ON THE LOCAL LANGLAND CORRESPONDENCE AND ARTHUR CONJECTURE FOR EVEN ORTHOGONAL GROUPS

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Abstract. In this paper, we highlight and state precisely the local Langlands correspondence for quasi-split O_{2n} established by Arthur. We give two applications: Prasad's conjecture and Gross–Prasad conjecture for O_n. Also, we discuss the Arthur conjecture for O_{2n}, and establish the Arthur multiplicity formula for O_{2n}.

Contents

1. Introduction

In his long-awaited book [Ar], Arthur obtained a classification of irreducible representations of quasi-split symplectic and special orthogonal groups over local fields of characteristic 0 (the local Langlands correspondence LLC) as well as a description of the automorphic discrete spectrum of these groups over number fields (the Arthur conjecture). He proved these results by establishing the twisted endoscopic transfer of automorphic representations from these classical groups to GL_N by exploiting the stabilization of the twisted trace formula of GL_N (which has now been completed by Waldspurger and Mœglin). However, for the quasi-split special even orthogonal groups SO_{2n}, the result is not as definitive as one hopes. More precisely, for a p-adic field F, Arthur only gave a classification of the irreducible representations of $\mathrm{SO}_{2n}(F)$ up to conjugacy by $\mathrm{O}_{2n}(F)$, instead of by $\mathrm{SO}_{2n}(F)$. Likewise, over a number field $\mathbb{F}$, he does not distinguish between a square-integrable automorphic representation π and its twist by the outer automorphism corresponding to an element of $\mathrm{O}_{2n}(\mathbb{F}) \setminus \mathrm{SO}_{2n}(\mathbb{F})$.

The reason for this less-than-optimal result for SO_{2n} is that, for the purpose of the twisted endoscopic transfer, it is more natural to work with the orthogonal groups O_{2n} instead of SO_{2n}. In fact, Arthur has obtained in [Ar, Theorems 2.2.1, 2.2.4] a full classification of the irreducible representations of $\mathrm{O}_{2n}(F)$. It is from this that he deduced the weak LLC for $\mathrm{SO}_{2n}(F)$ alluded to above. Indeed, the weak LLC for $\mathrm{SO}_{2n}(F)$ is equivalent to the classification of irreducible representations of $\mathrm{O}_{2n}(F)$ modulo twisting by the determinant character det.

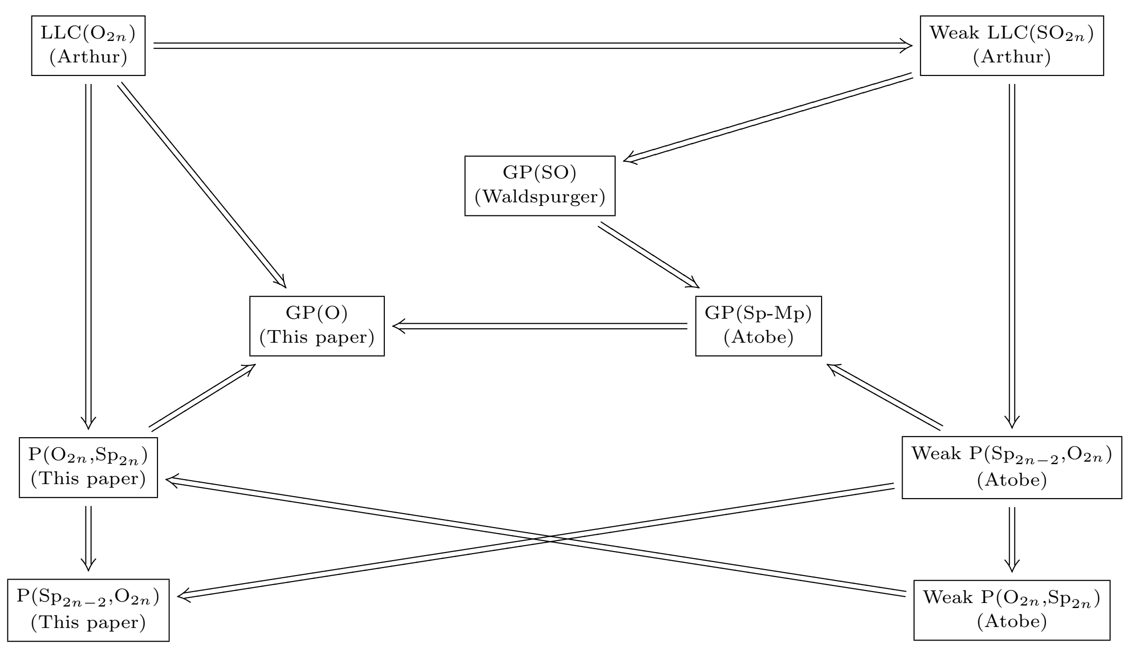
Unfortunately, this rather complete result for $\mathrm{O}_{2n}(F)$ was not highlighted in [Ar]. One possible reason is that O_{2n} is a disconnected linear algebraic group and so does not fit in the framework of the classical Langlands program; for example, one does not have a systematic definition of the L-group of a disconnected reductive group and so one does not have the notion of Langlands parameters. In choosing to stick to the context of the Langlands program, Arthur has highlighted the results for SO_{2n} instead. However, it has been observed that a suitable L-group for O_{2n} is the group $\mathrm{O}_{2n}(\mathbb{C})$ and an L-parameter for $\mathrm{O}_{2n}(F)$ should be an orthogonal representation of the Weil–Deligne group WD_F. A precise statement to this effect seems to be first formulated in the paper [P2] of D. Prasad.

The main goal of this paper is to highlight and state precisely the local Langlands correspondence for quasi-split $\mathrm{O}_{2n}(F)$ established by Arthur in [Ar] using this notion of L-parameters, and to establish various desiderata of this LLC. We also formulate the natural extension of the LLC to the pure inner forms (using Vogan L-packets [V]). The statements can be found in Desiderata 3.6 and 3.9. We especially note the key role played by the local intertwining relation in Hypothesis 3.10. This local intertwining relation was established in [Ar] for quasi-split groups but is conjectural for pure inner forms.

Our main motivation for formulating a precise LLC for O_{2n} is that the representations of $\mathrm{O}_{2n}(F)$ arise naturally in various context, such as in the theory of theta correspondence. If one wants to describe the local theta correspondence for the dual pair $\mathrm{O}_{2n}(F) \times \mathrm{Sp}_{2m}(F)$, one would need a classification of irreducible representations of $\mathrm{O}_{2n}(F)$. Thus, this paper lays the groundwork needed for our paper [AG] in which we determine the local theta lifting of tempered representations in terms of the local Langlands correspondence. Having described the LLC for O_{2n}, we give two applications:

- (Prasad's conjecture) We complete the results in the first author's PhD thesis [At1], in which the local theta correspondences for the almost equal rank dual pairs $\mathrm{O}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$ and $\mathrm{O}_{2n}(F)\times\mathrm{Sp}_{2n-2}(F)$ were determined in terms of the weak LLC for $\mathrm{O}_{2n}(F)$. In particular, we describe these theta correspondences completely in terms of the LLC for $\mathrm{O}_{2n}(F)$, thus completing the proof of Prasad's conjecture (Conjectures 4.4 and 4.8). The result is contained in Theorem 4.6.
- (Gross–Prasad conjecture for O_n) In [At1], these theta correspondences were used to prove the Fourier–Jacobi case of the local Gross–Prasad (GP) conjecture for symplectic-metaplectic groups, by relating the Fourier–Jacobi case with the Bessel case of (GP) for SO_n (which has been established by Waldspurger). For this purpose, the weaker version of Prasad's conjecture (based on the weak LLC for O_{2n}) is sufficient. Now that we have the full Prasad's conjecture, we use the Fourier–Jacobi case of (GP) to prove a version of (GP) for O_n. In other words, we shall extend and establish the Gross-Prasad conjecture to the context of orthogonal groups (Conjecture 5.3). The result is contained in Theorem 5.7.

These results are related in a complicated manner. The following diagram is a summary of the situation:



Here,

- LLC(O_{2n}) means the LLC for O_{2n}, which has been established by Arthur [Ar, Theorems 2.2.1, 2.2.4];
- Weak LLC(SO_{2n}) means the weak LLC for SO_{2n} or O_{2n};
- Weak P($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$) means the weaker version of Prasad's conjecture for ($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$), which was proven in [At1, §7];
- Weak P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) means the weaker version of Prasad's conjecture for ($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$), which follows from Weak P($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$) (see [At1, §5.5]);
- P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) means Prasad's conjecture for ($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) (Conjecture 4.4), which follows from Weak P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) and Hypothesis 3.10 (see Theorem 4.6);
- P($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$) means Prasad's conjecture for ($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$) (Conjecture 4.8), which follows from Weak P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) and P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) (see Theorem 4.9);
- GP(SO) means the local Gross–Prasad (GP) conjecture for special orthogonal groups, which was established by Waldspurger [W2], [W3], [W5], [W6];
- GP(Sp-Mp) means the local Gross–Prasad (GP) conjecture for symplectic-metaplectic groups, which was proven in [At1, Theorem 1.3] by using GP(SO) and Weak P($\mathrm{Sp}_{2n-2}, \mathrm{O}_{2n}$);
- GP(O) means the local Gross–Prasad (GP) conjecture for orthogonal groups (Conjecture 5.3), which follows from GP(Sp-Mp) and P($\mathrm{O}_{2n}, \mathrm{Sp}_{2n}$) (see Theorem 5.7).

Finally, we discuss how the Arthur conjecture for the automorphic discrete spectrum of SO_{2n} over a number field $\mathbb{F}$ can be extended to an analogous statement for O_{2n}. This is implicitly in [Ar], but not precisely formulated. In particular, we describe the automorphic discrete spectrum of O_{2n} in terms of local and global Arthur packets, and establish the Arthur multiplicity formula (Theorem 7.1). Also we show that the tempered part of the automorphic discrete spectrum of O_{2n} has multiplicity 1 (Proposition 7.2, see also Remark 7.3). This should lay the groundwork for a more precise study of the global theta correspondence for symplectic-orthogonal dual pairs.

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Notations. Let F be a non-archimedean local field with characteristic zero, $\mathfrak{o}$ be the ring of integers of F, ϖ be a uniformizer, q be the number of elements in the residue class field $\mathfrak{o}/\varpi\mathfrak{o}$ and $|\cdot|_F$ be the normalized absolute value on F so that $|\varpi|_F = q^{-1}$. We denote by W_F and $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil and Weil–Deligne groups of F, respectively. Fix a non-trivial additive character ψ of F. For $c \in F^\times$, we define an additive character ψ_c or $c\psi$ of F by

$$\psi_c(x) = c\psi(x) = \psi(cx).$$

We set $\chi_c = (\cdot, c)$ to be the quadratic character of $F^\times$ associated to $c \in F^\times/F^{\times 2}$. Here, $(\cdot,\cdot)$ is the quadratic Hilbert symbol of F. For a totally disconnected locally compact group G, we denote the set of equivalence classes of irreducible smooth representations of G by $\mathrm{Irr}(G)$. If G is the group of F-points of a linear algebraic group over F, we denote by $\mathrm{Irr}_{\mathrm{temp}}(G)$ (resp. $\mathrm{Irr}_{\mathrm{disc}}(G)$) the subset of $\mathrm{Irr}(G)$ of classes of irreducible tempered representations (resp. irreducible discrete series representations). For a topological group H, we define the component group of H by $\pi_0(H) = H/H^\circ$, where H° is the identity component of H. The Pontryagin dual (i.e., the character group) of a finite abelian group A is denoted by A^D or $\widehat{A}$.

2. Quasi-split orthogonal groups

In this section, we summarize facts about quasi-split orthogonal groups and their representations.

2.1. **Orthogonal spaces.** Let $V = V_{2n}$ be a vector space of dimension $2n$ over F and

$$\langle \cdot, \cdot \rangle_V : V \times V \to F$$

be a non-degenerate symmetric bilinear form. We take a basis $\{e_1, \dots, e_{2n}\}$ of V, and define the discriminant of V by

$$\mathrm{disc}(V) = (-1)^n \det((\langle e_i, e_j \rangle_V)_{i,j}) \bmod F^{\times 2} \in F^\times / F^{\times 2}.$$

Let $\chi_V = (\cdot, \mathrm{disc}(V))$ be the character of $F^\times$ associated with $F(\sqrt{\mathrm{disc}(V)})$. We call χ_V the discriminant character of V. The orthogonal group $\mathrm{O}(V)$ associated to V is defined by

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V) \mid \langle gv, gv' \rangle_V = \langle v, v' \rangle_V \text{ for any } v, v' \in V\}.$$

Fix $c, d \in F^\times$. Let

$$V_{(d,c)} = F[X]/(X^2 - d)$$

be a 2-dimensional vector space equipped with a bilinear form

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_{V_{(d,c)}} \coloneqq c \cdot \mathrm{tr}(\alpha \overline{\beta}),$$

where $\beta \mapsto \overline{\beta}$ is the involution on $F[X]/(X^2 - d)$ induced by $a + bX \mapsto a - bX$. This involution is regarded as an element $\epsilon \in \mathrm{O}(V_{(d,c)})$. The images of $1, X \in F[X]$ in $V_{(d,c)}$ are denoted by e, e', respectively.

For $n > 1$, we say that V_{2n} is associated to (d, c) if

$$V_{2n} \cong V_{(d,c)} \oplus \mathbb{H}^{n-1}$$

as orthogonal spaces, where $\mathbb{H}$ is the hyperbolic plane, i.e., $\mathbb{H} = Fv_i + Fv_i^*$ with $\langle v_i, v_i \rangle = \langle v_i^*, v_i^* \rangle = 0$ and $\langle v_i, v_i^* \rangle = 1$. Note that $\mathrm{disc}(V_{2n}) = d \bmod F^{\times 2}$. The orthogonal group $\mathrm{O}(V_{2n})$ is quasi-split, and any quasi-split orthogonal group can be obtained in this way. Note that $V_{2n} = V_{(d,c)} \oplus \mathbb{H}^{n-1} \cong V'_{2n} = V_{(d',c')} \oplus \mathbb{H}^{n-1}$ as orthogonal spaces if and only if $d \equiv d' \bmod F^{\times 2}$ so that $E \coloneqq F(\sqrt{d}) = F(\sqrt{d'})$ and $c \equiv c' \bmod N_{E/F}(E^\times)$.

2.2. **Generic representations.** Suppose that $n > 1$ in this subsection. Let $V = V_{2n}$ be an orthogonal space associated to (d, c). We set

$$X_k = Fv_1 + \cdots + Fv_k \quad \text{and} \quad X_k^* = Fv_1^* + \cdots + Fv_k^*$$

for $1 \le k \le n-1$. We denote by $B_0 = TU_0$ the F-rational Borel subgroup of $\mathrm{SO}(V_{2n})$ stabilizing the complete flag

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_{n-1} \rangle = X_{n-1},$$

where T is the F-rational torus stabilizing the lines Fv_i for $i = 1, \dots, n-1$. We identify $\mathrm{O}(V_{(d,c)})$ as a subgroup of $\mathrm{O}(V_{2n})$ which fixes $\mathbb{H}^{n-1}$. Via the canonical embedding $\mathrm{O}(V_{(d,c)}) \hookrightarrow \mathrm{O}(V_{2n})$, we regard ϵ as an element in $\mathrm{O}(V_{2n})$. Note that ϵ depends on (d, c). We define a generic character μ_c of U_0 by

$$\mu_c(u) = \psi(\langle uv_2, v_1^* \rangle_V + \cdots + \langle uv_n, v_{n-2}^* \rangle_V + \langle ue, v_{n-1}^* \rangle_V).$$

Note that ϵ normalizes U_0 and fixes μ_c.

Put $E = F(\sqrt{d})$. If $c' \in cN_{E/F}(E^\times)$, then we have an isomorphism $V_{2n} = V_{(d,c)} \oplus \mathbb{H}^{n-1} \to V'_{2n} = V_{(d,c')} \oplus \mathbb{H}^{n-1}$, and so that we obtain an isomorphism

$$f\colon \mathrm{O}(V_{2n}) \to \mathrm{O}(V'_{2n}).$$

Moreover, we can take an isomorphism $f\colon \mathrm{O}(V_{2n}) \to \mathrm{O}(V'_{2n})$ such that $f(B_0) = B'_0$, $f(T) = T'$ and $f(\mathrm{O}(V_{(d,c)})) = \mathrm{O}(V_{(d,c')})$, where $B'_0 = T'U'_0$ and T' are the Borel subgroup and maximal torus of $\mathrm{SO}(V'_{2n})$, respectively, defined as above. Let $\mathcal{F}$ be the set of such isomorphisms. Then the group $T' \rtimes \langle \epsilon' \rangle \cong \mathrm{O}(V_{(d,c')}) \times (F^\times)^{n-1}$ acts on $\mathcal{F}$ by

$$(t' \cdot f)(g) = t' f(g) t'^{-1}$$

for $t' \in T' \rtimes \langle \epsilon' \rangle$ and $g \in \mathrm{O}(V_{2n})$. Here, $\epsilon' \in \mathrm{O}(V_{(d,c')})$ is an analogue to $\epsilon \in \mathrm{O}(V_{(d,c)})$. Since $n > 1$, this action of $T' \rtimes \langle \epsilon' \rangle$ is transitive.

Choosing $f \in \mathcal{F}$, we regard $\mu_{c'}$ as a generic character of U_0 by

$$U_0 \xrightarrow{f} U'_0 \xrightarrow{\mu_{c'}} \mathbb{C}^\times.$$

Note that the T-orbit of $\mu_{c'}$ is independent of the choice of f since ϵ' fixes $\mu_{c'}$.

We consider a 4-tuple (V, B_0, T, μ), where

- $V = V_{2n}$ is an orthogonal space associated to some (d, c);
- B_0 is an F-rational Borel subgroup of $\mathrm{SO}(V)$;
- T is a maximal F-torus contained in B_0;
- μ is a generic character of U_0, where U_0 is the unipotent radical of B_0.

We say that two tuples (V, B_0, T, μ) and (V', B_0', T', μ') is equivalent if the following conditions hold:

(1) there exists an isomorphism $V \to V'$ as orthogonal spaces, which induces a group isomorphism $f \colon \mathrm{O}(V) \to \mathrm{O}(V')$;
(2) $f(B_0) = B_0'$ and $f(T) = T'$ (so that $f(U_0) = U_0'$);
(3) there exists $t \in T$ such that $\mu' \circ f = \mu \circ \mathrm{Int}(t)$.

Proposition 2.1. *Fix $d \in F^\times/F^{\times 2}$. For $c \in F^\times$, we associate the 4-tuple (V, B_0, T, μ), where*

- *$V = V_{2n}$ is an orthogonal space associated to (d, c);*
- *B_0 and T are as above;*
- *$\mu = \mu_c$.*

Then the map $c \mapsto (V, B_0, T, \mu)$ gives a canonical bijection (not depending on ψ)

$$F^\times/F^{\times 2} \to \{\textit{equivalence classes of tuples } (V, B_0, T, \mu) \textit{ with } \mathrm{disc}(V) = d\}.$$

Proof. Let $V = V_{2n}$ be an orthogonal space associated to (d, c). By [GGP, §12], the map $c' \mapsto \mu_{c'} \circ f$ for $f \in \mathcal{F}$ gives a well-defined bijection

$$cN_{E/F}(E^\times)/F^{\times 2} \to \{T\text{-orbits of generic characters of } U_0\},$$

where $E = F(\sqrt{d})$. For $c' \in cN_{E/F}(E^\times)$, let $V' = V_{2n}'$ be an orthogonal space associated to (d, c'). Then two tuples $(V, B_0, T, \mu_{c'} \circ f)$ and $(V', B_0', T', \mu_{c'})$ are equivalent each other. This implies that the map

$$F^\times \to \{\text{equivalence classes of tuples } (V, B_0, T, \mu) \text{ with } \mathrm{disc}(V) = d\}.$$

is surjective. Also, we note that for a generic character μ, two tuples (V, B_0, T, μ_c) and (V, B_0, T, μ) are equivalent if and only if $\mu = \mu_c \circ \mathrm{Int}(t)$ for some $t \in T$. This implies that the above map induces the bijection

$$F^\times/F^{\times 2} \to \{\text{equivalence classes of tuples } (V, B_0, T, \mu) \text{ with } \mathrm{disc}(V) = d\},$$

as desired. □

Remark 2.2. *Let $(V, \langle \cdot, \cdot \rangle_V)$ be an orthogonal space associated to (d, c). Fix $a \in F^\times$. We define a new orthogonal $(V', \langle \cdot, \cdot \rangle_{V'})$ by $V' = V$ as vector spaces and by*

$$\langle x, y \rangle_{V'} = a \cdot \langle x, y \rangle_V.$$

Then $(V', \langle \cdot, \cdot \rangle_{V'})$ is associated to (d, ac). As subgroup of $\mathrm{GL}(V) = \mathrm{GL}(V')$, we have identifications

$$\mathrm{O}(V) = \mathrm{O}(V') \quad \textit{and} \quad \mathrm{SO}(V) = \mathrm{SO}(V').$$

These identifications preserve F-rational Borel subgroups and maximal F-tori. Moreover, the generic character μ_c of a maximal unipotent subgroup of $\mathrm{SO}(V)$ transfers μ_{ac}. More precisely, see [At1, Appendix A.5].

Since ϵ' stabilizes $\mu_{c'}$, we can extend $\mu_{c'}$ to $U' = U_0' \rtimes \langle \epsilon' \rangle$. There are exactly two such extensions $\mu_{c'}^\pm \colon U' \to \mathbb{C}^\times$ which are determined by

$$\mu_{c'}^\pm(\epsilon') = \pm 1.$$

We say that an irreducible smooth representation σ of $\mathrm{O}(V_{2n})$ is $\mu_{c'}^\pm$-generic if

$$\mathrm{Hom}_{f^{-1}(U')}(\sigma, \mu_{c'}^\pm) \neq 0$$

for some $f \in \mathcal{F}$. For $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$, the $\mu_{c'}$-genericity is defined similarly. The $\mu_{c'}^\pm$-genericity and the $\mu_{c'}$-genericity are independent of the choice of f. Note that $f^{-1}(U_0') = U_0$ for $f \in \mathcal{F}$, and

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{U_0}(\sigma_0, \mu_{c'})) \leq 1$$

for $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$.

Lemma 2.3. *Let $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$.*

(1) *Assume that* σ_0 *can be extended to* $\mathrm{O}(V_{2n})$. *Then there are exactly two such extensions. Moreover, the following are equivalent:*
   (A) σ_0 *is* $\mu_{c'}$*-generic;*
   (B) *exactly one of two extensions is* $\mu_{c'}^{+}$*-generic but not* $\mu_{c'}^{-}$*-generic, and the other is* $\mu_{c'}^{-}$*-generic but not* $\mu_{c'}^{+}$*-generic.*

(2) *Assume that* σ_0 *can not be extended to* $\mathrm{O}(V_{2n})$. *Then* $\sigma = \mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$ *is irreducible. Moreover, the following are equivalent:*
   (A) σ_0 *is* $\mu_{c'}$*-generic;*
   (B) σ *is both* $\mu_{c'}^{+}$*-generic and* $\mu_{c'}^{-}$*-generic.*

Proof. The first assertions of (1) and (2) follow from the Clifford theory. It is easy that (B) implies (A) in both (1) and (2).

We show (A) implies (B). Let $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$ be a $\mu_{c'}$-generic representation, i.e.,

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{U_0}(\sigma_0, \mu_{c'} \circ f)) = 1.$$

for some $f \in \mathcal{F}$. By the Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Hom}_{f^{-1}(U')}(\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0), \mu_{c'}^{\pm} \circ f) &\cong \mathrm{Hom}_{\mathrm{O}(V_{2n})}(\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0), \mathrm{Ind}_{f^{-1}(U')}^{\mathrm{O}(V_{2n})}(\mu_{c'}^{\pm} \circ f))\\ &\cong \mathrm{Hom}_{\mathrm{SO}(V_{2n})}(\sigma_0, \mathrm{Ind}_{f^{-1}(U')}^{\mathrm{O}(V_{2n})}(\mu_{c'}^{\pm} \circ f)|\mathrm{SO}(V_{2n}))\\ &\cong \mathrm{Hom}_{\mathrm{SO}(V_{2n})}(\sigma_0, \mathrm{Ind}_{U_0}^{\mathrm{SO}(V_{2n})}(\mu_{c'} \circ f)) \cong \mathrm{Hom}_{U_0}(\sigma_0, \mu_{c'} \circ f).\end{aligned}$$

In particular, if $\sigma = \mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$ is irreducible, then σ is both $\mu_{c'}^{+}$-generic and $\mu_{c'}^{-}$-generic. This shows that (A) implies (B) in (2). If $\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0) \cong \sigma_1 \oplus \sigma_2$, then

$$\mathrm{Hom}_{U_0}(\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)|\mathrm{SO}(V_{2n}), \mu_{c'} \circ f) \cong \mathrm{Hom}_{U_0}(\sigma_1|\mathrm{SO}(V_{2n}), \mu_{c'} \circ f) \oplus \mathrm{Hom}_{U_0}(\sigma_2|\mathrm{SO}(V_{2n}), \mu_{c'} \circ f),$$

and $\mathrm{Hom}_{U_0}(\sigma_i|\mathrm{SO}(V_{2n}), \mu_{c'} \circ f)$ is $f^{-1}(\epsilon')$-stable for $i = 1, 2$. Hence this subspace is an eigenspace of $f^{-1}(\epsilon)$. Since both ± 1 are eigenvalues of $f^{-1}(\epsilon)$ in $\mathrm{Hom}_{U_0}(\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0), \mu_{c'} \circ f)$, exactly one of σ_1 and σ_2 is $\mu_{c'}^{+}$-generic, and the other is μ_{c}^{-}-generic. This shows that (A) implies (B) in (1). □

2.3. **Unramified representations.** Let $V = V_{2n}$ be an orthogonal space associated to (d, c). We say that $\mathrm{O}(V_{2n})$ (or $\mathrm{SO}(V_{2n})$) is unramified if $c, d \in \mathfrak{o}^{\times}$. In this subsection, we assume this condition. Recall that V_{2n} has a decomposition

$$V_{2n} = Fv_1 + \dots Fv_{n-1} + V_{(d,c)} + Fv_1^* + \cdots + Fv_{n-1}^*$$

with $V_{(d,c)} = Fe + Fe'$. We set

$$\begin{cases} v_0 = \dfrac{e + u^{-1}e'}{2}, \ v_0^* = \dfrac{e - u^{-1}}{2c} & \text{if } d = u^2,\\ \mathfrak{o}_E \text{ to be the ring of integers of } E = F(\sqrt{d}) \cong V_{(d,c)} & \text{if } d \notin \mathfrak{o}^{\times 2}.\end{cases}$$

Note that $\langle v_0, v_0\rangle_V = \langle v_0^*, v_0^*\rangle_V = 0$ and $\langle v_0, v_0^*\rangle_V = 1$. Let L_{2n} be the $\mathfrak{o}$-lattice of V_{2n} defined by

$$L_{2n} = \begin{cases} \mathfrak{o}v_1 + \cdots + \mathfrak{o}v_{n-1} + \mathfrak{o}v_0 + \mathfrak{o}v_0^* + \mathfrak{o}v_1^* + \cdots + \mathfrak{o}v_{n-1}^* & \text{if } d \in \mathfrak{o}^{\times 2},\\ \mathfrak{o}v_1 + \cdots + \mathfrak{o}v_{n-1} + \mathfrak{o}_E + \mathfrak{o}v_1^* + \cdots + \mathfrak{o}v_{n-1}^* & \text{if } d \notin \mathfrak{o}^{\times 2}.\end{cases}$$

Let K be the maximal compact subgroup of $\mathrm{O}(V_{2n})$ which preserves the lattice L_{2n}. Note that K contains ϵ and satisfies

$$K = K_0 \rtimes \langle \epsilon \rangle,$$

where $K_0 = K \cap \mathrm{SO}(V_{2n})$ is a maximal compact subgroup of $\mathrm{SO}(V_{2n})$.

Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$ and $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$. We say that σ (resp. σ_0) is unramified (with respect to K (resp. K_0)) if σ (resp. σ_0) has a nonzero K-fixed (resp. K_0-fixed) vector. In this case, it is known that $\dim(\sigma^K) = \dim(\sigma_0^{K_0}) = 1$.

Lemma 2.4. *Let $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$ be an unramified representation. Then $\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$ has a unique irreducible unramified constituent.*

Proof. Note that the map

$$\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)^K \to \sigma_0^{K_0},\ f \mapsto f(1)$$

is a $\mathbb{C}$-linear isomorphism. Hence the assertion holds if $\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$ is irreducible. Now suppose that $\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$ is reducible. Then it decomposes into direct sum

$$\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0) \cong \sigma_1 \oplus \sigma_2.$$

We may assume that σ_1 and σ_2 are realized on the same space $\mathcal{V}$ as σ_0. Since $\sigma_i(\epsilon)$ preserve the one dimension subspace $\mathcal{V}^{K_0}$, we have $\sigma_i(\epsilon) = \pm\mathrm{id}$ on $\mathcal{V}^{K_0}$. Since $\sigma_1(\epsilon) = -\sigma_2(\epsilon)$, exactly one $i \in \{1, 2\}$ satisfies that $\sigma_i(\epsilon) = +1$. Then σ_i is the unique irreducible unramified constituent of $\mathrm{Ind}_{\mathrm{SO}(V_{2n})}^{\mathrm{O}(V_{2n})}(\sigma_0)$. □

3. Local Langlands correspondence for $\mathrm{SO}(V_{2n})$ and $\mathrm{O}(V_{2n})$

In this we explain the LLC for $\mathrm{SO}(V_{2n})$ and $\mathrm{O}(V_{2n})$.

3.1. **Orthogonal representations of WD_F and its component groups.** Let M be a finite dimensional vector space over $\mathbb{C}$. We say that a homomorphism $\phi\colon WD_F \to \mathrm{GL}(M)$ is a representation of $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ if

- $\phi(\mathrm{Frob}_F)$ is semi-simple, where Frob_F is a geometric Frobenius element in W_F;
- the restriction of ϕ to W_F is smooth;
- the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic.

We call ϕ tempered if the image of W_F is bounded.

