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The Local Langlands Conjecture for Sp(4)

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We show that the local Langlands conjecture for $\mathrm{Sp}_{2n}(F)$ follows from that for $\mathrm{GSp}_{2n}(F)$, where F is a non-Archimedean local field of characteristic 0. In particular, we prove the local Langlands conjecture for $\mathrm{Sp}_4(F)$ based on our previous work [6] on the local Langlands conjecture for $\mathrm{GSp}_4(F)$. We also determine the possible sizes of L-packets for $\mathrm{Sp}_4(F)$.

1 Introduction

Let F be a non-Archimedean local field of characteristic 0 and residue characteristic p. Let W_F be the Weil group of F, and let $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group. It was shown by Harris–Taylor [8] and Henniart [10] that there is a natural bijection between the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$ and the set of conjugacy classes of L-parameters for GL_n, i.e. admissible homomorphisms

$$\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

This bijection satisfies a number of natural conditions which determines it uniquely.

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For a general connected reductive group G over F, which we assume to be split for simplicity, Langlands conjectures that there is a surjective finite-to-one map from the set $\Pi(G)$ of (equivalence classes of) irreducible admissible representations of $G(F)$ to the set $\Phi(G)$ of (equivalence classes of) admissible homomorphisms

$$W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \hat{G},$$

where $\hat{G}$ is the Langlands dual group of G and the homomorphisms are taken up to $\hat{G}$-conjugacy. This leads to a partition of the set $\Phi(G)$ into a disjoint union of finite subsets, which are the fibers of the map and are called L-packets.

In an earlier paper [6], we have demonstrated the local Langlands conjecture for the group GSp_4. In this paper, we shall show how the local Langlands conjecture for GSp_{2n}, together with some of its expected properties, implies the local Langlands conjecture for Sp_{2n}. Together with [6], this proves the local Langlands conjecture for Sp_4. Our main theorem is:

Main Theorem

There is a surjective finite-to-one map

$$\mathcal{L} : \Pi(\mathrm{Sp}_4) \longrightarrow \Phi(\mathrm{Sp}_4)$$

such that for an L-parameter φ, its fiber $\mathcal{L}_\varphi$ is of the same size (i.e. the same cardinality) as the character group of the (Abelian) component group

$$A_\varphi = \pi_0(Z(Im(\varphi))),$$

where $Z(Im(\varphi))$ is the centralizer of the image of φ in $\widehat{\mathrm{Sp}_4} = \mathrm{SO}_5(\mathbb{C})$. Moreover, the map $\mathcal{L}$ satisfies the following properties:

(i) $\mathcal{L}$ preserves local γ-factors, L-factors, and ϵ-factors of pairs whenever those factors are defined and satisfy some standard properties; see Section 3 for the details. More precisely, suppose that σ is an irreducible representation of $\mathrm{GL}_r(F)$ and ϖ is an irreducible representation of $\mathrm{Sp}_4(F)$ which we assume to be either generic or non-supercuspidal if $r > 1$. Let ϕ_σ and φ_ϖ be the L-parameters of σ and ϖ, respectively. Then we have

$$\begin{cases} \gamma(s, \varpi \times \sigma, \mathrm{std} \otimes \mathrm{std}, \psi) = \gamma(s, \varphi_\varpi \otimes \phi_\sigma, \psi) \\ L(s, \varpi \times \sigma, \mathrm{std} \otimes \mathrm{std}) = L(s, \varphi_\varpi \otimes \phi_\sigma) \\ \epsilon(s, \varpi \times \sigma, \mathrm{std} \otimes \mathrm{std}, \psi) = \epsilon(s, \varphi_\varpi \otimes \phi_\sigma, \psi). \end{cases}$$

Here, the functions on the right-hand side (RHS) are the local factors of Artin type associated to the relevant representations of the Weil–Deligne group WD_F. Those on the left-hand side (LHS) are the local factors attached by Shahidi [17] to the generic representations of $\mathrm{Sp}_4(F) \times \mathrm{GL}_r(F)$ and the standard representation std ⊗ std of the dual group $\mathrm{SO}_5(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$ and extended to all non-generic non-supercuspidal representations using the Langlands classification and multiplicativity. When $r = 1$, the local factors on the LHS can (alternatively) be defined for all representations using the doubling method of Piatetski-Shapiro and Rallis [15].

(ii) Suppose that ϖ is a non-generic supercuspidal representation. For any irreducible supercuspidal representation σ of $\mathrm{GL}_r(F)$, let $\mu(s, \varpi \boxtimes \sigma)$ denote the Plancherel measure associated to the family of induced representations $I_P(\varpi \boxtimes \sigma, s)$ on $\mathrm{Sp}_{2r+4}(F)$, where we have regarded $\varpi \boxtimes \sigma$ as a representation of the Levi subgroup $\mathrm{Sp}_4(F) \times \mathrm{GL}_r(F)$. Then $\mu(s, \varpi \boxtimes \sigma)$ is equal to

$$\gamma(s, \varphi_\varpi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \varphi_\varpi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \textstyle\bigwedge^2 \phi_\sigma, \psi) \cdot \gamma(-2s, \textstyle\bigwedge^2 \phi_\sigma^\vee, \overline{\psi}).$$

(iii) The representation ϖ is a discrete series representation if and only if (iff) its L-parameter $\mathcal{L}(\varpi)$ does not factor through any proper parabolic subgroup of $\mathrm{SO}_5(\mathbb{C})$.

(iv) An L-packet $\mathcal{L}_\varphi$ contains a generic representation iff the adjoint L-factor $L(s, \mathrm{Ad} \circ \varphi)$ is holomorphic at $s = 1$. Here, Ad denotes the adjoint representation of $\mathrm{SO}_5(\mathbb{C})$ on the complex Lie algebra $\mathfrak{so}(5)$. Moreover, $\mathcal{L}_\varphi$ contains a tempered generic representation iff φ is a tempered L-parameter, i.e. $\varphi|_{W_F}$ has bounded image in $\mathrm{SO}_5(\mathbb{C})$.

(v) The map $\mathcal{L} : \Pi(\mathrm{Sp}_4) \longrightarrow \Phi(\mathrm{Sp}_4)$ is the unique one, such that one has the following commutative diagram:

$$\begin{tikzcd}
\Pi(\mathrm{GSp}_4) \arrow[r, "L"] \arrow[d] & \Phi(\mathrm{GSp}_4) \arrow[d, "\mathrm{std}"] \\
\Pi(\mathrm{Sp}_4) \arrow[r, "\mathcal{L}"] & \Phi(\mathrm{Sp}_4)
\end{tikzcd}$$

Here, note that the left vertical arrow is not a map at all: it is a correspondence defined by the subset of $\Pi(\mathrm{GSp}_4) \times \Pi(\mathrm{Sp}_4)$ consisting of pairs (π, ϖ) such that ϖ is a constituent of the restriction of π to Sp_4. Moreover, the map L is given by the local Langlands correspondence for GSp_4 given in [6, Main Theorem], and the right vertical arrow is defined by composition with the natural projection

$$\mathrm{std} : \mathrm{GSp}_4(\mathbb{C}) \to \mathrm{PGSp}_4(\mathbb{C}) \cong \mathrm{SO}_5(\mathbb{C}),$$

which gives the standard representation of $\mathrm{GSp}_4(\mathbb{C})$.

(vi) The map $\mathcal{L}$ is uniquely determined by properties (i) (with $r \leq 3$), (ii) (with $r \leq 4$), and (iii). □

Our proof is essentially a modification of the earlier work by Gelbart and Knapp [4], in which they derived the local Langlands conjecture for SL_n from that for GL_n. As suggested by property (v) of the Main Theorem, the definition of $\mathcal{L}$ is given as follows. For each $\varpi \in \Pi(\mathrm{Sp}_4)$ one can find $\pi \in \Pi(\mathrm{GSp}_4)$ such that the restriction $\pi|_{\mathrm{Sp}_4}$ contains ϖ. If $\phi : \mathrm{WD}_F \to \mathrm{GSp}_4(\mathbb{C})$ is the L-parameter of π attached by [6], then the composite

$$\varphi := \mathrm{std}(\phi)$$

of ϕ with the projection std gives the L-parameter for ϖ. After checking that this is well defined, the main point is to verify that the fiber $\mathcal{L}_\varphi$ of $\mathcal{L}$ over φ has size $\#A_\varphi$. This is shown by establishing the following key short exact sequence:

$$1 \longrightarrow A_\phi \longrightarrow A_\varphi \longrightarrow I(\phi) \longrightarrow 1,$$

where

$$I(\phi) = \{\text{characters } \chi \text{ of } W_F \colon \phi \otimes \chi \cong \phi \text{ as elements of } \Phi(\mathrm{GSp}_4)\}.$$

We then give a more precise determination of the size of an L-packet of Sp_4. Unlike the case of GSp_4, where an L-packet is of size 1 or 2, for each L-parameter $\varphi \in \Phi(\mathrm{Sp}_4)$, we have:

$$\#L_\varphi = \begin{cases} 1, 2, 4, \text{ or } 8, & \text{if } p \neq 2; \\ 1, 2, 4, 8, \text{ or } 16, & \text{if } p = 2. \end{cases}$$

In Section 6, we give precise conditions for each of these possibilities in terms of the Galois theoretic properties of a GSp_4-parameter ϕ for which $\varphi = \mathrm{std}(\phi)$. Our result is the analog of the well-known fact that an L-packet of SL_2 associated to an irreducible representation π of $\mathrm{GL}_2(F)$ has size 1, 2, or 4, depending on whether π is primitive, dihedral with respect to (wrt) a unique quadratic field, or dihedral wrt three quadratic fields.

2 From GSp_{2n} to Sp_{2n}

For any connected split reductive group G over a non-Archimedean local field F, let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$ and $\Phi(G)$ the set of equivalence classes of admissible homomorphisms $\mathrm{WD}_F \to \hat{G}$. We will derive the local Langlands conjecture for Sp_{2n} from that for GSp_{2n}. First, let us note that

$$\widehat{\mathrm{GSp}_{2n}} = \mathrm{GSpin}_{2n+1}(\mathbb{C}) \quad \text{and} \quad \widehat{\mathrm{Sp}_{2n}} = \mathrm{PGSpin}_{2n+1}(\mathbb{C}) \cong \mathrm{SO}_{2n+1}(\mathbb{C})$$

(we refer the reader to [2, Section 2] for the structure of the group GSpin). Moreover, we have the canonical projection

$$\mathrm{std} : \mathrm{GSpin}_{2n+1}(\mathbb{C}) \twoheadrightarrow \mathrm{PGSpin}_{2n+1}(\mathbb{C}) \cong \mathrm{SO}_{2n+1}(\mathbb{C}),$$

which is simply the standard $(2n+1)$-dimensional representation of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$. We also let

$$\mathrm{sim} : \mathrm{GSp}_{2n} \to \mathbb{G}_m$$

be the similitude character. Using sim, we shall regard a character of $F^\times$ as a character of $\mathrm{GSp}_{2n}(F)$.

For the rest of this section, we assume the following working hypotheses, which we have verified for the case $n = 2$ in [6].

Working hypotheses

(H1) There is a surjective finite-to-one map $L : \Pi(\mathrm{GSp}_{2n}) \to \Phi(\mathrm{GSp}_{2n})$ with the property that for each $\phi \in \Phi(\mathrm{GSp}_{2n})$, the fiber L_ϕ is of the same size as the component group

$$A_\phi = \pi_0(Z(Im(\phi))),$$

where $Z(Im(\phi))$ is the centralizer of the image of ϕ.

(H2) For each quasicharacter χ of $F^\times$, we have

$$L(\pi \otimes \chi) = L(\pi) \otimes \chi,$$

where on the RHS, χ is viewed as a character of WD_F via local class field theory.

(H3) For each $\phi \in \Phi(\mathrm{GSp}_{2n})$, let

$$I(\phi) = \{\chi \in \widehat{(F^\times)} : \phi \cong \phi \otimes \chi\}.$$

Note that $I(\phi)$ consists of quadratic (and trivial) characters, since $\mathrm{sim}(\phi \otimes \chi) = \mathrm{sim}(\phi) \cdot \chi^2$. Similarly, for each $\pi \in \Pi(\mathrm{GSp}_{2n})$, let

$$I(\pi) = \{\chi \in \widehat{(F^\times)} : \pi \cong \pi \otimes \chi\},$$

which consists also of quadratic characters. Then for any $\pi \in L_\phi$,

$$I(\phi) = I(\pi). \qquad \square$$

We also note the following:

(H3a) For $\pi, \pi' \in L_\phi$, we have $I(\pi) = I(\pi')$.

(H3b) For each $\pi \in L_\phi$ and a quasicharacter χ,

$$\pi \otimes \chi \in L_\phi \Longrightarrow \pi \otimes \chi \cong \pi.$$

The following is easy to verify.

Lemma 2.1. Assume (H1) and (H2). Then (H3) holds iff (H3a) and (H3b) hold. $\square$

Now we have the following:

Proposition 2.2. If $n = 2$, i.e. for GSp_4, all of the above working hypotheses are satisfied. □

Proof. The hypotheses (H1) and (H2) are contained in the Main Theorem of [6]. For the hypothesis (H3), note that (H2) already implies that $I(\pi) \subset I(\phi)$ if $\pi \in L_\phi$. If $\chi \in I(\phi)$, then twisting by χ gives rise to a permutation of L_ϕ, but L_ϕ has size 1 or 2, and in the latter case, L_ϕ contains a unique generic representation. Since twisting by χ preserves genericity, the induced permutation must be trivial so that $\chi \in I(\pi)$ for any $\pi \in L_\phi$. This proves the proposition. ■

For each $\pi \in \Pi(\mathrm{GSp}_{2n})$, we define

$$\mathrm{JH}(\pi) := \{\text{constituents of } \pi|_{\mathrm{Sp}_{2n}}\}.$$

It has been shown by [1] that for each $\pi \in \Pi(\mathrm{GSp}_{2n})$, the restriction $\pi|_{\mathrm{Sp}_{2n}}$ is multiplicity-free. Hence, in the set $\mathrm{JH}(\pi)$, constituents can also be taken as constituents up to equivalence. Then the main result of this section is:

Theorem 2.3. Suppose that all of the above working hypotheses are satisfied (for example if $n = 2$). Then there is a surjective finite-to-one map

$$\mathcal{L} : \Pi(\mathrm{Sp}_{2n}) \to \Phi(\mathrm{Sp}_{2n})$$

so that for each $\varphi \in \Phi(\mathrm{Sp}_{2n})$, the fiber $\mathcal{L}_\varphi$ is given by

$$\mathcal{L}_\varphi = \bigcup_{\pi \in L_\phi} \mathrm{JH}(\pi) \quad \text{(disjoint union)},$$

for some (and any) lift ϕ of φ, i.e. such that $\mathrm{std}(\phi) = \varphi$. Moreover, the size of $\mathcal{L}_\varphi$ is equal to the size of the component group

$$A_\varphi = \pi_0(Z(Im(\varphi))).$$ □

We will prove this theorem step by step. As we mentioned in the introduction, our proof is a modification of the work by Gelbart and Knapp [4], so let us first quote a couple of general lemmas from [4] which we need for our proof.

