phi = \{\text{irreducible reps of } A_\phi\}.$$

Now we have:

Local Langlands Conjecture for G

There is a natural bijection

$$\boxed{\text{irreducible representations of } G(\mathbb{Q}_p)}$$

$$\downarrow$$

$$\boxed{\begin{array}{c} (\phi, \rho): \\ \phi : WD_{\mathbb{Q}_p} \to G^{\vee}(\mathbb{C}) \\ \rho \in \widehat{A}_{\phi} \end{array}}$$

The word "natural" here refers to the requirement that this bijection is suppsoed to be compatible with natural operations on both sides and to preserve natural invariants that one can attach to both sides.

Work of Debacker-Reeder

One direction of progress is to show the above bijection for certain subsets of the two sides for *general* G. For example, a result of Debacker-Reeder gives a construction

certain depth zero supercuspidal representations of $G(\mathbb{Q}_p)$

$$\downarrow$$

(ϕ, ρ): ϕ is tamely ramified

The Case of $\mathbf{GSp}_4$ Another is to establish the full conjecture for specific G.

We will hencforth focus on $G = \mathsf{GSp}_4$ over $\mathbb{Q}_p$.

More precisely, given a 4-dim $\mathbb{Q}_p$-vector space V equipped with a nondegenerate skew-symmetric form B,

$$G \subset \mathsf{GL}(V) \times \mathsf{GL}_1$$

is the subgroup defined by

$$\{(g, \lambda) : B(gv_1, gv_2) = \lambda \cdot B(v_1, v_2)\}.$$

The second projection gives the similitude character

$$sim : G \longrightarrow \mathsf{GL}_1$$

In this case, we have

$$G^\vee = \mathsf{GSpin}_5(\mathbb{C}) \cong \mathsf{GSp}_4(\mathbb{C})$$

Theorem (joint with S. Takeda)

There is a **unique** surjective, finite-to-one map

$$\boxed{\text{irreducible representations of } G(\mathbb{Q}_p)}$$
$$\Big\downarrow rec$$
$$\boxed{\rho : WD_{\mathbb{Q}_p} \to G^\vee(\mathbb{C})}$$

satisfying:

- the fiber of rec over ϕ are in natural bijection with $\widehat{A}_\phi$, and has at most 2 elements.

- (basic case) π is a discrete seris rep iff $rec(\pi)$ fdoes not factor through a proper parabolic.

- (twisting) $rec(\pi \otimes \chi \circ sim) = rec(\pi) \otimes rec_1(\chi)$.

- (central character) $rec_1(\omega_\pi) = sim \circ rec(\pi)$.

- (local factors) rec respects L- and $\epsilon-$factors of pairs, for generic or non-supercuspidal reps of $\mathrm{GSp}_4(\mathbb{Q}_p)$.

- (Plancherel) rec respects the Plancherel measure for pairs associated to non-generic supercuspidal reps of $\mathrm{GSp}_4(\mathbb{Q}_p)$.

The local factors of pairs have not been defined for non-generic supercuspidal reps. The Plancherel measure for pairs is a weaker invariant which is defined for all reps.

Remarks for Experts:

(i) In another paper with S. Takeda, we show how the LLC for Sp_4 follows from that for GSp_4.

(ii) In a paper with W. Tantono, we prove the LLC for the unique inner form of GSp_4. This is the group $GU_2(D)$, where D is the quaternion division algebra over $\mathbb{Q}_p$.

(iii) In a paper with P. S. Chan, we prove that the L-packets for GSp_4 and its inner form are stable and satisfy the desired character identities required by the theory of (twisted) endoscopy.

(iv) In her PhD thesis at UCSD, J. Lust has verified that the Debacker-Reeder packets agree with those given by the LLC for GSp_4.

Idea of Proof

One possible idea is to imitate the proof for GL_n, i.e. find a space X which has an action of $\mathrm{GSp}_4(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$. Such a candidate space exists, but it is difficult to analyze its cohomology.

Here's an alternative.

Let

$$\Pi(G) =$$

$$\{\text{irred. smooth representations of } G(\mathbb{Q}_p)\}$$

and

$$\Phi(G) = \{\text{L -parameters for } G\}$$

The proof then consists of relating these data for GSp_4 with those of $\mathrm{GL}_4 \times \mathrm{GL}_1$.