We say that ϕ is orthogonal if there exists a non-degenerate bilinear form $B\colon M \times M \to \mathbb{C}$ such that

$$\begin{cases} B(\phi(w)x, \phi(w)y) = B(x, y), \\ B(y, x) = B(x, y) \end{cases}$$

for $x, y \in M$ and $w \in WD_F$. In this case, ϕ is equivalent to its contragredient $\phi^\vee$. More precisely, see [GGP, §3].

For an irreducible representation ϕ_0 of WD_F, we denote the multiplicity of ϕ_0 in ϕ by $m_\phi(\phi_0)$. We can decompose

$$\phi = m_1\phi_1 + \cdots + m_s\phi_s + \phi' + \phi'^\vee,$$

where $\phi_1, \ldots, \phi_s$ are distinct irreducible orthogonal representations of WD_F, $m_i = m_\phi(\phi_i)$, and ϕ' is a sum of irreducible representations of WD_F which are not orthogonal. We say that a parameter ϕ is discrete if $m_i = 1$ for any $i = 1, \ldots, s$ and $\phi' = 0$, i.e., ϕ is a multiplicity-free sum of irreducible orthogonal representations of WD_F.

For a representation ϕ of WD_F, the L-factor and the ε-factor associated to ϕ, which are defined in [T], are denoted by $L(s, \phi)$ and $\varepsilon(s, \phi, \psi)$, respectively. If (ϕ, M) is an orthogonal representation with WD_F-invariant symmetric bilinear form B, then we define the adjoint L-function $L(s, \phi, \mathrm{Ad})$ associated to ϕ to be the L-function associated to

$$\mathrm{Ad} \circ \phi\colon WD_F \to \mathrm{GL}(\mathrm{Lie}(\mathrm{Aut}(M, B))).$$

We say that ϕ is generic if $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$. Note that $\mathrm{Ad} \circ \phi \cong \wedge^2\phi$ since B is symmetric. Hence the adjoint L-function $L(s, \phi, \mathrm{Ad})$ is equal to the exterior square L-function $L(s, \phi, \wedge^2) = L(s, \wedge^2\phi)$.

Let ϕ be a representation of $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$. We denote the inertia subgroup of W_F by I_F. We say that ϕ is unramified if ϕ is trivial on $I_F \times \mathrm{SL}_2(\mathbb{C})$. In this case, ϕ is a direct sum of unramified characters of $W_F^{\mathrm{ab}} \cong F^\times$.

Let (ϕ, M) be an orthogonal representation of WD_F with invariant symmetric bilinear form B. Let

$$C_\phi = \{g \in \mathrm{GL}(M) \mid B(gx, gy) = B(x, y) \text{ for any } x, y \in M, \text{ and } g\phi(w) = \phi(w)g \text{ for any } w \in WD_F\}$$

be the centralizer of $\mathrm{Im}(\phi)$ in $\mathrm{Aut}(M,B) \cong \mathrm{O}(\dim(M),\mathbb{C})$. Also we put

$$C_\phi^+ = C_\phi \cap \mathrm{SL}(M).$$

Finally, we define the large component group A_ϕ by

$$A_\phi = \pi_0(C_\phi).$$

The image of C_ϕ^+ under the canonical map $C_\phi \to A_\phi$ is denoted by A_ϕ^+, and called the component group of ϕ. By [GGP, §4], A_ϕ and A_ϕ^+ are described explicitely as follows:

Let $\phi = m_1\phi_1 + \cdots + m_s\phi_s + \phi' + \phi'^\vee$ be an orthogonal representation as above. Then we have

$$A_\phi = \bigoplus_{i=1}^{s} (\mathbb{Z}/2\mathbb{Z})a_i \cong (\mathbb{Z}/2\mathbb{Z})^s.$$

Namely, A_ϕ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank s and $\{a_1,\dots,a_s\}$ is a basis of A_ϕ with a_i associated to ϕ_i. For $a = a_{i_1} + \cdots + a_{i_k} \in A_\phi$ with $1 \le i_1 < \cdots < i_k \le s$, we put

$$\phi^a = \phi_{i_1} \oplus \cdots \oplus \phi_{i_k}.$$

Also, we put

$$z_\phi \coloneqq \sum_{i=1}^{s} m_\phi(\phi_i)\cdot a_i = \sum_{i=1}^{s} m_i \cdot a_i \in A_\phi.$$

This is the image of $-\mathbf{1}_M \in C_\phi$. We call z_ϕ the central element in A_ϕ. The determinant map $\det\colon \mathrm{GL}(M) \to \mathbb{C}^\times$ gives a homomorphism

$$\det\colon A_\phi \to \mathbb{Z}/2\mathbb{Z}, \quad \sum_{i=1}^{s}\varepsilon_i a_i \mapsto \sum_{i=1}^{s}\varepsilon_i\cdot\dim(\phi_i) \pmod 2,$$

where $\varepsilon_i \in \{0,1\} = \mathbb{Z}/2\mathbb{Z}$. Then we have $A_\phi^+ = \ker(\det)$.

By [GGP, §4], for each $c \in F^\times$, we can define a character $\eta_{\phi,c}$ of A_ϕ by

$$\eta_{\phi,c}(a) = \det(\phi^a)(c).$$

Note that $\eta_{\phi,c}(z_\phi) = 1$ if and only if $c \in \ker(\det(\phi))$.

3.2. **L-group and L-parameters of $\mathrm{SO}(V_{2n})$.** Let V_{2n} be an orthogonal space associated to (d,c) for some $c,d \in F^\times$. We put $E = F(\sqrt{d})$. Then the Langlands dual group of $\mathrm{SO}(V_{2n})$ is the complex Lie group $\mathrm{SO}(2n,\mathbb{C})$. We use

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

to define $\mathrm{O}(2n,\mathbb{C})$, i.e., $\mathrm{O}(2n,\mathbb{C}) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t gJg = J\}$. We denote the L-group of $\mathrm{SO}(V_{2n})$ by ${}^L(\mathrm{SO}(V_{2n})) = \mathrm{SO}(2n,\mathbb{C}) \rtimes W_F$. The action of W_F on the dual group $\mathrm{SO}(2n,\mathbb{C})$ factors through $W_F/W_E \cong \mathrm{Gal}(E/F)$. If $E \ne F$, i.e., $\mathrm{SO}(V_{2n})$ is not split, then the generator $\gamma \in \mathrm{Gal}(E/F)$ acts on $\mathrm{SO}(2n,\mathbb{C})$ by the inner automorphism of

$$\epsilon = \begin{pmatrix} \mathbf{1}_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathbf{1}_{n-1} \end{pmatrix} \in \mathrm{O}(2n,\mathbb{C}).$$

Hence by $\gamma \mapsto \epsilon$, we have the homomorphism

$${}^L(\mathrm{SO}(V_{2n})) = \mathrm{SO}(2n,\mathbb{C}) \rtimes W_F \twoheadrightarrow \mathrm{SO}(2n,\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \xrightarrow{\cong} \mathrm{O}(2n,\mathbb{C}).$$

On the other hand, if $E = F$, i.e., $\mathrm{SO}(V_{2n})$ is split, then W_F acts on $\mathrm{SO}(2n,\mathbb{C})$ trivially so that we have the homomorphism

$${}^L(\mathrm{SO}(V_{2n})) = \mathrm{SO}(2n,\mathbb{C}) \rtimes W_F \twoheadrightarrow \mathrm{SO}(2n,\mathbb{C}) \hookrightarrow \mathrm{O}(2n,\mathbb{C}).$$

An L-parameter of $\mathrm{SO}(V_{2n})$ is an admissible homomorphism

$$\underline{\phi}\colon WD_F \to {}^L(\mathrm{SO}(V_{2n})) = \mathrm{SO}(2n,\mathbb{C}) \rtimes W_F.$$

We put

$$\Phi(\mathrm{SO}(V_{2n})) = \{\mathrm{SO}(2n,\mathbb{C})\text{-conjugacy classes of } L\text{-parameters of } \mathrm{SO}(V_{2n})\}.$$

For an L-parameter $\underline{\phi}\colon WD_F \to {}^L(\mathrm{SO}(V_{2n}))$, by composing with the above map ${}^L(\mathrm{SO}(V_{2n})) \to \mathrm{O}(2n,\mathbb{C})$, we obtain a homomorphism

$$\phi\colon WD_F \to \mathrm{O}(2n,\mathbb{C}).$$

We may regard ϕ as an orthogonal representation of WD_F. Note that $\det(\phi) = \chi_V$ is the discriminant character of V_{2n}. The map $\underline{\phi} \mapsto \phi$ gives an identification

$$\Phi(\mathrm{SO}(V_{2n})) = \{\phi\colon WD_F \to \mathrm{O}(2n,\mathbb{C}) \mid \det(\phi) = \chi_V\}/(\mathrm{SO}(2n,\mathbb{C})\text{-conjugacy}).$$

Namely, we may regard $\Phi(\mathrm{SO}(V_{2n}))$ as the set of $\mathrm{SO}(M)$-conjugacy classes of orthogonal representations (ϕ, M) of WD_F with $\dim(M) = 2n$ and $\det(\phi) = \chi_V$.

We denote the subset of $\Phi(\mathrm{SO}(V_{2n}))$ consisting of $\mathrm{SO}(M)$-conjugacy classes of tempered (resp. discrete, generic) representations (ϕ, M) by $\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n}))$ (resp. $\Phi_{\mathrm{disc}}(\mathrm{SO}(V_{2n}))$, $\Phi_{\mathrm{gen}}(\mathrm{SO}(V_{2n}))$). Then we have a sequence

$$\Phi_{\mathrm{disc}}(\mathrm{SO}(V_{2n})) \subset \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n})) \subset \Phi_{\mathrm{gen}}(\mathrm{SO}(V_{2n})).$$

We define $\Phi^{\epsilon}(\mathrm{SO}(V_{2n}))$ to be the subset of $\Phi(\mathrm{SO}(V_{2n}))$ consisting of ϕ which contains an irreducible orthogonal representation of WD_F with odd dimension. We put $\Phi^{\epsilon}_*(\mathrm{SO}(V_{2n})) = \Phi^{\epsilon}(\mathrm{SO}(V_{2n})) \cap \Phi_*(\mathrm{SO}(V_{2n}))$ for $* \in \{\mathrm{disc}, \mathrm{temp}, \mathrm{gen}\}$.

3.3. **Local Langlands correspondence for** $\mathrm{SO}(V_{2n})$**.** Let V_{2n} be an orthogonal space associated to (d, c) for some $c, d \in F^{\times}$. The discriminant character is denoted by $\chi_V := \chi_d$. We set V'_{2n} to be the orthogonal space such that

$$\dim(V'_{2n}) = 2n \quad \text{and} \quad \mathrm{disc}(V'_{2n}) = \mathrm{disc}(V_{2n})$$

but $V'_{2n} \not\cong V_{2n}$ as orthogonal spaces. Such V'_{2n} exists uniquely up to isomorphisms unless $n = 1$ and $d \in F^{\times 2}$ By a companion space of V_{2n}, we mean V_{2n} or V'_{2n}.

Now we describe the desiderata for the local Langlands correspondence for $\mathrm{SO}(V_{2n})$.

Desideratum 3.1 (LLC for $\mathrm{SO}(V_{2n})$)**.** *Let V_{2n} be an orthogonal space associated to (d, c), and $\chi_V = (\cdot, d)$ be the discriminant character of V_{2n}.*

(1) *There exists a canonical surjection*

$$\bigsqcup_{V^{\bullet}_{2n}} \mathrm{Irr}(\mathrm{SO}(V^{\bullet}_{2n})) \to \Phi(\mathrm{SO}(V_{2n})).$$

where $V^{\bullet}_{2n}$ runs over all companion spaces of V_{2n}. For $\phi \in \Phi(\mathrm{SO}(V_{2n}))$, we denote by Π^0_ϕ the inverse image of ϕ under this map, and call Π^0_ϕ the L-packet of ϕ.

(2) *We have*

$$\bigsqcup_{V^{\bullet}_{2n}} \mathrm{Irr}_*(\mathrm{SO}(V^{\bullet}_{2n})) = \bigsqcup_{\phi \in \Phi_*(\mathrm{SO}(V_{2n}))} \Pi^0_\phi$$

for $* \in \{\mathrm{temp}, \mathrm{disc}\}$.

(3) *For each $c' \in F^{\times}$, there exists a suitable bijection*

$$\iota_{c'}\colon \Pi^0_\phi \to \widehat{A^+_\phi}.$$

(4) *For $\sigma_0 \in \Pi^0_\phi$ and $c' \in F^{\times}$, the following are equivalent:*
   - $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$;
   - $\iota_{c'}(\sigma_0)(z_\phi) = \chi_V(c'/c)$.

Note that $(\phi, M) \in \Phi(\mathrm{SO}(V_{2n}))$ is not an equivalence class but an $\mathrm{SO}(M)$-conjugacy class, Because of this difference, Desideratum 3.1 has not been established. Arthur [Ar] has established LLC for $\mathrm{O}(V_{2n})$, and deduced a weaker version of Desideratum 3.1 as follows.

We introduce an equivalence relation $\sim_\epsilon$ on $\mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))$. Choose an element ϵ in $\mathrm{O}(V_{2n}^\bullet)$ such that $\det(\epsilon) = -1$. For $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))$, we define its conjugate σ_0^ϵ by $\sigma_0^\epsilon(g) = \sigma_0(\epsilon^{-1} g \epsilon)$. Then the equivalence relation $\sim_\epsilon$ on $\mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))$ is defined by

$$\sigma_0 \sim_\epsilon \sigma_0^\epsilon.$$

The canonical map $\mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet)) \rightarrow \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))/\sim_\epsilon$ is denoted by $\sigma_0 \mapsto [\sigma_0]$. We say that $[\sigma_0] \in \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))/\sim_\epsilon$ is tempered (resp. discrete, $\mu_{c'}$-generic, unramified) if so is some (and hence any) representative σ_0.

Also, we introduce an equivalence relation $\sim_\epsilon$ on $\Phi(\mathrm{SO}(V_{2n}))$. For $\phi, \phi' \in \Phi(\mathrm{SO}(V_{2n}))$, we write $\phi \sim_\epsilon \phi'$ if ϕ is $\mathrm{O}(2n, \mathbb{C})$-conjugate to ϕ', i.e., ϕ is equivalent to ϕ' as representations of WD_F. The equivalence class of ϕ is also denoted by ϕ.

The desiderata for the weaker version of the local Langlands correspondence for $\mathrm{SO}(V_{2n})$ is described as follows:

Desideratum 3.2 (Weak LLC for $\mathrm{SO}(V_{2n})$)**.** *Let V_{2n} be an orthogonal space associated to (d, c), and $\chi_V = (\cdot, d)$ be the discriminant character of V_{2n}.*

(1) *There exists a canonical surjection*

$$\bigsqcup_{V_{2n}^\bullet} \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))/\sim_\epsilon \rightarrow \Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon .$$

where $V_{2n}^\bullet$ runs over all companion spaces of V_{2n}. For $\phi \in \Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon$, we denote by Π_ϕ^0 the inverse image of ϕ under this map, and call Π_ϕ^0 the L-packet of ϕ.

(2) *We have*

$$\bigsqcup_{V_{2n}^\bullet} \mathrm{Irr}_*(\mathrm{SO}(V_{2n}^\bullet))/\sim_\epsilon = \bigsqcup_{\phi \in \Phi_*(\mathrm{SO}(V_{2n}))/\sim_\epsilon} \Pi_\phi^0$$

for $* \in \{\mathrm{temp}, \mathrm{disc}\}$.

(3) *The following are equivalent:*
- *$\phi \in \Phi^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$;*
- *some $[\sigma_0] \in \Pi_\phi^0$ satisfies $\sigma_0^\epsilon \cong \sigma_0$;*
- *all $[\sigma_0] \in \Pi_\phi^0$ satisfy $\sigma_0^\epsilon \cong \sigma_0$.*

Here, $\Phi^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ is the subset of $\Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ consisting of ϕ which contains an irreducible orthogonal representation of WD_F with odd dimension.

(4) *For each $c' \in F^\times$, there exists a bijection (not depending on ψ)*

$$\iota_{c'} : \Pi_\phi^0 \rightarrow \widehat{A_\phi^+},$$

which satisfies the endoscopic and twisted endoscopic character identities.

(5) *For $[\sigma_0] \in \Pi_\phi^0$ and $c' \in F^\times$, the following are equivalent:*
- *$\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))$;*
- *$\iota_{c'}([\sigma_0])(z_\phi) = \chi_V(c'/c)$.*

(6) *Assume that $\phi = \phi_\tau + \phi_0 + \phi_\tau^\vee$, where ϕ_0 is an element in $\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n_0}))/\sim_\epsilon$ and ϕ_τ is an irreducible tempered representation of WD_F which corresponds to $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(F))$ with $n = n_0 + k$. Then the induced representation*

$$\mathrm{Ind}_P^{\mathrm{SO}(V_{2n})}(\tau \otimes \sigma_0)$$

is a multiplicity-free direct sum of tempered representations of $\mathrm{SO}(V_{2n})$, where P is a parabolic subgroup of $\mathrm{SO}(V_{2n})$ with Levi subgroup $M_P = \mathrm{GL}_k(F) \times \mathrm{SO}(V_{2n_0})$ and σ_0 is a representative of an element in $\Pi_{\phi_0}^0$. The L-packet Π_ϕ^0 is given by

$$\Pi_\phi^0 = \{[\sigma] \mid \sigma \subset \mathrm{Ind}_P^{\mathrm{SO}(V_{2n})}(\tau \otimes \sigma_0) \text{ for some } [\sigma_0] \in \Pi_{\phi_0}^0\}.$$

Moreover if $\sigma \subset \mathrm{Ind}_P^{\mathrm{SO}(V_{2n})}(\tau \otimes \sigma_0)$, then $\iota_{c'}([\sigma]) | A_{\phi_0}^+ = \iota_{c'}([\sigma_0])$ for $c' \in F^\times$.

(7) *Assume that*

$$\phi = \phi_{\tau_1}|\cdot|_F^{s_1} + \cdots + \phi_{\tau_r}|\cdot|_F^{s_r} + \phi_0 + (\phi_{\tau_1}|\cdot|_F^{s_1} + \cdots + \phi_{\tau_r}|\cdot|_F^{s_r})^\vee,$$

where ϕ_0 *is an element in* $\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n_0}))/\sim_\epsilon$, ϕ_{τ_i} *is an irreducible tempered representation of* WD_F *which corresponds to* $\tau_i \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_{k_i}(F))$ *with* $n = n_0 + k_1 + \cdots + k_r$ *and* s_i *is a real number with* $s_1 \geq \cdots \geq s_r > 0$. *Then the L-packet* Π_ϕ^0 *consists of the equivalence classes of the unique irreducible quotients* σ *of the standard modules*

$$\mathrm{Ind}_P^{\mathrm{SO}(V_{2n})}(\tau_1|\det|_F^{s_1} \otimes \cdots \otimes \tau_r|\det|_F^{s_r} \otimes \sigma_0),$$

where σ_0 *runs over representatives of elements of* $\Pi_{\phi_0}^0$ *and* P *is a parabolic subgroup of* $\mathrm{SO}(V_{2n})$ *with Levi subgroup* $M_P = \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F) \times \mathrm{SO}(V_{2n_0})$. *Moreover if* σ *is the unique irreducible quotient of* $\mathrm{Ind}_P^{\mathrm{SO}(V_{2n})}(\tau_1|\det|_F^{s_1} \otimes \cdots \otimes \tau_r|\det|_F^{s_r} \otimes \sigma_0)$, *then* $\iota_{c'}([\sigma])|A_{\phi_0}^+ = \iota_{c'}([\sigma_0])$ *for* $c' \in F^\times$.

In this paper, we take the position that the stabilization of the twisted trace formula used in [Ar] is complete. See the series of papers [W, I], [W, II], [W, III], [W, IV], [W, V], [MW, VI], [W, VII], [W, VIII], [W, IX] and [MW, X] of Waldspurger and Mœglin–Waldspurger, and papers of Chaudouard–Laumon [CL1] and [CL2]. Then the following theorem holds.

Theorem 3.3 ([Ar])**.** *Let* V_{2n} *be an orthogonal space associated to* (d, c). *Put* $E = F(\sqrt{d})$. *Then there exist a surjective map*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{SO}(V_{2n}))/\sim_\epsilon \to \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n}))/\sim_\epsilon$$

with the inverse image Π_ϕ^0 *of* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n}))/\sim_\epsilon$, *and a bijection*

$$\iota_{c'} : \Pi_\phi^0 \to (A_\phi^+/\langle z_\phi\rangle)\widehat{\ }$$

for $c' \in cN_{E/F}(E^\times)$ *satisfying Desideratum 3.2 (2), (3), (4), and (6). Moreover, using the Langlands classification, we can extend the map* $[\sigma] \mapsto \phi$ *to a surjective map*

$$\mathrm{Irr}(\mathrm{SO}(V_{2n}))/\sim_\epsilon \to \Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon$$

which satisfies Desideratum 3.2 (7).

Remark 3.4. (1) *Mœglin's work in* [M2, §1.4, Theorem 1.4.1] *may have extended Theorem 3.3 to the pure inner forms as well, in which case Desideratum 3.2 would be known in general. However, we are not sure how her work fits with the general theory of T. Kaletha on the normalization of transfer factors for inner forms* [Ka2]. *In particular, we are not sure if the local character relation of Arthur* [Ar, Theorems 2.2.1, 2.2.4] *(the analog of Hypothesis 3.10 below) holds in her work.*

(2) *If* $d \notin F^{\times 2}$, *then* $\mathrm{SO}(V_{2n}^\bullet)$ *is quasi-split for any companion space of* V_{2n}, *so that we may define L-packets* Π_ϕ^0 *and bijections*

$$\iota_{c_1} : \Pi_\phi^0 \cap \mathrm{Irr}(\mathrm{SO}(V_{2n}))/\sim_\epsilon \to (A_\phi^+/\langle z_\phi\rangle)\widehat{\ },$$
$$\iota_{c_2} : \Pi_\phi^0 \cap \mathrm{Irr}(\mathrm{SO}(V'_{2n}))/\sim_\epsilon \to (A_\phi^+/\langle z_\phi\rangle)\widehat{\ }$$

for $c_1, c_2 \in F^\times$ *with* $c_1 \in cN_{E/F}(E^\times)$ *and* $c_2 \notin cN_{E/F}(E^\times)$. *We define* $\iota_{c_2}([\sigma])$ *for* $[\sigma] \in \mathrm{Irr}(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ *by*

$$\iota_{c_2}([\sigma]) \coloneqq \iota_{c_1}([\sigma]) \otimes \eta_{\phi, c_1/c_2},$$

and define $\iota_{c_1}([\sigma'])$ *for* $[\sigma'] \in \mathrm{Irr}(\mathrm{SO}(V'_{2n}))/\sim_\epsilon$ *similarly. Then the character relations and local intertwining relations would continue to hold after modifying the transfer factor and the normalization of intertwining operators. See also* [KMSW], [Ka2, §5.4] *and* [At1, Appendix A].

There are some properties of Π_ϕ^0.

Proposition 3.5. *Assume Weak LLC for* $\mathrm{SO}(V_{2n})$ *(Desideratum 3.2).*

(1) *For* $c_1, c_2 \in F^\times$, *we have*

$$\iota_{c_2}([\sigma_0]) = \iota_{c_1}([\sigma_0]) \otimes \eta_{\phi, c_2/c_1}$$

as a character of A_ϕ^+.

(2) *ϕ is generic, i.e., $L(s,\phi,\mathrm{Ad})$ is regular at $s=1$ if and only if Π_ϕ^0 contains a $\mu_{c'}$-generic class $[\sigma_0]$ for each $c' \in F^\times$. Note that if $c' \notin cN_{E/F}(E^\times)$, then $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V'_{2n}))$.*
(3) *If ϕ is generic, then for each $c' \in F^\times$, $[\sigma_0] \in \Pi_\phi^0$ is $\mu_{c'}$-generic if and only if $\iota_{c'}([\sigma_0])$ is the trivial representation of A_ϕ^+.*
(4) *If both $\mathrm{SO}(V_{2n})$ and ϕ are unramified, then Π_ϕ^0 contains a unique unramified class $[\sigma_0]$, and it corresponds to the trivial representation of A_ϕ^+ under ι_c.*

In particular, these properties hold for the L-packets $\Pi_\phi^0 \cap \mathrm{Irr}(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ of quasi-split $\mathrm{SO}(V_{2n})$ and $c_1, c_2, c' \in cN_{E/F}(E^\times)$ unconditionally.

Proof. (1) is given in [Ka1, Theorem 3.3]. (2) is a conjecture of Gross–Prasad and Rallis ([GP, Conjecture 2.6]), and has been proven by Gan–Ichino ([GI2, Appendix B]). For (3), it is shown in [Ar, Proposition 8.3.2 (a)] supplemented by some results of many others that the class $[\sigma_0]$ corresponding to the trivial representation of A_ϕ^+ under $\iota_{c'}$ is $\mu_{c'}$-generic. A simple proof of the other direction is given by the first author [At2]. Also (3) is a special case of Gross–Prasad conjecture [GGP, Conjecture 17.1], which is proven by Waldspurger [W2], [W3], [W5] and [W6]. Finally, (4) is proven by Mœglin [M1]. □

3.4. **Local Langlands correspondence for $\mathrm{O}(V_{2n})$.** Let V_{2n} be an orthogonal space associated to (d,c), and $\epsilon \in \mathrm{O}(V_{2n})$ be as in §2.1. Put $\theta = \mathrm{Int}(\epsilon)$. It is an element in $\mathrm{Aut}(\mathrm{SO}(V_{2n}))$. In [Ar], Arthur has established the local Langlands correspondence for not $\mathrm{O}(V_{2n})$ but for

$$\mathrm{SO}(V_{2n}) \rtimes \langle\theta\rangle.$$

As topological groups, $\mathrm{O}(V_{2n})$ and $\mathrm{SO}(V_{2n}) \rtimes \langle\theta\rangle$ are isomorphic. However, it is not canonical. There are exactly two isomorphism $\mathrm{O}(V_{2n}) \cong \mathrm{SO}(V_{2n}) \rtimes \langle\theta\rangle$ which are identity on $\mathrm{SO}(V_{2n})$, and they are determined by $\pm\epsilon \leftrightarrow \theta$. We use the isomorphism such that $\epsilon \leftrightarrow \theta$. Via this isomorphism, we translate LLC for $\mathrm{SO}(V_{2n}) \rtimes \langle\theta\rangle$ into LLC for $\mathrm{O}(V_{2n})$. Note that the changing of the choice of the isomorphism corresponds to the automorphism

$$\mathrm{O}(V_{2n}) \rightarrow \mathrm{O}(V_{2n}),\ g \mapsto \det(g)\cdot g = \begin{cases} g & \text{if } g \in \mathrm{SO}(V_{2n}), \\ -g & \text{otherwise.} \end{cases}$$

Hence it induces the bijection

$$\mathrm{Irr}(\mathrm{O}(V_{2n})) \rightarrow \mathrm{Irr}(\mathrm{O}(V_{2n})),\ \sigma \mapsto (\omega_\sigma \circ \det) \otimes \sigma,$$

where ω_σ is the central character of σ, which is regarded as a character of $\{\pm 1\}$.

Let $V_{2n}^\bullet$ be a companion space of V_{2n}. We define an equivalence relation $\sim_{\det}$ on $\mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet))$ by

$$\sigma \sim_{\det} \sigma \otimes \det$$

for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet))$. The restriction and the induction give a canonical bijection

$$\mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet))/\sim_{\det} \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))/\sim_\varepsilon .$$

Put $\Phi(\mathrm{O}(V_{2n})) = \Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ and $\Phi_*(\mathrm{O}(V_{2n})) = \Phi_*(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ for $* \in \{\mathrm{temp}, \mathrm{disc}, \mathrm{gen}\}$. Also, we define $\Phi^\epsilon(\mathrm{O}(V_{2n})) = \Phi^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$. Namely, $\Phi(\mathrm{O}(V_{2n}))$ is the set of equivalence classes of orthogonal representations of WD_F with dimension $2n$ and determinant χ_V. We call an element in $\Phi(\mathrm{O}(V_{2n}))$ an L-parameter for $\mathrm{O}(V_{2n})$.

We describe the local Langlands correspondence for $\mathrm{O}(V_{2n})$.

Desideratum 3.6 (LLC for $\mathrm{O}(V_{2n})$)**.** *Let V_{2n} be an orthogonal space associated to (d,c), and $\chi_V = (\cdot, d)$ be the discriminant character of V_{2n}.*

(1) *There exists a canonical surjection*

$$\bigsqcup_{V_{2n}^\bullet} \mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet)) \rightarrow \Phi(\mathrm{O}(V_{2n})).$$

where $V_{2n}^\bullet$ runs over all companion spaces of V_{2n}. For $\phi \in \Phi(\mathrm{O}(V_{2n}))$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-packet of ϕ.

(2) *We have*

$$\bigsqcup_{V_{2n}^\bullet} \mathrm{Irr}_*(\mathrm{O}(V_{2n}^\bullet)) = \bigsqcup_{\phi\in\Phi_*(\mathrm{O}(V_{2n}))} \Pi_\phi$$

for $* \in \{\mathrm{temp}, \mathrm{disc}\}$*.*

(3) *The following are equivalent:*
- $\phi \in \Phi^\epsilon(\mathrm{O}(V_{2n}))$*;*
- *some* $\sigma \in \Pi_\phi$ *satisfies* $\sigma \otimes \det \ncong \sigma$*;*
- *all* $\sigma \in \Pi_\phi$ *satisfy* $\sigma \otimes \det \ncong \sigma$*.*

Here, $\Phi^\epsilon(\mathrm{O}(V_{2n}))$ *is the subset of* $\Phi(\mathrm{O}(V_{2n}))$ *consisting of* ϕ *which contains an irreducible orthogonal representation of* WD_F *with odd dimension.*

(4) *For each* $c' \in F^\times$*, there exists a bijection (not depending on* ψ*)*

$$\iota_{c'} \colon \Pi_\phi \to \widehat{A_\phi},$$

which satisfies the (twisted) endoscopic character identities.