Lemma 2.4. Let G be a totally disconnected locally compact group and H an open normal subgroup of G such that G/H is finite Abelian. Also let ϖ be an irreducible admissible representation of H. Suppose that π and π' are irreducible admissible representations of G whose restrictions to H are multiplicity-free and contain ϖ. Then

$$\pi|_H \cong \pi'|_H$$

and

$$\pi \cong \pi' \otimes \chi$$

for some one-dimensional character χ on G which is trivial on H. □

Proof. This is Lemma 2.4 of [4]. ■

Lemma 2.5. Let G and H be as in the above lemma and π an irreducible admissible representation of G. Assume that the restriction $\pi|_H$ is multiplicity-free and written as

$$\pi|_H = \varpi_1 \oplus \cdots \oplus \varpi_m,$$

where each ϖ_i is an irreducible admissible representation of H. Set

$$I_H(\pi) = \{\chi : G/H \longrightarrow \mathbb{C}^\times : \pi \otimes \chi = \pi\}$$

and

$$N_\pi = \bigcap_{\chi \in I_H(\pi)} \ker \chi$$

so that $I_H(\pi)$ is the Pontryagin dual of G/N_π. Then G/N_π acts simply transitively on the set $\{\varpi_1, \ldots, \varpi_m\}$ of irreducible constituents of $\pi|_H$. In particular, we have

$$m = \# I_H(\pi).$$ □

Proof. This is contained in Lemma 2.1 and Corollary 2.2 of [4]. Note that our $I_H(\pi)$ is denoted by $X_H(\pi)$ there. ■

Definition of the map $\mathcal{L}$

First, we consider how a representation π of $\mathrm{GSp}_{2n}(F)$ restricts to a representation of the group

$$H := F^{\times} \cdot \mathrm{Sp}_{2n}(F).$$

Note that H is an open normal subgroup of $\mathrm{GSp}_{2n}(F)$ such that the quotient $\mathrm{GSp}_4(F)/H \cong F^{\times}/F^{\times 2}$ is a finite elementary Abelian two-group. As we have mentioned before, it is shown in [1, Theorem 1.4] that for each $\pi \in \Pi(\mathrm{GSp}_{2n})$, the restriction $\pi|_{\mathrm{Sp}_{2n}}$ is multiplicity-free and hence, so is $\pi|_H$. Therefore, we can apply Lemmas 2.4 and 2.5 for $G = \mathrm{GSp}_{2n}(F)$ and $H = F^{\times} \cdot \mathrm{Sp}_{2n}(F)$.

Let ϖ be an irreducible admissible representation of $\mathrm{Sp}_{2n}(F)$ and μ a quasicharacter of $F^{\times}$. Then consider the representation $\mu \boxtimes \varpi$ of $F^{\times} \times \mathrm{Sp}_{2n}(F)$. If $\mu(-1) = \varpi(-1)$, then the representation factors through the surjection $F^{\times} \times \mathrm{Sp}_{2n}(F) \to H$, so we write a representation of H as $\mu \boxtimes \varpi$, with the assumption that $\mu(-1) = \varpi(-1)$.

Now for each $\varpi \in \Pi(\mathrm{Sp}_{2n})$, let us define

$$E(\varpi) = \{\pi \in \Pi(\mathrm{GSp}_{2n}) : \pi|_{\mathrm{Sp}_{2n}} \text{contains } \varpi\},$$

so that $E(\varpi)$ is the set of representations of $\mathrm{GSp}_{2n}(F)$ "lying above" ϖ. This set is nonempty, as can be seen as follows. For each given ϖ, let us pick a character μ of $F^{\times}$ such that $\mu(-1) = \varpi(-1)$. Then any irreducible constituent of $\mathrm{Ind}_H^{\mathrm{GSp}_{2n}(F)} \mu \boxtimes \varpi$ is an element of $E(\varpi)$. We need the following. □

Lemma 2.6. If $\pi, \pi' \in E(\varpi)$, then $\pi' \cong \pi \otimes \chi$ for some character χ. □

Proof. Firstly, for each irreducible representation $\mu \boxtimes \varpi$ of H, let us define

$$E(\mu \boxtimes \varpi) = \{\pi \in \Pi(\mathrm{GSp}_{2n}) : \pi|_H \text{ contains } \mu \boxtimes \varpi\}.$$

Secondly, assume that $\pi, \pi' \in E(\varpi)$. Then $\pi|_H$ (resp. $\pi'|_H$) contains $\mu \boxtimes \varpi$ (resp. $\mu' \boxtimes \varpi$) for some μ (resp. μ'), so we have

$$\pi \in E(\mu \boxtimes \varpi) \quad \text{and} \quad \pi' \in E(\mu' \boxtimes \varpi).$$

Then μ'/μ is trivial on $\{\pm 1\} \subset F^\times$ because

$$\mu(-1) = \mu'(-1) = \varpi(-1).$$

Hence, $\mu'/\mu = \nu^2$ for some quasicharacter ν, and thus $\mu' \boxtimes \varpi \cong (\mu\nu^2) \boxtimes \varpi$. Now consider $\pi \otimes \nu|_H$. This contains $(\mu\nu^2) \boxtimes \varpi \cong \mu' \boxtimes \varpi$ as a constituent since $\pi \in E(\mu \boxtimes \varpi)$, and so $\pi \otimes \nu \in E(\mu' \boxtimes \varpi)$. Thus, by Lemma 2.4, $\pi' \cong \pi \otimes \nu\chi$ for some quasicharacter χ. This proves the lemma. ■

Now we can define the map $\mathcal{L} : \Pi(\mathrm{Sp}_{2n}) \to \Phi(\mathrm{Sp}_{2n})$ by

$$\mathcal{L}(\varpi) = \mathrm{std}(L(\pi)), \text{ for any } \pi \in E(\varpi).$$

This is well defined. Indeed, if $\pi, \pi' \in E(\varpi)$, then the above lemma implies that $\pi' \cong \pi \otimes \chi$ for some χ, but by (H2), $L(\pi') = L(\pi) \otimes \chi$ and so

$$\mathrm{std}(L(\pi')) = \mathrm{std}(L(\pi) \otimes \chi) = \mathrm{std}(L(\pi)).$$

Surjectivity of the map $\mathcal{L}$

The surjectivity of $\mathcal{L}$ follows from the following. □

Lemma 2.7. Any (continuous) projective representation of the Weil–Deligne group

$$\rho : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

can be lifted to a representation

$$\tilde{\rho} : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}).$$

Moreover, if $\tilde{\rho}'$ is another lift of ρ, then $\tilde{\rho}' = \tilde{\rho} \otimes \chi$ for some character χ. □

Proof. Let $r_1 = \rho|_{W_F}$ and $r_2 = \rho|_{\mathrm{SL}_2(\mathbb{C})}$ so that r_1 and r_2 are projective representations of W_F and $\mathrm{SL}_2(\mathbb{C})$, respectively. Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected, r_2 has a unique lift

$$\tilde{r_2} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}).$$

Also, Henniart [9] has shown that r_1 has a lift

$$\tilde{r_1} : W_F \to \mathrm{GL}_n(\mathbb{C}).$$

Now let us define

$$\tilde{\rho} : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

by

$$\tilde{\rho}(g, h) = \tilde{r_1}(g)\tilde{r_2}(h).$$

To show that $\tilde{\rho}$ is indeed a representation of WD_F, it suffices to show that $\tilde{r_1}(g)\tilde{r_2}(h) = \tilde{r_2}(h)\tilde{r_1}(g)$. For this, let us define

$$c : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$$

by

$$c(g, h) = \tilde{r_1}(g) \cdot \tilde{r_2}(h) \cdot \tilde{r_1}(g)^{-1} \cdot \tilde{r_2}(h)^{-1}.$$

We will show that $c(g, h) = 1$ for all $g \in W_F$ and $h \in \mathrm{SL}_2(\mathbb{C})$. Note that $c(g, h)$ is in the center of $\mathrm{GL}_n(\mathbb{C})$ because the image of $c(g, h)$ in $\mathrm{PGL}_n(\mathbb{C})$ is trivial. Also notice that by taking the determinant of $c(g, h)$, we see that $c(g, h)^n = 1$, so the image of c is in the set of n-th roots of 1. Now let us fix g. Then the map $c(g, -)$ is a continuous map from $\mathrm{SL}_2(\mathbb{C})$ to the n-th roots of 1, so $c(g, -)$ must be a constant map since $\mathrm{SL}_2(\mathbb{C})$ is connected, but clearly, $c(g, 1) = 1$ and so $c(g, h) = 1$.

The latter part of the lemma is obvious. ■

Hence, we have

Proposition 2.8. Any $\varphi \in \Phi(\mathrm{Sp}_{2n})$ can be lifted to some $\phi \in \Phi(\mathrm{GSp}_{2n})$ and so $\mathcal{L}$ is surjective. Moreover, if ϕ' is another lift of φ, then $\phi' = \phi \otimes \chi$ for some character χ on WD_F. □

Proof. The group $\mathrm{GSpin}_{2n+1}(\mathbb{C})$ has a faithful 2^n-dimensional spin representation so that one has a commutative diagram:

$$\begin{tikzcd}
\mathrm{GSpin}_{2n+1}(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{GL}_{2^n}(\mathbb{C}) \arrow[d] \\
\mathrm{PGSpin}_{2n+1}(\mathbb{C}) \arrow[r] & \mathrm{PGL}_{2^n}(\mathbb{C}).
\end{tikzcd}$$

The proposition is now immediate from the above lemma. ■

The L-packet and the component group

Let us investigate the fiber $\mathcal{L}_\varphi$ for each $\varphi \in \Phi(\mathrm{Sp}_{2n})$. We begin by showing that

$$\mathcal{L}_\varphi = \bigcup_{\pi \in L_\phi} \mathrm{JH}(\pi) \quad \text{(disjoint union)},$$

where ϕ is any fixed lift of φ.

First, let us examine the RHS of the above equality. If ϕ and ϕ' are two different lifts of φ, then $\phi' = \phi \otimes \chi$ for some character χ on WD_F, but (H2) implies that we have a bijection $L_\phi \to L_{\phi'}$ given by $\pi \mapsto \pi \otimes \chi$ and clearly $\mathrm{JH}(\pi) = \mathrm{JH}(\pi \otimes \chi)$. Hence, the union on the RHS is independent of the choice of the lift ϕ.

Also, we can see that the union is disjoint. For if $\pi, \pi' \in L_\phi$ are such that there is a $\varpi \in \mathrm{JH}(\pi) \cap \mathrm{JH}(\pi')$, then by Lemma 2.6, we have $\pi' \cong \pi \otimes \chi$ for some character χ. Then by (H3b), we must have $\pi' \cong \pi$. It is now quite easy to see that

$$\mathcal{L}_\varphi = \bigcup_{\pi \in L_\phi} \mathrm{JH}(\pi).$$

Namely, if $\varpi \in \mathcal{L}_\varphi$, then by definition of $\mathcal{L}_\varphi$, we must have $\varpi \in \mathrm{JH}(\pi)$ with $\pi \in L_\phi$ for some lift ϕ. Conversely, if $\varpi \in \mathrm{JH}(\pi)$ with $\pi \in L_\phi$ for some lift ϕ, then once again by the definition of $\mathcal{L}$, we see that $\varpi \in \mathcal{L}_\varphi$.

Finally, we show that the fiber $\mathcal{L}_\varphi$ has the same size as the component group

$$A_\varphi = \pi_0(Z(Im(\varphi))) = Z(Im(\varphi))/Z^\circ(Im(\varphi)),$$

where $Z(Im(\varphi))$ is the centralizer of $Im(\varphi)$ and $Z^\circ(Im(\varphi))$ its connected component. By the above discussion, we see that

$$\#\mathcal{L}_\varphi = \#L_\phi \cdot \#\mathrm{JH}(\pi),$$

where ϕ is a lift of φ and $\pi \in L_\phi$. Now Lemma 2.5 and hypothesis (H3) imply that

$$\#\mathrm{JH}(\pi) = \#I(\pi) = \#I(\phi),$$

and (H1) says that

$$\#L_\phi = \#A_\phi = \#Z(Im(\phi))/Z^0(Im(\phi)).$$

Hence, we need to show that

$$\#A_\varphi = \#A_\phi \cdot \#I(\phi). \qquad \square$$

This is accomplished by the following key proposition:

Proposition 2.9. Let $\phi \in \Phi(\mathrm{GSp}_{2n})$, and set $\varphi = \mathrm{std}(\phi)$. Then there is a short exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & A_\phi \arrow[r, "\beta"] & A_\varphi \arrow[r, "\alpha"] & I(\phi) \arrow[r] & 1. \end{tikzcd} \qquad \square$$

Before giving the of the proof the proposition, let us quote a couple of basic facts about Lie groups, which we shall use for our proof.

Lemma 2.10. Let G be a Lie group and $H \subset G$ a closed subgroup. Then

(a) for each $s \in G/H$, there exists an open neighborhood U of s with a smooth (hence continuous) section $U \to G$, i.e. there exist local smooth sections of G/H in G;
(b) if H and G/H are connected, then G is connected. $\square$

Proof. For (a) see [21, Theorem 3.58(b)], and for (b) see [21, Proposition 3.66]. ■

Now we are ready to prove the proposition.