This is done by showing:

(i) There is a natural map

$$\theta : \Pi(\mathrm{GSp}_4) \longrightarrow \Pi(\mathrm{GL}_4 \times \mathrm{GL}_1)$$

whose fibers have size at most 2. This map is given by the **theta correspondence**.

(ii) There is a natural injection

$$\Phi(\mathrm{GSp}_4) \hookrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$$

This is given by composition with the natural embedding

$$\mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Moreover, the image can be characterized.

Given $\pi \in \Pi(\mathsf{GSp}_4)$, one has

$$\begin{array}{c} \pi \in \Pi(\mathsf{GSp}_4) \\ \downarrow \\ \theta(\pi) \in \Pi(\mathsf{GL}_4 \times \mathsf{GL}_1) \\ \downarrow \\ rec_{\mathsf{GL}_4 \times \mathsf{GL}_1}(\theta(\pi)) \\ \| \\ (\phi, \chi) \end{array}$$

$$\in \Phi(\mathsf{GL}_4 \times \mathsf{GL}_1)$$

The characterization of the image shows that ϕ factors through $\mathsf{GSp}_4(\mathbb{C})$ with $sim \circ \phi = \chi$, so that

$$(\phi, \chi) \in \Phi(\mathsf{GSp}_4).$$

Theta Correspondence

The map θ is found in nature.

Given a quadratic space V and a symplectic space $W = X + Y$ (X and Y totally isotropic), the space

$$\Omega = \{\text{Functions on } F^\times + V \otimes X\}$$

carries an action of

$$\mathrm{GSp}(W) \times \mathrm{GSO}(V).$$

This is the **Weil representation**, which gives a representation theoretic formulation of the classical **theta functions**.

One decomposes this:

$$\Omega = \bigoplus_{\pi \in \Pi(\mathrm{GSp}(W))} \pi \boxtimes \theta(\pi)$$

where $\theta(\pi)$ is the multiplicity space which carries an action of $\mathsf{GSO}(V)$. If one shows that $\theta(\pi)$ is always irreducible, one gets a map

$$\theta : \Pi(\mathsf{GSp}(W)) \longrightarrow \Pi(\mathsf{GSO}(V)),$$

called the theta correspondence.

In our case, we take $\dim W = 4$ and $\dim V = 6$ and note that

$$\mathsf{GSO}_6(\mathbb{Q}_p) =$$

$$(\mathsf{GL}_4(\mathbb{Q}_p) \times \mathsf{GL}_1(\mathbb{Q}_p))/\{(t, t^{-2}) : t \in \mathbb{Q}_p^\times\}$$

Work of Arthur and Moeglin

A forthcoming book of Arthur, together with the work of Moeglin, will prove, among other things, the LLC for all quasi-split classical groups, i.e.

- the symplectic groups $\mathrm{Sp}(W)$,
- the special orthogonal groups $\mathrm{SO}(V)$,
- the unitary groups $\mathbb{U}(V)$.

The proof relies also on relating the representations of these groups to those of GL_n. The analog of the map θ is provided by using the twisted Arthur-Selberg trace formula.

Further Developments

(i) (Mod ℓ LLC, $\ell \neq p$)

Replace $\mathbb{C}$ by $\overline{\mathbb{F}_\ell}$

$$\boxed{\text{irreducible } \overline{\mathbb{F}_\ell}\text{-representations of } G(\mathbb{Q}_p)}$$

$$\downarrow$$

$$\boxed{\rho : WD_{\mathbb{Q}_p} \to G^\vee(\overline{\mathbb{F}_\ell})}$$

The case $G = \mathrm{GL}_n$ is a theorem of Vigneras. The case $G = \mathrm{GSp}_4$ is ongoing work with A. Minguez.

(ii) (Mod p and p-adic LLC)

Replace $\mathbb{C}$ by $\overline{\mathbb{F}}_p$ or $\overline{\mathbb{Q}}_p$.

$$\boxed{\text{irreducible } \overline{\mathbb{F}}_p\text{-representations of } G(\mathbb{Q}_p)}$$
$$\downarrow$$
$$\boxed{\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G^{\vee}(\overline{\mathbb{F}}_p)}$$

Only the case of GL_2 is known, primarily through the work of C. Breuil, P. Colmez, M. Emerton, Schneider-Teitelbaum....

(iii) (Geometric LLC)

Work of E. Frenkel, D. Gaitsgory, S. Lysenko, V. Lafforgue........