(5) *For* $\sigma \in \Pi_\phi$ *and* $c' \in F^\times$*, the following are equivalent:*
- $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$*;*
- $\iota_{c'}(\sigma)(z_\phi) = \chi_V(c'/c)$*.*

(6) *Assume that* $\phi = \phi_\tau + \phi_0 + \phi_\tau^\vee$*, where* ϕ_0 *is an element in* $\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n_0}))$ *and* ϕ_τ *is an irreducible tempered representation of* WD_F *which corresponds to* $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(F))$ *with* $n = n_0 + k$*. Then the induced representation*

$$\mathrm{Ind}_P^{\mathrm{O}(V_{2n})}(\tau \otimes \sigma_0)$$

is a multiplicity-free direct sum of tempered representations of $\mathrm{O}(V_{2n})$*, where* P *is a parabolic subgroup of* $\mathrm{O}(V_{2n})$ *with Levi subgroup* $M_P = \mathrm{GL}_k(F) \times \mathrm{O}(V_{2n_0})$ *and* $\sigma_0 \in \Pi_{\phi_0}$*. The L-packet* Π_ϕ *is given by*

$$\Pi_\phi = \{\sigma \mid \sigma \subset \mathrm{Ind}_P^{\mathrm{O}(V_{2n})}(\tau \otimes \sigma_0) \text{ for some } \sigma_0 \in \Pi_{\phi_0}\}.$$

Moreover if $\sigma \subset \mathrm{Ind}_P^{\mathrm{O}(V_{2n})}(\tau \otimes \sigma_0)$*, then* $\iota_{c'}(\sigma)|A_{\phi_0} = \iota_{c'}(\sigma_0)$ *for* $c' \in F^\times$*.*

(7) *Assume that*

$$\phi = \phi_{\tau_1}|\cdot|_F^{s_1} + \cdots + \phi_{\tau_r}|\cdot|_F^{s_r} + \phi_0 + (\phi_{\tau_1}|\cdot|_F^{s_1} + \cdots + \phi_{\tau_r}|\cdot|_F^{s_r})^\vee,$$

where ϕ_0 *is an element in* $\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n_0}))$*,* ϕ_{τ_i} *is an irreducible tempered representation of* WD_F *which corresponds to* $\tau_i \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_{k_i}(F))$ *with* $n = n_0 + k_1 + \cdots + k_r$ *and* s_i *is a real number with* $s_1 \geq \cdots \geq s_r > 0$*. Then the L-packet* Π_ϕ *consists of the unique irreducible quotients* σ *of the standard modules*

$$\mathrm{Ind}_P^{\mathrm{O}(V_{2n})}(\tau_1|\det|_F^{s_1} \otimes \cdots \otimes \tau_r|\det|_F^{s_r} \otimes \sigma_0),$$

where σ_0 *runs over elements of* Π_{ϕ_0} *and* P *is a parabolic subgroup of* $\mathrm{O}(V_{2n})$ *with Levi subgroup* $M_P = \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F) \times \mathrm{O}(V_{2n_0})$*. Moreover if* σ *is the unique irreducible quotient of* $\mathrm{Ind}_P^{\mathrm{O}(V_{2n})}(\tau_1|\det|_F^{s_1} \otimes \cdots \otimes \tau_r|\det|_F^{s_r} \otimes \sigma_0)$*, then* $\iota_{c'}(\sigma)|A_{\phi_0} = \iota_{c'}(\sigma_0)$ *for* $c' \in F^\times$*.*

(8) *For* $\phi \in \Phi(\mathrm{O}(V_{2n})) = \Phi(\mathrm{SO}(V_{2n}))/\sim_\epsilon$*, the image of* Π_ϕ *under the map*

$$\mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet)) \to \mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet))/\sim_{\det} \to \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))/\sim_\epsilon$$

is the packet Π_ϕ^0 *in Weak LLC for* $\mathrm{SO}(V_{2n})$*, and the diagram*

$$\begin{array}{ccc} \Pi_\phi & \xrightarrow{\iota_{c'}} & \widehat{A_\phi} \\ \downarrow & & \downarrow \\ \Pi_\phi^0 & \xrightarrow{\iota_{c'}} & \widehat{A_\phi^+} \end{array}$$

is commutative for $c' \in F^\times$*.*

(9) *For* $c' \in F^\times$ *and* $\sigma \in \Pi_\phi$*, the determinant twist* $\sigma \otimes \det$ *also belongs to* Π_ϕ*, and*

$$\iota_{c'}(\sigma \otimes \det)(a) = \iota_{c'}(a) \cdot (-1)^{\det(a)}$$

for $a \in A_\phi$*.*

As Weak LLC for $\mathrm{SO}(V_{2n})$, the following theorem holds.

Theorem 3.7 ([Ar])**.** *Let V_{2n} be an orthogonal space associated to (d,c). Put $E = F(\sqrt{d})$. Then there exist a surjective map*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_{2n})) \to \Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$$

with the inverse image Π_ϕ of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$, and a bijection

$$\iota_{c'} \colon \Pi_\phi \to (A_\phi/\langle z_\phi\rangle)\widehat{}$$

for $c' \in cN_{E/F}(E^\times)$ satisfying Desideratum 3.6 (2), (3), (4), (6), (8), and (9). Moreover, using the Langlands classification, we can extend the map $\sigma \mapsto \phi$ to a surjective map

$$\mathrm{Irr}(\mathrm{O}(V_{2n})) \to \Phi(\mathrm{O}(V_{2n}))$$

which satisfies Desideratum 3.6 (7).

In fact, Arthur established Theorem 3.7 first and by using Desideratum 3.6 (8), he then defined the L-packets Π_ϕ^0 for $\mathrm{SO}(V_{2n})$ (Theorem 3.3).

Remark 3.8. *As we mentioned in Remark 3.4, Mœglin's work in* [M2, §1.4, Theorem 1.4.1] *seems to extend Theorem 3.7 to the pure inner forms as well. Also, when $d \notin F^{\times 2}$, we can define L-packets Π_ϕ and a bijection $\iota_{c'} \colon \Pi_\phi \to \widehat{A_\phi}$ for any $c' \in F^\times$ similar to Remark 3.4. However, motivated by Prasad conjecture (Conjecture 4.4 below), we should define $\iota_{c'}(\sigma)$ for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$ by*

$$\iota_{c'}(\sigma) = \iota_c(\sigma) \otimes \eta_{\phi\chi_V, c'/c}.$$

See also Desideratum 3.9 and Hypothesis 3.10 below.

The following is an analogue of Proposition 3.5.

Desideratum 3.9. *Let V_{2n} be an orthogonal space associated to (d,c). Let $\phi \in \Phi(\mathrm{O}(V_{2n}))$ and $\sigma \in \Pi_\phi$. We write $\phi\chi_V = \phi \otimes \chi_V$.*

(1) *For $c_1, c_2 \in F^\times$, we have*
$$\iota_{c_2}(\sigma) = \iota_{c_1}(\sigma) \otimes \eta_{\phi\chi_V, c_2/c_1}$$
as a character of A_ϕ.

(2) *ϕ is generic, i.e., $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$ if and only if Π_ϕ contains a $\mu_{c'}^\varepsilon$-generic representation σ for each $c' \in F^\times$ and $\varepsilon \in \{\pm 1\}$.*

(3) *If ϕ is generic, then for each $c' \in F^\times$,*
   - *$\sigma^+ \in \Pi_\phi$ is $\mu_{c'}^+$-generic if and only if $\iota_{c'}(\sigma^+)$ is the trivial representation of A_ϕ;*
   - *$\sigma^- \in \Pi_\phi$ is $\mu_{c'}^-$-generic if and only if $\iota_{c'}(\sigma^-)$ is given by $A_\phi \ni a \mapsto (-1)^{\det(a)}$.*

(4) *If both $\mathrm{O}(V_{2n})$ and ϕ are unramified, then Π_ϕ contains a unique unramified representation σ, and it corresponds to the trivial representation of A_ϕ under ι_c.*

Under Desideratum 3.6, Proposition 3.5 and Hypothesis 3.10, Desideratum 3.9 will be proven in §3.6 below. Note that $\eta_{\phi\chi_V, c_2/c_1}|A_\phi^+ = \eta_{\phi, c_2/c_1}$ since $\dim(\phi^a)$ is even for $a \in A_\phi^+$.

3.5. **Hypothesis.** To establish Desideratum 3.9 and two main local theorems, we will use a very delicate hypothesis, which is an intertwining relation.

Let V_{2n} be an orthogonal space associated to (d,c), and V be a companion space of V_{2n}. For a fixed positive integer k, we set

$$X = Fv_1 \oplus \cdots \oplus Fv_k, \quad X^* = Fv_1^* \oplus \cdots \oplus Fv_k^*$$

to be k-dimensional vector spaces over F. Let $V' = V \oplus X \oplus X^*$ be the orthogonal space define by

$$\langle v_i, v_j\rangle_{V'} = \langle v_i^*, v_j^*\rangle_{V'} = \langle v_i, v_0\rangle_{V'} = \langle v_i^*, v_0\rangle_{V'} = 0, \quad \langle v_i, v_j^*\rangle_{V'} = \delta_{i,j}$$

for any $i, j = 1, \ldots, k$ and $v_0 \in V$. Let $P = M_P U_P$ be the maximal parabolic subgroup of $\mathrm{O}(V')$ stabilizing X, where M_P is the Levi component of P stabilizing X^*. Hence

$$M_P \cong \mathrm{GL}(X) \times \mathrm{O}(V).$$

Using the basis $\{v_1, ..., v_k\}$ of X, we obtain an isomorphism $m_P \colon \mathrm{GL}_k(F) \to \mathrm{GL}(X)$. Let ϕ_τ be an orthogonal tempered representation of WD_F of dimension k, and τ be the tempered representation of $\mathrm{GL}_k(F)$ on a space $\mathcal{V}_\tau$ associated to ϕ_τ. For $s \in \mathbb{C}$, we realize the representation $\tau_s := \tau \otimes |\det|_F^s$ on $\mathcal{V}_\tau$ by setting $\tau_s(a) := |\det(a)|_F^s \tau(a)v$ for $v \in \mathcal{V}_\tau$ and $a \in \mathrm{GL}_k(F)$. Let $\sigma \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V))$. Assume that $\sigma \in \Pi_{\phi_\sigma}$ with $\phi_\sigma \in \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$, i.e., $\sigma \not\cong \sigma \otimes \det$ and $\sigma|\mathrm{SO}(V)$ is irreducible. We define a normalized intertwining operator

$$R_{c'}(w, \tau_s \otimes \sigma) \colon \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma) \to \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma)$$

by (the meromorphic continuations of) the integral

$$R_{c'}(w, \tau_s \otimes \sigma) f_s(h') = e(V)^k \cdot \chi_V(c'/c)^k \cdot |c'|_F^{k\rho_P} \cdot r(\tau_s \otimes \sigma)^{-1} \cdot \mathcal{A}_w \left(\int_{U_P} f_s(\widetilde{w}_{c'}^{-1} u_P h') du_P \right)$$

for $f_s \in \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma)$. Here,

- w is the non-trivial element in the relative Weyl group $W(M_P) (\cong \mathbb{Z}/2\mathbb{Z})$ for M_P;
- $\widetilde{w}_{c'} \in \mathrm{O}(V)$ is the representative of w given by

$$\widetilde{w}_{c'} = w_P \cdot m_P(c' \cdot a) \cdot ((-1)^k \mathbf{1}_V),$$

  where $w_P \in \mathrm{O}(V')$ is defined by $w_P v_i = -v_i^*$, $w_P v_i^* = -v_i$ and $w_P|V = \mathbf{1}_V$, and $a \in \mathrm{GL}_k(F)$ is given by

$$a = \begin{pmatrix} & & (-1)^{n-k+1} \\ & \iddots & \\ (-1)^n & & \end{pmatrix};$$

- $e(V) = \iota_c(z_{\phi_\sigma}) \in \{\pm 1\}$, i.e.,

$$e(V) = \begin{cases} 1 & \text{if } V \text{ is associated to } (d, c), \\ -1 & \text{otherwise;} \end{cases}$$

- $\rho_P = m + (k-1)/2$, so that the modulus character δ_P of P satisfies that $\delta_P(m_P(a)) = |\det(a)|_F^{2\rho_P}$ for $a \in \mathrm{GL}_k(F)$;
- $r(\tau_s \otimes \sigma)$ is the normalizing factor given by

$$r(\tau_s \otimes \sigma) = \lambda(E/F, \psi)^k \frac{L(s, \phi_\tau \otimes \phi_\sigma)}{\varepsilon(s, \phi_\tau \otimes \phi_\sigma, \psi) L(1+s, \phi_\tau \otimes \phi_\sigma)} \frac{L(-2s, (\wedge_2)^\vee \circ \phi_\tau)}{\varepsilon(-2s, (\wedge_2)^\vee \circ \phi_\tau) L(1-2s, (\wedge_2)^\vee \circ \phi_\tau)},$$

  where $\wedge_2$ is the representation of $\mathrm{GL}_k(\mathbb{C})$ on the space of skew-symmetric (k,k)-matrices, and $\lambda(E/F, \psi)$ is the Langlands λ-factor associated to $E = F(\sqrt{\mathrm{disc}(V)}) = F(\sqrt{d})$.
- du_P is the Haar measure of U_P given in [Ar] (see also [At1, §6.1]);
- $\mathcal{A}_w \colon w(\tau \otimes \sigma) \to \tau \otimes \sigma$ is the intertwining isomorphism defined in [Ar] (see also [At1, §6.3]), where $w(\tau \otimes \sigma)(m) := (\tau \otimes \sigma)(\widetilde{w}_{c'}^{-1} m \widetilde{w}_{c'})$ for $m \in M_P$.

We expect that the intertwining operators and the local Langlands correspondence are related as follows:

Hypothesis 3.10. *Notation is as above.*

(1) *The normalized intertwining operator $R_{c'}(w, \tau_s \otimes \sigma)$ is holomorphic at $s = 0$. We put $R_{c'}(w, \tau \otimes \sigma) := R_{c'}(w, \tau_0 \otimes \sigma)$.*

(2) *Suppose that ϕ_τ is a multiplicity-free sum of irreducible orthogonal tempered representations. Put $\phi_{\sigma'} = \phi_\tau \oplus \phi_\sigma \oplus \phi_\tau$, and denote by $a \in A_{\phi_{\sigma'}}$ the element corresponding to ϕ_τ. Let $\sigma' \in \Pi_{\phi_{\sigma'}}$ be an irreducible constituent of $\mathrm{Ind}_P^{\mathrm{O}(V')}(\tau \otimes \sigma)$. Then we have*

$$R_{c'}(w, \tau \otimes \sigma)|\sigma' = \iota_{c'}(\sigma')(a)$$

for any $c' \in F^\times$.

In special cases, Hypothesis 3.10 has been established:

Theorem 3.11. *Hypothesis 3.10 holds in the following cases:*

- *The case when $V = V_{2n}$ and $c' \in cN_{E/F}(E^\times)$.*
- *The case when k is even and $d \neq 1$ in $F^\times/F^{\times 2}$ under assuming Desideratum 3.2.*

Proof. In the first case, Hypothesis 3.10 is Proposition 2.3.1 and Theorems 2.2.1, 2.2.4, 2.4.1 and 2.4.4 in [Ar]. The second case follows from Arthur's results above and [At1, Proposition 3.3]. □

The cases when Hypothesis 3.10 has not yet been verified are

- the non-quasi-split even orthogonal case; and
- the case when k is odd and $d \neq 1$ in $F^\times/F^{\times 2}$.

In general, Hypothesis 3.10 would follow from similar results to [Ar] and [KMSW].

Remark 3.12. *Recall that we need to choose an isomorphism*

$$\mathrm{O}(V) \longleftrightarrow \mathrm{SO}(V) \rtimes \langle\theta\rangle$$

to translate Arthur's result. There exist two choices of isomorphisms, which are determined by $\pm\epsilon \leftrightarrow \theta$. We have chosen the isomorphism such that $\epsilon \leftrightarrow \theta$. If one chooses the other isomorphism $-\epsilon \leftrightarrow \theta$, one should replace the representative $\widetilde{w}_{c'}$ of $w \in W(M_P)$ as

$$\widetilde{w}'_{c'} = -w_P \cdot m_P(c' \cdot a) \cdot ((-1)^k \mathbf{1}_V) = -\widetilde{w}_{c'}.$$

Note that

$$f_s(\widetilde{w}'_{c'} u_P h') = \omega_s(-1) \cdot f_s(\widetilde{w}_{c'} u_P h')$$

for $f_s \in \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma)$, where ω_s is the central character of $\mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma)$. This is compatible the bijection

$$\mathrm{Irr}(\mathrm{O}(V)) \to \mathrm{Irr}(\mathrm{O}(V)),\ \sigma \mapsto (\omega_\sigma \circ \det) \otimes \sigma$$

as in §3.4. Hence all results below are independent of the choice of the isomorphism $\mathrm{O}(V) \cong \mathrm{SO}(V) \rtimes \langle\theta\rangle$.

3.6. **Proof of Desideratum 3.9.** In this subsection, we prove Desideratum 3.9 under Hypothesis 3.10. First, we treat the tempered case.

Theorem 3.13. *Assume Desiderata 3.2, 3.6 and Hypothesis 3.10. Then Desideratum 3.9 holds for $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V))$. In particular, it holds for the L-packets $\Pi_\phi \cap \mathrm{Irr}(\mathrm{O}(V_{2n}))$ of quasi-split $\mathrm{O}(V_{2n})$ and $c_1, c_2, c' \in cN_{E/F}(E^\times)$ unconditionally.*

Proof. Note that Proposition 3.5 holds since we assume Desideratum 3.2.

First, we consider (1). Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V))$ and $\sigma \in \Pi_\phi$. We have to show that

$$\iota_{c_1}(\sigma)(a) = \iota_{c_2}(\sigma)(a) \cdot \det(\phi^a \chi_V)(c_1/c_2)$$

for any $a \in A_\phi$ and $c_1, c_2 \in F^\times$. Fix $a \in A_\phi$ and consider the parameter

$$\phi' = \phi^a \oplus \phi \oplus \phi^a.$$

Let $\tau \in \mathrm{Irr}(\mathrm{GL}_k(F))$ be the representation corresponding to ϕ^a, where $k = \dim(\phi^a)$, and put $\sigma' = \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau \otimes \sigma)$ as above. Then $A_\phi = A_{\phi'}$ since ϕ contains ϕ^a. Hence σ' is irreducible and $\sigma' \in \Pi_{\phi'}$ by Desideratum 3.6 (6). Moreover, we have

$$\iota_{c'}(\sigma')|A_\phi = \iota_{c'}(\sigma)$$

for any $c' \in F^\times$. By Hypothesis 3.10, $R_{c_i}(w, \tau \otimes \sigma)$ is the scalar operator with eigenvalue $\iota_{c_i}(\sigma)(a)$ for $i = 1, 2$. By definition, we have

$$R_{c_1}(w, \tau \otimes \sigma) = \chi_V(c_1/c_2)^k \cdot \omega_\tau(c_1/c_2) \cdot R_{c_2}(w, \tau \otimes \sigma),$$

where ω_τ is the central character of τ, which is equal to $\det(\phi^a)$. Since

$$\chi_V(c_1/c_2)^k \cdot \omega_\tau(c_1/c_2) = \det(\phi^a \chi_V)(c_1/c_2),$$

we have

$$\iota_{c_1}(\sigma)(a) = \det(\phi^a \chi_V)(c_1/c_2) \cdot \iota_{c_2}(\sigma)(a),$$

as desired.

The assertion (2) follows from Lemma 2.3 and Proposition 3.5 (2).

Next, we consider (3). Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V))$. Note that ϕ is generic. By Proposition 3.5 (2) and Desideratum 3.9 (2), for each $c' \in F^\times$, there exists a $\mu_{c'}^+$-generic representation $\sigma \in \Pi_\phi$ such that $\iota_{c'}(\sigma)|A_\phi^+ = \mathbf{1}$. We may assume that σ is a representation of $\mathrm{O}(V)$ with V associated to (d, c'). We have to show that $\iota_{c'}(\sigma) = \mathbf{1}$. If $\phi \notin \Phi^\epsilon(\mathrm{O}(V))$, then $A_\phi^+ = A_\phi$ so that we have nothing to prove. Hence we may assume that ϕ contains an irreducible orthogonal representation ϕ_0 with odd dimension k. Let $a_0 \in A_\phi$ be the element corresponding to ϕ_0. For $s \in \mathbb{C}$, consider the parameter

$$\phi' = \phi_0 \oplus \phi \oplus \phi_0.$$

Let $\tau_s = \tau|\cdot|_F^s \in \mathrm{Irr}(\mathrm{GL}_k(F))$ be the representation corresponding to $\phi_0|\cdot|_F^s$. We may assume that τ_s is realized on a space $\mathcal{V}_\tau$, which is independent of $s \in \mathbb{C}$. Put $\sigma'_s = \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau_s \otimes \sigma)$ as above. Then by Desideratum 3.6 (6), σ'_0 is irreducible and $\sigma'_0 \in \Pi_{\phi'}$. Moreover, the canonical injection $A_\phi \hookrightarrow A_{\phi'}$ is bijective, and we have

$$\iota_{c'}(\sigma'_0)|A_\phi = \iota_{c'}(\sigma).$$

Note that $\tau = \tau_0$ is tempered, so that generic. Fix a nonzero homomorphism

$$l\colon \tau \otimes \sigma \to \mathbb{C}$$

such that

$$l(\sigma(u)v) = \mu_{c'}^+(u)\ell(v)$$

for $u \in U' \cap M_P$ and $v \in \tau \otimes \sigma$, where $U' = U'_0 \rtimes \langle \epsilon \rangle$ with the maximal unipotent subgroup U'_0 of $\mathrm{SO}(V')$ as in §2.2, and $M_P = \mathrm{GL}_k(F) \times \mathrm{O}(V)$ is the Levi subgroup of P. For $f_s \in \sigma'_s$, we put

$$l_s(f_s) = \int_{U_0} l(f_s(\widetilde{w}_{c'}^{-1} u_0))\mu_{c'}^{-1}(u_0) du_0,$$

where $\widetilde{w}_{c'} \in \mathrm{SO}(V)$ is the representative of w defined in §3.5. Then by [CS, Proposition 2.1] and [S1, Proposition 3.1], $l_s(f_s)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and holomorphic continuation to $\mathbb{C}$. Moreover, l_0 gives a nonzero map

$$l_0\colon \sigma'_0 \to \mathbb{C}$$

such that

$$l_0(\sigma'_0(u')f_0) = \mu_{c'}^+(u')l_0(f_0)$$

for $u' \in U'$ and $f_0 \in \sigma'_0$. By a result of Shahidi ([S2, Theorem 3.5]), we have

$$l_0 \circ R_{c'}(w, \tau \otimes \sigma) = l_0.$$

See also [Ar, Theorem 2.5.1]. This equation together with Hypothesis 3.10 shows that

$$\iota_{c'}(\sigma)(a_0) = \iota_{c'}(\sigma'_0)(a_0) = 1.$$

Since $[A_\phi : A_\phi^+] = 2$, we have $\iota_{c'}(\sigma) = \mathbf{1}$, as desired.

Finally, we consider (4). Suppose that $\mathrm{O}(V)$ and $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V))$ are unramified. By Desideratum 3.2 and Lemma 2.4, Π_ϕ contains a unique unramified representation σ, which satisfies that $\iota_c(\sigma)|A_\phi^+ = \mathbf{1}$. We have to show that $\iota_c(\sigma) = \mathbf{1}$. We may assume that $\phi \in \Phi^\epsilon(\mathrm{O}(V))$. Since $\iota_c(\sigma)|A_\phi^+ = \mathbf{1}$, by Proposition 3.5 (3), we see that σ is μ_c-generic, i.e., there is a nonzero homomorphism $l\colon \sigma \to \mathbb{C}$ such that $l(\sigma(u_0)v) = \mu_c(u_0)l(v)$ for $u_0 \in U_0$ and $v \in \sigma$. By the Casselman–Shalika formula [CS, Theorem 5.4], we have $l|\sigma^{K_0} \neq 0$, i.e., if $v \in \sigma$ is a nonzero K_0-fixed vector, then $l(v) \neq 0$. Since $\sigma \otimes \det \not\cong \sigma$, we have

$$l \in \mathrm{Hom}_U(\sigma, \mu_c^\delta)$$

for some $\delta \in \{\pm 1\}$. However, if $v \in \sigma$ is a nonzero K-fixed vector, then we have

$$\delta \cdot l(v) = \mu_c^\delta(\epsilon) \cdot l(v) = l(\sigma(\epsilon)v) = l(v).$$

This shows that σ is μ_c^+-generic, and so that $\iota_c(\sigma) = \mathbf{1}$ by Lemma 2.3 and Desideratum 3.9 (3). □

Now we treat the general case.

Corollary 3.14. *Assume Desiderata 3.2, 3.6 and Hypothesis 3.10. Then Desideratum 3.9 holds in general. In particular, it holds for the L-packets $\Pi_\phi \cap \mathrm{Irr}(\mathrm{O}(V_{2n}))$ of quasi-split $\mathrm{O}(V_{2n})$ and $c_1, c_2, c' \in cN_{E/F}(E^\times)$ unconditionally.*

Proof. This follows from the compatibility of LLC and the Langlands quotients (Desideratum 3.6 (7)). □

Remark 3.15. (1) *Kaletha proved Proposition 3.5 (1) in* [Ka1, Theorem 3.3] *by comparing transfer factors. One may feel that the proof of Theorem 3.13 (1) differs from Kaletha's proof. However to prove Hypothesis 3.10, one would need a similar argument to* [Ka1]. *Hence the proof of Theorem 3.13 (1) would be essentially the same as the one of* [Ka1, Theorem 3.3].

(2) *In* [At2]*, the first author gave a proof of "only if" part of Proposition 3.5 (3). This proof is essentially the same as the proof of Desideratum 3.9 (3) (Theorem 3.13).*

4. Prasad's conjecture

Prasad's conjecture describes precisely the local theta correspondence for $(\mathrm{O}(V_{2n}), \mathrm{Sp}(W_{2n}))$ in terms of the local Langlands correspondence for $\mathrm{O}(V_{2n})$ and $\mathrm{Sp}(W_{2n})$. A weaker version of this conjecture has been proven by the first author [At1]. In this section, we state Prasad's conjecture and give a proof for the full version.

4.1. **Local Langlands correspondence for** $\mathrm{Sp}(W_{2m})$**.** Let W_{2m} be a symplectic space over F with dimension $2m$. The associated symplectic group is denoted by $\mathrm{Sp}(W_{2m})$. Fix an F-rational Borel subgroup $B' = T'U'$ of $\mathrm{Sp}(W_{2m})$. By [GGP, §12], there is a canonical bijection (depending on the choice of ψ)

$$F^\times/F^{\times 2} \rightarrow \{T'\text{-orbits of generic characters of } U'\},\ c \mapsto \mu'_c.$$

The Langlands dual group of $\mathrm{Sp}(W_{2m})$ is the complex Lie group $\mathrm{SO}(2m+1,\mathbb{C})$, and W_F acts on $\mathrm{SO}(2m+1,\mathbb{C})$ trivially. We denote the L-group of $\mathrm{Sp}(W_{2m})$ by ${}^L(\mathrm{Sp}(W_{2m})) = \mathrm{SO}(2m+1,\mathbb{C}) \times W_F$. An L-parameter of $\mathrm{Sp}(W_{2m})$ is an admissible homomorphism

$$\underline{\phi}\colon WD_F \rightarrow {}^L(\mathrm{Sp}(W_{2m})) = \mathrm{SO}(2m+1,\mathbb{C}) \times W_F.$$

We put

$$\Phi(\mathrm{Sp}(W_{2m})) = \{\mathrm{SO}(2m+1,\mathbb{C})\text{-conjugacy classes of } L\text{-parameters of } \mathrm{Sp}(W_{2m})\}.$$

For an L-parameter $\underline{\phi}\colon WD_F \rightarrow {}^L(\mathrm{Sp}(W_{2m}))$, by composing with the projection $\mathrm{SO}(2m+1,\mathbb{C}) \times W_F \rightarrow \mathrm{SO}(2m+1,\mathbb{C})$, we obtain a map

$$\phi\colon WD_F \rightarrow \mathrm{SO}(2m+1,\mathbb{C}).$$

The map $\underline{\phi} \mapsto \phi$ gives an identification

$$\Phi(\mathrm{Sp}(W_{2m})) = \{\phi\colon WD_F \rightarrow \mathrm{SO}(2m+1,\mathbb{C})\}/(\mathrm{SO}(2m+1,\mathbb{C})\text{-conjugacy}).$$

Namely, we regard $\Phi(\mathrm{Sp}(W_{2m}))$ as the set of equivalence classes of orthogonal representations of WD_F with dimension $2m+1$ and trivial determinant. We denote the subset of $\Phi(\mathrm{Sp}(W_{2m}))$ consisting of equivalence classes of tempered (resp. discrete, generic) representations by $\Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2n}))$ (resp. $\Phi_{\mathrm{disc}}(\mathrm{Sp}(W_{2m}))$, $\Phi_{\mathrm{gen}}(\mathrm{Sp}(W_{2m}))$). Then we have a sequence

$$\Phi_{\mathrm{disc}}(\mathrm{Sp}(W_{2m})) \subset \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2n})) \subset \Phi_{\mathrm{gen}}(\mathrm{Sp}(W_{2m})).$$

The following theorem are due to Arthur [Ar] supplemented by some results of many others (c.f., see the proof of Proposition 3.5). See also [At1, §3, §6.3] and [Ka1, Theorem 3.3].

Theorem 4.1. *There exist a surjective map*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{Sp}(W_{2m})) \rightarrow \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2m}))$$

with the inverse image Π_ϕ of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2m}))$, and a bijection

$$\iota'_c\colon \Pi_\phi \rightarrow \widehat{A_\phi^+}$$

for $c \in F^\times$ which satisfy analogues of Desideratum 3.6 (2), (4), and (6). Moreover, using the Langlands classification, we can extend the map $\pi \mapsto \phi$ to a surjective map

$$\mathrm{Irr}(\mathrm{Sp}(W_{2m})) \to \Phi(\mathrm{Sp}(W_{2m}))$$

which satisfies an analogue of Desideratum 3.6 (7). In addition, an analogue to Proposition 3.5 holds. In particular, we have

$$\iota'_{c_1}(\pi) = \iota'_{c_2}(\pi) \cdot \eta_{\phi, c_1/c_2}$$

for $\pi \in \Pi_\phi$ and $c_1, c_2 \in F^\times$.