Proof of Proposition 2.9. The proof follows the lines of the proof of [4, Theorem 4.3]. We shall define a group homomorphism

$$Z(Im(\varphi)) \to I(\phi), \quad s \mapsto \chi_s,$$

as follows. First of all, let us define a function $\chi_s : \mathrm{WD}_F \to \mathrm{GSpin}_{2n+1}(\mathbb{C})$ by

$$\chi_s(w) = \tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} \cdot \phi(w)^{-1} \quad \text{for } w \in \mathrm{WD}_F,$$

where $\tilde{s}$ is a lift of s to $\mathrm{GSpin}_{2n+1}(\mathbb{C})$. It is clear that χ_s is well defined. We now note the following properties:

- χ_s takes values in $\{\pm 1\}$ in the center of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$. Indeed, since $\mathrm{std} \circ \chi_s = 1$, we have $\chi_s(w) \in Z_{\mathrm{GSpin}_{2n+1}} = \mathbb{C}^\times$. On the other hand, $\chi_s(w)$ is a commutator and thus lies in the derived group $\mathrm{Spin}_{2n+1}(\mathbb{C})$ of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$. Hence,

$$\chi_s(w) \in Z_{\mathrm{GSpin}_{2n+1}} \cap \mathrm{Spin}_{2n+1}(\mathbb{C}) = \{\pm 1\}.$$

- χ_s is a homomorphism and hence by the above assertion, a quadratic character of WD_F. This is because

$$\begin{aligned}\chi_s(ww') &= \tilde{s} \cdot \phi(ww') \cdot \tilde{s}^{-1} \cdot \phi(ww')^{-1} \\ &= \tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} \cdot \left(\tilde{s}\phi(w')\tilde{s}^{-1}\phi(w')^{-1}\right)\phi(w)^{-1} \\ &= \tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} \cdot \phi(w)^{-1}\left(\tilde{s}\phi(w')\tilde{s}^{-1}\phi(w')^{-1}\right), \text{ because } \tilde{s} \cdot \phi(w') \cdot \tilde{s}^{-1} \cdot \phi(w')^{-1} \in \mathbb{C}^\times \\ &= \chi_s(w) \cdot \chi_s(w').\end{aligned}$$

- $\chi_s \in I(\phi)$. Indeed, since

$$\tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} = \chi_s(w) \cdot \phi(w),$$

we have

$$\phi \cong \phi \otimes \chi_s \quad \text{as elements of } \Phi(\mathrm{GSp}_4),$$

and thus, $\chi_s \in I(\phi)$. The assignment $s \mapsto \chi_s$ thus gives a map

$$\alpha : Z(Im(\varphi)) \longrightarrow I(\phi).$$

- The map α is a group homomorphism. This is proved in a similar way as the proof above that χ_s is a character.
- The homomorphism α factors through A_φ so that we have

  $$\alpha : A_\varphi \longrightarrow I(\phi).$$

  To see this, note that for each $w \in \mathrm{WD}_F$, the map $s \mapsto \chi_s(w)$ is a continuous homomorphism

  $$Z(Im(\varphi)) \longrightarrow \{\pm 1\}$$

  because by Lemma 2.10 (a) one can choose a local continuous section $s \mapsto \tilde{s}$, and thus, it must be trivial on $Z^0(Im(\varphi))$.
- The homomorphism α is surjective. This is similar to [4, Theorem 4.3]. Namely, if $\chi \in I(\phi)$, then $\phi \cong \phi \otimes \chi$, i.e. there exists $\tilde{s} \in \mathrm{GSpin}_{2n+1}(\mathbb{C})$ such that

  $$\tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} = \chi(w) \cdot \phi(w)$$

  for all $w \in \mathrm{WD}_F$. If we let $s = \mathrm{std}(\tilde{s})$, we have $\chi_s = \chi$.

We now examine the kernel of the map α. Observe that via the projection map std, one has a natural map

$$\beta : Z(Im(\phi)) \longrightarrow Z(Im(\varphi)) \longrightarrow A_\varphi.$$

The map β has the following properties:

- The image of β is precisely the kernel of α. Indeed, if $\tilde{s} \in Z(Im(\phi))$, then $\tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} = \phi(w)$ for all $w \in \mathrm{WD}_F$ so that

  $$\chi_{\mathrm{std}(\tilde{s})}(w) = 1 \quad \text{for all } w \in \mathrm{WD}_F.$$

  Hence, the image of β is contained in the kernel of α. Conversely, let $s \in \ker\alpha$ so that $\chi_s(w) = 1$ for all $w \in \mathrm{WD}_F$. Then $\tilde{s} \cdot \phi(w) \cdot \tilde{s}^{-1} = \phi(w)$ for a lift $\tilde{s}$ of s, so $\tilde{s} \in Z(Im(\varphi))$ and $\beta(\tilde{s}) = s$. This shows that the image of β is precisely $\ker\alpha$.

- The kernel of β is equal to $Z^0(Im(\phi))$ so that β induces an injection

$$\beta : A_\phi \longrightarrow A_\varphi.$$

It is clear that $Z^0(Im(\phi)) \subset \ker \beta$ since std maps $Z^0(Im(\phi))$ into $Z^0(Im(\varphi))$. It remains to show the reverse containment. Let $\tilde{s} \in \ker \beta$ so that $\mathrm{std}(\tilde{s}) \in Z^0(Im(\varphi))$. Now observe that

$$Z^0(Im(\phi)) \subseteq \mathrm{std}^{-1}(Z^0(Im(\varphi))) \subseteq Z(Im(\phi)).$$

The first containment is what we have just observed. For the second, suppose that

$$\tilde{s} \in \mathrm{std}^{-1}(Z^0(Im(\varphi))) \text{ so that } s = \mathrm{std}(\tilde{s}) \in Z^0(Im(\varphi)).$$

Then we have seen that χ_s is trivial so that $\tilde{s}$ commutes with $Im(\phi)$, as desired. Note that since $\mathbb{C}^\times \subset \mathrm{std}^{-1}(Z^0(Im(\varphi)))$ is closed and $\mathrm{std}^{-1}(Z^0(Im(\varphi)))/\mathbb{C}^\times = Z^0(Im(\varphi))$ is connected, by Lemma 2.10 (b), we know that $\mathrm{std}^{-1}(Z^0(Im(\varphi)))$ is connected. Hence, the first containment is an equality and so $\tilde{s} \in Z^0(Im(\phi))$.

The proposition is proved. ■

This completes the proof of Theorem 2.3.

It is natural to ask if one can naturally index the elements of the L-packet $\mathcal{L}_\varphi$ by the Pontrjagin dual $\widehat{A_\varphi}$ of A_φ, say in the case of Sp_4. It is well known that to fix such a parametrization, one has to choose a generic character ψ of the unipotent radical of a Borel subgroup of $\mathrm{Sp}_4(F)$. Suppose that one has fixed such a choice of generic character ψ. Proposition 2.9 implies that

$$1 \longrightarrow \widehat{I(\phi)} \longrightarrow \widehat{A_\varphi} \longrightarrow \widehat{A_\phi} \longrightarrow 1.$$

Moreover, we already know from [6] that $\widehat{A_\phi}$ naturally indexes the elements in the L-packet L_ϕ (which has size 1 or 2). Thus, it is completely natural to insist that for a character η of A_ϕ, the constituents of $\pi_\eta|_{\mathrm{Sp}_4}$ correspond to those elements of $\widehat{A_\varphi}$ which restrict to η. If L_ϕ contains a generic representation π of $\mathrm{GSp}_4(F)$ (which corresponds to the trivial character of A_ϕ), then we can further insist that the trivial character of A_φ indexes the unique ψ-generic constituent in $\pi|_{\mathrm{Sp}_4}$. Since $\widehat{I(\phi)} = \mathrm{GSp}_4(F)/N_\pi$ acts transitively on the constituents of $\pi|_{\mathrm{Sp}_4}$, we then obtain a parametrization of the constituents of $\pi|_{\mathrm{Sp}_4}$ by the elements of $\widehat{A_\varphi}$ which restricts trivially to A_ϕ.

If L_ϕ contains another representation π' (which is necessarily non-generic), then we do not know how to parameterize the constituents of $\pi'|_{\mathrm{Sp}_4}$ by the remaining characters of A_φ. Indeed, the set of constituents of $\pi'|_{\mathrm{Sp}_4}$ and the set of characters of A_φ with non-trivial restriction to A_ϕ are both principal homogeneous spaces over $\widehat{I(\phi)}$, and the choice of a base point in each will provide a bijection between them. However, it seems that the only natural way to do this is via the character relations predicted by the theory of (twisted) endoscopy. The recent work of Hiraga–Saito [13] on the local Langlands correspondence for inner forms of SL_n may shed some light on this matter.

Remark 2.1. Let us mention here that our derivation of the local Langlands conjecture for Sp_{2n} from that for GSp_{2n} works for other pairs of groups such as $(\mathrm{GSO}_n, \mathrm{SO}_n)$ and $(\mathrm{GSpin}_{2n}, \mathrm{Spin}_{2n})$ as long as the multiplicity-free result of [1] is obtained for the relevant groups. □

3 Properties of $\mathcal{L}$

To prove the Main Theorem, it remains to verify the properties (i)–(vi) in the Main Theorem. In fact, many of these follow immediately from the construction of $\mathcal{L}$ and the analogous properties of the map L for GSp_4. We treat each property in turn:

(i) Let us treat the preservation of local factors described in property (i). We begin by recalling the precise definition of the local factors on the LHS of property (i). Let ϖ and σ be irreducible admissible representations of $\mathrm{Sp}_4(F)$ and $\mathrm{GL}_r(F)$, respectively. If ϖ and σ are both generic, then Shahidi has defined in [17, Theorem 3.5] the local factors associated to $\varpi \boxtimes \sigma$ of $\mathrm{Sp}_4(F) \times \mathrm{GL}_r(F)$ and the standard representations of the dual group $\mathrm{SO}_5(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$, which appears on the LHS of (i) (by an analysis of the Plancherel measure treated in (ii) below). One of the key properties of these local factors of Shahidi is that of multiplicativity of the local gamma factors under parabolic induction; for a discussion of this property in the case of interest, the reader can consult [18, Lecture 3, p. 318, Example 1]. The L-factor and ϵ-factor, on the other hand, only satisfy multiplicativity for the formation of standard modules.

To extend the definition of the local factors beyond generic representations, we make use of the Langlands classification and multiplicativity. If $\varpi \boxtimes \sigma$ is non-generic and non-supercuspidal, then by the Langlands classification, it is a quotient of a standard module $\mathrm{Ind}_P^{\mathrm{Sp}_4\times\mathrm{GL}_r}\tau$ with τ a discrete series representation up to twist. Since the Levi factor of any proper parabolic subgroup of $\mathrm{Sp}_4\times\mathrm{GL}_r$ is a product of GL_n's and perhaps SL_2, τ is a generic representation (as any discrete series representation of GL_n or SL_2 is generic). Thus, we may define the local factors associated to $\varpi\boxtimes\sigma$ by insisting that multiplicativity holds for such standard modules. Again, we refer the reader to [18, p. 318, Example 1] for the precise statements.

In the case $r=1$, there is an alternative definition of the local factors on the LHS of (i) via the doubling method of Piatetski-Shapiro and Rallis; a definitive treatment of this can be found in [15]. Though it covers only the case $r=1$, it has the advantage that the local factors can be defined for all representations $\varpi\boxtimes\sigma$, and not just for the generic ones. Moreover, these local factors also satisfy multiplicativity and agree with Shahidi's local factors when the representations involved are generic.

With the above definitions, the preservation of the local factors follows readily from [6, Theorem 8.3] and [6, Lemma 4.2].

(ii) To establish property (ii) in the Main Theorem, let us briefly recall the notion of the Plancherel measure in our context and prove a few facts on the Plancherel measure necessary for our purposes.

Let ϖ be an irreducible admissible representation of $\mathrm{Sp}_4(F)$ and σ a representation of $\mathrm{GL}_r(F)$ so that $\varpi\boxtimes\sigma$ is a representation of

$$M_r(F) := \mathrm{Sp}_4(F)\times\mathrm{GL}_r(F).$$

Now M_r is the Levi factor of a maximal parabolic subgroup $P_r = M_r\cdot N_r$ of $G_r = \mathrm{Sp}_{2r+4}$ so that one can form the generalized principal series representation

$$I_{P_r}(s,\varpi\boxtimes\sigma) = \mathrm{Ind}_{P_r}^{G_r}\varpi\boxtimes\sigma|\det|^s \quad \text{(normalized induction)},$$

where det is the determinant character of GL_r. If $\bar{P}_r = M_r\cdot\bar{N}_r$ is the opposite parabolic, then we similarly have the induced representation $I_{\bar{P}_r}(s,\varpi\boxtimes\sigma)$. There is a standard intertwining operator

$$A_\psi(s,\varpi\boxtimes\sigma,N_r,\bar{N}_r) : I_{P_r}(s,\varpi\boxtimes\sigma)\to I_{\bar{P}_r}(s,\varpi\boxtimes\sigma),$$

defined by

$$A_\psi(s, \varpi \boxtimes \sigma, N_r, \bar{N}_r) f(g) = \int_{\bar{N}_r} f(\bar{n}g)\, \mathrm{d}\bar{n}_\psi$$

for $f \in I_{P_r}(s, \varpi \boxtimes \sigma)$. Then the composite $A_\psi(s, \varpi \boxtimes \sigma, \bar{N}_r, N_r) \circ A_\psi(s, \varpi \boxtimes \sigma, N_r, \bar{N}_r)$ is a scalar operator on $I_{P_r}(s, \varpi \boxtimes \sigma)$ for generic s, and the Plancherel measure is the scalar-valued meromorphic function defined by

$$\mu(s, \varpi \boxtimes \sigma, \psi)^{-1} = A(s, \varpi \boxtimes \sigma, \bar{N}_r, N_r) \circ A(s, \varpi \boxtimes \sigma, N_r, \bar{N}_r).$$

By results of Shahidi [17, Theorem 3.5] and Henniart [12], we have

Proposition 3.1. Suppose that $\varpi \boxtimes \sigma$ is a generic representation of $M_r(F) = \mathrm{Sp}_4(F) \times \mathrm{GL}_r(F)$. Then for appropriate measures $\mathrm{d}n_\psi$ and $\mathrm{d}\bar{n}_\psi$ on N_r and $\bar{N}_r$, respectively, $\mu(s, \varpi \boxtimes \sigma, \psi)$ is equal to

$$\begin{aligned}&\gamma(s, \varpi^\vee \boxtimes \sigma, \psi) \cdot \gamma(-s, \varpi \boxtimes \sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \sigma, \textstyle\bigwedge^2, \psi) \cdot \gamma(-2s, \sigma^\vee, \textstyle\bigwedge^2, \overline{\psi})\\ &\quad = \gamma(s, \varphi_\varpi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \varphi_\varpi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \textstyle\bigwedge^2 \phi_\sigma, \psi) \cdot \gamma(-2s, \textstyle\bigwedge^2 \phi_\sigma^\vee, \overline{\psi}),\end{aligned}$$

where φ_ϖ is the L-parameter for ϖ by our construction and ϕ_σ is that for σ. □

Now the same consideration can be applied to the group

$$M_r'(F) = \mathrm{GSp}_4(F) \times \mathrm{GL}_r(F),$$

which is viewed as the Levi factor of a maximal parabolic subgroup $P_r' = M_r' \cdot N_r'$ of $G_r' = \mathrm{GSp}_{2r+4}$. Thus, for an irreducible admissible representation $\pi \boxtimes \sigma$, we have the principal series representations $I_{P_r'}(s, \pi \boxtimes \sigma)$ and $I_{\bar{P}_r'}(s, \pi \boxtimes \sigma)$ and the intertwining operator

$$A_\psi(s, \pi \boxtimes \sigma, N_r', \bar{N}'_r) : I_{P_r}(s, \pi \boxtimes \sigma) \to I_{\bar{P}'_r}(s, \pi \boxtimes \sigma),$$

defined by

$$A_\psi(s, \pi \boxtimes \sigma, N_r', \bar{N}_r') f(g) = \int_{\bar{N}_r'} f(\bar{n}g)\, \mathrm{d}\bar{n}_\psi.$$