Note that for $\phi \in \Phi(\mathrm{Sp}(W_{2n}))$, we have

$$A_\phi = A_\phi^+ \oplus \langle z_\phi \rangle.$$

Hence we may identify $\widehat{A_\phi^+}$ with

$$(A_\phi/\langle z_\phi \rangle)\widehat{\ } \subset \widehat{A_\phi}.$$

Via this identification, we regard ι'_c as an injection

$$\iota'_c \colon \Pi_\phi \to \widehat{A_\phi}.$$

Let ϕ_τ be an orthogonal tempered representation of WD_F, and $\tau \in \mathrm{Irr}(\mathrm{GL}_k(F))$ be the tempered representation corresponding to ϕ_τ. In [Ar], Arthur has defined a normalized intertwining operator $R(w', \tau \otimes \pi)$ on $\mathrm{Ind}_Q^{\mathrm{Sp}(W_{2m'})}(\tau \otimes \pi)$ for $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{Sp}(W_{2m}))$, where $m' = m + k$ and Q is a parabolic subgroup of $\mathrm{Sp}(W_{2m'})$ whose Levi subgroup is $M_Q \cong \mathrm{GL}_k(F) \times \mathrm{Sp}(W_{2m})$. See also [At1, §6. 3]. Note that $\mathrm{Ind}_Q^{\mathrm{Sp}(W_{2m'})}(\tau \otimes \pi)$ is multiplicity-free. An analogue of Hypothesis 3.10 is given as follows:

Proposition 4.2. *Let $\phi_\pi \in \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2m}))$ and $\pi \in \Pi_{\phi_\pi}$. We put $\phi_{\pi'} = \phi_\tau \oplus \phi_\pi \oplus \phi_\tau$. We denote by $a' \in A_{\phi_{\pi'}}$ the element corresponding to ϕ_τ. Let π' be an irreducible constituent of $\mathrm{Ind}_Q^{\mathrm{Sp}(W_{2m'})}(\tau \otimes \pi)$. Then we have*

$$R(w', \tau \otimes \pi)|\pi' = \iota'_1(\pi')(a').$$

Proof. This follows from Theorems 2.2.1 and 2.4.1 in [Ar]. □

4.2. **Local theta correspondence.** We introduce the local theta correspondence induced by a Weil representation $\omega_{W,V,\psi}$ of $\mathrm{Sp}(W_{2m}) \times \mathrm{O}(V_{2n})$, and recall some basic general results.

We have fixed a non-trivial additive character ψ of F. We denote a Weil representation of $\mathrm{Sp}(W_{2m}) \times \mathrm{O}(V_{2n})$ by $\omega = \omega_{W,V,\psi}$. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$. Then the maximal σ-isotypic quotient of ω is of the form

$$\Theta(\sigma) \boxtimes \sigma,$$

where $\Theta(\sigma) = \Theta_{W,V,\psi}(\sigma)$ is a smooth representation of $\mathrm{Sp}(W_{2m})$. It was shown by Kudla [Ku] that $\Theta(\sigma)$ has finite length (possibly zero). The maximal semi-simple quotient of $\Theta(\sigma)$ is denoted by $\theta(\sigma) = \theta_{W,V,\psi}(\sigma)$.

Similarly, for $\pi \in \mathrm{Irr}(\mathrm{Sp}(W_{2m}))$, we obtain smooth finite length representations $\Theta(\pi) = \Theta_{V,W,\psi}(\pi)$ and $\theta(\pi) = \theta_{V,W,\psi}(\pi)$ of $\mathrm{O}(V_{2n})$. The Howe duality conjecture, which was proven by Waldspurger [W1] if the residue characteristic is not 2 and by Gan–Takeda [GT1], [GT2] in general, says that $\theta(\sigma)$ and $\theta(\pi)$ are irreducible (if they are nonzero).

4.3. **Prasad's conjecture.** Let V be an orthogonal space associated to (d, c), and W be a symplectic space with $\dim(V) = \dim(W) = 2n$. We denote the discriminant character of V by χ_V. Let $\phi \in \Phi(\mathrm{O}(V))$, and put

$$\phi' = (\phi \oplus \mathbf{1}) \otimes \chi_V.$$

Then we have $\phi' \in \Phi(\mathrm{Sp}(W))$. Moreover we have a canonical injection $A_\phi \hookrightarrow A_{\phi'}$. We denote the image of $a \in A_\phi$ by $a' \in A_{\phi'}$. One should not confuse z'_ϕ with $z_{\phi'}$. They satisfy $z'_\phi = e'_1 + z_{\phi'}$, where $e'_1 \in A_{\phi'}$ is the element corresponding to $\chi_V \subset \phi'$.

Lemma 4.3. *For any $\phi \in \Phi(\mathrm{O}(V))$, the map*

$$A_\phi \hookrightarrow A_{\phi'} \twoheadrightarrow A_{\phi'}/\langle z_{\phi'} \rangle$$

is surjective. It is not injective if and only if ϕ contains $\mathbf{1}$. In this case, the kernel of this map is generated by $e_1 + z_\phi$, where $e_1 \in A_\phi$ is the element corresponding to $\mathbf{1}$.

Proof. The map $A_\phi \hookrightarrow A_{\phi'}$ is not surjective if and only if $\phi \in \Phi(\mathrm{O}(V))$ and ϕ does not contain $\mathbf{1}$. In this case the cokernel of this map is generated by e_1'. Since $z_\phi' = e_1' + z_{\phi'}$, we have the surjectivity of $A_\phi \hookrightarrow A_{\phi'} \twoheadrightarrow A_{\phi'}/\langle z_{\phi'} \rangle$.

By comparing the order of A_ϕ with the one of $A_{\phi'}/\langle z_{\phi'} \rangle$, we see that $A_\phi \hookrightarrow A_{\phi'} \twoheadrightarrow A_{\phi'}/\langle z_{\phi'} \rangle$ is not injective if and only if ϕ contains $\mathbf{1}$. In this case, the order of the kernel is 2. Since $(e_1 + z_\phi)' = e_1' + z_{\phi'} = z_{\phi'}$ the kernel is generated by $e_1 + z_\phi$. □

Prasad's conjecture is stated as follows:

Conjecture 4.4 (Prasad's conjecture for $(\mathrm{O}(V_{2n}), \mathrm{Sp}(W_{2n}))$). *Let V and W be an orthogonal space associated to (d, c) and a symplectic space with $\dim(V) = \dim(W) = 2n$, respectively. We denote by $\chi_V = \chi_d$ the discriminant character of V. Let $\phi \in \Phi(\mathrm{O}(V))$ and put $\phi' = (\phi \oplus \mathbf{1}) \otimes \chi_V \in \Phi(\mathrm{Sp}(W))$. For $\sigma \in \Pi_\phi$, we have the following:*

1. *$\Theta_{W,V^\bullet,\psi}(\sigma)$ is zero if and only if ϕ contains $\mathbf{1}$ and $\iota_{c'}(\sigma)(z_\phi + e_1) = -1$, where $e_1 \in A_\phi$ is the element corresponding to $\mathbf{1} \subset \phi$.*
2. *Assume that $\pi = \theta_{W,V^\bullet,\psi}(\sigma)$ is nonzero. Then $\pi \in \Pi_{\phi'}$ and $\iota'_{c'}(\pi)|A_\phi = \iota_{c'}(\sigma)$ for $c' \in F^\times$.*

Remark 4.5. (1) *Recall that for $\pi \in \Pi_{\phi'} \subset \mathrm{Irr}(\mathrm{Sp}(W_m))$, the character $\iota'_{c'}(\pi)$ of $A_{\phi'}$ factors through $A_{\phi'}/\langle z_{\phi'} \rangle$. By Lemma 4.3, we see that $\iota'_{c'}(\pi)$ is determined completely by its restriction to A_ϕ.*

(2) *By* [GI1, Theorem C.5], *we know that*
- *if ϕ does not contain $\mathbf{1}$, then both $\Theta_{W,V^\bullet,\psi}(\sigma)$ and $\Theta_{W,V^\bullet,\psi}(\sigma \otimes \det)$ are nonzero;*
- *if ϕ contains $\mathbf{1}$, then exactly one of $\Theta_{W,V^\bullet,\psi}(\sigma)$ or $\Theta_{W,V^\bullet,\psi}(\sigma \otimes \det)$ is nonzero;*
- *if $\pi = \theta_{W,V^\bullet,\psi}(\sigma)$ is nonzero, then $\pi \in \Pi_{\phi'}$.*

Hence Conjecture 4.4 (1) follows from (2) since $z_\phi' = e_1' + z_{\phi'}$.

The first main theorem is as follows:

Theorem 4.6. *Assume Desideratum 3.6 and Hypothesis 3.10. Then Prasad's conjecture for $(\mathrm{O}(V_{2n}), \mathrm{Sp}(W_{2n}))$ (Conjecture 4.4) holds. In particular, it holds unconditionally when $V^\bullet = V$ and $c' \in cN_{E/F}(E^\times)$ with $E = F(\sqrt{d})$.*

A weaker version of Prasad's conjecture (Conjecture 4.4), which is formulated by using Weak LLC for $\mathrm{SO}(V)$ or its translation into $\mathrm{O}(V)$ (i.e., by using A_ϕ^+), was proven by [At1] under Desideratum 3.2 and Hypothesis 3.10.

Theorem 4.7 ([At1, §5.5]). *Assume Desideratum 3.2 and Hypothesis 3.10 for even k. Let $\phi \in \Phi(\mathrm{O}(V))$ and put $\phi' = (\phi \oplus \mathbf{1}) \otimes \chi_V \in \Phi(\mathrm{Sp}(W))$ as in Conjecture 4.4. For $\sigma \in \Pi_\phi$, if $\pi = \theta_{W,V^\bullet,\psi}(\sigma)$ is nonzero, then $\pi \in \Pi_{\phi'}$ and*

$$\iota'_{c'}(\pi)|A_\phi^+ = \iota_{c'}(\sigma)|A_\phi^+$$

for $c' \in F^\times$. In particular, the same unconditionally holds when $V^\bullet = V$ and $c' \in cN_{E/F}(E^\times)$ with $E = F(\sqrt{d})$.

We may consider the theta correspondence for $(\mathrm{Sp}(W_{2n-2}), \mathrm{O}(V_{2n}))$. There is also Prasad's conjecture for $(\mathrm{Sp}(W_{2n-2}), \mathrm{O}(V_{2n}))$.

Conjecture 4.8 (Prasad's conjecture for $(\mathrm{Sp}(W_{2n-2}), \mathrm{O}(V_{2n}))$). *Let V be an orthogonal space associated to (d, c) with $\dim(V) = 2n$, and W be a symplectic space with $\dim(W) = 2n-2$. We denote by $\chi_V = \chi_d$ the discriminant character of V. Let $\phi' \in \Phi(\mathrm{Sp}(W))$ and put $\phi = (\phi' \otimes \chi_V) \oplus \mathbf{1} \in \Phi(\mathrm{O}(V))$. For a companion space $V^\bullet$ of V, we put*

$$e(V^\bullet) = \begin{cases} \chi_V(c'/c) & \text{if } V^\bullet = V, \\ -\chi_V(c'/c) & \text{if } V^\bullet \neq V \end{cases}$$

Let $\pi \in \Pi_{\phi'}$.

(1) $\Theta_{V^\bullet,W,\psi}(\pi) = 0$ *if and only if* ϕ' *contains* χ_V *and* $\iota'_{c'}(\pi)(e_1 + z_{\phi'}) = -e(V^\bullet)$.

(2) *Assume that* $\sigma = \theta_{V^\bullet,W,\psi}(\pi)$ *is nonzero. Then* $\sigma \in \Pi_\phi$ *and so that there is a canonical injection* $A_{\phi'} \hookrightarrow A_\phi$*. Moreover,* $\iota_{c'}(\sigma)$ *satisfies that*
- $\iota_{c'}(\sigma)(z_\phi) = e(V^\bullet)$;
- $\iota_{c'}(\sigma)|A_{\phi'} = \iota'_{c'}(\pi)$ *for* $c' \in F^\times$.

The following theorem shows that Conjecture 4.4 implies Conjecture 4.8.

Theorem 4.9. *Assume Desideratum 3.6 and Hypothesis 3.10 (so that Conjecture 4.4 holds by Theorem 4.6). Then Prasad's conjecture for* $(\mathrm{Sp}(W_{2n-2}), \mathrm{O}(V_{2n}))$ *(Conjecture 4.8) holds. In particular, it holds unconditionally when* $V^\bullet = V$ *and* $c' \in cN_{E/F}(E^\times)$ *with* $E = F(\sqrt{d})$.

Proof. The equation $\iota_{c'}(\sigma)(z_\phi) = e(V^\bullet)$ follows from Desideratum 3.6 (5) and Proposition 3.5 (1). Under assuming Desideratum 3.2 and Hypothesis 3.10 for even k, the first author showed that $\iota_{c'}(\sigma)|A^+_{\phi'} = \iota'_{c'}(\pi)|A^+_{\phi'}$ for $c' \in F^\times$ ([At1, Theorem 1.7]). Hence it suffices to show the equation

$$\iota_{c'}(\sigma)(e_1 + z_\phi) = 1,$$

where e_1 is the element of A_ϕ corresponding to $\mathbf{1}$. This equation follows from Conjecture 4.4 (Theorem 4.6) together with the tower property (see [Ku]). □

4.4. **Proof of Prasad's conjecture.** In this subsection, we prove Theorem 4.6.

Recall that there is a sequence

$$\Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V)) \subset \Phi_{\mathrm{temp}}(\mathrm{O}(V)) \subset \Phi(\mathrm{O}(V)).$$

First, we reduce Conjecture 4.4 to the case when $\phi \in \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V))$.

Lemma 4.10. *If Prasad's conjecture (Conjecture 4.4) holds for any* $\phi_0 \in \Phi_{\mathrm{temp}}(\mathrm{O}(V))$*, then it holds for any* $\phi \in \Phi(\mathrm{O}(V))$.

Proof. This follows from a compatibility of LLC, Langlands quotients and theta lifts (Desideratum 3.1 (7) and [GI1, Proposition C.4]). □

Lemma 4.11. *Assume Desideratum 3.2 and Hypothesis 3.10. Then Prasad's conjecture (Conjecture 4.4) holds for any* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V)) \setminus \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V))$.

Proof. Since $A^+_\phi = A_\phi$, this follows from Theorem 4.7. □

Hence Prasad's conjecture (Conjecture 4.4) is reduced to the case when $\phi \in \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V))$. For this case, the following is the key proposition:

Proposition 4.12. *Let* V_{2n} *and* W_{2n} *be an orthogonal space associated to* (d, c) *and a symplectic space with* $\dim(V_{2n}) = \dim(W_{2n}) = 2n$*, respectively. Fix a positive integer* k*. For a companion space* V *of* V_{2n}*, put* $V' = V \oplus \mathbb{H}^k$*. Also we set* $V_{2n+2k} = (V_{2n})'$, $W = W_{2n}$ *and* $W' = W_{2n+2k} = W \oplus \mathbb{H}^k$*. Let* ϕ_τ *be an irreducible orthogonal tempered representation of* WD_F*, and* $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(F))$ *be the corresponding representation. For* $\phi_\sigma \in \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$*, put*

$$\begin{aligned}\phi_{\sigma'} &= \phi_\tau \oplus \phi_\sigma \oplus \phi_\tau \in \Phi^\epsilon_{\mathrm{temp}}(\mathrm{O}(V_{2n+2k})),\\ \phi_\pi &= (\phi_\sigma \oplus \mathbf{1}) \otimes \chi_V \in \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2n})) \text{ and}\\ \phi_{\pi'} &= (\phi_{\sigma'} \oplus \mathbf{1}) \otimes \chi_V = \phi_\tau\chi_V \oplus \phi_\pi \oplus \phi_\tau\chi_V \in \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_{2n+2k})).\end{aligned}$$

Let $\sigma \in \Pi_\sigma$, $\sigma' \in \Pi_\sigma$, $\pi \in \Pi_{\phi_\pi}$ *and* $\pi' \in \Pi_{\phi'_\pi}$ *such that* $\sigma \in \mathrm{Irr}(\mathrm{O}(V))$ *for a companion space* V *of* V_{2n}*,* $\sigma' \subset \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau \otimes \sigma)$ *and* $\pi' \subset \mathrm{Ind}_Q^{\mathrm{Sp}(W')}(\tau\chi_V \otimes \pi)$*, where* $P \subset \mathrm{O}(V')$ *and* $Q \subset \mathrm{Sp}(W')$ *are suitable parabolic subgroups. Suppose that*
- ϕ_τ *is not the trivial representation of* WD_F;
- $\pi' = \theta_{W',V',\psi}(\sigma')$.

We denote by $a \in A_{\phi_{\sigma'}}$ and $a' \in A_{\phi_{\pi'}}$ the elements corresponding to ϕ_τ and $\phi_\tau\chi_V$, respectively. Then we have

$$\iota_{c'}(\sigma')(a) = \iota'_{c'}(\pi')(a')$$

for $c' \in F^\times$.

Proof. The argument is similar to those of [GI2, §8] and [At1, §7], but it has one difference. So we shall give a sketch of the proof.

Let $\omega = \omega_{W,V,\psi}$ and $\omega' = \omega_{W',V',\psi}$. We use a mixed model $\mathcal{S}' = \mathcal{S}(V' \otimes Y^*) \otimes \mathcal{S} \otimes \mathcal{S}(X^* \otimes W)$ for ω', where $\mathcal{S}$ is a space of ω (see [At1, §6.2]). For $\varphi \in \mathcal{S}'$, we define a map $\hat{f}(\varphi) \colon \mathrm{Sp}(W') \times \mathrm{O}(V') \to \mathcal{S}$ as in [GI2, §8.1] and [At1, §7.1]. By a similar argument to the proof of [GS, Theorem 8.1], we have $\pi = \theta_{W,V,\psi}(\sigma)$ (see also [GI1, Proposition C.4]). Fix a nonzero $\mathrm{Sp}(W) \times \mathrm{O}(V)$-equivariant map

$$\mathcal{T}_{00} \colon \omega \times \sigma^\vee \to \pi.$$

For $\varphi \in \mathcal{S}'$, $\Phi_s \in \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau|\cdot|_F^s \otimes \sigma^\vee)$, $g \in \mathrm{Sp}(W')$, $\check{v} \in \tau^\vee$ and $\check{v}_0 \in \pi^\vee$, consider the integral

$$\langle \mathcal{T}_s(\varphi, \Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle := L(s+1, \tau)^{-1} \cdot \int_{U_P \mathrm{O}(V) \backslash \mathrm{O}(V')} \langle \mathcal{T}_{00}(\hat{f}(\varphi)(g,h), \langle \Phi_s(h), \check{v} \rangle), \check{v}_0 \rangle dh.$$

Then one can show that

(1) the integral $\langle \mathcal{T}_s(\varphi, \Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle$ is absolutely convergent for $\mathrm{Re}(s) > -1$ and admits a holomorphic continuation to $\mathbb{C}$;

(2) $\mathcal{T}_s$ gives an $\mathrm{Sp}(W') \times \mathrm{O}(V')$-equivalent map

$$\mathcal{T}_s \colon \omega' \otimes \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau|\cdot|_F^s \otimes \sigma^\vee) \to \mathrm{Ind}_Q^{\mathrm{Sp}(W')}(\tau\chi_V|\cdot|_F^s \otimes \pi).$$

See [GI2, Lemmas 8.1–8.2] and [At1, Proposition 7.2]. The one difference is that our case does not satisfy an analogue of [GI2, Lemma 8.3]. So we have to modify this lemma. One can show that

(3) if $L(-s, \tau^\vee)$ is regular at $s = 0$, then for any $\Phi \in \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau \otimes \sigma^\vee)$ with $\Phi \neq 0$, there exists $\varphi \in \omega'$ such that $\mathcal{T}_0(\varphi', \Phi) \neq 0$.

Since ϕ_τ is irreducible and tempered, $L(-s, \tau^\vee)$ is regular at $s = 0$ if and only if ϕ_τ is not the trivial representation of WD_F.

By the same calculation as [GI2, Proposition 8.4] and [At1, Corollary 7.4], one can show that

(4) for $\Phi \in \mathrm{Ind}_P^{\mathrm{O}(V')}(\tau \otimes \sigma^\vee)$ and $\varphi \in \omega'$, we have

$$R(w', \tau\chi_V \otimes \pi)\mathcal{T}_0(\varphi, \Phi) = \omega_{\tau\chi_V}(c') \cdot \mathcal{T}_0(\varphi, R_{c'}(w, \tau \otimes \sigma^\vee)\Phi).$$

Here, we use the fact that

$$\gamma_V^{-1} \cdot \lambda(E/F, \psi) = e(V) \cdot \chi_V(c),$$

where γ_V is the Weil constant associated to V which appears on the explicit formula for ω', and $\lambda(E/F, \psi)$ is the Langlands constant which appears on the normalizing factor of $R_{c'}(w, \tau \otimes \sigma^\vee)$.

By the same argument as [At1, Lemma 7.5], (3) and (4) together with Hypothesis 3.10 and Proposition 4.2 imply that

(5) $\iota_{c'}(\sigma')(a) = \omega_{\tau\chi_V}(c') \cdot \iota'_1(\pi')(a')$.

Since $\omega_{\tau\chi_V}(c') = \det(\phi_\tau\chi_V)(c') = \det(\phi_{\pi'}^{a'})(c')$, we have

(6) $\omega_{\tau\chi_V}(c') \cdot \iota'_1(\pi')(a') = \iota'_{c'}(\pi')(a')$.

The equations (5) and (6) imply the desired equation. □

Theorem 4.7 and Proposition 4.12 imply Prasad's conjecture (Theorem 4.6).

Proof of Theorem 4.6. By Remark 4.5 (2) and Lemmas 4.10 and 4.11, we only consider Conjecture 4.4 (2) for $\phi_\sigma \in \Phi_{\mathrm{temp}}^\epsilon(\mathrm{O}(V))$. Hence ϕ_σ contains an irreducible orthogonal representation ϕ_0 with odd dimension k_0. Put $\phi_\pi = (\phi_\sigma \oplus \mathbf{1}) \otimes \chi_V$. Let $\sigma \in \Pi_{\phi_\sigma}$ and assume that $\pi = \theta_{W,V^\bullet,\psi}(\sigma)$ is nonzero. Hence we have $\pi \in \Pi_{\phi_\pi}$.

Let $a_0 \in A_{\phi_\sigma}$ (resp. $a'_0 \in A_{\phi_\pi}$) be the element corresponding to ϕ_0 (resp. $\phi_0\chi_V$). Since $[A_{\phi_\sigma} : A^+_{\phi_\sigma}] = 2$, by Theorem 4.7, it is enough to show that

$$\iota_{c'}(\sigma)(a_0) = \iota'_{c'}(\pi)(a'_0)$$

for $c' \in F^\times$. We choose an irreducible orthogonal tempered representation ϕ_τ of WD_F such that

- ϕ_τ is not the trivial representation;
- ϕ_τ is not contained in ϕ_σ;
- $k = \dim(\phi_\tau)$ is odd.

Put $\phi_{\sigma'} = \phi_\tau \oplus \phi_\sigma \oplus \phi_\tau$ and $\phi_{\pi'} = \phi_\tau\chi_V \oplus \phi_\pi \oplus \phi_\tau\chi_V$. Let $a_\tau \in A_{\phi_{\sigma'}}$ (resp. $a'_\tau \in A_{\phi_{\pi'}}$) be the element corresponding to ϕ_τ (resp. $\phi_\tau\chi_V$). The claims (2) and (3) in the proof of Proposition 4.12, there exist $\sigma' \subset \mathrm{Ind}_P^{\mathrm{O}(V^{\bullet\prime})}(\tau \otimes \sigma)$ and $\pi' \subset \mathrm{Ind}_Q^{\mathrm{Sp}(W')}(\tau\chi_V \otimes \pi)$ such that $\pi' = \theta_{W',V^{\bullet\prime},\psi}(\sigma')$. By Proposition 4.12, we have

$$\iota_{c'}(\sigma')(a_\tau) = \iota'_{c'}(\pi')(a'_\tau).$$

On the other hand, we know

$$\iota_{c'}(\sigma')(a_\tau + a_0) = \iota'_{c'}(\pi')(a'_\tau + a'_0)$$

by Theorem 4.7. Hence we have

$$\iota_{c'}(\sigma')(a_0) = \iota'_{c'}(\pi')(a'_0).$$

Since $\iota_{c'}(\sigma')|A_{\phi_\sigma} = \iota_{c'}(\sigma)$ and $\iota'_{c'}(\pi')|A_{\phi_\pi} = \iota'_{c'}(\pi)$, we have

$$\iota_{c'}(\sigma)(a_0) = \iota'_{c'}(\pi)(a_0)'.$$

This completes the proof. □

Remark 4.13. *One may feel that Prasad's conjecture (Conjecture 4.4) can be proven by a similar way to* [At1, Theorem 1.7 (§7)] *without assuming the weaker version of Prasad's conjecture (Theorem 4.7). However, because of the lack of an analogue of* [GI2, Lemma 8.3]*, the same method as* [At1] *can not be applied to Prasad's conjecture for* $(\mathrm{O}(V_{2n}), \mathrm{Sp}(W_{2n}))$ *when* $\phi \in \Phi(\mathrm{O}(V_{2n}))$ *contains* $\mathbf{1}$.

5. Gross–Prasad conjecture

Gross and Prasad gave a conjectural answer for a restriction problem for special orthogonal groups. For the tempered case, this conjecture has been proven by Waldspurger [W2], [W3], [W5], [W6]. In this section, we recall the Gross–Prasad conjecture and consider an analogous restriction problem for orthogonal groups.

5.1. Local Langlands correspondence for $\mathrm{O}(V_{2n+1})$.

Let V_m be an orthogonal space of dimension m. Recall that the discriminant of V_m is defined by

$$\mathrm{disc}(V_m) = 2^{-m}(-1)^{\frac{m(m-1)}{2}} \det(V_m) \in F^\times/F^{\times 2}.$$

An orthogonal space $V_m^\bullet$ is a companion space of V_m if $\dim(V_m^\bullet) = \dim(V_m)$ and $\mathrm{disc}(V_m^\bullet) = \mathrm{disc}(V_m)$.

Let V_{2n+1} be an orthogonal space over F with dimension $2n+1$. We denote the orthogonal group and the special orthogonal group associated to V_{2n+1} by $\mathrm{O}(V_{2n+1})$ and $\mathrm{SO}(V_{2n+1})$, respectively. Suppose that $\mathrm{O}(V_{2n+1})$ is split.

We say that a representation ϕ of WD_F is symplectic if ϕ admits a non-degenerate symplectic bilinear form which is WD_F-invariant. More precisely, see [GGP, §3].

The Langlands dual group of $\mathrm{SO}(V_{2n+1})$ is the complex Lie group $\mathrm{Sp}(2n,\mathbb{C})$, and W_F acts on $\mathrm{Sp}(2n,\mathbb{C})$ trivially. We denote the L-group of $\mathrm{SO}(V_{2n+1})$ by ${}^L(\mathrm{SO}(V_{2n+1})) = \mathrm{Sp}(2n,\mathbb{C}) \times W_F$. An L-parameter of $\mathrm{SO}(V_{2n+1})$ is an admissible homomorphism

$$\underline{\phi} \colon WD_F \to {}^L(\mathrm{SO}(V_{2n+1})) = \mathrm{Sp}(2n,\mathbb{C}) \times W_F.$$

We put

$$\Phi(\mathrm{SO}(V_{2n+1})) = \{\mathrm{Sp}(2n,\mathbb{C})\text{-conjugacy classes of } L\text{-parameters of } \mathrm{SO}(V_{2n+1})\}.$$

For an L-parameter $\underline{\phi} \colon WD_F \to {}^L(\mathrm{SO}(V_{2n+1}))$, by composing with the projection $\mathrm{Sp}(2n,\mathbb{C}) \times W_F \to \mathrm{Sp}(2n,\mathbb{C})$, we obtain a map

$$\phi \colon WD_F \to \mathrm{Sp}(2n,\mathbb{C}).$$

The map $\underline{\phi} \mapsto \phi$ gives an identification

$$\Phi(\mathrm{SO}(V_{2n+1})) = \{\phi\colon WD_F \to \mathrm{Sp}(2n,\mathbb{C})\}/(\mathrm{Sp}(2n,\mathbb{C})\text{-conjugacy}).$$

Namely, we regard $\Phi(\mathrm{SO}(V_{2n+1}))$ as the set of equivalence classes of symplectic representations of WD_F with dimension $2n$. We denote the subset of $\Phi(\mathrm{SO}(V_{2n+1}))$ consisting of equivalence classes of tempered (resp. discrete, generic) representations by $\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n+1}))$ (resp. $\Phi_{\mathrm{disc}}(\mathrm{SO}(V_{2n+1}))$, $\Phi_{\mathrm{gen}}(\mathrm{SO}(V_{2n+1}))$). Then we have a sequence

$$\Phi_{\mathrm{disc}}(\mathrm{SO}(V_{2n+1})) \subset \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n+1})) \subset \Phi_{\mathrm{gen}}(\mathrm{SO}(V_{2n+1})).$$

We expect a similar desideratum for $\mathrm{SO}(V_{2n+1})$ to Desideratum 3.6. Namely, for $\phi \in \Phi(\mathrm{SO}(V_{2n+1}))$, we expect there are an L-packet

$$\Pi_\phi^0 \subset \bigsqcup_{V_{2n+1}^\bullet} \mathrm{Irr}(\mathrm{SO}(V_{2n+1}^\bullet)),$$

and a canonical bijection

$$\iota\colon \Pi_\phi^0 \to \widehat{A_\phi},$$

which satisfy similar properties to Desideratum 3.6. Here, $V_{2n+1}^\bullet$ runs over all companion spaces of $V_{2n+1}^\bullet$ Note that $A_\phi = A_\phi^+$ for $\phi \in \Phi(\mathrm{SO}(V_{2n+1}))$.

For the (quasi-)split case, it is known by Arthur [Ar].

Theorem 5.1. *There exist a surjective map*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{SO}(V_{2n+1})) \to \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n+1}))$$

with the inverse image Π_ϕ^0 *of* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(V_{2n+1}))$*, and a canonical bijection*

$$\iota\colon \Pi_\phi^0 \to (A_\phi^+/\langle z_\phi\rangle)^\wedge$$

which satisfy analogues of Desideratum 3.6 (2), (4), and (6). Moreover, using the Langlands classification, we can extend the map $\tau \mapsto \phi$ *to a surjective map*

$$\mathrm{Irr}(\mathrm{SO}(V_{2n+1})) \to \Phi(\mathrm{SO}(V_{2n+1}))$$

which satisfies an analogue of Desideratum 3.6 (7).

Mœglin's work in [M2, §1.4, Theorem 1.4.1] seems to extend This theorem to the pure inner forms as well.

Since $\mathrm{O}(V_{2n+1}^\bullet)$ is the direct product

$$\mathrm{O}(V_{2n+1}^\bullet) = \mathrm{SO}(V_{2n+1}^\bullet) \times \{\pm \mathbf{1}_{V_{2n+1}^\bullet}\},$$

any $\tau \in \mathrm{Irr}(\mathrm{O}(V_{2n+1}^\bullet))$ is determined by its restriction $\tau|\mathrm{SO}(V_{2n+1}^\bullet) \in \mathrm{Irr}(\mathrm{SO}(V_{2n+1}^\bullet))$ and its central character $\omega_\tau \in \{\pm\mathbf{1}_{V_{2n+1}^\bullet}\}^\wedge \cong \{\pm 1\}$. We define $\Phi(\mathrm{O}(V_{2n+1}))$ by

$$\Phi(\mathrm{O}(V_{2n+1})) \coloneqq \Phi(\mathrm{SO}(V_{2n+1})) \times \{\pm 1\}.$$

For $(\phi, b) \in \Phi(\mathrm{O}(V_{2n+1}))$, we put

$$\Pi_{\phi,b} = \{\tau \in \mathrm{Irr}(\mathrm{O}(V_{2n+1}^\bullet)) \mid \tau|\mathrm{SO}(V_{2n+1}^\bullet) \in \Pi_\phi^0,\ \omega_\tau(-1) = b\}.$$

Then we have a canonical bijection

$$\Pi_{\phi,b} \xrightarrow{\mathrm{Res}} \Pi_\phi^0 \xrightarrow{\iota} \widehat{A_\phi}.$$

which is also denoted by ι. Also we have

$$\Pi_{\phi,-b} = \Pi_{\phi,b} \otimes \det.$$

5.2. **Gross–Prasad conjecture for special orthogonal groups.** In this subsection, we recall the Gross–Prasad conjecture.

Let V_{m+1} be an orthogonal space of dimension $m+1$, and V_m be a non-degenerate subspace of V_{m+1} with codimension 1. We denote by V_{even} (resp. V_{odd}) the space V_m or V_{m+1} such that $\dim(V_{\mathrm{even}})$ is even (resp. $\dim(V_{\mathrm{odd}})$ is odd). Suppose that $\mathrm{SO}(V_m)\times\mathrm{SO}(V_{m+1})$ is quasi-split. We put $c=-\mathrm{disc}(V_{\mathrm{odd}})/\mathrm{disc}(V_{\mathrm{even}})\in F^\times/F^{\times 2}$. Then V_{even} is associated to $(\mathrm{disc}(V_{\mathrm{even}}),c)$. We say that a pair $(V_m^\bullet, V_{m+1}^\bullet)$ of companion spaces of (V_m,V_{m+1}) is relevant if $V_m^\bullet\subset V_{m+1}^\bullet$. Then we have a diagonal map

$$\Delta\colon \mathrm{O}(V_m^\bullet)\to \mathrm{O}(V_m^\bullet)\times \mathrm{O}(V_{m+1}^\bullet).$$

By [AGRS] and [W4], for $\sigma_0\in\mathrm{Irr}(\mathrm{SO}(V_{\mathrm{even}}^\bullet))$ and $\tau_0\in\mathrm{Irr}(\mathrm{SO}(V_{\mathrm{odd}}^\bullet))$, we have

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0\boxtimes\tau_0,\mathbb{C})\leq 1.$$

Choose $\epsilon\in\mathrm{O}(V_m^\bullet)$ such that $\det(\epsilon)=-1$. We extend τ_0 to an irreducible representation τ of $\mathrm{O}(V_{\mathrm{odd}}^\bullet)$. For $\varphi\in\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0\boxtimes\tau_0,\mathbb{C})$, we put

$$\varphi'=\varphi\circ(\mathbf{1}\boxtimes\tau(\epsilon)).$$

Then we have $\varphi'\in\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0^\epsilon\boxtimes\tau_0,\mathbb{C})$, and the map $\varphi\mapsto\varphi'$ gives an isomorphism

$$\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0\boxtimes\tau_0,\mathbb{C})\cong\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0^\epsilon\boxtimes\tau_0,\mathbb{C}).$$

Therefore, $\dim_{\mathbb{C}}\mathrm{Hom}_{\Delta\mathrm{SO}(V_m^\bullet)}(\sigma_0\boxtimes\tau_0,\mathbb{C})$ depends only on

$$([\sigma_0],\tau_0)\in\mathrm{Irr}(\mathrm{SO}(V_{\mathrm{even}}))/\sim_\epsilon\times\mathrm{Irr}(\mathrm{SO}(V_{\mathrm{odd}})).$$

The Gross–Prasad conjecture determines this dimension in terms of Weak LLC for $\mathrm{SO}(V_{\mathrm{even}})$ and LLC for $\mathrm{SO}(V_{\mathrm{odd}})$.

Let $\phi\in\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{\mathrm{even}}))/\sim_\epsilon$ and $\phi'\in\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{\mathrm{odd}}))$. Following [GGP, §6], for semi-simple elements $a\in C_\phi$ and $a'\in C_{\phi'}$, we put

$$\chi_{\phi'}(a)=\varepsilon(\phi^a\otimes\phi')\cdot\det(\phi^a)(-1)^{\frac{1}{2}\dim(\phi')},$$
$$\chi_\phi(a')=\varepsilon(\phi\otimes\phi'^{a'})\cdot\det(\phi)(-1)^{\frac{1}{2}\dim(\phi'^{a'})}.$$

Here, $\varepsilon(\phi^a\otimes\phi')=\varepsilon(1/2,\phi^a\otimes\phi',\psi)$ and $\varepsilon(\phi\otimes\phi'^{a'})=\varepsilon(1/2,\phi\otimes\phi'^{a'},\psi)$ are the local root numbers, which are independent of the choice of ψ. By [GP, Proposition 10.5], $\chi_{\phi'}$ and χ_ϕ define characters on A_ϕ and on $A_{\phi'}$, respectively.

The following is a result of Waldspurger [W2], [W3], [W5], [W6].

Theorem 5.2 (Gross–Prasad conjecture for special orthogonal groups)**.** *Let V_{m+1} be an orthogonal space of dimension $m+1$, and V_m be a non-degenerate subspace of V_{m+1} with codimension 1. Suppose that $\mathrm{SO}(V_m)\times\mathrm{SO}(V_{m+1})$ is quasi-split. We put $c=-\mathrm{disc}(V_{\mathrm{odd}})/\mathrm{disc}(V_{\mathrm{even}})\in F^\times/F^{\times 2}$, so that V_{even} is associated to $(\mathrm{disc}(V_{\mathrm{even}}),c)$. Assume*

- *Weak LLC for $\mathrm{SO}(V_{\mathrm{even}})$ (Desideratum 3.2);*
- *LLC for $\mathrm{SO}(V_{\mathrm{odd}})$ (an analogue of Desideratum 3.6 for $\mathrm{SO}(V_{\mathrm{odd}})$).*

Let $\phi\in\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{\mathrm{even}}))/\sim_\epsilon$ and $\phi'\in\Phi_{\mathrm{temp}}(\mathrm{SO}(V_{\mathrm{odd}}))$. Then there exists a unique pair $([\sigma_0],\tau_0)\in\Pi_\phi^0\times\Pi_{\phi'}^0$ such that $\sigma_0\boxtimes\tau_0$ is a representation of $\mathrm{SO}(V_{\mathrm{even}}^\bullet)\times\mathrm{SO}(V_{\mathrm{odd}}^\bullet)$ with a relevant pair $(V_{\mathrm{even}}^\bullet,V_{\mathrm{odd}}^\bullet)$ of companion spaces of $(V_{\mathrm{even}},V_{\mathrm{odd}})$, and

$$\mathrm{Hom}_{\Delta\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0\boxtimes\tau_0,\mathbb{C})\neq 0.$$

Moreover, $\iota_c([\sigma_0])\times\iota(\tau_0)$ satisfies that

$$\iota_c([\sigma_0])\times\iota(\tau_0)=(\chi_{\phi'}|A_\phi^+)\times\chi_\phi.$$

In particular, the same unconditionally holds for quasi-split $\mathrm{SO}(V_m)\times\mathrm{SO}(V_{m+1})$.

5.3. **Gross–Prasad conjecture for orthogonal groups.** Let $V_{m+1}^\bullet$ be an orthogonal space of dimension $m+1$, and $V_m^\bullet$ be a non-degenerate subspace of V_{m+1} with codimension 1. In [AGRS], Aizenbud, Gourevitch, Rallis and Schiffmann showed that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta \mathrm{O}(V_m^\bullet)}(\sigma \boxtimes \tau, \mathbb{C}) \leq 1$$

for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{\mathrm{even}}^\bullet))$ and $\tau \in \mathrm{Irr}(\mathrm{O}(V_{\mathrm{odd}}^\bullet))$. The following conjecture determines this dimension for $(\sigma, \tau) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{even}}^\bullet)) \times \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{odd}}^\bullet))$.

Let $\phi \in \Phi(\mathrm{O}(V_{2n}))$. For $b \in \{\pm 1\}$ and $a \in C_\phi$, we put

$$d_{\phi,b}(a) = b^{\dim(\phi^a)}.$$

By [GGP, §4], $d_{\phi,b}$ defines a character on A_ϕ. Note that $d_{\phi,b}$ is trivial on A_ϕ^+.

Conjecture 5.3 (Gross–Prasad conjecture for orthogonal groups)**.** *Let V_{m+1} be an orthogonal space of dimension $m+1$, and V_m be a non-degenerate subspace of V_{m+1} with codimension 1. Suppose that $\mathrm{O}(V_m) \times \mathrm{O}(V_{m+1})$ is quasi-split. We put $c = -\mathrm{disc}(V_{\mathrm{odd}})/\mathrm{disc}(V_{\mathrm{even}}) \in F^\times/F^{\times 2}$, so that V_{even} is associated to $(\mathrm{disc}(V_{\mathrm{even}}), c)$. Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{even}}))$ and $(\phi', b) \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{odd}}))$. Then there exists a unique pair $(\sigma, \tau) \in \Pi_\phi \times \Pi_{\phi',b}$ such that $\sigma \boxtimes \tau$ is a representation of $\mathrm{O}(V_{\mathrm{even}}^\bullet) \times \mathrm{O}(V_{\mathrm{odd}}^\bullet)$ with a relevant pair $(V_{\mathrm{even}}^\bullet, V_{\mathrm{odd}}^\bullet)$ of companion spaces of $(V_{\mathrm{even}}, V_{\mathrm{odd}})$, and*

$$\mathrm{Hom}_{\Delta \mathrm{O}(V_m^\bullet)}(\sigma \boxtimes \tau, \mathbb{C}) \neq 0.$$

Moreover, $\iota_c(\sigma) \times \iota(\tau)$ satisfies that

$$\iota_c(\sigma) \times \iota(\tau) = (\chi_{\phi'} \cdot d_{\phi,b}) \times \chi_\phi.$$

Remark 5.4. *Let $V_{2n+1} = V_{2n} \oplus L$ be an orthogonal space of dimension $2n+1$, and V_{2n} be a non-degenerate subspace of V_{2n+1} with codimension 1. The stabilizer of the line L in $\mathrm{SO}(V_{2n+1})$ is the subgroup:*

$$\mathrm{S}(\mathrm{O}(V_{2n}) \times \mathrm{O}(L)) = \{(g_1, g_2) \in \mathrm{O}(V_{2n}) \times \mathrm{O}(L) \mid \det(g_1) = \det(g_2)\},$$

which is isomorphic to $\mathrm{O}(V_{2n})$ by the first projection. Then the restriction problem of $\mathrm{O}(V_{2n}) \subset \mathrm{O}(V_{2n+1})$ is equivalent to the one of $\mathrm{S}(\mathrm{O}(V_{2n}) \times \mathrm{O}(L)) \subset \mathrm{SO}(V_{2n+1})$. Indeed, let τ be an irreducible representation of $\mathrm{SO}(V_{2n+1})$, and τ^b be the extension of τ to $\mathrm{O}(V_{2n+1})$ satisfying $\tau^b(-\mathbf{1}_{V_{2n+1}}) = b \cdot \mathrm{id}$ for $b \in \{\pm\}$. For $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$, define $\sigma^b \in \mathrm{Irr}(\mathrm{O}(V_{2n}))$ by

$$\sigma^b(g) = \begin{cases} \sigma(g) & \text{if } \det(g) = 1, \\ b \cdot \omega_\sigma(-1) \cdot \sigma(g) & \text{if } \det(g) = -1. \end{cases}$$

Here, ω_σ denotes the central character of σ, which is regarded as a character of $\{\pm 1\}$. We regard σ^b as an irreducible representation of $\mathrm{S}(\mathrm{O}(V_{2n}) \times \mathrm{O}(L))$ by pulling back via the first projection. Then we have an identification

$$\mathrm{Hom}_{\mathrm{O}(V_{2n})}(\tau^b \otimes \sigma, \mathbb{C}) = \mathrm{Hom}_{\mathrm{S}(\mathrm{O}(V_{2n}) \times \mathrm{O}(L))}(\tau \otimes \sigma^b, \mathbb{C}).$$

In §5.4, we review a result of Prasad [P3] for a low rank case and check that it is compatible with Conjecture 5.3. In §5.5, we will prove Conjecture 5.3 under assuming LLC for $\mathrm{O}(V_m) \times \mathrm{O}(V_{m+1})$ and Hypothesis 3.10.

5.4. **Low rank cases.** In [P3], D. Prasad extended a theorem on trilinear forms of three representations of $\mathrm{GL}_2(F)$ ([P1, Theorem 1.4]). In this subsection, we check this theorem follows from Conjecture 5.3.

First, we recall a theorem on trilinear forms. Let D be the (unique) quaternion division algebra over F. For an irreducible representation π of $\mathrm{GL}_2(F)$, let π' be the Jacquet–Langlands lift of π if π is an essentially discrete series representation, and put $\pi' = 0$ otherwise. Also, for a representation ϕ of WD_F, if $\det(\phi) = \mathbf{1}$, we write $\varepsilon(\phi) = \varepsilon(1/2, \phi, \psi)$, which is independent of a non-trivial additive character ψ of F.

Theorem 5.5 ([P1, Theorem 1.4])**.** *For $i = 1, 2, 3$, let π_i be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$ with central character ω_{π_i}. Assume that $\omega_{\pi_1}\omega_{\pi_2}\omega_{\pi_3} = \mathbf{1}$. We denote the representation of WD_F corresponding to π_i by ϕ_i. Then:*

- *there exists a nonzero $\mathrm{GL}_2(F)$-invariant linear form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ if and only if $\varepsilon(\phi_1 \otimes \phi_2 \otimes \phi_3) = 1$;*
- *there exists a nonzero $D^\times$-invariant linear form on $\pi_1' \otimes \pi_2' \otimes \pi_3'$ if and only if $\varepsilon(\phi_1 \otimes \phi_2 \otimes \phi_3) = -1$.*

When π_i is tempered for each i and $\omega_{\pi_1}\omega_{\pi_2} = \omega_{\pi_3} = \mathbf{1}$, this theorem is a special case of GP conjecture for special orthogonal groups (Theorem 5.2). Recall that there exist two exact sequences

$$1 \longrightarrow F^\times \xrightarrow{\Delta_1} \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \xrightarrow{\rho_1} \mathrm{GSO}(V_4) \longrightarrow 1,$$
$$1 \longrightarrow F^\times \xrightarrow{\Delta_2} D^\times \times D^\times \xrightarrow{\rho_2} \mathrm{GSO}(V_4') \longrightarrow 1,$$

where

- $V_4 = \mathrm{M}_2(F)$ which is regarded as an orthogonal space with
$$(\begin{pmatrix} a_1, b_1 \\ c_1, d_1 \end{pmatrix}, \begin{pmatrix} a_2, b_2 \\ c_2, d_2 \end{pmatrix})_{V_4} = \mathrm{tr}(\begin{pmatrix} a_1, b_1 \\ c_1, d_1 \end{pmatrix} \begin{pmatrix} d_2, -b_2 \\ -c_2, a_2 \end{pmatrix});$$
- $V_4' = D$ is regarded as an orthogonal space with
$$(x, y)_{V_4'} = \tau(xy^*),$$
where $y \mapsto y^*$ is the main involution and $\tau(x)$ is the reduced trace of x;
- for an orthogonal space V_{2n} with even dimension $2n$, the similitude special orthogonal group $\mathrm{GO}(V_{2n})$ is defined by
$$\mathrm{GO}(V_{2n}) = \{g \in \mathrm{GL}(V_{2n}) \mid \langle gv_1, gv_2\rangle_{V_{2n}} = \nu(g)\langle v_1, v_2\rangle_{V_{2n}},\ \nu(g) \in F^\times \text{ for any } v_1, v_2 \in V_{2n}\}$$
and $\mathrm{GSO}(V_{2n})$ is defined by
$$\mathrm{GSO}(V_{2n}) = \{g \in \mathrm{GO}(V_{2n}) \mid \det(g) = \nu(g)^n\};$$
- Δ_1 and Δ_2 are the diagonal embeddings;
- ρ_1 and ρ_2 are given by
$$\rho_1(g_1, g_2)x = g_1 x g_2^{-1}, \quad \rho_2(g_1', g_2')x' = g_1' x' {g_2'}^{-1}$$
for $g_1, g_2 \in \mathrm{GL}_2(F)$, $x \in V_4$, $g_1', g_2' \in D^\times$ and $x' \in V_4'$.

Hence if $\omega_{\pi_1}\omega_{\pi_2} = \omega_{\pi_3} = \mathbf{1}$, then $\pi_1 \otimes \pi_2$ (resp. $\pi_1' \otimes \pi_2'$) is regarded as a representation $\widetilde{\sigma}$ of $\mathrm{GSO}(V_4)$ (resp. $\widetilde{\sigma}'$ of $\mathrm{GSO}(V_4')$). The restriction of $\widetilde{\sigma}$ to $\mathrm{SO}(V_4)$ (resp. $\widetilde{\sigma}'$ to $\mathrm{SO}(V_4')$) decomposes into a direct sum of irreducible representations, i.e.,
$$\widetilde{\sigma}|\mathrm{SO}(V_4) = \sigma_1 \oplus \cdots \oplus \sigma_r \quad (\textit{resp. } \widetilde{\sigma}'|\mathrm{SO}(V_4') = \sigma_1' \oplus \cdots \oplus \sigma_{r'}').$$
Let $\phi_\sigma = \phi_1 \otimes \phi_2$. Then we have $\phi_\sigma \in \Phi(\mathrm{SO}(V_4))/\sim_\epsilon$, and the L-packet $\Pi^0_{\phi_\sigma}$ is given by
$$\Pi^0_{\phi_\sigma} = \{[\sigma_i] \mid i = 1, \ldots, r\} \cup \{[\sigma_i'] \mid i = 1, \ldots, r'\}.$$
Here, if $\pi_1' \otimes \pi_2' = 0$, we neglect $\{[\sigma_i']\}$.

On the other hand, if $\omega_{\pi_3} = \mathbf{1}$, then π_3 (resp. π_3') is regarded as a representation τ of $\mathrm{SO}(V_3)$ (resp. τ' of $\mathrm{SO}(V_3')$), where V_3 (resp. V_3') is the orthogonal space of dimension 3, discriminant -1, and such that $\mathrm{SO}(V_3)$ is split (resp. $\mathrm{SO}(V_3')$ is not split). We identify V_3 (resp. V_3') with the subspace of V_4 (resp. V_4') consisting of trace zero elements (resp. reduced trace zero elements). Let $\phi_\tau = \phi_3\colon WD_F \to \mathrm{SL}_2(\mathbb{C})$. Then we have $\phi_\tau \in \Phi(\mathrm{SO}(V_3))$, and the L-packet $\Pi^0_{\phi_\tau}$ is given by
$$\Pi^0_{\phi_\tau} = \{\tau,\ \tau'\}.$$
Here, if $\pi_3' = 0$, we neglect τ'.

We embed $\mathrm{GL}_2(F)$ into $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ (resp. $D^\times$ into $D^\times \times D^\times$) as the diagonal subgroup. This embedding induces the inclusion $\mathrm{SO}(V_3) \hookrightarrow \mathrm{SO}(V_4)$ (resp. $\mathrm{SO}(V_3') \hookrightarrow \mathrm{SO}(V_4')$). Then we conclude that

- there exists a nonzero $\mathrm{GL}_2(F)$-invariant linear form on $\pi_1 \otimes \pi_2 \otimes \pi_3$ if and only if $\mathrm{Hom}_{\mathrm{SO}(V_3)}(\sigma_i \otimes \tau, \mathbb{C}) \neq 0$ for some $i = 1, \ldots, r$;
- there exists a nonzero $D^\times$-invariant linear form on $\pi_1' \otimes \pi_2' \otimes \pi_3'$ if and only if $\mathrm{Hom}_{\mathrm{SO}(V_3')}(\sigma_i' \otimes \tau', \mathbb{C}) \neq 0$ for some $i = 1, \ldots, r'$;
- $\varepsilon(\phi_1 \otimes \phi_2 \otimes \phi_3) = \varepsilon(\phi_\sigma \otimes \phi_\tau) = \chi_{\phi_\tau}(z_{\phi_\sigma}) = \chi_{\phi_\sigma}(z_{\phi_\tau})$.

Hence Theorem 5.2 implies Theorem 5.5 for tempered π_1, π_2 and π_3 when $\omega_{\pi_1}\omega_{\pi_2} = \omega_{\pi_3} = \mathbf{1}$.

Next, we recall [P3, Theorem 3], which is an extension of Theorem 5.5. This is the case when $\pi_1 = \pi_2$. Note that $\pi_1 \otimes \pi_1 = \mathrm{Sym}^2(\pi_1) \oplus \wedge^2(\pi_1)$. Also, for a representation ϕ_1 of WD_F, we have $\phi_1 \otimes \phi_1 = \mathrm{Sym}^2(\phi_1) \oplus \wedge^2(\phi_1)$.

Theorem 5.6 ([P3, Theorem 3])**.** *Let π_1 and π_3 be irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(F)$. Assume that $\omega_{\pi_1}^2\omega_{\pi_3} = \mathbf{1}$. We denote the representation of WD_F corresponding to π_i by ϕ_i. Then:*

- $\mathrm{Sym}^2(\pi_1)\otimes\pi_3$ *has a $\mathrm{GL}_2(F)$-invariant linear form if and only if $\varepsilon(\mathrm{Sym}^2(\phi_1)\otimes\phi_3) = \omega_{\pi_1}(-1)$ and $\varepsilon(\wedge^2(\phi_1)\otimes\phi_3) = \omega_{\pi_1}(-1)$;*
- $\wedge^2(\pi_1)\otimes\pi_3$ *has a $\mathrm{GL}_2(F)$-invariant linear form if and only if $\varepsilon(\mathrm{Sym}^2(\phi_1)\otimes\phi_3) = -\omega_{\pi_1}(-1)$ and $\varepsilon(\wedge^2(\phi_1)\otimes\phi_3) = -\omega_{\pi_1}(-1)$.*

We check that GP conjecture (Conjecture 5.3) implies this theorem for tempered π_1 and π_3 such that $\omega_{\pi_1}^2 = \mathbf{1}$ and $\omega_{\pi_3} = \mathbf{1}$. Consider the group

$$(\mathrm{GL}_2(F)\times\mathrm{GL}_2(F))\rtimes\langle c\rangle,$$

where $c^2 = 1$ and c acts on $\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$ by the exchange of the two factors of $\mathrm{GL}_2(F)$. Then $\rho_1\colon \mathrm{GL}_2(F)\times\mathrm{GL}_2(F)\to\mathrm{GSO}(V_4)$ gives a surjection

$$\rho_1\colon (\mathrm{GL}_2(F)\times\mathrm{GL}_2(F))\rtimes\langle c\rangle \to \mathrm{GSO}(V_4)\rtimes\langle c\rangle\cong\mathrm{GO}(V_4).$$

Here, we identify c as the element in $\mathrm{O}(V_4)$ which acts on V_3 by -1 and on the orthogonal complement of V_3 by $+1$. There are two extensions of the representation $\widetilde{\sigma}|\mathrm{SO}(V_4) = \sigma_1\oplus\cdots\oplus\sigma_r$ of $\mathrm{SO}(V_4)$ on $\pi_1\otimes\pi_1 = \mathrm{Sym}^2(\pi_1)\oplus\wedge^2(\pi_1)$ to $\mathrm{O}(V_4)$. We denote by $\widetilde{\sigma}^\pm|\mathrm{O}(V_4) = \sigma_1^\pm\oplus\cdots\oplus\sigma_r^\pm$ the extension such that c acts on $\mathrm{Sym}^2(\pi_1)$ by ±1 and on $\wedge^2(\pi_1)$ by ∓1, respectively. On the other hand, the representation τ of $\mathrm{SO}(V_3)$ on π_3 has two extensions $\tau^\pm$ to $\mathrm{O}(V_3)$, which satisfies that $\tau^\pm(-\mathbf{1}_{V_3}) = \pm\mathrm{id}$. Since c centralizes the diagonal subgroup $\mathrm{GL}_2(F)$ of $\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$, we see that

- $\mathrm{Sym}^2(\pi_1)\otimes\pi_3$ has a $\mathrm{GL}_2(F)$-invariant linear form if and only if $\mathrm{Hom}_{\mathrm{O}(V_3)}(\sigma_i^+\otimes\tau^+,\mathbb{C})\neq0$ for some $i = 1,\dots,r$;
- $\wedge^2(\pi_1)\otimes\pi_3$ has a $\mathrm{GL}_2(F)$-invariant linear form if and only if $\mathrm{Hom}_{\mathrm{O}(V_3)}(\sigma_i^+\otimes\tau^-,\mathbb{C})\neq0$ for some $i = 1,\dots,r$.

Since π_1 is generic, $\sigma_i^\pm$ is μ_a^b-generic for some $a\in F^\times$ and $b\in\{\pm\}$. We claim that $b = \pm\omega_{\pi_1}(a)$. Fix a nonzero ψ-Whittaker functional $l\colon\pi_1\to\mathbb{C}$ and $x\in\pi_1$ such that $l(x)\neq0$. For each $a\in F^\times$, put

$$l_a = l\circ\pi_1(\begin{pmatrix}a&0\\0&1\end{pmatrix}),\quad x_a = \pi_1(\begin{pmatrix}a&0\\0&1\end{pmatrix}^{-1})x.$$

We define a basis $\{v, e_a, e'_a, v'\}$ of $V_4 = \mathrm{M}_2(F)$ by

$$v = \begin{pmatrix}0&1\\0&0\end{pmatrix},\ e_a = \begin{pmatrix}1&0\\0&a\end{pmatrix},\ e'_a = \begin{pmatrix}1&0\\0&-a\end{pmatrix},\ v = \begin{pmatrix}0&0\\-1&0\end{pmatrix}.$$

This basis makes V_4 the orthogonal space associated to $(1,a)$. Also we have

$$\rho_1(\begin{pmatrix}1&n_1\\0&1\end{pmatrix},\begin{pmatrix}1&n_2\\0&1\end{pmatrix})(v,e_a,e'_a,v') = (v,e_a,e'_a,v')\begin{pmatrix}1&n_1a-n_2&-n_1a-n2&-n_1n_2\\0&1&0&\frac{-n_1+n_2a^{-1}}{2}\\0&0&1&\frac{-n_1-n_2a^{-1}}{2}\\0&0&0&1\end{pmatrix}.$$

Hence $l_a\otimes l_{-1}\colon\pi_1\otimes\pi_1\to\mathbb{C}$ gives a μ_a-Whittaker functional l_0 on $\widetilde{\sigma}^\pm$. Let $v_0 = x_a\otimes x_{-1}\in\widetilde{\sigma} = \pi_1\otimes\pi_1$. Note that V_3 is the orthogonal complement of Fe_1 in V_4. Since we regard c as the nontrivial element in the center of $\mathrm{O}(V_3)$, it acts on v, e_1, e'_1, and v' by

$$cv = -v,\ ce_1 = e_1,\ ce'_1 = -e'_1,\ cv' = -v'.$$

We define $\epsilon_a\in\mathrm{O}(V_4)$ so that $\epsilon_a v = v$, $\epsilon_a e_a = e_a$, $\epsilon_a e'_a = -e'_a$, and $\epsilon_a v' = v'$. Then

$$\epsilon_a = \rho_1(\begin{pmatrix}-1&0\\0&a\end{pmatrix},\begin{pmatrix}-a&0\\0&1\end{pmatrix})\cdot c.$$

Therefore we have

$$b\cdot l_0(v_0) = l_0(\epsilon_a v_0)$$

$$= (l_a \otimes l_{-1}) \circ \pi_1(\begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}) \otimes \pi_1(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}) \circ c(x_a \otimes x_{-1})$$
$$= \pm\omega_{\pi_1}(a) \cdot (l_a \otimes l_{-1}) \circ \pi_1(\begin{pmatrix} -a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) \otimes \pi_1(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix})(x_{-1} \otimes x_a)$$
$$= \pm\omega_{\pi_1}(a) \cdot (l_a \otimes l_{-1})(x_a \otimes x_{-1})$$
$$= \pm\omega_{\pi_1}(a) \cdot l_0(v_0).$$

Hence $b = \pm\omega_{\pi_1}(a)$, as desired.