Then the Plancherel measure $\mu(s, \pi \boxtimes \sigma, \psi)$ is defined in the same way as the Sp_4 case. Then we have

Lemma 3.2. Let ϖ and π be irreducible admissible representations of $\mathrm{Sp}_4(F)$ and $\mathrm{GSp}_4(F)$, respectively, such that the restriction $\pi|_{\mathrm{Sp}_4(F)}$ contains ϖ as a constituent. Then for any irreducible admissible representation σ of $\mathrm{GL}_r(F)$, we have

$$\mu(s, \varpi \boxtimes \sigma, \psi) = \mu(s, \pi \boxtimes \sigma, \psi). \qquad \square$$

Proof. Let $\pi|_{\mathrm{Sp}_4(F)} = \oplus_i \varpi_i$, where each ϖ_i is irreducible. Then we have

$$I_{P_r'}(s, \pi \boxtimes \sigma)|_{\mathrm{Sp}_{2r+4}(F)} = \bigoplus_i I_{P_r}(s, \varpi_i \boxtimes \sigma).$$

Now looking at the integrals defining $A_\psi(s, \varpi_i \boxtimes \sigma, N_r, \bar{N}_r)$ and $A_\psi(s, \pi \boxtimes \sigma, N_r', \bar{N}'r)$, one immediately knows that

$$A_\psi(s, \pi \boxtimes \sigma, N_r', \bar{N}'r)|_{I_{P_r}(s, \varpi_i \boxtimes \sigma)} = A_\psi(s, \varpi_i \boxtimes \sigma, N_r, \bar{N}_r).$$

Hence, we have

$$\mu(s, \varpi_i \boxtimes \sigma, \psi) = \mu(s, \pi \boxtimes \sigma, \psi)$$

for each i. Since $\varpi = \varpi_i$ for some i, the lemma follows. ■

Then we have:

Proposition 3.3. Let ϖ and ϖ' be in the same L-packet of our construction with ϖ non-generic supercuspidal, i.e. $\mathcal{L}(\varpi) = \mathcal{L}(\varpi')$. Then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$ for any r, we have

$$\mu(s, \varpi \boxtimes \sigma, \psi) = \mu(s, \varpi' \boxtimes \sigma, \psi). \qquad \square$$

Proof. By our construction of L-packets, we can find irreducible admissible representations π and π' of $\mathrm{GSp}_4(F)$ so that $\varpi \subset \pi$, $\varpi' \subset \pi'$, and π and π' are in the same L-packet (possibly $\pi = \pi'$) as in [6]. Then by [6, Theorem 9.6], one knows that

$$\mu(s, \pi \boxtimes \sigma, \psi) = \mu(s, \pi' \boxtimes \sigma, \psi).$$

Hence, the proposition follows by the previous lemma. ■

Finally, the property (ii) of the Main Theorem follows from Propositions 3.1 and 3.3.

(iii) If ϖ is a constituent of $\pi|_{\mathrm{Sp}_4}$, then ϖ is discrete series iff π is essentially discrete series. Similarly, $\varphi = \mathrm{std} \circ \phi \in \Pi(\mathrm{Sp}_4)$ does not factor through any proper parabolic subgroup of $\mathrm{SO}_5(\mathbb{C})$ iff ϕ does not factor through any proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$. From these, the property (iii) of the Main Theorem is an immediate consequence of [6, Main Theorem (i)].

(iv) The property (iv) is a direct consequence of [6, Main Theorem (vii)] and the definition of $\mathcal{L}$.

(v) The property (v) follows immediately by the construction of the map $\mathcal{L}$; the uniqueness of $\mathcal{L}$ satisfying (v) is clear.

The remaining property (vi), i.e. the characterization of the map $\mathcal{L}$, will be shown in the next section.

4 Characterization of the Map $\mathcal{L}$

In this section, we show that our map $\mathcal{L} : \Pi(\mathrm{Sp}_4) \to \Phi(\mathrm{Sp}_4)$ is uniquely characterized by some of the properties of $\mathcal{L}$. Namely, we prove

Theorem 4.1. There is at most one map

$$\mathcal{L} : \Pi(\mathrm{Sp}_4) \to \Phi(\mathrm{Sp}_4)$$

satisfying:

(a) ϖ is a discrete series representation iff $\varphi_\varpi := \mathcal{L}(\varpi)$ does not factor through any proper Levi subgroup of $\mathrm{SO}_5(\mathbb{C})$.

(b) if ϖ is generic or non-supercuspidal, then for any irreducible representation σ of $\mathrm{GL}_r(F)$ with $r \leq 3$,

$$\begin{cases} L(s, \varpi \times \sigma) = L(s, \varphi_\varpi \otimes \phi_\sigma) \\ \epsilon(s, \varpi \times \sigma, \psi) = \epsilon(s, \varphi_\varpi \otimes \phi_\sigma, \psi). \end{cases}$$

(c) if ϖ is non-generic supercuspidal, then for any supercuspidal representation σ of $\mathrm{GL}_r(F)$ with $r \leq 4$, the Plancherel measure $\mu(s, \varpi \boxtimes \sigma, \psi)$ is equal to

$$\gamma(s, \varphi_\varpi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \varphi_\varpi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \bigwedge^2 \phi_\sigma, \psi) \cdot \gamma(-2s, \bigwedge^2 \phi_\sigma^\vee, \overline{\psi}). \qquad \square$$

As one can see from the theorem, we have to resort to the Plancherel measure for the non-generic supercuspidal representations. This is due to the lack of a theory of the local factors for these representations, but as in the theorem, the Plancherel measure turns out to be sufficient to characterize the correspondence.

To prove our theorem, we consider the two separate cases.

Case 1: ϖ is generic or non-supercuspidal

First, we consider the case where φ is generic or non-supercuspidal. The proof for this case is almost identical to the analogous case given in [6, Thm. 10.1]; so we omit the details here.

In fact, let us mention that recently, it has been shown by the second author that by combining the results of Henniart [11, Corollary 1.4 and Theorem 1.7], in which he characterized the local Langlands correspondence of GL_n by twists up to GL_{n-1}, and Chen's $n \times (n-2)$ local converse theorem for supercuspidal representations of GL_n [3], one can characterize the local Langlands conjecture of GL_n by twists only up to GL_{n-2}. The proof of this result will appear elsewhere. $\square$

Case 2: ϖ is non-generic supercuspidal

Let π be a non-generic supercuspidal representation of $\mathrm{GSp}_4(F)$ such that $\pi|_{\mathrm{Sp}_4(F)}$ contains ϖ as a constituent. By [6], we know that the L-parameter $\phi := L(\pi)$ of π is of the form $\phi_1 \oplus \phi_2$ where each ϕ_i is a two-dimensional irreducible representation of the Weil–Deligne group WD_F with $\det \phi_1 = \det \phi_2$. Now set

$$\Phi := \mathrm{std}(\phi) = \mathbf{1} \oplus (\phi_1^\vee \otimes \phi_2).$$

Then we have shown in the previous section that the Plancherel measure $\mu(s, \varphi \boxtimes \sigma, \psi)$ is equal to

$$\gamma(s, \Phi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \Phi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(2s, \bigwedge^2 \phi_\sigma, \psi) \cdot \gamma(-2s, \bigwedge^2 \phi_\sigma^\vee, \overline{\psi}).$$

Hence, if $\mathcal{L}$ is a map verifying the requirement (c), with $\varphi = \mathcal{L}(\varpi)$, then we have

$$\gamma(s, \Phi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \Phi \otimes \phi_\sigma^\vee, \overline{\psi}) = \gamma(s, \varphi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(-s, \varphi \otimes \phi_\sigma^\vee, \overline{\psi}). \tag{4.2}$$

We will show that together with the requirement (a), this forces $\varphi = \Phi$, which completes the proof of the theorem. □

Case I:

First, assume that ϕ and hence Φ is a representation of the Weil group W_F (with the $\mathrm{SL}_2(\mathbb{C})$ in WD_F acting trivially). Since ϕ is a discrete series parameter, Φ is a multiplicity-free direct sum of irreducible representations, each of which is an orthogonal representation. Let us write $\Phi = \oplus_i \Phi_i$, where the Φ_i's are distinct. Now let us take $\phi_\sigma = \Phi_i$ for any fixed i. Then it is easy to see that the LHS of (4.2) has a zero at $s = 0$. Hence, for some irreducible constituent $\varphi_i = \rho_i \boxtimes S_{r_i}$ of φ, the function

$$L(s, \varphi_i^\vee \otimes \Phi_i) \cdot L(s, \varphi_i \otimes \Phi_i^\vee)$$

has a pole at $s = 0$. This happens iff

$$\rho_i = \Phi_i \otimes |-|^{\pm(r_i-1)/2},$$

but the requirement (a) implies that each irreducible constituent of φ is an orthogonal representation and so $\det \rho_i = \pm 1$. This implies that $r_i = 1$ and so $\varphi = \Phi$.

Case II:

Next, assume that $\phi_1 = \chi \boxtimes S_2$ and ϕ_2 is a representation of the Weil group W_F such that $\det \phi_2 = \chi^2$. Then

$$\Phi = \mathbf{1} \oplus (\phi_1^\vee \otimes \phi_2) = \mathbf{1} \oplus (\chi^{-1} \cdot \phi_2 \boxtimes S_2),$$

so the LHS of (4.2) with $\phi_\sigma = \chi^{-1}\cdot\phi_2$ becomes

$$\begin{aligned}&\gamma(s,(\chi^{-1}\cdot\phi_2\boxtimes S_2)^\vee\otimes(\chi^{-1}\cdot\phi_2),\psi)\cdot\gamma(-s,(\chi^{-1}\cdot\phi_2\boxtimes S_2)\otimes(\chi^{-1}\cdot\phi_2)^\vee,\psi)\\&\qquad\cdot\gamma(s,(\chi^{-1}\cdot\phi_2)^\vee,\psi)\cdot\gamma(-s,\chi^{-1}\cdot\phi_2,\psi)\\&=(\epsilon\text{ factors})\cdot\frac{\zeta(\frac{1}{2}+1-s)\cdot\zeta(\frac{1}{2}+1+s)}{\zeta(\frac{1}{2}+s)\zeta(\frac{1}{2}-s)},\end{aligned}\tag{4.3}$$

which has a zero at $s=1/2$. Hence, the RHS of (4.2) with $\phi_\sigma=\chi^{-1}\cdot\phi_2$ must also have a zero at $s=1/2$, i.e. φ has a constituent $\rho\boxtimes S_r$ such that

$$\begin{aligned}&\gamma(s,(\rho\boxtimes S_r)^\vee\otimes(\chi^{-1}\cdot\phi_2),\psi)\cdot\gamma(-s,(\rho\boxtimes S_r)\otimes(\chi^{-1}\cdot\phi_2)^\vee,\psi)\\&\quad=(\epsilon\text{ factors})\cdot\frac{L(\frac{r-1}{2}+1-s,\rho\otimes(\chi^{-1}\cdot\phi_2)^\vee)L(\frac{r-1}{2}+1+s,\rho^\vee\otimes(\chi^{-1}\cdot\phi_2))}{L(\frac{r-1}{2}+s,\rho^\vee\otimes(\chi^{-1}\cdot\phi_2))L(\frac{r-1}{2}-s,\rho\otimes(\chi^{-1}\cdot\phi_2)^\vee)}\end{aligned}$$

has a zero at $s=1/2$, i.e. the denominator of this fraction must have a pole at $s=1/2$. This happens iff

$$\rho=\chi^{-1}|-|^{r/2}\cdot\phi_2\quad\text{or}\quad\chi^{-1}|-|^{-(r-2)/2}\cdot\phi_2,$$

but once again, the requirement (a) implies that $\det\rho=\pm1$, which implies $r=0$ or $r=2$. Since $r>0$, we conclude that $r=2$ and $\rho=\chi^{-1}\cdot\phi_2$, which gives $\varphi=\mu\oplus(\chi^{-1}\cdot\phi_2\boxtimes S_2)$ for some one-dimensional μ. The fact that φ takes value in $\mathrm{SO}_5(\mathbb{C})$ implies that μ is trivial so that $\varphi=\Phi$, as desired.

Case III:
Finally, assume that $\phi_1=\chi\boxtimes S_2$ and $\phi_2=\mu\boxtimes S_2$ with $\chi^2=\mu^2$ and $\chi\neq\mu$ so that

$$\Phi=\mathbf{1}\oplus(\phi_1^\vee\otimes\phi_2)=\mathbf{1}\oplus\chi^{-1}\mu\oplus(\chi^{-1}\mu\boxtimes S_3).$$

Then by setting $\phi_\sigma=\mathbf{1}$ and arguing as in Case II, one sees that the RHS has a zero at $s=0$, which implies that φ contains

$$|-|^{\pm(t-1)/2}\boxtimes S_t$$

as a constituent, but once again, the requirement (a) implies that the determinant of this constituent is ±1, i.e. $t=1$ and so φ contains $\mathbf{1}$. Similarly by taking $\phi_\sigma=\chi^{-1}\mu$ in (4.2), the LHS is, up to ϵ factors, equal to

$$\begin{aligned}&\frac{L(1-s,\chi\mu^{-1})\cdot L(1+s,\chi\mu^{-1})\cdot\zeta(1-s)\cdot\zeta(1+s)\cdot\zeta(1-s+1)\cdot\zeta(1+s+1)}{L(s,\chi^{-1}\mu)\cdot L(-s,\chi^{-1}\mu)\cdot\zeta(s)\cdot\zeta(-s)\cdot\zeta(s+1)\cdot\zeta(-s+1)}\\&=\frac{L(1-s,\chi\mu^{-1})\cdot L(1+s,\chi\mu^{-1})\cdot\zeta(2-s)\cdot\zeta(2+s)}{L(s,\chi^{-1}\mu)\cdot L(-s,\chi^{-1}\mu)\cdot\zeta(s)\cdot\zeta(-s)}.\end{aligned}\tag{4.4}$$

Note that $\chi^{-1}\mu$ is a non-trivial quadratic character so that $L(s,\chi^{-1}\mu)$ has no poles on $\mathbb{R}$, so the above fraction has a zero at $s=0$ and a pole at $s=2$. The zero at $s=0$ implies that φ contains

$$\chi^{-1}\mu|-|^{\pm(r-1)/2}\boxtimes S_r,$$

and again, the requirement (a) implies $r=1$ so that φ contains $\chi^{-1}\mu$. Similarly, the pole at $s=2$ implies that it contains

$$\chi^{-1}\mu|-|^{-(q-3)/2}\boxtimes S_q \quad \text{or} \quad \chi^{-1}\mu|-|^{(q-5)/2}\boxtimes S_q,$$

and the requirement (a) gives $q=3$ for the former and $q=5$ for the latter, but for dimension reasons, the latter cannot occur here, so φ contains $\chi^{-1}\mu\boxtimes S_3$. All these considerations imply that $\varphi=\Phi$.