By Desideratum 3.9 (1) and (3), we have

$$\iota_1(\sigma_i^{\pm})(s) = (\pm\omega_{\pi_1}(a))^{\det(s)} \cdot \eta_{\phi_\sigma,a}(s)$$

for $s \in A_{\phi_\sigma}$, In particular, if we denote the element in A_{ϕ_σ} corresponding to $\wedge^2(\phi_1) = \det(\phi_1) = \omega_{\pi_1}$ by s_0, we have

$$\iota_1(\sigma_i^{\pm})(s_0) = \pm 1.$$

Also we have

$$(\chi_{\phi_\tau} \cdot d_{\phi_\sigma,\pm1})(s_0) = \pm\varepsilon(\wedge(\phi_1) \otimes \phi_3) \cdot \omega_{\pi_1}(-1).$$

Hence by GP conjecture (Conjecture 5.3), we see that:

- $\mathrm{Hom}_{\mathrm{O}(V_3)}(\sigma_i^{+} \otimes \tau^{\pm}, \mathbb{C}) \neq 0$ for some $i = 1, \ldots, r$ if and only if $\pm\varepsilon(\wedge(\phi_1) \otimes \phi_3) \cdot \omega_{\pi_1}(-1) = +1$.

This implies Theorem 5.6 for tempered π_1 and π_3 such that $\omega_{\pi_1}^2 = \omega_{\pi_3} = \mathbf{1}$.

In fact, there is an analogous theorem ([P4, Theorem 6]) for the quaternion algebra case (in which case the product of the two root numbers is -1). If one knew $\iota_1({\sigma'}_i^{\pm})$ explicitly (as we have done for $\sigma_i^{\pm}$ by using Desideratum 3.9 (3)), one would show that this theorem follows from Conjecture 5.3. Conversely, by using Prasad's theorem [P4, Theorem 6] and Conjecture 5.3, we may conclude that

$$\iota_1({\sigma'}_i^{\pm})(s_0) = \mp 1.$$

Similarly, [P3, Theorem 4] follows from Conjecture 5.3 for $m = 2$ (using Remark 5.4).

5.5. **Proof of Conjecture 5.3.** In this subsection, we prove that Prasad's conjecture (Conjecture 4.4) implies the Gross–Prasad conjecture (Conjecture 5.3). The second main theorem is as follows:

Theorem 5.7. *Assume*

- *LLC for* $\mathrm{O}(V_m) \times \mathrm{O}(V_{m+1})$ *(Desideratum 3.6 and the analogue of Desideratum 3.1 for* $\mathrm{SO}(V_{\mathrm{odd}})$*);*
- *Hypothesis 3.10 (which implies Prasad's conjecture 4.4 by Theorem 4.6).*

Then the Gross–Prasad conjecture (Conjecture 5.3) holds. In particular, it unconditionally holds for quasi-split $\mathrm{O}(V_m) \times \mathrm{O}(V_{m+1})$*.*

First, we consider the case when $m = 2n$ is even. We need the following lemma for this case:

Lemma 5.8. *Let* $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n}^\bullet))$ *and* $\tau_0 \in \mathrm{Irr}(\mathrm{SO}(V_{2n+1}^\bullet))$*. For* $b \in \{\pm1\}$*, we denote by* τ^b *the extension of* τ *to* $\mathrm{O}(V_{2n+1}^\bullet)$ *such that* $\tau^b(-\mathbf{1}_{V_{2n+1}}) = b \cdot \mathrm{id}$*. Assume that*

$$\mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0 \otimes \tau_0, \mathbb{C}) \neq 0.$$

(1) *There exists a unique irreducible constituent* σ^b *of* $\mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0)$ *such that*

$$\mathrm{Hom}_{\mathrm{O}(V_{2n}^\bullet)}(\sigma^b \otimes \tau^b, \mathbb{C}) \neq 0.$$

(2) *If* $\sigma_0^\epsilon \cong \sigma_0$*, then the correspondence*

$$\{\pm1\} \ni b \mapsto \sigma^b \in \{\textit{irreducible constituents of } \mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0)\}$$

is bijective.

Proof. By the Frobenius reciprocity, we have

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0 \otimes \tau_0, \mathbb{C}) &\cong \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\tau_0, \sigma_0^\vee) \\ &\cong \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\tau^b|\mathrm{SO}(V_{2n}^\bullet), \sigma_0^\vee) \cong \mathrm{Hom}_{\mathrm{O}(V_{2n}^\bullet)}(\tau^b|\mathrm{O}(V_{2n}^\bullet), \mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0^\vee)) \\ &\cong \mathrm{Hom}_{\mathrm{O}(V_{2n}^\bullet)}(\mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0) \otimes \tau^b, \mathbb{C}) \end{aligned}$$

for any $b \in \{\pm 1\}$. Hence if $\sigma_0^\epsilon \not\cong \sigma_0$, then $\mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0)$ is irreducible, so that the first assertion is trivial.

Next, suppose that $\sigma_0^\epsilon \cong \sigma_0$. Then $\mathrm{Ind}_{\mathrm{SO}(V_{2n}^\bullet)}^{\mathrm{O}(V_{2n}^\bullet)}(\sigma_0) \cong \sigma_1 \oplus \sigma_2$, where $\sigma_1, \sigma_2 \in \mathrm{Irr}(\mathrm{O}(V_{2n}^\bullet))$ satisfy $\sigma_1 \not\cong \sigma_2$ and $\sigma_i|\mathrm{SO}(V_{2n}^\bullet) = \sigma_0$ for $i \in \{1,2\}$. Let $\epsilon \in \mathrm{O}(V_{2n}^\bullet)$ be as in §2.1. It satisfies that $\det(\epsilon) = -1$ and $\epsilon^2 = \mathbf{1}_{V_{2n}}$. Note that $\sigma_2(\epsilon) = -\sigma_1(\epsilon)$. Fix a nonzero homomorphism $f \in \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0 \otimes \tau_0, \mathbb{C})$, and put

$$f_i = f \circ (\sigma_i(\epsilon) \otimes \tau^b(\epsilon)).$$

Then $f_i \in \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0 \otimes \tau_0, \mathbb{C})$, and we have $(f_i)_i = f$ and $(f_1)_2 = -f$. Since $\dim \mathrm{Hom}_{\mathrm{SO}(V_{2n}^\bullet)}(\sigma_0 \otimes \tau_0, \mathbb{C}) = 1$ by [W4], there exists $c_i \in \{\pm 1\}$ such that $f_i = c_i \cdot f$ and $c_1 \neq c_2$. If $c_i = +1$, then $f \in \mathrm{Hom}_{\mathrm{O}(V_{2n}^\bullet)}(\sigma_i \otimes \tau^b, \mathbb{C})$. In this case, we have $\sigma^b = \sigma_i$. Also, if we replace b with $-b$, then c_i must be replaced by $-c_i$. Hence the second assertion holds. □

Proof of Theorem 5.7. First, we consider the case when $m = 2n$ is even. Let $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$ and $(\phi', b) \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n+1}))$. Theorem 5.2 and Lemma 5.8 imply that there exists a unique $(\sigma, \tau) \in \Pi_\phi \times \Pi_{\phi', b}$ such that $\sigma \boxtimes \tau$ is a representation of $\mathrm{O}(V_{2n}^\bullet) \times \mathrm{O}(V_{2n+1}^\bullet)$ with a relevant pair $(V_{2n}^\bullet, V_{2n+1}^\bullet)$ of companion spaces of (V_{2n}, V_{2n+1}), and

$$\mathrm{Hom}_{\Delta\mathrm{O}(V_{2n}^\bullet)}(\sigma \boxtimes \tau, \mathbb{C}) \neq 0.$$

Moreover, we have $\iota(\tau) = \chi_\phi$.

We show that $\iota_c(\sigma) = \chi_{\phi'} \cdot d_{\phi,b}$. If $\phi \notin \Phi^\epsilon(\mathrm{O}(V_{2n}))$, then $A_\phi = A_\phi^+$ and so that $d_{\phi,b} = \mathbf{1}$. Hence the desired equation follows from Theorem 5.2.

Now we assume that $\phi \in \Phi_{\mathrm{temp}}^\epsilon(\mathrm{O}(V_{2n}))$. Then $\sigma \not\cong \sigma \otimes \det$. Since $d_{\phi,-1}$ is the non-trivial character of A_ϕ which is trivial on A_ϕ^+, by Lemma 5.8 (2), it suffices to show that if

$$b = \varepsilon(\frac{1}{2}, \tau, \psi) \cdot e(V_{2n+1}^\bullet)$$

and $(\sigma, \tau) \in \Pi_\phi \times \Pi_{\phi', b}$ satisfies that $\mathrm{Hom}_{\Delta\mathrm{O}(V_{2n}^\bullet)}(\sigma \boxtimes \tau, \mathbb{C}) \neq 0$, then $\iota_c(\sigma) = \chi_{\phi'} \cdot d_{\phi,b}$. Here, $\varepsilon(s, \tau, \psi)$ is the standard ε-factor defined by the doubling method (see [LR, §10]) and

$$e(V_{2n+1}^\bullet) = \begin{cases} 1 & \text{if } \mathrm{O}(V_{2n+1}^\bullet) \text{ is split,} \\ -1 & \text{otherwise.} \end{cases}$$

Note that

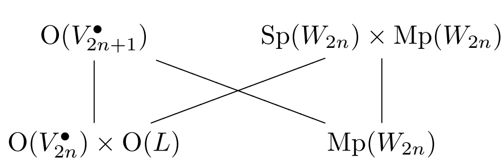
$$e(V_{2n+1}^\bullet) = \iota(\tau)(z_{\phi'}) = \chi_\phi(z_{\phi'}) = \varepsilon(\phi \otimes \phi') \cdot \chi_{V_{2n}}(-1)^n$$

by Theorem 5.2.

To obtain the desired formula, we shall use the theta correspondence as in §4.2. Let W_{2n} be a symplectic space of dimension $2n$. We consider the theta correspondence for $(\mathrm{O}(V_{2n}^\bullet), \mathrm{Sp}(W_{2n}))$ and $(\mathrm{O}(V_{2n+1}^\bullet), \mathrm{Mp}(W_{2n}))$. Here, $\mathrm{Mp}(W_{2n})$ is the metaplectic group associated to W_{2n}, i.e., the unique topological double cover of $\mathrm{Sp}(W_{2n})$. Since $\omega_\tau(-1) = b = \varepsilon(1/2, \tau, \psi) e(V_{2n+1}^\bullet)$, by [GI1, Theorem 11.1], we have

$$\Theta_{W_{2n}, V_{2n+1}^\bullet, \psi}(\tau) \neq 0,$$

so that $\rho := \theta_{W_{2n}, V_{2n+1}^\bullet, \psi}(\tau)$ is an irreducible genuine representation of $\mathrm{Mp}(W_{2n})$. Let L be the orthogonal complement of $V_{2n}^\bullet$ in $V_{2n+1}^\bullet$. Note that $\mathrm{disc}(L) = -c$. Considering the following see-saw



we see that

$$\mathrm{Hom}_{\mathrm{Mp}(W_{2n})}(\Theta_{W_{2n},V_{2n}^\bullet,\psi}(\sigma)\otimes\omega_{\psi_{-c}},\rho)\neq 0.$$

In particular, $\pi \coloneqq \Theta_{W_{2n},V_{2n}^\bullet,\psi}(\sigma)$ is nonzero. Since σ is tempered, by [GI1, Proposition C.4], π is irreducible, so that $\pi=\theta_{W_{2n},V_{2n}^\bullet,\psi}(\sigma)$. By Prasad's conjecture (Conjecture 4.4), we have $\pi\in\Pi_{\phi_\pi}$ with $\phi_\pi=(\phi\oplus\mathbf{1})\otimes\chi_{V_{2n}}$ and

$$\iota_c'(\pi)|A_\phi=\iota_c(\sigma).$$

Here, we regard A_ϕ as a subgroup of A_{ϕ_π} via the canonical injection $A_\phi\hookrightarrow A_{\phi_\pi}$. On the other hand, by the Gross–Prasad conjecture for the symplectic-metaplectic case, which has established by [At1, Theorem 1.3] (using Theorems 5.2, 4.9 and results of [GS]) and [GGP, Proposition 18.1], we have

$$\iota_c'(\pi)(a')=\varepsilon(\phi_\pi^{a'}\chi_c\otimes\phi_{\rho^\vee})\cdot\varepsilon(\phi_\pi\chi_c\otimes\phi_{\rho^\vee})^{\det(a')}\cdot\det(\phi_\pi^{a'})(-1)^n$$

for $a'\in A_{\phi_\pi}$. Here, $\phi_{\rho^\vee}$ is the L-parameter for $\rho^\vee$. It is given by $\phi_{\rho^\vee}=\phi'\otimes\chi_{-1}\chi_{V_{2n+1}}$ (see [At1, §3.6]). Recall that $\iota_c'(\pi)$ is a priori a character of $A_{\phi_\pi}^+$, but by using the isomorphism $A_{\phi_\pi}^+\cong A_{\phi_\pi}/\langle z_{\phi_\pi}\rangle$, we regard $\iota_c'(\pi)$ as a character of A_{ϕ_π} which is trivial at z_{ϕ_π}. If $a\in A_\phi\subset A_{\phi_\pi}$, then we have $\phi_\pi^a=\phi^a\otimes\chi_{V_{2n}}$, so that

$$\det(\phi_\pi^a)(-1)=\det(\phi^a)(-1)\cdot\chi_{V_{2n}}(-1)^{\dim(\phi^a)}.$$

Since $\det(a)=\dim(\phi^a)\pmod 2$ and $\chi_{V_{2n}}\chi_{V_{2n+1}}\chi_c\chi_{-1}=\mathbf{1}$, we have $\varepsilon(\phi_\pi^a\chi_c\otimes\phi_{\rho^\vee})=\varepsilon(\phi^a\otimes\phi')$ and

$$\varepsilon(\phi_\pi\chi_c\otimes\phi_{\rho^\vee})^{\det(a)}=\big(\varepsilon(\phi\otimes\phi')\cdot\varepsilon(\phi')\big)^{\dim(\phi^a)}.$$

Since $\chi_{\phi'}(a)=\varepsilon(\phi^a\otimes\phi')\cdot\det(\phi^a)(-1)^n$ and $\varepsilon(\phi')\cdot\varepsilon(\phi\otimes\phi')\cdot\chi_{V_{2n}}(-1)^n=\varepsilon(\phi')\cdot e(V_{2n+1}^\bullet)=b$, we have

$$\iota_c(\sigma)(a)=\iota_c'(\pi)(a)=\chi_{\phi'}(a)\cdot b^{\dim(\phi^a)}$$

for $a\in A_\phi$. This completes the proof of Theorem 5.7 when $m=2n$.

Next, we consider the case when $m=2n-1$ is odd. The proof is similar to that of [GGP, Theorem 19.1]. Let $\phi\in\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$ and $(\phi',b)\in\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n-1}))$. Suppose that $V_{2n-1}^\bullet\subset V_{2n}^\bullet$. Let $L^\bullet=Fe_0$ be the orthogonal complement to $V_{2n-1}^\bullet$ in $V_{2n}^\bullet$ and $e_0\in L^\bullet$ such that $\langle e_0,e_0\rangle_{V_{2n}^\bullet}=2c$. Set

$$V_{2n+1}^\bullet=V_{2n}^\bullet\oplus(-L^\bullet)=V_{2n}^\bullet\oplus Ff_0,$$

where $\langle f_0,f_0\rangle_{V_{2n+1}^\bullet}=-2c$. Put $v_0=e_0+f_0$ and $X^\bullet=Fv_0$. Then we have

$$V_{2n+1}^\bullet=X^\bullet\oplus V_{2n-1}^\bullet\oplus(X^\bullet)^*.$$

Let $P=M_PU_P$ be the parabolic subgroup of $\mathrm{O}(V_{2n+1}^\bullet)$ stabilizing the line $X^\bullet$, where $M_P\cong\mathrm{GL}(X^\bullet)\times\mathrm{O}(V_{2n-1}^\bullet)$ is the Levi subgroup of P stabilizing $(X^\bullet)^*$. Choose a unitary character χ of $F^\times\cong\mathrm{GL}(X^\bullet)$ which satisfies the condition of [GGP, Theorem 15,1], and such that the induced representation

$$\mathrm{Ind}_{P(X^\bullet)}^{\mathrm{O}(V_{2n+1}^\bullet)}(\chi\otimes\tau)$$

is irreducible for any $\tau\in\Pi_{\phi',b}$. Note that in [GGP], one consider the unnormalized induction, but in this paper, we consider the normalized induction. Then by [GGP, Theorem 15,1], we have

$$\mathrm{Hom}_{\mathrm{O}(V_{2n}^\bullet)}(\mathrm{Ind}_{P(X^\bullet)}^{\mathrm{O}(V_{2n+1}^\bullet)}(\chi\otimes\tau)\otimes\sigma,\mathbb{C})\cong\mathrm{Hom}_{\mathrm{O}(V_{2n-1}^\bullet)}(\tau\otimes\sigma,\mathbb{C}).$$

Put $\phi''=\chi\oplus\phi'\oplus\chi^{-1}$. Then

$$\Pi_{\phi'',\chi(-1)b}=\{\mathrm{Ind}_{P(X^\bullet)}^{\mathrm{O}(V_{2n+1}^\bullet)}(\chi\otimes\tau)\mid\tau\in\Pi_{\phi'}\}$$

and

$$\iota(\mathrm{Ind}_{P(X^\bullet)}^{\mathrm{O}(V_{2n+1}^\bullet)}(\chi\otimes\tau))=\iota(\tau)$$

as a character of $A_{\phi''}=A_{\phi'}$. Applying the even case above to $\phi\in\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n}))$ and $(\phi'',\chi(-1)b)\in\Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n+1}))$, we see that there exists a unique pair $(\sigma,\tau)\in\Pi_\phi\times\Pi_{\phi',b}$ such that $\sigma\boxtimes\tau$ is a representation of $\mathrm{O}(V_{2n}^\bullet)\times\mathrm{O}(V_{2n-1}^\bullet)$ with a relevant pair $(V_{2n}^\bullet,V_{2n-1}^\bullet)$ of companion spaces of (V_{2n},V_{2n-1}), and

$$\mathrm{Hom}_{\Delta\mathrm{O}(V_{2n-1}^\bullet)}(\sigma\boxtimes\tau,\mathbb{C})\neq 0.$$

Moreover, we have

$$\iota_c(\sigma)(a) = \chi_{\phi''}(a) \cdot (\chi(-1)b)^{\dim(\phi^a)},$$
$$\iota(\tau) = \iota(\mathrm{Ind}_{P(X^\bullet)}^{\mathrm{O}(V_{2n+1}^\bullet)}(\chi \otimes \tau)) = \chi_\phi.$$

By the definition, we have

$$\frac{\chi_{\phi''}(a)}{\chi_{\phi'}(a)} = \varepsilon(\phi^a \otimes (\chi \oplus \chi^{-1})) \cdot \det(\phi^a)(-1) = \det(\phi^a \otimes \chi)(-1) \cdot \det(\phi^a)(-1) = \chi(-1)^{\dim(\phi^a)}.$$

Hence we have

$$\iota_c(\sigma)(a) = \chi_{\phi'}(a) \cdot b^{\dim(\phi^a)},$$

as desired. This completes the proof of Theorem 5.7 when $m = 2n - 1$. □

By Theorem 5.7, we have established the Gross–Prasad conjecture (Conjecture 5.3) under Prasad's conjecture (Conjecture 4.4). As in [GGP], one may consider the general codimension case, and may prove this for tempered L-parameters similarly. Also, one may consider the generic case, i.e., the L-parameters ϕ and ϕ' are generic. It would follow from a similar argument to [MW].

6. Arthur's multiplicity formula for $\mathrm{SO}(V_{2n})$

The final main theorem is the so-called Arthur's multiplicity formula, which describes a spectral decomposition of the discrete automorphic spectrum for $\mathrm{O}(V_{2n})$. In this section, we recall the local and global A-parameters, and Arthur's multiplicity formula for $\mathrm{SO}(V_{2n})$. Then we will establish an analogous formula for $\mathrm{O}(V_{2n})$ in the next section.

6.1. **Notation and measures.** Let $\mathbb{F}$ be a number field, $\mathbb{A}$ be the ring of adeles of $\mathbb{F}$. We denote by $\mathbb{A}_{\mathrm{fin}} = \prod'_{v<\infty} \mathbb{F}_v$ and $\mathbb{F}_\infty = \prod_{v|\infty} \mathbb{F}_v$ the ring of finite adeles and infinite adeles, respectively. As in the precious sections, we write V_{2n} for an orthogonal space associated to (d, c) for some $c, d \in \mathbb{F}^\times$. Let $\mathrm{O}(V_{2n})$ (resp. $\mathrm{SO}(V_{2n})$) be a quasi-split orthogonal (resp. special orthogonal) group over $\mathbb{F}$. We denote by $\chi_V = \otimes_v \chi_{V,v} \colon \mathbb{A}^\times/\mathbb{F}^\times \to \mathbb{C}^\times$ the discriminant character.

For each v, we fix a maximal compact subgroup K_v of $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ such that K_v is special if v is non-archimedean. Moreover, if $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ is unramified, we choose K_v as in §2.3, which is hyperspecial. Also, we take $\epsilon_v \in K_v$ such that $\det(\epsilon_v) = -1$, $\epsilon_v^2 = \mathbf{1}_{V_{2n}}$ and that $\epsilon = (\epsilon_v)_v \in \mathrm{O}(V_{2n})(\mathbb{A})$ is in $\mathrm{O}(V_{2n})(\mathbb{F})$. Put $K_{0,v} = K_v \cap \mathrm{SO}(V_{2n})(\mathbb{F}_v)$. Note that $\epsilon_v^{-1} K_{0,v} \epsilon_v = K_{0,v}$.

Let $\mu_2 = \{\pm 1\}$ be the group of order 2. We regard μ_2 as an algebraic group over $\mathbb{F}$. There exists an exact sequence of algebraic group over $\mathbb{F}$:

$$1 \longrightarrow \mathrm{SO}(V_{2n}) \longrightarrow \mathrm{O}(V_{2n}) \xrightarrow{\det} \mu_2 \longrightarrow 1.$$

For $t = (t_v)_v \in \mu_2(\mathbb{A})$, we define $\epsilon_t = (\epsilon_{t,v})_v \in \mathrm{O}(V_{2n})(\mathbb{A})$ by

$$\epsilon_{t,v} = \begin{cases} \mathbf{1}_{V_{2n}} & \text{if } t_v = 1, \\ \epsilon_v & \text{if } t_v = -1. \end{cases}$$

We take the Haar measures dg_v, dh_v, and dt_v on $\mathrm{O}(V_{2n})(\mathbb{F}_v)$, $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$ and $\mu_2(\mathbb{F}_v)$, respectively, so that

$$\mathrm{vol}(K_v, dg_v) = \mathrm{vol}(K_{0,v}, dh_v) = \mathrm{vol}(\mu_2(\mathbb{F}_v), dt_v) = 1.$$

Then they induce the Haar measures dg, dh and dt on $\mathrm{O}(V_{2n})(\mathbb{A})$, $\mathrm{SO}(V_{2n})(\mathbb{A})$ and $\mu_2(\mathbb{A})$, respectively, satisfying that

$$\int_{\mathrm{O}(V_{2n})(\mathbb{F})\backslash \mathrm{O}(V_{2n})(\mathbb{A})} f(g)dg = \int_{\mu_2(\mathbb{F})\backslash\mu_2(\mathbb{A})} \left(\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash \mathrm{SO}(V_{2n})(\mathbb{A})} f(h\epsilon_t)dh \right) dt$$

for any smooth function f on $\mathrm{O}(V_{2n})(\mathbb{F})\backslash \mathrm{O}(V_{2n})(\mathbb{A})$.

6.2. **Local A-parameters.** In this subsection, we fix a place v of $\mathbb{F}$, and introduce local A-parameters for $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ and $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$.

Let $W_{\mathbb{F}_v}$ be the Weil group and $WD_{\mathbb{F}_v}$ be the Weil–Deligne group of $\mathbb{F}_v$, i.e.,

$$WD_{\mathbb{F}_v} = \begin{cases} W_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C}) & \text{if } v \text{ is non-archimedean,} \\ W_{\mathbb{F}_v} & \text{if } v \text{ is archimedean.} \end{cases}$$

A local A-parameter for $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$ is an admissible homomorphism

$$\underline{\psi} \colon WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L(\mathrm{SO}(V_{2n})) = \mathrm{SO}(2n, \mathbb{C}) \rtimes W_{\mathbb{F}_v}$$

such that $\underline{\psi}(W_{\mathbb{F}_v})$ projects a relatively compact subset of $\mathrm{SO}(2n, \mathbb{C})$. We put

$$\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v)) = \{\mathrm{SO}(2n, \mathbb{C})\text{-conjugacy classes of local } A\text{-parameters of } \mathrm{SO}(V_{2n})(\mathbb{F}_v)\}.$$

In §3.2, we have defined a map ${}^L(\mathrm{SO}(V_{2n})) \to \mathrm{O}(2n, \mathbb{C})$. By composing with this map, $\underline{\psi} \in \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ gives a representation

$$\psi \colon WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(2n, \mathbb{C}).$$

We may regard ψ as an orthogonal representation of $WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$. The map $\underline{\psi} \mapsto \psi$ gives an identification

$$\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v)) = \{\psi \colon WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(2n, \mathbb{C}) \mid \det(\psi) = \chi_{V,v}\}/(\mathrm{SO}(2n, \mathbb{C})\text{-conjugacy}).$$

Namely, we regard $\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ as the set of $\mathrm{SO}(M)$-conjugacy classes of orthogonal representations (ψ, M) of $WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$ with $\dim(M) = 2n$ and $\det(\psi) = \chi_{V,v}$. We say that $\psi \in \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ is tempered if $\psi|\mathrm{SL}_2(\mathbb{C}) = \mathbf{1}$, i.e., ψ is a tempered representation of $WD_{\mathbb{F}_v}$. We denote by $\Psi_{\mathrm{temp}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ the subset of $\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ consisting of the classes of tempered representation. Also we put $\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ to be the set of equivalence classes of orthogonal representations (ψ, M) of $WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$ with $\dim(M) = 2n$ and $\det(\psi) = \chi_{V,v}$. Then there exists a canonical surjection

$$\Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v)) \twoheadrightarrow \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$$

such that the order of each fiber is one or two. We also denote by $\Psi_{\mathrm{temp}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ the image of $\Psi_{\mathrm{temp}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$.

On the other hand, we put

$$\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \{\mathrm{O}(2n, \mathbb{C})\text{-conjugacy classes of local } A\text{-parameters of } \mathrm{SO}(V_{2n})(\mathbb{F}_v)\}.$$

We call an element in $\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ an A-parameter of $\mathrm{O}(V_{2n})(\mathbb{F}_v)$. Then we have a canonical identification $\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$. Under this identification, we put $\Psi_{\mathrm{temp}}(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \Psi_{\mathrm{temp}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$.

Let $\psi \in \Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ be a local A-parameter. We put

$$\mathcal{S}_\psi = \pi_0(\mathrm{Cent}(\mathrm{Im}(\psi), \mathrm{O}_{2n}(\mathbb{C}))/\{\pm \mathbf{1}_{2n}\}) \quad \text{and} \quad \mathcal{S}_\psi^+ = \pi_0(\mathrm{Cent}(\mathrm{Im}(\psi), \mathrm{SO}_{2n}(\mathbb{C}))/\{\pm \mathbf{1}_{2n}\}).$$

6.3. **Local A-packets.** We denote by $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n}))(\mathbb{F}_v)$ be the set of equivalence classes of irreducible unitary representations of $\mathrm{O}(V_{2n})(\mathbb{F}_v)$. By Theorems 2.2.1 and 2.2.4 in [Ar], there exist a finite set Π_ψ with maps

$$\Pi_\psi \to \mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$$

and

$$\iota_c \colon \Pi_\psi \to \widehat{\mathcal{S}_\psi},$$

which satisfy certain character identities. Using the multiplicity function

$$m_1 \colon \mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v)) \to \mathbb{Z}_{\geq 0}$$

which gives the order of the fibers in Π_ψ, we may regard Π_ψ as a multiset on $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$. We call Π_ψ the local A-packets for $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ associated to ψ. Note that $m_1(\sigma) = m_1(\sigma \otimes \det)$ by [Ar, Theorem 2.2.4]. If ψ is tempered, Π_ψ coincides with the L-packet described in §3.4. In particular, if ψ is tempered, then Π_ψ is multiplicity-free.

We denote by Π_ψ^0 the image of Π_ψ under the canonical map

$$\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v)) \to \mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))/\sim_{\det} \to \mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon .$$

Namely, Π_ψ^0 is a multiset on $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ with the multiplicity function

$$m_0\colon \mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon \to \mathbb{Z}_{\geq 0}$$

such that

$$m_0([\sigma]) = m_1(\sigma) = m_1(\sigma\otimes\det),$$

where $[\sigma]\in \mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ is the image of $\sigma\in\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$. We call Π_ψ^0 the local A-packets for $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$ associated to ψ. Moreover there exist a map

$$\iota_c\colon \Pi_\psi^0 \to \widehat{\mathcal{S}_\psi^+}$$

which satisfies certain character identities and such that the diagram

$$\begin{array}{ccc} \Pi_\psi & \xrightarrow{\iota_c} & \widehat{\mathcal{S}_\psi} \\ \big\downarrow & & \big\downarrow \\ \Pi_\psi^0 & \xrightarrow{\iota_c} & \widehat{\mathcal{S}_\psi^+} \end{array}$$

is commutative.

Recall that the local A-packets Π_ψ and Π_ψ^0 are multisets. However, it is expected that the A-packets are sets.

Conjecture 6.1. *Let $\psi\in\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$ be a local A-parameter, and Π_ψ and Π_ψ^0 be the local A-packets. Then for $\sigma\in\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n}))$ and $[\sigma]\in\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$, we have*

$$m_1(\sigma)\leq 1 \quad \text{and} \quad m_0([\sigma])\leq 1.$$

In other words, Π_ψ and Π_ψ^0 are subsets of $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n}))$ and $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$, respectively.

Proposition 6.2. *Conjecture 6.1 for ψ holds for the following cases:*
- *when $\psi=\phi$ is tempered A-parameter;*
- *when $\mathbb{F}_v$ is non-archimedean.*

Proof. For a tempered A-parameter $\psi=\phi$, the local A-packet Π_ψ coincides with the local L-packet Π_ϕ, which are (multiplicity-free) sets. When $\mathbb{F}_v$ is non-archimedean, Conjecture 6.1 is proven by Mœglin [M2]. See also Xu's paper [X, Theorem 8.10]. □

Remark 6.3. *By Proposition 6.2, only when $\mathbb{F}_v$ is archimedean and ψ is non-tempered, Conjecture 6.1 for ψ is not verified. However, it is known by* [AMR] *that a part of local A-packets (in the archimedean case) coincides with Adams–Johnson A-packets, which are (multiplicity-free) sets.*

Let $\psi\in\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v)) = \Psi(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$. Then ψ gives a local L-parameter ϕ_ψ defined by

$$\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|_{\mathbb{F}_v}^{\frac{1}{2}} & 0 \\ 0 & |w|_{\mathbb{F}_v}^{-\frac{1}{2}} \end{pmatrix})$$

for $w\in WD_{\mathbb{F}_v}$. Here, $|w|_{\mathbb{F}_v}$ is the extension to $WD_{\mathbb{F}_v}$ of the absolute value on $W_{\mathbb{F}_v}$, which is trivial on $\mathrm{SL}_2(\mathbb{C})$. We put

$$\mathcal{S}_{\phi_\psi} = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi_\psi), \mathrm{O}_{2n}(\mathbb{C}))/\{\pm\mathbf{1}_{2n}\}) \quad \text{and} \quad \mathcal{S}_{\phi_\psi}^+ = \pi_0(\mathrm{Cent}(\mathrm{Im}(\phi_\psi), \mathrm{SO}_{2n}(\mathbb{C}))/\{\pm\mathbf{1}_{2n}\}).$$

Then we have L-packets

$$\Pi_{\phi_\psi}\subset\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v)) \quad \text{and} \quad \Pi_{\phi_\psi}^0\subset\mathrm{Irr}_{\mathrm{unit}}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$$

and bijections

$$\iota_c\colon \Pi_{\phi_\psi}\to\widehat{\mathcal{S}_{\phi_\psi}} \quad \text{and} \quad \iota_c\colon \Pi_{\phi_\psi}^0\to\widehat{\mathcal{S}_{\phi_\psi}^+}.$$

Moreover we have a canonical surjection

$$\mathcal{S}_\psi\twoheadrightarrow\mathcal{S}_{\phi_\psi} \quad \text{and} \quad \mathcal{S}_\psi^+\twoheadrightarrow\mathcal{S}_{\phi_\psi}^+.$$

Proposition 7.4.1 in [Ar] says that $\Pi^0_{\phi_\psi}$ is contained in Π^0_ψ and the diagram

$$\begin{array}{ccc} \Pi^0_{\phi_\psi} & \longrightarrow & \Pi^0_\psi \\ {\scriptstyle \iota_c}\downarrow & & \downarrow{\scriptstyle \iota_c} \\ \widehat{\mathcal{S}^+_{\phi_\psi}} & \longrightarrow & \widehat{\mathcal{S}^+_\psi} \end{array}$$

is commutative. We prove an analogue of this statement.

Proposition 6.4. *Assume Conjecture 6.1. Let* $\psi \in \Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$. *We denote by* ϕ_ψ *the L-parameter given by* ψ. *Let* Π_ψ *be the local A-packet of* ψ, *which is a multiset of* $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$, *and* Π_{ϕ_ψ} *be the local L-packet of* ϕ_ψ, *which is a subset of* $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$. *Then* Π_{ϕ_ψ} *is contained in* Π_ψ *and the diagram*

$$\begin{array}{ccc} \Pi_{\phi_\psi} & \longrightarrow & \Pi_\psi \\ {\scriptstyle \iota_c}\downarrow & & \downarrow{\scriptstyle \iota_c} \\ \widehat{\mathcal{S}_{\phi_\psi}} & \longrightarrow & \widehat{\mathcal{S}_\psi} \end{array}$$

is commutative.

Proof. The first assertion follows from the result for $\Pi^0_{\phi_\psi}$ and Π^0_ψ. We write $\iota^L_c = \iota_c\colon \Pi_{\phi_\psi} \to \widehat{\mathcal{S}_{\phi_\psi}}$ for the left arrow in the diagram, and $\iota^A_c = \iota_c\colon \Pi_\psi \to \widehat{\mathcal{S}_\psi}$ for the right arrow in the diagram. Let $a \in \mathcal{S}_\psi$ and $s \in \mathrm{O}_{2n}(\mathbb{C})$ be a (suitable) semi-simple representative of a. We denote by ψ^a the (-1)-eigenspace of s in ψ, which is a representation of $WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$. For the last assertion, we may assume that $d = \dim(\psi^a)$ is odd. Put $\psi' = \psi \oplus \psi^a \oplus (\psi^a)^\vee$. Then we have

$$\phi_{\psi'} = \phi_\psi \oplus \phi^a_\psi \oplus (\phi^a_\psi)^\vee.$$

Take $\sigma \in \Pi_{\phi_\psi}$. Let τ be an irreducible representation of $\mathrm{GL}_d(\mathbb{F}_v)$ corresponding to ϕ^a_ψ. We denote the normalized intertwining operator of $\mathrm{Ind}_P^{\mathrm{O}(V_{2n+2d})(\mathbb{F}_v)}(\tau \boxtimes \sigma)$ by $R_c(w, \tau \boxtimes \sigma)$, where $P = MN$ be a suitable parabolic subgroup of $\mathrm{O}(V_{2n+2d})(\mathbb{F}_v)$ with the Levi factor $M \cong \mathrm{GL}_d(\mathbb{F}_v) \times \mathrm{O}(V_{2n})(\mathbb{F}_v)$, and w is a suitable representative of an element in the relative Weyl group $W(M)$ of M. Let σ' be an irreducible constituent of $\mathrm{Ind}_P^{\mathrm{O}(V_{2n+2d})(\mathbb{F}_v)}(\tau \boxtimes \sigma)$. Then $\sigma' \in \Pi_{\phi_{\psi'}} \subset \Pi_{\psi'}$. Regarding σ' as an element in $\Pi_{\phi_{\psi'}}$, by Theorems 2.2.4 and 2.4.4 in [Ar] together with Conjecture 6.1, $R_c(w, \sigma \boxtimes \tau)$ induces a scalar operator on σ' with eigenvalue $\iota^L_c(\sigma')(a)$. On the other hand, regarding σ' as an element in $\Pi_{\psi'}$, by the same theorems and conjecture, $R_c(w, \sigma \boxtimes \tau)$ induces a scalar operator on σ' with eigenvalue $\iota^A_c(\sigma')(a)$. Hence

$$\iota^L_c(\sigma')(a) = \iota^A_c(\sigma')(a).$$

Since $\iota^L_c(\sigma')(a) = \iota^L_c(\sigma)(a)$ and $\iota^A_c(\sigma')(a) = \iota^A_c(\sigma)(a)$, we have

$$\iota^L_c(\sigma)(a) = \iota^A_c(\sigma)(a),$$

as desired. □

We remark on unramified representations. Suppose that $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ is unramified, which is equivalent that v is non-archimedean and $c, d \in \mathfrak{o}_v^\times$, where $\mathfrak{o}_v$ is the ring of integers of $\mathbb{F}_v$. We say that $\psi \in \Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ is unramified if $\psi|WD_{\mathbb{F}_v}$ is trivial on $I_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$, where $I_{\mathbb{F}_v}$ is the inertia subgroup of $W_{\mathbb{F}_v}$.

Corollary 6.5. *If* $\psi \in \Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ *is unramified, then* Π_ψ *has a unique unramified representation* σ. *It satisfies* $\iota_c(\sigma) = \mathbf{1}$.

Proof. The uniqueness follows from [M1, p. 18 Proposition] and Lemma 2.4. The unique unramified representation $\sigma \in \Pi_\psi$ belongs to the subset Π_{ϕ_ψ}. By Proposition 6.4 and Desideratum 3.9 (4), we have $\iota_c(\sigma) = 1$. □

6.4. **Hypothetical Langlands group and its substitute.** We denote the set of irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ by $\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_m)$. Let $\mathcal{L}_{\mathbb{F}}$ be the hypothetical Langlands group of $\mathbb{F}$. It is expected that there exists a canonical bijection

$$\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_m) \longleftrightarrow \{m\text{-dimensional irreducible unitary representations of } \mathcal{L}_{\mathbb{F}}\}.$$

We want to use $\mathcal{L}_{\mathbb{F}}$ as a global analogue of Weil–Deligne group $WD_{\mathbb{F}_v}$. Namely, for a connected reductive group G, we want to define a global A-parameters of G by an admissible homomorphism

$$\psi\colon \mathcal{L}_{\mathbb{F}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G.$$

In this paper, we do not assume the existence of $\mathcal{L}_{\mathbb{F}}$. So we have to modify the definition of global A-parameters. For the definition of global A-parameters, we use elements in $\mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_m)$ instead of m-dimensional irreducible unitary representations of $\mathcal{L}_{\mathbb{F}}$. For this reason, we will define not $\Psi_2(\mathrm{SO}(V_{2n}))$ but only $\Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ as well as $\Psi_2(\mathrm{O}(V_{2n}))$ in the next subsection.

6.5. **Global A-parameters and localization.** Let V_{2n} be an orthogonal space associated to (d,c) for some $c,d \in \mathbb{F}^\times$. We denote by $\chi_V : \mathbb{A}^\times/\mathbb{F}^\times \to \mathbb{C}^\times$ the discriminant character of V_{2n}. A discrete global A-parameter for $\mathrm{SO}(V_{2n})$ and $\mathrm{O}(V_{2n})$ is a symbol

$$\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l],$$

where

- $1 \leq l \leq 2n$ is an integer;
- $\Sigma_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{m_i})$;
- d_i is a positive integer such that $\sum_{i=1}^l m_i d_i = 2n$;
- if d_i is odd, then $L(s, \Sigma_i, \mathrm{Sym}^2)$ has a pole at $s=1$;
- if d_i is even, then $L(s, \Sigma_i, \wedge^2)$ has a pole at $s=1$;
- if we denote the central character of Σ_i by ω_i, then $\omega_1^{d_1} \cdots \omega_l^{d_l} = \chi_V$;
- if $i \neq j$ and $\Sigma_i \cong \Sigma_j$, then $d_i \neq d_j$.

Two global A-parameters $\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l]$ and $\Sigma' = \Sigma'_1[d'_1] \boxplus \cdots \boxplus \Sigma'_{l'}[d'_{l'}]$ are said equivalent if $l = l'$ and there exists a permutation $\sigma \in \mathfrak{S}_l$ such that $d'_i = d_{\sigma(i)}$ and $\Sigma'_i \cong \Sigma_{\sigma(i)}$ for each i. We denote by $\Psi_2(\mathrm{O}(V_{2n})) = \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ the set of equivalence classes of discrete global A-parameters for $\mathrm{SO}(V_{2n})$ and $\mathrm{O}(V_{2n})$. Let $\Psi_2^\epsilon(\mathrm{O}(V_{2n}))$ be the subset of $\Psi_2(\mathrm{O}(V_{2n}))$ consisting of $\Sigma = \boxplus_{i=1}^l \Sigma_i[d_i]$ as above such that $m_i d_i$ is odd for some i. Also, we put

$$\Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n})) = \{\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l] \in \Psi_2(\mathrm{O}(V_{2n})) \mid d_i = 1 \text{ for any } i\},$$

and $\Psi_{2,\mathrm{temp}}^\epsilon(\mathrm{O}(V_{2n})) = \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n})) \cap \Psi_2^\epsilon(\mathrm{O}(V_{2n}))$. We define $\Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ and $\Psi_{2,\mathrm{temp}}(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ similarly.

Let $\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l] \in \Psi_2(\mathrm{O}(V_{2n}))$. For each place of v of $\mathbb{F}$, we denote the m_i-dimensional representation of $WD_{\mathbb{F}_v}$ corresponding to $\Sigma_{i,v} \in \mathrm{Irr}(\mathrm{GL}_{m_i}(\mathbb{F}_v))$ by $\phi_{i,v}$. Because of the lack of the generalized Ramanujan conjecture, $\phi_{i,v}$ is not necessarily a tempered representation. We define a representation $\Sigma_v\colon WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$ by

$$\Sigma_v = (\phi_{1,v} \boxtimes S_{d_1}) \oplus \cdots \oplus (\phi_{l,v} \boxtimes S_{d_l}),$$

where S_d is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d. We call Σ_v the localization of Σ at v. By [Ar, Proposition 1.4.2], the representation Σ_v factors through $\mathrm{O}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$. In particular, if $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$, then $\Sigma_v \in \Phi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$.

Let $\Sigma = \Sigma_1[d_1] \boxplus \cdots \boxplus \Sigma_l[d_l] \in \Psi_2(\mathrm{O}(V_{2n}))$ be a global A-parameter with $\Sigma_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{m_i})$, and $\Sigma_v = (\phi_{1,v} \boxtimes S_{d_1}) \oplus \cdots \oplus (\phi_{l,v} \boxtimes S_{d_l})$ be the localization at v. We also write $\Sigma_{i,v} = \phi_{i,v} \otimes S_{d_i}$. We put

$$A_\Sigma = \bigoplus_{i=1}^l (\mathbb{Z}/2\mathbb{Z}) a_{\Sigma_i[d_i]} \cong (\mathbb{Z}/2\mathbb{Z})^l.$$

Namely, A_Σ is a free $(\mathbb{Z}/2\mathbb{Z})$-module of rank l and $\{a_{\Sigma_1[d_1]}, \dots, a_{\Sigma_l[d_l]}\}$ is a basis of A_Σ with $a_{\Sigma_i[d_i]}$ associated to $\Sigma_i[d_i]$. We define A_Σ^+ by the kernel of the map $A_\Sigma \ni a_{\Sigma_i[d_i]} \mapsto (-1)^{m_i d_i} \in \{\pm 1\}$. Also, we put

$$z_\Sigma = \sum_{i=1}^{l} m_i d_i \cdot a_{\Sigma_i[d_i]} \in A_\Sigma^+ \subset A_\Sigma.$$

We define the global component groups $\mathcal{S}_\Sigma$ and $\mathcal{S}_\Sigma^+$ by

$$\mathcal{S}_\Sigma = A_\Sigma/\langle z_\phi \rangle \quad \text{and} \quad \mathcal{S}_\Sigma^+ = A_\Sigma^+/\langle z_\phi \rangle.$$

On the other hand, as in §6.2, we put

$$\mathcal{S}_{\Sigma_v} = \pi_0(\mathrm{Cent}(\mathrm{Im}(\Sigma_v), \mathrm{O}_{2n}(\mathbb{C}))/\{\pm \mathbf{1}_{2n}\}) \quad \text{and} \quad \mathcal{S}_{\Sigma_v}^+ = \pi_0(\mathrm{Cent}(\mathrm{Im}(\Sigma_v), \mathrm{SO}_{2n}(\mathbb{C}))/\{\pm \mathbf{1}_{2n}\}).$$

Then we have a map

$$\mathcal{S}_\Sigma \to \mathcal{S}_{\Sigma_v}, \ a_{\Sigma_i[d_i]} \mapsto -\mathbf{1}_{\Sigma_{i,v}}$$

for each place v, where $-\mathbf{1}_{\Sigma_{i,v}}$ is the image of the element in $\mathrm{Cent}(\mathrm{Im}(\Sigma_v), \mathrm{O}_{2n}(\mathbb{C}))$ which acts on $\Sigma_{i,v}$ by $-\mathrm{id}$ and acts on $\Sigma_{j,v}$ trivially for any $j \neq i$. Hence we obtain the diagonal maps

$$\Delta\colon \mathcal{S}_\Sigma \to \prod_v \mathcal{S}_{\Sigma_v} \quad \text{and} \quad \Delta\colon \mathcal{S}_\Sigma^+ \to \prod_v \mathcal{S}_{\Sigma_v}^+.$$

Let $\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))$ be a global A-parameter and $\psi = \Sigma_v$ be the localization at v. We emphasize that ψ does not necessarily belong to $\Psi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ defined in §6.2. We can decompose

$$\psi = \psi_1 |\cdot|_{\mathbb{F}_v}^{s_1} \oplus \cdots \oplus \psi_r |\cdot|_{\mathbb{F}_v}^{s_r} \oplus \psi_0 \oplus \psi_r^\vee |\cdot|_{\mathbb{F}_v}^{-s_r} \oplus \cdots \oplus \psi_1^\vee |\cdot|_{\mathbb{F}_v}^{-s_1},$$

where

- ψ_i is an irreducible representation of $WD_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$ of dimension d_i such that $\psi_i(W_{\mathbb{F}_v})$ is bounded;
- $\psi_0 \in \Psi(\mathrm{O}(V_{2n_0})(\mathbb{F}_v))$;
- $d_1 + \cdots + d_r + n_0 = n$ and $s_1 \geq \cdots \geq s_r > 0$.

We define a representation ϕ_{ψ_i} of $WD_{\mathbb{F}_v}$ by

$$\phi_{\psi_i}(w) = \psi_i(w, \begin{pmatrix} |w|_{\mathbb{F}_v}^{\frac{1}{2}} & 0 \\ 0 & |w|_{\mathbb{F}_v}^{-\frac{1}{2}} \end{pmatrix}),$$

and we denote by τ_{ψ_i} the irreducible representation of $\mathrm{GL}_{d_i}(\mathbb{F}_v)$ corresponding to ϕ_{ψ_i}. Let Π_{ψ_0} be the local A-packet associated to ψ_0, which is a multiset of $\mathrm{Irr}_{\mathrm{unit}}(\mathrm{O}(V_{2n_0})(\mathbb{F}_v))$. For $\sigma_0 \in \Pi_{\psi_0}$, we put

$$I(\sigma_0) = \mathrm{Ind}_{P(\mathbb{F}_v)}^{\mathrm{O}(V_{2n})(\mathbb{F}_v)}(\tau_{\psi_1}|\cdot|_{\mathbb{F}_v}^{s_1} \otimes \cdots \otimes \tau_{\psi_r}|\cdot|_{\mathbb{F}_v}^{s_r} \otimes \sigma_0),$$

where P is a parabolic subgroup of $\mathrm{O}(V_{2n})$ with Levi subgroup $M_P = \mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_r} \times \mathrm{O}(V_{2n_0})$. Note that $I(\sigma_0)$ depends on not only σ_0 but also ψ.

To establish the global main result (Theorem 7.1 below), we need the following conjecture (see [Ar, Conjecture 8.3.1]).

Conjecture 6.6. *Let $\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))$ be a global A-parameter. We decompose Σ_v as above, so that $\psi_0 \in \Psi(\mathrm{O}(V_{2n_0})(\mathbb{F}_v))$. Then the induced representation $I(\sigma_0)$ is irreducible for $\sigma_0 \in \Pi_{\psi_0}$. Moreover, if $I(\sigma_0) \cong I(\sigma_0')$ for $\sigma_0, \sigma_0' \in \Pi_{\psi_0}$, then $\sigma_0 \cong \sigma_0'$.*

For tempered A-parameters, the irreducibility and the disjointness follows from a result of Heiermann [H].

Proposition 6.7. *Conjecture 6.6 holds for the following cases:*

- *when $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$;*
- *when $\mathbb{F}_v$ is non-archimedean.*

Proof. The second case is [M3, Proposition 5.1]. So we shall prove the first case.

Let Σ be an element in $\Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$, $\phi = \Sigma_v$ be the localization of Σ at v, and $\phi_0 = \psi_0$ be as above. Note that $\phi \in \Phi(\mathrm{O}(V_{2n})(\mathbb{F}_v))$. Then the L-packet Π_ϕ is equal to the set of the Langlands quotients of the standard module $I(\sigma_0)$, where σ_0 runs over Π_{ϕ_0}. Heiermann's result [H] (together with the Clifford theory) asserts that if $\phi \in \Phi_{\mathrm{gen}}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$, then $I(\sigma_0)$ is irreducible for any $\sigma_0 \in \Pi_{\phi_0}$. Namely, to show the irreducibility of $I(\sigma_0)$, it suffices to prove that $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$.

We decompose

$$\phi = \phi_1|\cdot|_{\mathbb{F}_v}^{s_1} \oplus \cdots \oplus \phi_r|\cdot|_{\mathbb{F}_v}^{s_r} \oplus \phi_0 \oplus \phi_r^\vee|\cdot|_{\mathbb{F}_v}^{-s_r} \oplus \cdots \oplus \phi_1^\vee|\cdot|_{\mathbb{F}_v}^{-s_1},$$

where

- ϕ_i is an irreducible tempered representation of $WD_{\mathbb{F}_v}$ of dimension d_i;
- $\phi_0 \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{2n_0})(\mathbb{F}_v))$;
- $d_1 + \cdots + d_r + n_0 = n$ and $s_1 \geq \cdots \geq s_r > 0$.

Then $L(s, \phi, \mathrm{Ad})$ is equal to

$$L(s, \phi_0, \mathrm{Ad}) \left(\prod_{i=1}^r L(s, \phi_i, \mathrm{Ad}_{\mathrm{GL}}) L(s + s_i, \phi_0 \otimes \phi_i) L(s - s_i, \phi_0 \otimes \phi_i^\vee) L(s + 2s_i, \phi_i, \wedge^2) L(s - 2s_i, \phi_i^\vee, \wedge^2) \right)$$
$$\times \left(\prod_{1 \leq i < j \leq r} L(s + s_i + s_j, \phi_i \otimes \phi_j) L(s + s_i - s_j, \phi_i \otimes \phi_j^\vee) L(s - s_i + s_j, \phi_i^\vee \otimes \phi_j) L(s - s_i - s_j, \phi_i^\vee \otimes \phi_j^\vee) \right),$$

where $\mathrm{Ad}_{\mathrm{GL}}$ is the adjoint representation of $\mathrm{GL}_m(\mathbb{C})$ on $\mathrm{Lie}(\mathrm{GL}_m(\mathbb{C}))$ for a suitable m.

Since ϕ corresponds to an irreducible representation of $\mathrm{GL}_{2n}(\mathbb{F}_v)$, which is a local constituent of $\Sigma = \boxplus_{i=1}^l \Sigma_i$, we have $|s_i| < 1/2$ for any i. See e.g., [JS, (2.5) Corollary] and [RS, Appendix]. Note that for tempered representations, all L-functions appeared in the above equation are regular for $\mathrm{Re}(s) > 0$. Hence we conclude that $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$.

Since $I(\sigma_0)$ is a standard module, the last assertion of Conjecture 6.6 follows from the Langlands classification. □

Go back to the general situation. Let $\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))$ be a global A-parameter, and $\psi = \Sigma_v$, ψ_0 and $I(\sigma_0)$ for $\sigma_0 \in \Pi_{\psi_0}$ be as above. We define the local A-packet Π_ψ associated to $\psi = \Sigma_v$, which is a multiset on $\mathrm{Irr}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$, by

$$\Pi_\psi = \bigsqcup_{\sigma_0 \in \Pi_{\psi_0}} \{\sigma \mid \sigma \text{ is an irreducible constituent of } I(\sigma_0)\}.$$

Namely, Π_ψ is the disjoint union of the multisets of the Jordan–Hölder series of $I(\sigma_0)$. Similarly, we can define the local A-packet Π_ψ^0 associated to $\psi = \Sigma_v$, which is a multiset on $\mathrm{Irr}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))/\sim_\epsilon$. Since $\mathcal{S}_\psi = \mathcal{S}_{\psi_0}$ and $\mathcal{S}_\psi^+ = \mathcal{S}_{\psi_0}^+$, we can define maps

$$\iota_c \colon \Pi_\psi \to \widehat{\mathcal{S}_\psi} \quad \text{and} \quad \iota_c \colon \Pi_\psi^0 \to \widehat{\mathcal{S}_\psi}^+$$

by

$$\iota_c(\sigma) \coloneqq \iota_c(\sigma_0) \quad \text{and} \quad \iota_c([\sigma]) \coloneqq \iota_c([\sigma_0])$$

if σ is an irreducible constituent of $I(\sigma_0)$.

6.6. **Global A-packets.** Let $\mathcal{H}(\mathrm{O}(V_{2n})) = \otimes_v' \mathcal{H}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ (resp. $\mathcal{H}(\mathrm{SO}(V_{2n})) = \otimes_v' \mathcal{H}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$) be the global Hecke algebra on $\mathrm{O}(V_{2n})(\mathbb{A})$ (resp. $\mathrm{SO}(V_{2n})(\mathbb{A})$) with respect to the maximal compact subgroup $K = \prod_v K_v$ (resp. $K_0 = \prod_v K_{0,v}$) fixed in §6.1. Namely, $\mathcal{H}(\mathrm{O}(V_{2n})(\mathbb{F}_v))$ (resp. $\mathcal{H}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$) is the algebra of smooth, left and right K_v-finite (resp. $K_{0,v}$-finite) functions of compact support on $\mathrm{O}(V_{2n})(\mathbb{F}_v)$ (resp. $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$). We denote by $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ the subspace of functions in $\mathcal{H}(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$ which are invariant under ϵ_v. We put $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n})) = \otimes_v' \mathcal{H}^\epsilon(\mathrm{SO}(V_{2n})(\mathbb{F}_v))$. We say that two admissible representations of $\mathrm{SO}(V_{2n})(\mathbb{A})$ of the form

$$\sigma_0 = \otimes_v' \sigma_{0,v} \quad \text{and} \quad \sigma_0' = \otimes_v' \sigma_{0,v}'$$

are ϵ-equivalent if $\sigma_{0,v} \sim_{\epsilon_v} \sigma'_{0,v}$ for each v. The ϵ-equivalence class of σ_0 is denoted by $[\sigma_0] = \otimes'_v [\sigma_{0,v}]$. For $f \in \mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$ and any σ_0 as above, the operator $\sigma_0(f)$ depends only on the class $[\sigma_0]$.

Let $\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))$ be a global A-parameter. We attach a global A-packets

$$\Pi_\Sigma = \{\sigma = \otimes'_v \sigma_v \mid \sigma_v \in \Pi_{\Sigma_v},\ \iota_c(\sigma_v) = \mathbf{1} \text{ for almost all } v\}$$

of equivalence classes of irreducible representations of $\mathrm{O}(V_{2n})(\mathbb{A})$, and a global A-packets

$$\Pi^0_\Sigma = \{[\sigma_0] = [\otimes'_v \sigma_{0,v}] \mid [\sigma_{0,v}] \in \Pi^0_{\Sigma_v},\ \iota_c([\sigma_{0,v}]) = \mathbf{1} \text{ for almost all } v\}$$

of ϵ-equivalence classes of irreducible representations of $\mathrm{SO}(V_{2n})(\mathbb{A})$. Note that an element $\sigma \in \Pi_\Sigma$ and a representative σ_0 of $[\sigma_0] \in \Pi^0_\Sigma$ are not necessarily unitary. For $\sigma \in \Pi_\Sigma$ and $[\sigma_0] \in \Pi^0_\Sigma$, the operators $\sigma(f)$ and $\sigma_0(f_0)$ are well-defined for $f \in \mathcal{H}(\mathrm{O}(V_{2n}))$ and $f_0 \in \mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$, respectively.

For $\sigma = \otimes'_v \sigma_v \in \Pi_\Sigma$ and $[\sigma_0] = [\otimes'_v \sigma_{0,v}] \in \Pi^0_\Sigma$, we define characters $\iota_c(\sigma)$ of $\prod_v \mathcal{S}_{\Sigma_v}$ and $\iota_c([\sigma_0])$ of $\prod_v \mathcal{S}^+_{\Sigma_v}$ by

$$\iota_c(\sigma) = \prod_v \iota_c(\sigma_v) \quad \text{and} \quad \iota_c([\sigma_0]) = \prod_v \iota_c([\sigma_{0,v}]),$$

respectively.

6.7. **Arthur's multiplicity formula for** $\mathrm{SO}(V_{2n})$**.** We say that a function $\varphi \colon \mathrm{O}(V_{2n})(\mathbb{A}) \to \mathbb{C}$ is an automorphic form on $\mathrm{O}(V_{2n})(\mathbb{A})$ if φ satisfies the following conditions:

- φ is left $\mathrm{O}(V_{2n})(\mathbb{F})$-invariant;
- φ is smooth and moderate growth;
- φ is right K-finite, where $K = \prod_v K_v$ is the maximal compact subgroup of $\mathrm{O}(V_{2n})(\mathbb{A})$ fixed in §6.1;
- φ is $\mathfrak{z}$-finite, where $\mathfrak{z}$ is the center of the universal enveloping algebra of $\mathrm{Lie}(\mathrm{O}(V_{2n})(\mathbb{F}_\infty)) \otimes_\mathbb{R} \mathbb{C}$.

We define automorphic forms on $\mathrm{SO}(V_{2n})(\mathbb{A})$ similarly. More precisely, see [BJ, §4.2]. Let $\mathcal{A}(\mathrm{O}(V_{2n}))$ be the space of automorphic forms on $\mathrm{O}(V_{2n})(\mathbb{A})$. We denote by $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ the subspace of $\mathcal{A}(\mathrm{O}(V_{2n}))$ consisting of square-integrable automorphic forms on $\mathrm{O}(V_{2n})(\mathbb{A})$. Similarly, we define $\mathcal{A}(\mathrm{SO}(V_{2n}))$ and $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. We call $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ (resp. $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$) the automorphic discrete spectrum of $\mathrm{O}(V_{2n})$ (resp. $\mathrm{SO}(V_{2n})$).

The Hecke algebra $\mathcal{H}(\mathrm{O}(V_{2n}))$ (resp. $\mathcal{H}(\mathrm{SO}(V_{2n}))$) acts on $\mathcal{A}(\mathrm{O}(V_{2n}))$ (resp. $\mathcal{A}(\mathrm{SO}(V_{2n}))$) by

$$(f \cdot \varphi)(g) = \int_{\mathrm{O}(V_{2n})(\mathbb{A})} \varphi(gx) f(x) dx$$

$$(\text{resp.} \quad (f_0 \cdot \varphi_0)(g_0) = \int_{\mathrm{SO}(V_{2n})(\mathbb{A})} \varphi_0(g_0 x_0) f_0(x_0) dx_0)$$

for $f \in \mathcal{H}(\mathrm{O}(V_{2n}))$ and $\varphi \in \mathcal{A}(\mathrm{O}(V_{2n}))$ (resp. $f_0 \in \mathcal{H}(\mathrm{SO}(V_{2n}))$ and $\varphi_0 \in \mathcal{A}(\mathrm{SO}(V_{2n}))$). This action preserves $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ (resp. $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$).