Theorem 4.1 is proved.

5 Parameters of GSp$_4$

The rest of the paper is devoted to the determination of the sizes of the L-packets of Sp$_4$ in terms of Galois theoretic properties of their L-parameters. Before coming to that, it will be useful to have a better understanding of the L-parameters of GSp$_4$:

$$\phi : \mathrm{WD}_F \longrightarrow \mathrm{GSp}_4(\mathbb{C}).$$

We shall call such ϕ a **symplectic parameter**. In particular, we shall give a coarse classification of the most nondegenerate L-parameters of GSp$_4$, namely those

$$\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

which are irreducible as four-dimensional representations of the Weil group W_F.

For this purpose, let us recall that an irreducible representation ϕ of W_F is called **primitive** if it is not of the form $\mathrm{Ind}_{W_E}^{W_F}\sigma$ for a finite extension E/F, whereas ϕ is **dihedral** wrt a quadratic extension (quad. ext.) E/F if $\phi = \mathrm{Ind}_{W_E}^{W_F}\sigma$ or equivalently if $\phi \otimes \omega_{E/F} \cong \phi$. It has been shown by Koch [14] that a primitive representation exists only when p divides $\dim \phi$.

The main result of this section is the trichotomy of the following proposition.

Proposition 5.1. Let $\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$ be an irreducible four-dimensional representation with similitude character $\mathrm{sim}(\phi)$. Then we have the following possibilities:

(I) ϕ is primitive. In this case, the five-dimensional representation $\mathrm{std}(\phi)$ is irreducible.

(II) There is a quadratic extension E/F (with $\mathrm{Gal}(E/F) = \langle \tau \rangle$), a primitive representation σ of W_E, and a character χ of W_E such that

$$\phi = \mathrm{Ind}_{W_E}^{W_F}\sigma, \quad \sigma^{\tau} \cong \sigma \cdot \chi, \quad \chi^2 \neq 1, \quad \text{and} \quad \mathrm{sim}(\phi)|_{W_E} = \chi \cdot \det\sigma \neq \det\sigma.$$

Moreover, the five-dimensional representation $\mathrm{std}(\phi)$ is reducible but does not have a one-dimensional constituent; it decomposes as the sum of a two-dimensional irreducible constituent and a three-dimensional irreducible constituent.

(III) There is a quadratic extension E/F and an irreducible two-dimensional representation σ of W_E such that

$$\phi = \mathrm{Ind}_{W_E}^{W_F}\sigma \quad \text{and} \quad \mathrm{sim}(\phi)|_{W_E} = \det\sigma.$$

In this case, the five-dimensional standard representation $\mathrm{std}(\phi)$ contains at least one one-dimensional constituent. Indeed, it contains $\omega_{E/F}$.

The three situations above are mutually exclusive. When $p \neq 2$, only (III) can occur, but when $p = 2$, all the three situations can occur. □

The rest of the section is devoted to the proof of the proposition. We first study the primitive ϕ's. The following lemma proves the characterization of primitive ϕ's given in (I) of the proposition.

Lemma 5.2. Let $\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$ be an irreducible representation. Then ϕ is primitive iff $\mathrm{std}(\phi)$ is irreducible as a five-dimensional representation. □

Proof. Suppose that ϕ is primitive but $\mathrm{std}(\phi)$ is reducible. Then the image of $\mathrm{std}(\phi)$ must be contained in the subgroup $S(\mathrm{O}_2(\mathbb{C}) \times \mathrm{O}_3(\mathbb{C}))$ or $S(\mathrm{O}_1(\mathbb{C}) \times \mathrm{O}_4(\mathbb{C}))$ of $\mathrm{SO}_5(\mathbb{C})$. The preimage of each of these two groups in $\mathrm{GSp}_4(\mathbb{C})$ is, respectively, the normalizer of the Levi subgroup $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ of a Siegel parabolic or the normalizer of the subgroup $(\mathrm{GSp}_2(\mathbb{C}) \times \mathrm{GSp}_2(\mathbb{C}))^0$. In either case, the preimage is disconnected with two connected components, and its identity component acts reducibly. Thus, ϕ becomes reducible when restricted to a subgroup of index 2, which contradicts the assumption that ϕ is primitive. Thus, $\mathrm{std}(\phi)$ must be irreducible if ϕ is primitive.

Conversely, suppose that $\mathrm{std}(\phi)$ is irreducible. We need to rule out the possibility that ϕ is induced. Observe that as representations of W_F,

$$\left(\bigwedge^2 \phi\right) \otimes \mathrm{sim}(\phi)^{-1} = \mathrm{std}(\phi) \oplus \mathbb{C}.$$

Now we show that ϕ cannot be equal to $\mathrm{Ind}_{W_K}^{W_F} \chi$ with K/F quartic. Indeed, if ϕ has this form, the restriction of ϕ to W_K contains the one-dimensional submodule χ. It must then contain (at least) two one-dimensional submodules since it preserves a nondegenerate symplectic form up to scaling. From this, it follows that $\wedge^2\phi|_{W_K}$ contains at least two one-dimensional submodules. This would imply that $\mathrm{std}(\phi)|_{W_K}$ contains a one-dimensional submodule, and by Frobenius reciprocity, one would obtain a nonzero W_F-intertwining map from $\mathrm{std}(\phi)$ to a four-dimensional representation, which is a contradiction. On the other hand, suppose that $\phi = \mathrm{Ind}_{W_E}^{W_F} \rho$ with E/F quadratic. Then the restriction of ϕ to W_E is the sum of two two-dimensional submodules. Again, this implies that $\mathrm{std}(\phi)|_{W_E}$ contains a one-dimensional submodule which is again impossible. We have thus shown that ϕ is primitive. ■

Next, we describe a construction of primitive ϕ's which was shown to us by D. Prasad.

Proposition 5.3. When the residue characteristic p of F is equal to 2, there exists irreducible primitive representations $\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$. □

Proof. Suppose that $\varphi^{\#}$ is an irreducible self-dual five-dimensional representation of W_F; we shall show below that such a representation exists. Such a $\varphi^{\#}$ must necessarily preserve a quadratic form, and thus $\varphi^{\#}$ factors through $\mathrm{O}_5(\mathbb{C})$. By twisting by a quadratic character if necessary, we can ensure that $\varphi^{\#}$ factors through $\mathrm{SO}_5(\mathbb{C})$. As we saw earlier, such a $\varphi^{\#}$ admits a lifting

$$\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

so that $\mathrm{std}(\phi) = \varphi^{\#}$. By the previous lemma, we know that ϕ must be primitive.

Thus, it remains to show that an irreducible self-dual $\varphi^{\#}$ exists, or equivalently (by the local Langlands correspondence for GL_5) that there exists a self-dual supercuspidal representation of $\mathrm{GL}_5(F)$. By the Jacquet–Langlands correspondence, it is equivalent to showing that $D_5^{\times}$ has a self-dual irreducible representation of dimension > 1 (where D_5 is a division algebra of degree 5). The group $D_5^{\times}$ has a standard decreasing filtration

$$D_5^{\times} \supset D_5^{(1)} \supset D_5^{(2)} \supset \ldots.$$

by open compact subgroups so that, for $i \geq 1$, the successive quotients $D_5^{(i)}/D_5^{(i+1)}$ are equal to the additive group of a finite field of characteristic 2 (the degree 5 extension of the residue field of F). Since any irreducible representation of $D_5^{\times}$ factors through the finite group $D_5^{\times}/D_5^{(i+1)}$ for some i, we are reduced to showing that $D_5^{\times}/D_5^{(i+1)}$ has non-central elements of order 2. This is certainly the case, as one can readily see by examining the elements in $D_5^{(i)}/D_5^{(i+1)}$. ■

We may now focus on the non-primitive ϕ's. We first note the following lemma.

Lemma 5.4. Suppose that an irreducible symplectic parameter ϕ is of the form $\phi = \mathrm{Ind}_{W_K}^{W_F}\chi$ with K/F a quartic non-Galois extension which does not contain a quadratic subfield. Then one can find a quadratic extension E/F such that $\phi = \mathrm{Ind}_{W_E}^{W_F}\sigma$ for some σ. Note that this situation is possible only when $p = 2$ because if $p \neq 2$, any quartic extension contains an intermediate field. □

Proof. Consider the restriction of ϕ to W_K. By Frobenius reciprocity and the irreducibility of ϕ, $\phi|_{W_K}$ contains the character χ with multiplicity 1. Since ϕ is a symplectic parameter, the line affording the character χ is an isotropic line, and thus, $\phi|_{W_K}$ must also contain the character

$$\chi' = \mathrm{sim}(\phi)|_{W_K} \cdot \chi^{-1} \neq \chi$$

so that

$$\phi|_{W_K} = \chi \oplus \chi' \oplus V$$

with $\dim V = 2$ and $\det V = \chi \cdot \chi'$.

On the other hand, the fact that K/F has no quadratic subfield implies that the double coset space $W_K \backslash W_F / W_K$ has size 2. Thus, Mackey's lemma implies that

$$\phi|_{W_K} = \chi \oplus \mathrm{Ind}_{W_K \cap \tau^{-1} W_K \tau}^{W_K} \chi^{\tau},$$

where τ is any element of $W_F \smallsetminus W_K$. The latter summand must contain χ' and so by Frobenius reciprocity, we have

$$\chi'|_{W_K \cap \tau^{-1} W_K \tau} = \chi^{\tau}|_{W_K \cap \tau^{-1} W_K \tau}.$$

Hence,

$$\phi|_{W_K} = \chi \oplus \left(\chi' \cdot \mathrm{Ind}_{W_K \cap \tau^{-1} W_K \tau}^{W_K} 1 \right).$$

Now let L be the compositum of K and K^{τ} so that L/K is a cubic extension since $W_L = W_K \cap \tau^{-1} W_K \tau$. We have:

- L/K is non-Galois. If not, then $\phi|_{W_K} = \chi \oplus \chi' \cdot (1 \oplus \mu \oplus \mu^2)$ where μ is a cubic character of $\mathrm{Gal}(L/K) \cong \mathbb{Z}/3\mathbb{Z}$. This would imply that $\chi \cdot \chi' = \det V = \chi'^2$ so that $\chi' = \chi$. This is a contradiction.
- If M is the Galois closure of L/K so that $\mathrm{Gal}(M/K) \cong S_3$, then M is the Galois closure of K/F so that $\mathrm{Gal}(M/F) \cong S_4$. Indeed, on one hand, M is a degree 24 extension of F. On the other hand, the Galois closure of K/F has degree ≤ 24 and must contain L and hence M. This shows that M is the Galois closure of K/F and $\mathrm{Gal}(M/F) \cong S_4$.

Now the sign character ϵ of $\mathrm{Gal}(M/F) \cong S_4$ determines a quadratic extension E/F. Moreover, we have

$$\phi|_{W_K} = \chi \oplus \chi' \oplus \chi' \cdot V_0,$$

where V_0 is the (unique) irreducible two-dimensional representation of $\mathrm{Gal}(M/K) \cong S_3$. Note that $\det V_0$ is the sign character of $\mathrm{Gal}(M/K)$, which is $\epsilon|_{W_K}$. Since $V = \chi' \cdot V_0$ and

$\chi \cdot \chi' = \det V$, we deduce that

$$\chi' = \chi \cdot \epsilon|_{W_K},$$

which implies that

$$\phi \otimes \epsilon = \mathrm{Ind}_{W_K}^{W_F}(\chi \cdot \epsilon|_{W_K}) = \mathrm{Ind}_{W_K}^{W_F}\chi' = \phi.$$

This shows that

$$\phi \cong \mathrm{Ind}_{W_E}^{W_F}\sigma$$

for some σ. This finishes the proof of the lemma. ■

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. When $p \neq 2$, this has been shown by Vigneras [20]. We shall argue generally below. Let us recall that the similitude character $\mathrm{sim}(\phi)$ occurs in $\wedge^2\phi$, and

$$\mathrm{sim}(\phi)^{-1} \cdot \bigwedge^2 \phi = 1 \bigoplus \mathrm{std}(\phi). \tag{5.5}$$

The case of primitive ϕ's have been handled by Lemma 5.2 and Proposition 5.3. Thus, we may assume that ϕ is not primitive below. Hence, by the above lemma, we may suppose that

$$\phi = \mathrm{Ind}_{W_E}^{W_F}\sigma,$$

for some quadratic extension E/F with $\mathrm{Gal}(E/F) = \langle\tau\rangle$ and some irreducible two-dimensional representation σ. Then note that

$$\mathrm{sim}(\phi)^2 = \det\phi = \det\sigma|_{F^\times} \tag{5.6}$$

as characters of $F^\times$. Moreover, we have

$$\bigwedge^2 \phi = \mathrm{Ind}_{W_E}^{W_F}\det\sigma \bigoplus M(\sigma), \tag{5.7}$$

where $M(\sigma)$ is the multiplicative induction of σ to W_F, which is simply an extension of $\sigma \otimes \sigma^\tau$ from W_E to W_F and sometimes called the Asai lift of σ (see [16, §7] for this notion). Now note that in $\wedge^2\phi|_{W_E}$, any one-dimensional character occurs with multiplicity at most 1, except for the character $\det\sigma$ which may occur with multiplicity 2. To see this, observe that since σ is irreducible, any one-dimensional character χ occurs in $M(\sigma)|_{W_K} = \sigma \otimes \sigma^\tau$ with multiplicity at most 1 because

$$\dim \operatorname{Hom}_{W_E}(\sigma \otimes \sigma^\tau, \chi) = \dim \operatorname{Hom}_{W_E}(\sigma^\tau, \chi \otimes \sigma^\vee) \leq 1$$

by Schur's lemma. Also neither $\det\sigma$ nor $\det\sigma^\tau$ occurs in $\sigma \otimes \sigma^\tau$; if either one of them does, then we would have

$$\sigma^\tau \cong \sigma^\vee \otimes \det\sigma \cong \sigma,$$

which is a contradiction to the assumption that σ is not τ-invariant. Hence, in $\wedge^2\phi|_{W_E}$, any one-dimensional character occurs with multiplicity at most 1, except perhaps for the character $\det\sigma$, which may occur with multiplicity 2 in $(\operatorname{Ind}_{W_E}^{W_F} \det\sigma)|_{W_E}$.

Now if $\operatorname{sim}(\phi)$ occurs in the first summand on the RHS of (5.7), then

$$\operatorname{sim}(\phi)|_{W_E} = \det\sigma,$$

and we are in situation (III) of the proposition. Moreover, (5.5) and (5.7) imply that $\operatorname{std}(\phi)$ contains $\omega_{E/F}$.

Henceforth, we suppose that $\operatorname{sim}(\phi)$ occurs in $M(\sigma)$. Then we have

$$\sigma^\tau \cong \sigma^\vee \otimes \operatorname{sim}(\phi)|_{W_E} = \sigma \otimes (\operatorname{sim}(\phi)|_{W_E} / \det\sigma). \tag{5.8}$$

Thus, we have

$$\operatorname{sim}(\phi)^{-1} \cdot M(\sigma)|_{W_E} = 1 \bigoplus \operatorname{Ad}(\sigma). \tag{5.9}$$

Since the LHS is τ-invariant, τ must permute the one-dimensional constituents of Ad (σ) if there are any. These one-dimensional constituents are precisely those quadratic characters ω of W_E wrt which σ is dihedral, i.e. such that $\sigma \otimes \omega \cong \sigma$. If σ is dihedral, $\operatorname{Ad}(\sigma)$ contains one or three quadratic characters. In either case, we see that at least one of these quadratic characters must be fixed by τ. If we denote this τ-invariant quadratic character by ω_0, then (5.9) shows that $\mathrm{sim}(\phi)^{-1} \cdot M(\sigma)$ contains an extension of ω_0 to W_F, which may be a quadratic or quartic character.

For each of those cases, we will show below that if σ is primitive, either (II) or (III) holds, and if it is dihedral, then ω_0 can actually extend only to a quadratic character, and for this case (III) happens.

Case 1-a: σ is primitive, and $\chi := \mathrm{sim}(\phi)|_{W_E} / \det \sigma$ is not quadratic. Note that by (5.8),

$$\sigma^\tau = \sigma \otimes \chi$$

and so

$$\det \sigma^\tau = \chi^2 \cdot \det \sigma \neq \det \sigma.$$

Moreover, equations (5.5), (5.7), and (5.9) together imply that $\mathrm{std}(\phi)$ decomposes as the sum of an irreducible two-dimensional representation and an irreducible three-dimensional representation. Thus, we are in situation (II) of the proposition.

Case 1-b: σ is primitive, and $\chi := \mathrm{sim}(\phi)|_{W_E} / \det \sigma$ is quadratic. By (5.8), we see that $\det \sigma^\tau = \det \sigma$ so that $\det \sigma$ extends to W_F. Since $\mathrm{sim}(\phi)|_{W_E} = \chi \cdot \det \sigma$, we deduce that χ also extends to W_F, and one also has

$$\mathrm{sim}(\phi)^2 = \det \sigma|_{F^\times} \cdot \chi|_{F^\times} \quad \text{as characters on } F^\times.$$

Now the identity (5.6) implies that χ is trivial when restricted to $F^\times$. Thus, χ extends to a quadratic character of W_F and determines a quadratic extension E'/F. Moreover, $\phi \otimes \omega_{E'/F} \cong \phi$ since

$$\phi \otimes \omega_{E'/F} = \mathrm{Ind}_{W_E}^{W_F}(\sigma \otimes \chi) = \mathrm{Ind}_{W_E}^{W_F} \sigma^\tau = \phi.$$

Thus, $\wedge^2 \phi$ contains both $\mathrm{sim}(\phi)$ and $\mathrm{sim}(\phi) \cdot \omega_{E'/F}$ so that $\wedge^2 \phi|_{W_{E'}}$ contains $\mathrm{sim}(\phi)|_{W_{E'}}$ with multiplicity 2. Hence, we conclude that

$$\phi \cong \mathrm{Ind}_{W_{E'}}^{W_F} \sigma' \quad \text{with} \quad \mathrm{sim}(\phi)|_{W_{E'}} = \det \sigma',$$

so that we are in situation (III).

Case 2-a: σ is dihedral, and ω_0 extends to a quadratic character $\omega_{E'/F}$. In this case, it is clear that $\phi \otimes \omega_{E'/F} \cong \phi$ so that

$$\phi = \mathrm{Ind}_{W_K}^{W_F} \sigma'$$

for some irreducible two-dimensional σ'. Now $\wedge^2\phi$ contains the characters $\mathrm{sim}(\phi)$ and $\mathrm{sim}(\phi) \cdot \omega_{E'/F}$, and thus, $\wedge^2\phi|_{W_{E'}}$ contains $\mathrm{sim}(\phi)|_{W_{E'}}$ with multiplicity 2. As we observed above, the only character which may occur with multiplicity 2 in $\wedge^2\phi|_{W_{E'}}$ is $\det\sigma'$. Hence, we have

$$\mathrm{sim}(\phi)|_{W_{E'}} = \det\sigma',$$

and we are in situation (III) of the proposition.

Case 2-b: σ is dihedral, and ω_0 extends to a quartic character. In this case, the quadratic extension K/E determined by ω_0 is cyclic quartic over F, and E is the unique quadratic subfield of K/F. Set $\mathrm{Gal}(K/F) = \langle\tau\rangle$ so that $\mathrm{Gal}(K/E) = \langle\tau^2\rangle$. Then

$$\phi = \mathrm{Ind}_{W_E}^{W_F} \sigma = \mathrm{Ind}_{W_K}^{W_F} \chi.$$

Since $\sigma^\tau = \sigma^\vee \otimes \mathrm{sim}(\phi)|_{W_E}$, we see that

$$\mathrm{Ind}_{W_K}^{W_E}(\chi^\tau) = \mathrm{Ind}_{W_K}^{W_E}(\chi^{-1} \cdot \mathrm{sim}(\phi)|_{W_K}).$$

This implies that

$$\chi^\tau \cdot \chi = \mathrm{sim}(\phi)|_{W_K} \quad \text{or} \quad \chi^{\tau^3} \cdot \chi = \mathrm{sim}(\phi)|_{W_K}$$

so that $\chi \cdot \chi^\tau$ is τ-invariant, but this implies that $\chi^{\tau^2} = \chi$, which contradicts the irreducibility of σ. Hence, ω_0 cannot extend to a quartic character of W_F.

Therefore, we have thus shown that the situations (I), (II), and (III) of the proposition encompass all the possibilities for ϕ. Moreover, from the behavior of $\mathrm{std}(\phi)$, it is clear that these three situations are mutually exclusive. By the theorem by Koch [14] mentioned right before the proposition, only (III) can occur if $p \neq 2$, so the only thing we are left with is to show that all of the three possibilities actually happen if $p = 2$. We have constructed examples of primitive ϕ in Proposition 5.3. Also it is easy to see that (III) can be achieved. Thus, it remains to construct examples of situation (II).

For this, let E/F be quadratic extension with $\mathrm{Gal}(E/F) = \langle \tau \rangle$. Let ρ be a primitive two-dimensional representation of W_F, and let χ be a character of W_E such that $(\chi^\tau/\chi)^2 \neq 1$. Note that $\rho|_{W_E}$ is still primitive since $\mathrm{Ad}(\rho)$ is irreducible, and thus, $\mathrm{Ad}(\rho)|_{W_E}$ cannot contain a one-dimensional constituent and is thus irreducible also. Now consider the four-dimensional representation

$$\phi = \rho \otimes \mathrm{Ind}_{W_E}^{W_F} \chi = \mathrm{Ind}_{W_E}^{W_F} \rho|_{W_E} \otimes \chi.$$

This is an irreducible representation because if we let $\sigma = \rho|_{W_E} \cdot \chi$,

$$\sigma^\tau = \rho|_{W_E} \cdot \chi^\tau = \sigma \cdot (\chi^\tau/\chi) \neq \sigma.$$

Here, to show $\neq$, we have used the assumption that $(\chi^\tau/\chi)^2 \neq 1$.

To show that ϕ is symplectic, we need to show that ϕ preserves a nondegenerate symplectic form up to scaling, but

$$\wedge^2\phi = \wedge^2\rho \otimes \mathrm{Sym}^2(\mathrm{Ind}_{W_E}^{W_F} \chi) \bigoplus \mathrm{Sym}^2\rho \otimes \wedge^2\mathrm{Ind}_{W_E}^{W_F} \chi.$$

Now $\mathrm{Sym}^2(\rho)$ is irreducible so that the second summand contains no one-dimensional character. Moreover,

$$\mathrm{Ad}(\mathrm{Ind}_{W_E}^{W_F} \chi) = \omega_{E/F} \oplus \mathrm{Ind}_{W_E}^{W_F} \chi^\tau/\chi,$$

and the second summand is irreducible. Thus, we see that $\wedge^2\phi$ contains a unique one-dimensional character, namely $\det \rho \cdot \chi|_{F^\times}$ (regarded as a character of $F^\times$). In other words, ϕ preserves a unique symplectic form up to scaling (necessarily nondegenerate since ϕ is irreducible) and the similitude character $\mathrm{sim}(\phi)$ satisfies

$$\mathrm{sim}(\phi)|_{W_E} = \det \rho|_{W_E} \cdot \chi \cdot \chi^\tau = \det \sigma \cdot (\chi^\tau/\chi) \neq \det \sigma.$$

Thus, ϕ satisfies all the requirements of situation (II). ■

6 Sizes of L-Packets of Sp_4

In this section, we determine the sizes of the L-packets of Sp_4. More precisely, given an L-parameter $\phi : \mathrm{WD}_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$, we describe the size of the L-packet of Sp_4 associated to $\varphi = \mathrm{std}(\phi)$ in terms of Galois theoretic properties of ϕ.

First of all, it is quite elementary to see that the possible sizes of the L-packet are given by

$$\#\mathcal{L}_\varphi = \begin{cases} 1, 2, 4, \text{ or } 8, & \text{if } p \neq 2; \\ 1, 2, 4, 8, \text{ or } 16, & \text{if } p = 2. \end{cases}$$

When $p \neq 2$, this follows immediately from Proposition 2.9 and the fact that $\#A_\phi \leq 2$ and $\#I(\phi) = 1$, 2, or 4 since there are only three quadratic characters. To deal with the case $p = 2$, it is probably easier to work with the parameter φ. One has the following general statement [5, Corollary 6.6]:

Lemma 6.1. Let $\varphi : \mathrm{WD}_F \longrightarrow \mathrm{SO}_N(\mathbb{C})$ be an admissible homomorphism with N odd, and regard it as an N-dimensional representation of WD_F with isotypic decomposition

$$\varphi = \bigoplus_i n_i \cdot M_i.$$

Then

$$\pi_0(Z_{\mathrm{SO}_N}(Im(\varphi))) = (\mathbb{Z}/2\mathbb{Z})^{r-1},$$

where

$$r = \#\{i : M_i^\vee \cong M_i \text{ and } M_i \text{ is an orthogonal representation}\}.$$ □

The lemma immediately implies:

Corollary 6.2. Let $\varphi : \mathrm{WD}_F \longrightarrow \mathrm{SO}_5(\mathbb{C})$ be an L-parameter for $\mathrm{Sp}_4(F)$. Then

$$\#\mathcal{L}_\varphi = \begin{cases} 1, 2, 4, \text{ or } 8, & \text{if } p \neq 2; \\ 1, 2, 4, 8, \text{ or } 16, & \text{if } p = 2. \end{cases}$$ □

Of course, the corollary does not show that all possibilities for $\#L_\varphi$ occur. For the rest of the section, we will show that all of them do occur, together with precise information on when each case happens in terms of the Galois theoretic properties of any $\phi \in \Phi(\mathrm{GSp}_4)$ for which $\varphi = \mathrm{std}(\phi)$. Our result is then the analog of the following well-known result for SL_2 (cf. [19]).

Proposition 6.3. Let $\sigma : \mathrm{WD}_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be an irreducible representation and $\sigma_0 = \mathrm{Ad}(\sigma) : \mathrm{WD}_F \longrightarrow \mathrm{SO}_3(\mathbb{C})$ the associated discrete series L-parameter for SL_2. Then

$$\#L_{\sigma_0} = \begin{cases} 1, \text{ if } \sigma \text{ is primitive or } \sigma \text{ is non-trivial on } \mathrm{SL}_2(\mathbb{C}); \\ 2, \text{ if } \sigma \text{ is dihedral wrt one quadratic field;} \\ 4, \text{ if } \sigma \text{ is dihedral wrt three quadratic fields.} \end{cases}$$

In particular, the size of the L-packet $\#L_{\sigma_0}$ is equal to the number of characters ω such that $\sigma \otimes \omega \cong \sigma$, and such non-trivial characters are precisely those which occur in $\sigma_0 = \mathrm{Ad}(\sigma)$. Moreover, the third case above happens iff $\sigma = \mathrm{Ind}_{W_E}^{W_F} \rho$ for some quadratic E/F with ρ^c/ρ a non-trivial quadratic character for the non-trivial element $c \in \mathrm{Gal}(E/F)$. $\square$

We now consider the question of determining the size of the L-packet of Sp_4 associated to the L-parameter $\varphi = \mathrm{std}(\phi)$. By our main theorem, this is given by

$$N(\phi) := \text{size of the component group } A_{\mathrm{std}(\phi)}.$$

If ϕ is a discrete series parameter, then $\mathrm{std}(\phi)$ is a discrete series parameter for Sp_4 and is the multiplicity-free direct sum of orthogonal submodules as a five-dimensional representation, in which case by Lemma 6.1, we know

$$\log_2 N(\phi) = (\text{the number of irreducible constituents of } \mathrm{std}(\phi)) - 1.$$

This gives a convenient way of calculating $N(\phi)$ for discrete series parameters. Recall that we have defined

$$I(\phi) = \{\text{quadratic characters } \chi \text{ of } W_F \text{ with } \phi \otimes \chi \cong \phi\}$$

and have shown that

$$N(\phi) = \#L_\phi \cdot \#I(\phi) = \begin{cases} \#I(\phi), & \text{if } L_\phi \text{ is a singleton;} \\ 2 \cdot \#I(\phi), & \text{otherwise.} \end{cases}$$

In some cases (especially for non-discrete series parameters), it will be easier to compute $N(\phi)$ by directly determining $I(\phi)$. Note that $I(\phi)$ is an elementary Abelian two-group and we shall frequently specify $I(\phi)$ by writing down a set of generators for it. For example, we shall write $I(\phi) = \langle \chi_1, \chi_2, \ldots, \chi_k \rangle$ to indicate that it is generated by the χ_i's.

The following lemma says that one can determine $I(\phi)$ by regarding ϕ as an L-parameter for GL_4 and is useful for our purposes below:

Lemma 6.4. Let $\phi \in \Phi(\mathrm{GSp}_4)$, and consider the natural inclusion

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}).$$

Then

$$I(\phi) = \{\text{quadratic characters } \chi : \iota \circ \phi \cong (\iota \circ \phi) \otimes \chi\}. \qquad \square$$

Proof. It is obvious that the LHS is contained in the RHS. Conversely, suppose that χ lies in the RHS. By [6, Lemma 6.1], one knows that the natural map

$$\Phi(\mathrm{GSp}_4) \longrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$$

given by

$$\phi \mapsto (\iota \circ \phi, \mathrm{sim}(\phi))$$

is injective. However, since χ lies in the RHS, we have:

$$(\iota \circ (\phi \otimes \chi), \mathrm{sim}(\phi \otimes \chi)) = ((\iota \circ \phi) \otimes \chi, \mathrm{sim}(\phi)) = (\iota \circ \phi, \mathrm{sim}(\phi)).$$

Hence, we conclude by injectivity that $\phi \otimes \chi = \phi$, as desired. ■

The rest of this paper is devoted to the determination of the number $N(\phi)$ and the group $I(\phi)$.

Discrete series parameters

We begin with the most nondegenerate cases, namely those treated in Proposition 5.1. □

Theorem 6.5. Suppose that $\phi : W_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$ is an irreducible representation with similitude character $\mathrm{sim}(\phi)$.

(I) If ϕ is primitive, i.e. as in situation (I) of Proposition 5.1, then $N(\phi) = 1$.

(II) If ϕ is as in situation (II) of Proposition 5.1, then $N(\phi) = 2$ and $I(\phi) = \langle \omega_{E/F} \rangle$.

(III) If $\phi = \mathrm{Ind}_{W_E}^{W_F} \sigma$ is as in situation (III) of Proposition 5.1, then we have the following two cases, each of which is further divided into several subcases.

(a) Suppose that $\sigma^\tau \neq \sigma \otimes \chi$ for any character χ. Then, we have

$$N(\phi) = \begin{cases} 4, & \text{if } \sigma \text{ is dihedral wrt a quadratic } K/E \text{ such that } K/F \text{ is biquadratic;} \\ 2, & \text{otherwise,} \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \omega_{E/F}, \omega_{E'/F} \rangle, & \text{with } E' \neq E \text{ a quadratic extension in the biquadratic } K/F; \\ \langle \omega_{E/F} \rangle. \end{cases}$$

More precisely, we have:

(a1) if σ is primitive, then $N(\phi) = 2$.

(a2) if $\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho$ with $\mathrm{Gal}(K/F) = \mathbb{Z}/4\mathbb{Z}$, then $N(\phi) = 2$.

(a3) if $\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho$ with $\mathrm{Gal}(K/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $N(\phi) = 4$.

(a4) if $\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho$ with K/F non-Galois, then

$$N(\phi) = \begin{cases} 2, & \text{if } \sigma^\tau|_{W_K} \cdot \rho \text{ does not extend to } W_F; \\ 4, & \text{otherwise.} \end{cases}$$

Moreover, $\sigma^\tau|_{W_K} \cdot \rho$ is extendable to W_F iff σ is dihedral wrt a K'/E with K'/F biquadratic, in which case we are reduced to situation (a3) by replacing K by K'.

(b) Suppose that $\sigma^\tau = \sigma \otimes \chi$ (so that χ is necessarily quadratic).

(b1) If $\chi^\tau \neq \chi$, then σ is dihedral wrt $\chi \cdot \chi^\tau$, which determines a biquadratic extension K/F. In this case,

$$N(\phi) = 4$$

and

$$I(\phi) = \langle \omega_{E/F}, \omega_{E'/F} \rangle,$$

where $E' \neq E$ is a quadratic extension of F contained in K.

(b2) If $\chi^\tau = \chi$, then there is a character λ of W_E such that $\chi = \lambda^\tau/\lambda$ and $\sigma \otimes \lambda^{-1}$ extends to a representation π of W_F. In other words, χ determines a biquadratic extension K/F with quadratic subfields E, E', and E'', and

$$\sigma = \pi|_{W_E} \otimes \lambda \quad \text{and} \quad \phi = \pi \otimes \mathrm{Ind}_{W_E}^{W_F} \lambda.$$

Then we have:

$$N(\phi) = \begin{cases} 4, & \text{if } \pi \text{ is primitive;} \\ 8, & \text{if } \pi \text{ is dihedral wrt one quadratic field } M/F; \\ 16, & \text{if } \pi \text{ is dihedral wrt three quadratic fields } M_i/F, \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \omega_{E/F}, \omega_{E'/F} \rangle; \\ \langle \omega_{E/F}, \omega_{E'/F}, \omega_{M/F} \rangle; \\ \langle \omega_{E/F}, \omega_{E'/F}, \omega_{M_1/F}, \omega_{M_2/F} \rangle. \end{cases}$$

Moreover, situation (b2) can occur only when $p = 2$. □

Proof. The statements (I) and (II) are clear from Proposition 5.1 and so we focus on (III). Hence, we are in situation (III) of Proposition 5.1 so that

$$\phi = \mathrm{Ind}_{W_E}^{W_F} \sigma \quad \text{with} \quad \mathrm{sim}(\phi)|_{W_E} = \det \sigma.$$

In this case, we have

$$\mathrm{std}(\phi) = \omega_{E/F} \oplus \mathrm{sim}(\phi)^{-1} \cdot M(\sigma),$$

so we need to determine how many irreducible constituents $M(\sigma)$ has. As in the statement of the proposition, we consider the two cases (a) and (b).

(a) $\sigma^\tau \neq \sigma \otimes \chi$ *for any character* χ: in this case, we see that

$$M(\sigma)|_{W_E} = \sigma^\tau \otimes \sigma$$

does not contain any one-dimensional constituent so that

$$M(\sigma)|_{W_E} = 4 \quad \text{or} \quad 2+2,$$

where the RHS means that the representation in question is either irreducible or the sum of two two-dimensional irreducible constituents. We consider the following different cases:

(a1) σ primitive. Then we claim that $\sigma^\tau \otimes \sigma$ is irreducible. If not, then

$$\sigma^\tau \otimes \sigma = V_1 \oplus V_2, \quad \text{with } \dim V_i = 2,$$

which implies that

$$\wedge^2(\sigma^\tau \otimes \sigma) = \wedge^2 V_1 \oplus \wedge^2 V_2 \oplus (V_1 \otimes V_2)$$

contains one-dimensional constituents. However, we also have

$$\wedge^2(\sigma^\tau \otimes \sigma) = (\wedge^2\sigma^\tau \otimes \mathrm{Sym}^2(\sigma)) \bigoplus (\mathrm{Sym}^2(\sigma^\tau) \otimes \wedge^2\sigma) = 3+3.$$

This gives the desired contradiction. Hence, $M(\sigma)$ is irreducible in this case and $N(\phi) = 2$.

(a2) $\sigma = \mathrm{Ind}_{W_K}^{W_E}\rho$ with $\mathrm{Gal}(K/F) = \mathbb{Z}/4\mathbb{Z} = \langle\tau\rangle$. In this case, one has

$$M(\sigma)|_{W_E} = \sigma^\tau \otimes \sigma = \mathrm{Ind}_{W_K}^{W_E}\rho\cdot\rho^\tau \bigoplus \mathrm{Ind}_{W_K}^{W_E}\rho^\tau\cdot\rho^{\tau^2}$$

and τ switches these two components. Note that those two components are non-isomorphic because otherwise we would have $\rho\cdot\rho^\tau = \rho^\tau\cdot\rho^{\tau^2}$ or $(\rho^\tau\cdot\rho^{\tau^2})^{\tau^2}$, which immediately implies $\rho^{\tau^2} = \rho$, thus contradicting the irreducibility of σ, so $M(\sigma)$ is irreducible and $N(\phi) = 2$.

(a3) $\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho$ with $\mathrm{Gal}(K/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this case, let τ_1 and τ_2 be the two elements of $\mathrm{Gal}(K/F)$ which projects to the non-trivial element $\tau \in \mathrm{Gal}(E/F)$. One has

$$M(\sigma)|_{W_E} = \sigma^{\tau} \otimes \sigma = \mathrm{Ind}_{W_K}^{W_E} \rho \cdot \rho^{\tau_1} \bigoplus \mathrm{Ind}_{W_K}^{W_E} \rho \cdot \rho^{\tau_2},$$

and τ fixes each of the two components on the RHS, so

$$M(\sigma) = 2 + 2$$

and $N(\phi) = 4$.

(a4) $\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho$ with K/F non-Galois. In this case, if L denotes the composite of K and K^{τ}, then L is the Galois closure of K/F and $\mathrm{Gal}(L/F)$ is a non-Abelian group of order 8. Since it contains the Klein four-group $\mathrm{Gal}(L/E)$, it must in fact be the dihedral group of order 8. In any case,

$$\sigma^{\tau} \otimes \sigma|_{W_K} = \rho \cdot \sigma^{\tau}|_{W_K} \bigoplus \rho' \cdot \sigma^{\tau}|_{W_K},$$

where $\rho' \neq \rho$ is the conjugate of ρ by W_E. It is not difficult to see that

$$\sigma^{\tau} \neq \sigma \otimes \chi \iff \rho'/\rho \neq \omega_{L/K} \iff \sigma^{\tau}|_{W_K} \text{ is irreducible.}$$

Hence, $\sigma^{\tau} \otimes \sigma|_{W_K} = 2 + 2$, and we have

$$N(\phi) = \begin{cases} 2, & \text{if } \rho \cdot \sigma^{\tau}|_{W_K} \text{ does not extend to } W_F; \\ 4, & \text{otherwise.} \end{cases}$$

Let us examine this condition of extendability of $\sigma^{\tau}|_{W_K} \cdot \rho$ to W_F in greater detail. For this, we first consider the issue of extendability to W_E. Clearly, $\sigma^{\tau}|_{W_K} \cdot \rho$ can be extended to W_E iff

$$(\rho'/\rho) \otimes \sigma^{\tau}|_{W_K} \cong \sigma^{\tau}|_{W_K}.$$

This is equivalent to saying that ρ'/ρ defines a biquadratic M/E (different from L/E) containing quadratic subfields K, K', and K'' wrt which σ is dihedral and such that σ^{τ} is dihedral wrt K' (without loss of generality). In that case, K' is necessarily Galois over F. In other words, the mere requirement that $\sigma^{\tau}|_{W_K} \cdot \rho$ be extendable to W_E already forces σ to be dihedral wrt K'/E such that K'/F is Galois so that we are reduced to situation (a2) or (a3). Depending on whether K'/F is cyclic or biquadratic, we then obtain $N(\phi) = 2$ or 4, respectively. Hence, our result can be stated as follows.

$$N(\phi) = \begin{cases} 2, & \text{if } \rho \cdot \sigma^\tau|_{W_K} \text{ does not extend to } W_E; \\ 2, & \text{if } \rho \cdot \sigma^\tau|_{W_K} \text{ extends to } W_E \text{ but not to } W_F; \\ 4, & \text{if } \rho \cdot \sigma^\tau|_{W_K} \text{ extends to } W_F. \end{cases}$$

In the second case, we can be reduced to situation (a2), and in the third case, we can be reduced to (a3) by re-choosing the quartic extension K.

(b) $\sigma^\tau = \sigma \otimes \chi$ *for some* χ: since $\det \sigma^\tau = \det \sigma$, we see that

$$\chi^2 = 1 \quad \text{and} \quad \sigma = \sigma \otimes \chi \chi^\tau.$$

We thus have the following cases:

(b1) $\chi^\tau \neq \chi$. In this case, we need to show that $N(\phi) = 4$. Since σ is dihedral wrt the quadratic extension K of E defined by $\chi \cdot \chi^\tau$, it is not primitive. Moreover, since $\omega_{K/E} = \chi \cdot \chi^\tau$ is τ-invariant and trivial on $F^\times$, it extends to a quadratic character of W_F. Hence, the quartic field K/F is biquadratic. Let τ_1 and τ_2 be the two elements of $\mathrm{Gal}(K/F)$ which project to the element $\tau \in \mathrm{Gal}(E/F)$, and let $c = \tau_1 \cdot \tau_2$ so that $\mathrm{Gal}(K/E) = \langle c \rangle$. Writing

$$\sigma = \mathrm{Ind}_{W_K}^{W_E} \rho,$$

the fact that $\sigma^\tau = \sigma \otimes \chi$ implies (without loss of generality) that

$$\rho^{\tau_1}/\rho = \chi|_{W_K}.$$

Hence, we have

$$(\rho^c/\rho)^{\tau_1} = \rho^{\tau_1 c}/\rho^{\tau_1} = \rho^c \cdot \chi|_{W_K}/\rho \cdot \chi|_{W_K} = \rho^c/\rho.$$

Now we have

$$\left(\mathrm{sim}(\phi)^{-1} \cdot M(\sigma)\right)|_{W_E} = \chi \oplus \chi^\tau \oplus \chi \cdot \mathrm{Ind}_{W_K}^{W_E} \rho^c/\rho,$$

and the first two summands on the RHS are exchanged by τ. If ρ^c/ρ is not quadratic, then we see that

$$M(\sigma) = 2 + 2 \quad \text{and} \quad N(\phi) = 4.$$

On the other hand, suppose that ρ^c/ρ is quadratic. Then ρ^c/ρ is fixed by c and τ_1 and hence by $\mathrm{Gal}(K/F)$. If μ is an extension of ρ^c/ρ to W_E, then

$$\left(\mathrm{sim}(\phi)^{-1} \cdot M(\sigma)\right)|_{W_E} = \chi \oplus \chi^\tau \oplus \chi\mu \oplus \chi^\tau \mu,$$

and τ exchanges the last two summands as well since μ is τ-invariant, so we again have

$$M(\sigma) = 2 + 2 \quad \text{and} \quad N(\phi) = 4.$$

(b2) $\chi^\tau = \chi$. In this case, the quadratic character χ extends to W_F. We claim that it must extend to a quadratic character of W_F instead of quartic or, equivalently, that $\chi|_{F^\times} = 1$. Suppose on the contrary that χ extends to a quartic character $\tilde{\chi}$ with $\tilde{\chi}^2 = \omega_{E/F}$. Then we would have

$$\phi = \mathrm{Ind}_{W_E}^{W_F} \sigma^\tau = \mathrm{Ind}_{W_E}^{W_F} \sigma \cdot \chi = \phi \cdot \tilde{\chi},$$

and this would imply

$$\wedge^2 \phi = \wedge^2 \phi \cdot \tilde{\chi}^2 = \wedge^2 \phi \cdot \omega_{E/F}, \tag{6.6}$$

but $\mathrm{sim}(\phi)^{-1} \cdot \wedge^2 \phi|_{W_E}$ contains χ with multiplicity 1 so that $\mathrm{sim}(\phi)^{-1} \cdot \wedge^2 \phi$ contains precisely one extension of χ to W_F. Without loss of generality, one may suppose that $\mathrm{sim}(\phi)^{-1} \cdot \wedge^2 \phi$ contains $\tilde{\chi}$ but not $\tilde{\chi}^{-1} = \tilde{\chi} \cdot \omega_{E/F}$, but this contradicts equation (6.6). With this contradiction, we conclude that χ extends to a quadratic character of W_F or, equivalently, that $\chi|_{F^\times} = 1$. Thus, χ determines a biquadratic extension of F with quadratic subfields E, E_1, and E_2.

Now since $\chi|_{F^\times} = 1$, there exists a character of W_E such that

$$\chi = \lambda^\tau / \lambda.$$

Then we see that

$$(\sigma \cdot \lambda^{-1})^{\tau} = \sigma \cdot \lambda^{-1},$$

so there exists an irreducible two-dimensional representation π of W_F such that

$$\sigma = \pi|_{W_E} \cdot \lambda,$$

and hence,

$$\phi = \pi \otimes \mathrm{Ind}_{W_E}^{W_F} \lambda,$$

but then

$$\wedge^2 \phi = \det \pi \cdot \lambda|_{F^\times} \cdot \omega_{E/F} \cdot \left(\omega_{E/F} \oplus \omega_{E_1/F} \oplus \omega_{E_2/F} \oplus \mathrm{Ad}(\pi) \right),$$

so we conclude that

$$N(\phi) = \begin{cases} 4, & \text{if } \pi \text{ (or equivalently } \sigma\text{) is primitive;} \\ 8, & \text{if } \pi \text{ is dihedral wrt one quadratic field;} \\ 16, & \text{if } \pi \text{ is dihedral wrt three quadratic fields.} \end{cases}$$

Note that in the last two cases, the quadratic fields wrt which π is dihedral are necessarily different from E, E_1, and E_2 (by the irreducibility of ϕ). Moreover, it is clear that these three cases can occur only when $p = 2$. ■

Remark 6.7. Let us note that for the GSp_4-parameter ϕ in the above theorem, the L-packet L_ϕ consists of a unique supercuspidal representation π of $\mathrm{GSp}_4(F)$, and $N(\phi)$ is the number of irreducible constituents of the restriction $\pi|_{\mathrm{Sp}_4(F)}$. □

The following proposition determines $N(\phi)$ and $I(\phi)$ for the remaining discrete series parameters ϕ.

Proposition 6.8. Let S_n denote the n-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. We have

(i) If $\phi = \mu \boxtimes S_4$ with μ a one-dimensional character of W_F, then

$$N(\phi) = 1$$

and so, of course,

$$I(\phi) = \langle 1 \rangle.$$

(ii) If $\phi = \sigma \boxtimes S_2$ with σ an irreducible two-dimensional dihedral representation of W_F, then

$$N(\phi) = \begin{cases} 2, & \text{if } \sigma \text{ is dihedral wrt one quadratic field } E/F; \\ 4, & \text{if } \sigma \text{ is dihedral wrt three quadratic fields } E_i/F, \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \omega_{E/F} \rangle; \\ \langle \omega_{E_1/F}, \omega_{E_2/F} \rangle. \end{cases}$$

(iii) If $\phi = \phi_1 \oplus \phi_2$ where $\phi_1 \neq \phi_2$ are irreducible two-dimensional representations of WD_F with $\det \phi_1 = \det \phi_2$, then we have two cases:

(a) if $\phi_1 \neq \phi_2 \otimes \chi$ for any χ, then

$$N(\phi) = 2 \cdot \# I(\phi) = \begin{cases} 4, & \text{if } \phi_1 \text{ and } \phi_2 \text{ are dihedral wrt the same quadratic } E/F; \\ 2, & \text{otherwise,} \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \omega_{E/F} \rangle; \\ \langle 1 \rangle. \end{cases}$$

(b) if $\phi_1 = \phi_2 \otimes \chi$, with χ necessarily quadratic, then

$$N(\phi) = 2 \cdot \# I(\phi) = \begin{cases} 4, & \text{if } \phi_1 \text{ is primitive or non-trivial on } \mathrm{SL}_2(\mathbb{C}); \\ 8, & \text{if } \phi_1 \text{ is dihedral wrt one quadratic field } E/F; \\ 16, & \text{if } \phi_1 \text{ is dihedral wrt three quadratic fields } E_i/F, \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \chi \rangle; \\ \langle \chi, \omega_{E/F} \rangle; \\ \langle \chi, \omega_{E_1/F}, \omega_{E_2/F} \rangle. \end{cases} \qquad \square$$

Proof.

(i) It is easy to see that $\mathrm{std}(\phi) = S_5$ is irreducible.

(ii) This follows from

$$\det \sigma^{-1} \cdot \wedge^2 \phi = \mathrm{Ad}(\sigma) \bigoplus S_3.$$

(iii) Consider the two cases separately.

(a) If $\phi_1 \neq \phi_2 \otimes \chi$, then

$$\det \phi_1 \cdot \mathrm{std}(\phi) = \det \phi_1 \bigoplus \phi_1 \otimes \phi_2,$$

and $\phi_1 \otimes \phi_2 = 2 + 2$ or 4, so $N(\phi) = 2$ or 4. On the other hand, it is clear that

$$I(\phi) = \{\text{quadratic characters } \chi \colon \phi_i \otimes \chi = \phi_i\}$$

This gives the desired result.

Observe in particular that it is not possible for ϕ_1 and ϕ_2 to be dihedral wrt the same three quadratic fields. This is well known and can in fact be checked directly. Namely, suppose that $\phi_i = \mathrm{Ind}_{W_E}^{W_F} \rho_i$ with $\rho_1^c/\rho_1 = \rho_2^c/\rho_2$ quadratic.

Then ρ_1/ρ_2 is Gal(E/F)-invariant and

$$\det\phi_1 = \det\phi_2 \Longrightarrow \rho_1|_{F^\times} = \rho_2|_{F^\times}$$

so that ρ_1/ρ_2 is quadratic and trivial on $F^\times$. Hence, ρ_1/ρ_2 extends to a quadratic character ω, but this implies that $\phi_1 = \phi_2 \otimes \omega$, which contradicts our assumption that $\phi_1 \neq \phi_2 \otimes \chi$ for any χ.

(b) If $\phi_1 = \phi_2 \otimes \chi$, with $\chi^2 = 1$, then ϕ_1 is not dihedral wrt χ. Moreover, it is clear that $I(\phi)$ is generated by χ and the quadratic characters ω such that $\phi_1 = \phi_1 \otimes \omega$. This gives the desired result. ■

Remark 6.9. In the above proposition, in situations (i) and (ii), the GSp$_4$ L-packet L_ϕ consists of a unique non-supercuspidal representation. In situation (iii), it consists of two discrete series representations. Also in (ii), in order for ϕ to be symplectic, σ is necessarily dihedral (see [7]). □

Non-discrete series parameters

The following proposition determines $N(\phi)$ and $I(\phi)$ for the non-discrete series parameters ϕ of GSp$_4$. We omit the proof as it is quite simple. Indeed, one can easily determine $I(\phi)$ by using Lemma 6.4. □

Proposition 6.10. Consider the non-discrete series parameters ϕ of GSp$_4(F)$ which fall into 3 families according to the smallest Levi subgroup of GSp$_4(\mathbb{C})$ through which ϕ factors.

(i) If $Im(\phi)$ is contained in a Siegel parabolic (but not a Borel subgroup) so that $\phi = \sigma \oplus (\sigma \otimes \chi)$ with σ irreducible and $\mathrm{sim}(\phi) = \chi \cdot \det\sigma$, then we consider the following different cases:

(a) if $\chi^2 \neq 1$, then

$$N(\phi) = \#I(\phi) = \begin{cases} 1, & \text{if } \sigma \text{ is primitive or non-trivial on } \mathrm{SL}_2(\mathbb{C}); \\ 2, & \text{if } \sigma \text{ is dihedral wrt one quadratic field } E; \\ 4, & \text{if } \sigma \text{ is dihedral wrt three quadratic fields } E_i, \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle 1 \rangle; \\ \langle \omega_{E/F} \rangle; \\ \langle \omega_{E_1/F}, \omega_{E_2/F} \rangle. \end{cases}$$

(b) if $\chi^2 = 1$, then

$$N(\phi) = \begin{cases} \#I(\phi), & \text{if } \chi \neq 1; \\ 2 \cdot \#I(\phi), & \text{if } \chi = 1. \end{cases}$$

Moreover,

$$I(\phi) = \begin{cases} \langle \chi \rangle, & \text{if } \sigma \text{ is primitive or non-trivial on } \mathrm{SL}_2(\mathbb{C}); \\ \langle \chi, \omega_{E/F} \rangle, & \text{if } \sigma \text{ is dihedral wrt one quadratic } E/F; \\ \langle \chi, \omega_{E_1/F}, \omega_{E_2/F} \rangle, & \text{if } \sigma \text{ dihedral wrt three quadratic } E_i/F, \end{cases}$$

with χ suppressed if $\chi = 1$.

(ii) If $Im(\phi)$ is contained in a Klingen parabolic (but not a Borel subgroup) so that $\phi = \chi \cdot (1 \oplus \sigma \oplus \det \sigma)$ with σ irreducible and $\mathrm{sim}(\phi) = \chi^2 \cdot \det \sigma$, then

$$N(\phi) = \#I(\phi) = \begin{cases} 2, & \text{if } \sigma \text{ is dihedral wrt } E/F \text{ and } \det \sigma = \omega_{E/F}; \\ 1, & \text{otherwise,} \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \omega_{E/F} \rangle; \\ \langle 1 \rangle. \end{cases}$$

(iii) If $Im(\phi)$ is contained in a Borel subgroup so that $\phi = \chi \cdot (\chi_1\chi_2 \oplus \chi_1 \oplus \chi_2 \oplus 1)$, with $\mathrm{sim}(\phi) = \chi^2\chi_1\chi_2$, then the non-trivial quadratic characters in $I(\phi)$ are precisely the distinct non-trivial quadratic characters among χ_1, χ_2, and $\chi_1\chi_2$. More precisely,

$$N(\phi) = \#I(\phi) = \begin{cases} 4, & \text{if } \chi_1 \neq \chi_2 \text{ are non-trivial quadratic;} \\ 1, & \text{if none of } \chi_1, \chi_2, \text{ and } \chi_1\chi_2 \text{ is non-trivial quadratic;} \\ 2, & \text{if exactly one of } \chi_1, \chi_2, \text{ and } \chi_1\chi_2 \text{ is quadratic,} \end{cases}$$

and, respectively,

$$I(\phi) = \begin{cases} \langle \chi_1, \chi_2 \rangle; \\ \langle 1 \rangle; \\ \langle \omega \rangle, \end{cases}$$

where ω is the unique quadratic character among χ_1, χ_2, or $\chi_1\chi_2$ in the last case. □

Remark 6.11. In situation (b) in (i) above, if $\chi = 1$, the GSp_4 L-packet L_ϕ consists of two non-discrete series representations, namely in the notation of [7], $L_\phi = \{\pi_{\mathrm{gen}}(\tau), \pi_{\mathrm{ng}}(\tau)\}$ where τ is the irreducible admissible representation of $\mathrm{GL}_2(F)$ corresponding to σ. For all the other cases, L_ϕ consists of a unique non-discrete series representation. □

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Appendix A: Restrictions of Admissible Representations of $\mathrm{GSp}_4(F)$ to $\mathrm{Sp}_4(F)$

Recall that for each $\pi \in \Pi(\mathrm{GSp}_4)$, we define

$$\mathrm{JH}(\pi) := \{\text{constituents of } \pi|_{\mathrm{Sp}_4}\}.$$

Then the following tables (Table A1, Table A2, and Table A3) summarize the sizes of $\mathrm{JH}(\pi)$ for irreducible admissible representations π of $\mathrm{GSp}_4(F)$. Those tables are obtained simply by translating what we have obtained for the "Galois side" in the previous section to the "automorphic side." We use the notations of [7] to describe π (see Table 1 of [7]).

Table A1 Restriction from $\mathrm{GSp}_4(F)$ to $\mathrm{Sp}_4(F)$ (supercuspidal (SC))

		π		#JH(π)	
SC	a	Not a lift from GSO(2, 2) or GSO(4)		2, 4	$p \neq 2$
				1,2,4,8,16	$p = 2$
	b, c	$\theta(\tau_1 \boxtimes \tau_2)$ or $\theta(\tau_1^D \boxtimes \tau_2^D)$	$\tau_1 \neq \tau_2 \otimes \chi$	2	τ_1 and τ_2 are dihedral wrt same quad. ext.
				1	Otherwise
			$\tau_1 = \tau_2 \otimes \chi$	2	τ_1 is primitive or twisted Steinberg
				4	τ_1 is dihedral wrt one quad. ext.
				8	τ_1 is dihedral wrt three quad. ext.

Table A2 Restriction from $\mathrm{GSp}_4(F)$ to $\mathrm{Sp}_4(F)$ (discrete series (DS))

		π	#JH(π)	
DS	a	$St(\chi, \tau)$	2	τ is dihedral wrt one quad. ext.
			4	τ is dihedral wrt three quad. ext.
	b	$St(\tau, \mu)$	2	$\tau = st_\chi$ with χ non-trivial quadratic
			1	Otherwise
	c	$St_{\mathrm{PGSp}_4} \otimes \mu$	1	

Table A3 Restriction from $\mathrm{GSp}_4(F)$ to $\mathrm{Sp}_4(F)$ (non-discrete series (NDS))

		π		#JH(π)	
NDS	a	$J_{Q(Z)}(\chi, \tau)$ $\chi \neq 1$	$\chi^2 = 1$	2	τ is primitive or twisted Steinberg
				4	τ is dihedral wrt one quad. ext.
				8	τ is dihedral wrt three quad. ext.
			$\chi^2 \neq 1$	1	τ is primitive or twisted Steinberg
				2	τ is dihedral wrt one quad. ext.
				4	τ is primitive or twisted Steinberg
	b, c	$\pi_{\mathrm{gen}}(\tau)$ or $\pi_{\mathrm{ng}}(\tau)$		1	τ is primitive or twisted Steinberg
				2	τ is dihedral wrt one quad. ext.
				4	τ is dihedral wrt three quad. ext.
	d	$J_{P(Y)}(\tau, \chi)$		2	$\tau = \tau \otimes \omega_\tau$, $\omega_\tau \neq 1$
				1	Otherwise
	e	$J_B(\chi_1, \chi_2; \chi)$		4	$\chi_1 \neq \chi_2$ non-trivial quadratic
				1	None of χ_1, χ_2, or $\chi_1\chi_2$ non-trivial quadratic
				2	Otherwise

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