Arthur's multiplicity formula for $\mathrm{SO}(V_{2n})$ is formulated as follows:

Theorem 6.8 (Arthur's multiplicity formula ([Ar, Theorem 1.5.2]))**.** *Let V_{2n} be be the orthogonal space over a number field $\mathbb{F}$ associated to (d, c) for some $c, d \in \mathbb{F}^\times$. Then for each $\Sigma \in \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$, there exists a character*

$$\varepsilon_\Sigma \colon \mathcal{S}_\Sigma \to \{\pm 1\}$$

defined explicitly in terms of symplectic ε-factors such that

$$\mathcal{A}_2(\mathrm{SO}(V_{2n})) \cong \bigoplus_{\Sigma \in \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon} m_\Sigma \bigoplus_{[\sigma_0] \in \Pi^0_\Sigma(\varepsilon_\Sigma)} [\sigma_0]$$

as $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-modules. Here for $\Sigma = \boxplus_{i=1}^l \Sigma_i[d_i] \in \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ with $\Sigma_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{m_i})$, we put

$$m_\Sigma = \begin{cases} 1 & \text{if } \Sigma \in \Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon, \text{i.e., } m_i d_i \text{ is odd for some } i, \\ 2 & \text{otherwise,} \end{cases}$$

and we put

$$\Pi^0_\Sigma(\varepsilon_\Sigma) = \{[\sigma_0] \in \Pi^0_\Sigma \mid \iota_c([\sigma_0]) \circ \Delta = \varepsilon_\Sigma | \mathcal{S}^+_\Sigma\}.$$

Moreover, if $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{SO}(V_{2n}))$, then ε_Σ is the trivial representation of $\mathcal{S}_\Sigma$.

For the definition of ε_Σ, see the remark after [Ar, Theorem 1.5.2].

Remark 6.9. *In fact, Arthur described a spectral decomposition of* $L^2_{\mathrm{disc}}(\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A}))$. *However it is well understood (by Harish-Chadra, Langlands etc) that* $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ *is a dense subspace of* $L^2_{\mathrm{disc}}(\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A}))$. *So we shall work with* $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ *in this paper.*

Remark 6.10. *Taïbi* [Ta] *prove the multiplicity formula for* $\mathrm{SO}(V_{2n})$ *when* $\mathrm{SO}(V_{2n})(\mathbb{F}_\infty)$ *is compact and* $\mathrm{SO}(V_{2n})(\mathbb{F}_v)$ *is quasi-split at all finite places* v *of* $\mathbb{F}$.

6.8. **Arthur's multiplicity formula for** $\mathrm{SO}(V_{2n})(\mathbb{A})\cdot\mathrm{O}(V_{2n})(\mathbb{F})$**.** Theorem 6.8 follows from a more precise result. In this subsection, we recall this result.

Let $\epsilon\in\mathrm{O}(V_{2n})(\mathbb{F})$ be as in §6.1. We may consider $\mathcal{H}(\mathrm{SO}(V_{2n}))\rtimes\langle\epsilon\rangle$, where ϵ acts on $\mathcal{H}(\mathrm{SO}(V_{2n}))$ by

$$(\epsilon f\epsilon^{-1})(x) := f(\epsilon^{-1}x\epsilon)$$

for $f\in\mathcal{H}(\mathrm{SO}(V_{2n}))$. We say that $(\sigma,\mathcal{V}_\sigma)$ is an $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-module if

- $(\sigma,\mathcal{V}_\sigma)$ is an $\mathcal{H}(\mathrm{SO}(V_{2n}))$-module;
- there is an automorphism $\sigma(\epsilon)$ on $\mathcal{V}_\sigma$ such that $\sigma(\epsilon)^2=\mathbf{1}_{\mathcal{V}_\sigma}$;
- $\sigma(\epsilon f\epsilon^{-1})\circ\sigma(\epsilon)=\sigma(\epsilon)\circ\sigma(f)$ for any $f\in\mathcal{H}(\mathrm{SO}(V_{2n}))$.

We define an action of ϵ on $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ by

$$(\epsilon\cdot\varphi)(h_0)=\varphi(\epsilon^{-1}h_0\epsilon).$$

It makes $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ an $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-module.

For an $\mathcal{H}(\mathrm{SO}(V_{2n}))$-module $(\sigma,\mathcal{V}_\sigma)$, we define an $\mathcal{H}(\mathrm{SO}(V_{2n}))$-module $(\sigma^\epsilon,\mathcal{V}_{\sigma^\epsilon})$ by $\mathcal{V}_\sigma=\mathcal{V}_{\sigma^\epsilon}$ and

$$\sigma^\epsilon(f)v=\sigma(\epsilon^{-1}f\epsilon)v$$

for $f\in\mathcal{H}(\mathrm{SO}(V_{2n}))$ and $v\in\mathcal{V}_\sigma$.

Lemma 6.11. *If* $\mathcal{V}_0\subset\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ *is an irreducible* $\mathcal{H}(\mathrm{SO}(V_{2n}))$*-summand isomorphic to* σ*, then the subspace*

$$\mathcal{V}_0^\epsilon=\{\epsilon\cdot\varphi\mid\varphi\in\mathcal{V}_0\}$$

is an irreducible $\mathcal{H}(\mathrm{SO}(V_{2n}))$*-summand isomorphic to* σ^ϵ*.*

Proof. Obvious. □

Now, we recall a result of Arthur [Ar] which states a decomposition of $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ as an $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-module by using global A-parameters. Let $\Sigma=\boxplus_{i=1}^l\Sigma_i[d_i]\in\Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ and $[\sigma_0]\in\Pi_\Sigma^0$. We take a representative σ_0 which occurs in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$, and we denote by $\mathcal{V}_0$ a subspace of $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ which realizes σ_0. We distinguish 3 cases as follows:

(A) Suppose that $\Sigma\in\Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$. In this case, we have $m_\Sigma=1$. Hence $\mathcal{V}_0$ is stable under the action of ϵ, and so that $\sigma_0^\epsilon\cong\sigma_0$. The space $\mathcal{V}_0$ realizes a distinguished extension of σ_0 to an $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-module.

(B) Suppose that $\Sigma\notin\Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ and $\sigma_0^\epsilon\not\cong\sigma_0$. In this case, $m_\Sigma=2$ and $\mathcal{V}_0^\epsilon\not\cong\mathcal{V}_0$. This shows that both σ_0 and σ_0^ϵ occur in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. This explains why $m_\Sigma=2$.

(C) Suppose that $\Sigma\notin\Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ and $\sigma_0^\epsilon\cong\sigma_0$. Then there are exactly two extensions σ_1 and σ_2 of σ_0 to $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-modules. Moreover, [Ar, Theorem 4.2.2 (a)] implies that both σ_1 and σ_2 occur in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. This explains why $m_\Sigma=2$.

In the case (A), there are exactly two extensions σ_1 and σ_2 of σ_0 to $(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)$-modules. The above argument shows that exactly one of σ_1 or σ_2 occurs in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. The following theorem determines which extension occurs.

Theorem 6.12 ([Ar, Theorem 4.2.2])**.** *Let* $\Sigma\in\Psi_2^\epsilon(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ *and* $[\sigma_0]\in\Pi_\Sigma^0$. *Assume that an* $\mathcal{H}(\mathrm{SO}(V_{2n}))$*-module* $\sigma_0=\otimes'_v\sigma_{0,v}$ *occurs in* $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$*, so that* $\iota_c([\sigma_0])\circ\Delta=\varepsilon_\Sigma|\mathcal{S}_\Sigma^+$ *by Arthur's multiplicity formula. For each place* v *of* $\mathbb{F}$*, take an extension* σ_v *of* $\sigma_{0,v}$ *to an* $(\mathcal{H}(\mathrm{SO}(V_{2n})(\mathbb{F}_v)),\epsilon_v)$*-module such that* $\iota_c(\sigma_v)=\mathbf{1}$ *for almost all* v*. Put* $\sigma=\otimes'_v\sigma_v$*. Let* $\iota_c(\sigma)$ *be the character of* $\prod_v\mathcal{S}_{\Sigma_v}$ *defined by*

$$\iota_c(\sigma)=\prod_v\iota_c(\sigma_v).$$

Then as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$*-module,* σ *occurs in* $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ *if and only if* $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$*.*

7. Arthur's multiplicity formula for $\mathrm{O}(V_{2n})$

In this section, we prove Arthur's multiplicity formula for $\mathrm{O}(V_{2n})$, which is the third main theorem in this paper.

7.1. **Statements.** The global main theorem is Arthur's multiplicity formula for $\mathrm{O}(V_{2n})$, which is formulated as follows:

Theorem 7.1 (Arthur's multiplicity formula for $\mathrm{O}(V_{2n})$)**.** *Assume Conjectures 6.1 and 6.6. Let* $\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))$ *and*

$$\varepsilon_\Sigma \colon \mathcal{S}_\Sigma \to \{\pm 1\}$$

be the character of $\mathcal{S}_\Sigma$ *as in Theorem 6.8, which is trivial if* $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$*. Then we have a decomposition*

$$\mathcal{A}_2(\mathrm{O}(V_{2n})) \cong \bigoplus_{\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))} \bigoplus_{\sigma \in \Pi_\Sigma(\varepsilon_\Sigma)} \sigma$$

as $\mathcal{H}(\mathrm{O}(V_{2n}))$*-modules. Here we put*

$$\Pi_\Sigma(\varepsilon_\Sigma) = \{\sigma \in \Pi_\Sigma \mid \iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma\}.$$

Also, we will show the following:

Proposition 7.2. *Assume Conjectures 6.1 and 6.6. Then for an irreducible* $\mathcal{H}(\mathrm{O}(V_{2n}))$*-module* σ*, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \le 1.$$

In other words, $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ *is multiplicity-free as an* $\mathcal{H}(\mathrm{O}(V_{2n}))$*-module.*

Remark 7.3. *The arguments in the proofs of Theorem 7.1 and Proposition 7.2 work when we restrict to* $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$*. In this case, Proposition 6.7, which shows Conjecture 6.6 for* $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$*, also implies that the local A-packet* Π_{Σ_v} *associated to the localization* Σ_v *becomes a local L-packet, which is multiplicity-free, i.e., which satisfies Conjecture 6.1. Hence without assuming any conjectures, Theorem 7.1 and Proposition 7.2 hold for* $\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))$*. In particular, the tempered part of the automorphic discrete spectrum of* $\mathrm{O}(V_{2n})$

$$\mathcal{A}_{2,\mathrm{temp}}(\mathrm{O}(V_{2n})) \coloneqq \bigoplus_{\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{O}(V_{2n}))} \bigoplus_{\sigma \in \Pi_\Sigma(\varepsilon_\Sigma)} \sigma$$

is multiplicity-free as an $\mathcal{H}(\mathrm{O}(V_{2n}))$*-module unconditionally.*

The rest of this section is devoted to the proofs of Theorem 7.1 and Proposition 7.2.

7.2. **Restriction of automorphic forms.** Now we compare $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ with $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. To do this, we consider the restriction map

$$\mathrm{Res} \colon \mathcal{A}(\mathrm{O}(V_{2n})) \to \mathcal{A}(\mathrm{SO}(V_{2n})),\ \varphi \mapsto \varphi|\mathrm{SO}(V_{2n})(\mathbb{A}).$$

Lemma 7.4. *We have* $\mathrm{Res}(\mathcal{A}_2(\mathrm{O}(V_{2n}))) \subset \mathcal{A}_2(\mathrm{SO}(V_{2n}))$*.*

Proof. Let $\varphi \in \mathcal{A}_2(\mathrm{O}(V_{2n}))$. Then

$$\int_{\mathrm{O}(V_{2n})(\mathbb{F})\backslash \mathrm{O}(V_{2n})(\mathbb{A})} |\varphi(g)|^2 dg = \int_{\mu_2(\mathbb{F})\backslash \mu_2(\mathbb{A})} \left(\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash \mathrm{SO}(V_{2n})(\mathbb{A})} |\varphi(h\epsilon_t)|^2 dh \right) dt$$

is finite. By Fubini's theorem, we see that for almost everywhere $t \in \mu_2(\mathbb{F})\backslash \mu_2(\mathbb{A})$, the function $h \mapsto |\varphi(h\epsilon_t)|^2$ is integrable on $\mathrm{SO}(V_{2n})(\mathbb{F})\backslash \mathrm{SO}(V_{2n})(\mathbb{A})$.

Since $\varphi(g)$ is right K-finite, there exists a finite set S of finite places of $\mathbb{F}$ containing all infinite places of $\mathbb{F}$ such that φ is right ϵ_t-invariant for any $t \in \prod_{v \notin S} \mu_2(\mathbb{F}_v)$. Since $\mu_2(\mathbb{F}_\infty)$ is finite, we see that $\prod_{v \notin S} \mu_2(\mathbb{F}_v)$ is open in $\mu_2(\mathbb{A})$ (not only in $\mu_2(\mathbb{A}_{\mathrm{fin}})$). This implies that

$$\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash \mathrm{SO}(V_{2n})(\mathbb{A})} |\varphi(h\epsilon_t)|^2 dh < \infty$$

for some (hence any) $t \in \prod_{v \notin S} \mu_2(\mathbb{F}_v)$. In particular, $\mathrm{Res}(\varphi)$ is square-integrable. □

Proposition 7.5. *We have* $\mathrm{Res}(\mathcal{A}_2(\mathrm{O}(V_{2n}))) = \mathcal{A}_2(\mathrm{SO}(V_{2n}))$.

Proof. Let $\varphi_0 \in \mathcal{A}_2(\mathrm{SO}(V_{2n}))$. Since φ_0 is K_0-finite, there exists a compact open subgroup K_1 of $K_0 \cap \mathrm{SO}(V_{2n})(\mathbb{A}_{\mathrm{fin}})$ such that φ_0 is right K_1-invariant. We may assume that K_1 is of the form $K_1 = \prod_{v<\infty} K_{1,v}$ for some compact open subgroup $K_{1,v}$ of $K_{0,v}$ such that $\epsilon_v^{-1} K_{1,v} \epsilon_v = K_{1,v}$ for any $v < \infty$. Moreover, we can find a finite set S of places of $\mathbb{F}$ containing all infinite places of $\mathbb{F}$ such that $K_{1,v} = K_{0,v}$ for any $v \notin S$. We fix a complete system B of representative of

$$\mu_2(\mathbb{F})\backslash \left(\prod_{v\in S} \mu_2(\mathbb{F}_v) \right).$$

We may assume that B contains the identity element $1 \in \prod_{v \in S} \mu_2(\mathbb{F}_v)$.

We regard φ_0 as a function on $\mathrm{O}(V_{2n})(\mathbb{F}) \cdot \mathrm{SO}(V_{2n})(\mathbb{A})$, which is left $\mathrm{O}(V_{2n})(\mathbb{F})$-invariant. For $t \in \mu_2(\mathbb{A})$, we define a function $\varphi_t \colon \mathrm{O}(V_{2n})(\mathbb{F}) \cdot \mathrm{SO}(V_{2n})(\mathbb{A}) \to \mathbb{C}$ by

$$\varphi_t(h) = \begin{cases} \varphi_0(h) & \text{if } (t_v)_{v\in S} \in B, \\ \varphi_0(h\epsilon) & \text{if } (t_v)_{v\in S} \notin B \end{cases}$$

for $h \in \mathrm{O}(V_{2n})(\mathbb{F}) \cdot \mathrm{SO}(V_{2n})(\mathbb{A})$. Then we see that

$$\varphi_{ta}(h) = \varphi_t(h), \quad \varphi_{-t}(h) = \varphi_t(h\epsilon)$$

for $t \in \mu_2(\mathbb{A})$ and $a \in \prod_{v \notin S} \mu_2(\mathbb{F}_v)$ since $\epsilon^2 = \mathbf{1}_{V_{2n}}$. Moreover, φ_t is right K_1-invariant for any $t \in \mu_2(\mathbb{A})$.

Now we define a function $\varphi \colon \mathrm{O}(V_{2n})(\mathbb{A}) \to \mathbb{C}$ by

$$\varphi(g) = \varphi_{\det(g)}(g\epsilon_{\det(g)}^{-1})$$

for $g \in \mathrm{O}(V_{2n})(\mathbb{A})$. Then we have $\mathrm{Res}(\varphi) = \varphi_0$. We show that $\varphi \in \mathcal{A}_2(\mathrm{O}(V_{2n}))$.

Let $\gamma \in \mathrm{O}(V_{2n})(\mathbb{F})$. If $\det(\gamma) = 1$, we have

$$\varphi(\gamma g) = \varphi_{\det(g)}(\gamma g\epsilon_{\det(g)}^{-1}) = \varphi_{\det(g)}(g\epsilon_{\det(g)}^{-1}) = \varphi(g).$$

If $\det(\gamma) = -1$, we have

$$\varphi(\gamma g) = \varphi_{-\det(g)}(\gamma g\epsilon_{-\det(g)}^{-1}) = \varphi_{\det(g)}(g\epsilon_{-\det(g)}^{-1}\epsilon) = \varphi(g).$$

Hence φ is left $\mathrm{O}(V_{2n})(\mathbb{F})$-invariant. It is easy to see that φ is right $(K_1 \cdot \prod_{v\notin S} K_v)$-invariant and is a C^∞-function on $\mathrm{O}(V_{2n})(\mathbb{F}_\infty)$. Hence φ is a smooth function on $\mathrm{O}(V_{2n})(\mathbb{A})$.

We denote the space spanned by $k \cdot \varphi$ for $k \in K$ (resp. for $k \in K_0$) by $K\varphi$ (resp. $K_0\varphi$). Since any $\varphi' \in K\varphi$ is right $\prod_{v\notin S} K_v$-invariant, the finiteness of $\dim(K\varphi)$ is equivalent to the one of $\dim(K_0\varphi)$. So we shall prove that $\dim(K_0\varphi) < \infty$. Let

$$\varphi' = \sum_{i=1}^{r} c_i (k_i \cdot \varphi) \in K_0\varphi$$

with $c_i \in \mathbb{C}$ and $k_i \in K_0$. Then for $a \in \prod_{v\in S} \mu_2(\mathbb{F}_v)$ and $x \in \mathrm{SO}(V_{2n})(\mathbb{A})$, we have

$$(\epsilon_a \cdot \varphi')(x) = \varphi'(x\epsilon_a) = \sum_{i=1}^{r} c_i \varphi(x\epsilon_a k_i) = \sum_{i=1}^{r} c_i \varphi(x(\epsilon_a k_i \epsilon_a^{-1})\epsilon_a).$$

Since $\epsilon_a k_i \epsilon_a^{-1} \in K_0$, we have

$$(\epsilon_a \cdot \varphi')|\mathrm{SO}(V_{2n})(\mathbb{A}) \in K_0((\epsilon_a \cdot \varphi)|\mathrm{SO}(V_{2n})(\mathbb{A})).$$

Hence we may consider the map

$$\Phi\colon K_0\varphi \to \bigoplus_a K_0((\epsilon_a\cdot\varphi)|\mathrm{SO}(V_{2n})(\mathbb{A})),\ \varphi' \mapsto \oplus_a((\epsilon_a\cdot\varphi')|\mathrm{SO}(V_{2n})(\mathbb{A}))_a,$$

where a runs over $\prod_{v\in S}\mu_2(\mathbb{F}_v)$. Since any $\varphi'\in K_0\varphi$ is right $\prod_{v\notin S}K_v$-invariant and the map

$$\prod_{v\in S}\mu_2(\mathbb{F}_v)\to \mathrm{SO}(V_{2n})(\mathbb{A})\backslash \mathrm{O}(V_{2n})(\mathbb{A})/\prod_{v\notin S}K_v,\ a\mapsto \epsilon_a$$

is bijective, we see that Φ is injective. Since $(\epsilon_a\cdot\varphi')|\mathrm{SO}(V_{2n})(\mathbb{A})\in\mathcal{A}(\mathrm{SO}(V_{2n}))$, it is K_0-finite. Hence we have $\dim(K_0\varphi)<\infty$, and so that we get the K-finiteness of φ. Similarly, we obtain the $\mathfrak{z}$-finiteness of φ. Note that $\mathrm{Lie}(\mathrm{O}(V_{2n})(\mathbb{F}_\infty))=\mathrm{Lie}(\mathrm{SO}(V_{2n})(\mathbb{F}_\infty))$.

Now we show that $\|\varphi\|_{L^2(\mathrm{O}(V_{2n}))}<\infty$, where $\|\cdot\|_{L^2(\mathrm{O}(V_{2n}))}$ (resp. $\|\cdot\|_{L^2(\mathrm{SO}(V_{2n}))}$) is the L^2-norm on $\mathrm{O}(V_{2n})(\mathbb{F})\backslash\mathrm{O}(V_{2n})(\mathbb{A})$ (resp. $\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A})$). Let Ω be the characteristic function on $\prod_{v\notin S}\mu_2(\mathbb{F}_v)\times B\subset\mu_2(\mathbb{A})$. Then we have

$$\begin{aligned}
\|\varphi\|^2_{L^2(\mathrm{O}(V_{2n}))} &= \int_{\mathrm{O}(V_{2n})(\mathbb{F})\backslash \mathrm{O}(V_{2n})(\mathbb{A})}|\varphi(g)|^2dg\\
&= \int_{\mu_2(\mathbb{F})\backslash\mu_2(\mathbb{A})}\left(\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A})}|\varphi(h\epsilon_t)|^2dh\right)dt\\
&= \int_{\mu_2(\mathbb{A})}\Omega(t)\cdot\left(\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A})}|\varphi(h\epsilon_t)|^2dh\right)dt\\
&= \int_{\mu_2(\mathbb{A})}\Omega(t)\cdot\left(\int_{\mathrm{SO}(V_{2n})(\mathbb{F})\backslash\mathrm{SO}(V_{2n})(\mathbb{A})}|\varphi_0(h)|^2dh\right)dt\\
&= \mathrm{vol}(\prod_{v\notin S}\mu_2(\mathbb{F}_v)\times B)\cdot\|\varphi_0\|^2_{L^2(\mathrm{SO}(V_{2n}))}\\
&= 2^{-1}\|\varphi_0\|^2_{L^2(\mathrm{SO}(V_{2n}))}<\infty.
\end{aligned}$$

We have shown that φ is a smooth, K-finite, $\mathfrak{z}$-finite, square-integrable function on $\mathrm{O}(V_{2n})(\mathbb{F})\backslash\mathrm{O}(V_{2n})(\mathbb{A})$. Such functions are of moderate growth (see [BJ, §4.3]). Therefore we conclude that $\varphi\in\mathcal{A}_2(\mathrm{O}(V_{2n}))$. This completes the proof. □

7.3. **Near equivalence classes.** Let $\sigma=\otimes'_v\sigma_v$ and $\sigma'=\otimes'_v\sigma'_v$ be two admissible representations of $\mathcal{H}(\mathrm{O}(V_{2n}))$. We say that σ and σ' are nearly equivalent if $\sigma_v\cong\sigma'_v$ for almost all v. In this case, we write $\sigma\sim_{\mathrm{ne}}\sigma'$. Similarly, let $[\sigma_0]=\otimes'_v[\sigma_{0,v}]$ and $\sigma'_0=\otimes'_v[\sigma'_{0,v}]$ be two equivalence classes of admissible representations of $\mathcal{H}(\mathrm{SO}(V_{2n}))$. We say that $[\sigma_0]$ and $[\sigma'_0]$ are ϵ-nearly equivalent if $[\sigma_{0,v}]=[\sigma'_{0,v}]$, i.e., $\sigma'_{0,v}\cong\sigma_{0,v}$ or $\sigma^\epsilon_{0,v}$ for almost all v. In this case, we write $[\sigma_0]\sim_{\mathrm{ne}}[\sigma'_0]$.

By a near equivalence class in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$, we mean a maximal $\mathcal{H}(\mathrm{O}(V_{2n}))$-submodule $\mathcal{V}$ of $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ such that

- all irreducible constituents of $\mathcal{V}$ are nearly equivalent each other;
- any irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-submodule of $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ which is orthogonal to $\mathcal{V}$ is not nearly equivalent to the constituents of $\mathcal{V}$.

We define a ϵ-near equivalence class in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ similarly.

We relate ϵ-near equivalence classes in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ with elements in $\Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$.

Proposition 7.6. *The there exists a canonical bijection*

$$\{\epsilon\text{-}\textit{near equivalence classes in } \mathcal{A}_2(\mathrm{SO}(V_{2n}))\}\longleftrightarrow\Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon.$$

Proof. Suppose that $[\sigma_0]$ and $[\sigma'_0]$ appear in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. Let Σ and $\Sigma'\in\Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$ be A-parameters associated to $[\sigma_0]$ and $[\sigma'_0]$, i.e., $[\sigma_0]\in\Pi^0_\Sigma$ and $[\sigma'_0]\in\Pi^0_{\Sigma'}$, respectively. We claim that $[\sigma_0]\sim_{\mathrm{ne}}[\sigma'_0]$ if and only if $\Sigma=\Sigma'$. Suppose that $[\sigma_0],[\sigma'_0]\in\Pi^0_\Sigma$. Note that both of $\sigma_{0,v}$ and $\sigma'_{0,v}$ are unramified for almost all v. By [M1, §4.4 Proposition], for such v, we have $[\sigma_{0,v}]=[\sigma'_{0,v}]$. Hence $[\sigma_0]\sim_{\mathrm{ne}}[\sigma'_0]$.

Conversely, suppose that $[\sigma_0] \sim_{\mathrm{ne}} [\sigma_0']$. For each place v of $\mathbb{F}$, we decompose

$$\Sigma_v = (\phi_{1,v} \boxtimes S_{d_1}) \oplus \cdots \oplus (\phi_{l,v} \boxtimes S_{d_l}), \quad \Sigma_v' = (\phi_{1,v}' \boxtimes S_{d_1'}) \oplus \cdots \oplus (\phi_{l,v}' \boxtimes S_{d_{l'}'}),$$

where $\phi_{i,v}$ and $\phi_{j,v}'$ are representations of $WD_{\mathbb{F}_v} = W_{\mathbb{F}_v} \times \mathrm{SL}_2(\mathbb{C})$. Note that for almost all v, $\phi_{i,v}$ and $\phi_{j,v}'$ are trivial on $\mathrm{SL}_2(\mathbb{C})$. Hence by [M1, §4.2 Corollaire], we have $\Sigma_v \cong \Sigma_v'$ for almost all v. As in [Ar, §1.3], Σ and Σ' define isobaric sums of representations ϕ_Σ and $\phi_{\Sigma'}$ which belong to the discrete spectrum of $\mathrm{GL}(2n)$. It follows by the generalized strong multiplicity one theorem of Jacquet–Shalika ([JS2, (4.4) Theorem]) that $\phi_\Sigma \cong \phi_{\Sigma'}$. Since the map $\Sigma \mapsto \phi_\Sigma$ is injective, we conclude that $\Sigma = \Sigma'$. □

Corollary 7.7. *Let $\Sigma, \Sigma' \in \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$. If $\Sigma \neq \Sigma'$, then $\Pi_\Sigma^0 \cap \Pi_{\Sigma'}^0 = \emptyset$.*

This corollary together with Arthur's multiplicity formula (Theorem 6.8) and the argument in §6.8 gives the multiplicity in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$.

Corollary 7.8. *Assume Conjectures 6.1 and 6.6.*

(1) *For $[\sigma_0] \in \Pi_\Sigma^0$, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))}([\sigma_0], \mathcal{A}_2(\mathrm{SO}(V_{2n}))) = \begin{cases} m_\Sigma & \text{if } \iota_c([\sigma_0]) = \varepsilon_\Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For any irreducible $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module σ, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)}(\sigma, \mathcal{A}_2(\mathrm{SO}(V_{2n}))) \leq 1.$$

In other words, $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ is multiplicity-free as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module.

In particular, the same properties unconditionally hold for the tempered part of the automorphic discrete spectrum of $\mathrm{SO}(V_{2n})$

$$\mathcal{A}_{2,\mathrm{temp}}(\mathrm{SO}(V_{2n})) := \bigoplus_{\Sigma \in \Psi_{2,\mathrm{temp}}(\mathrm{SO}(V_{2n}))/\sim_\epsilon} \bigoplus_{[\sigma_0] \in \Pi_\Sigma^0(\varepsilon_\Sigma)} m_\Sigma [\sigma_0].$$

Proof. By Arthur's multiplicity formula (Theorem 6.8), we have

$$\mathcal{A}_2(\mathrm{SO}(V_{2n})) \cong \bigoplus_{\Sigma \in \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon} \bigoplus_{[\sigma_0] \in \Pi_\Sigma^0(\varepsilon_\Sigma)} m_\Sigma [\sigma_0]$$

as $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-modules. Since $\Pi_\Sigma^0 \cap \Pi_{\Sigma'}^0 = \emptyset$ for $\Sigma \neq \Sigma'$ by Corollary 7.7, and Π_Σ^0 is a multiplicity-free set by Conjectures 6.1 and 6.6, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))}([\sigma_0], \mathcal{A}_2(\mathrm{SO}(V_{2n}))) = \begin{cases} m_\Sigma & \text{if } \iota_c([\sigma_0]) = \varepsilon_\Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

This is (1). The proof of (2) is similar. □

Recall that by Proposition 7.5, there exists a surjective linear map

$$\mathrm{Res} \colon \mathcal{A}_2(\mathrm{O}(V_{2n})) \twoheadrightarrow \mathcal{A}_2(\mathrm{SO}(V_{2n})).$$

It is easy to see that Res is an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-homomorphism.

Proposition 7.9. *The map* Res *induces a bijection*

$$\{\textit{near equivalence classes in } \mathcal{A}_2(\mathrm{O}(V_{2n}))\}$$
$$\Big\downarrow r$$
$$\{\epsilon\textit{-near equivalence classes in } \mathcal{A}_2(\mathrm{SO}(V_{2n}))\}.$$

Proof. Let Φ be a near equivalence class in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$. Then $\mathrm{Res}(\Phi)$ will a priori meet several ϵ-near equivalence classes. Suppose that for an irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-module σ belonging to Φ, $\mathrm{Res}(\sigma)$ contains an irreducible $\mathcal{H}(\mathrm{SO}(V_{2n}))$-module σ_0 and $[\sigma_0]$ belongs to an ϵ-near equivalence class Φ_0. Then we claim that

$$\mathrm{Res}(\Phi) \subset \Phi_0.$$

Indeed, suppose that $\sigma' \in \Phi$, $\mathrm{Res}(\sigma') \supset \sigma'_0$, and $[\sigma'_0]$ belongs to an ϵ-near equivalence class Φ'_0. Then for almost all v, we have $\sigma'_v \cong \sigma_v$ so that $\sigma'_{0,v} \cong \sigma_{0,v}$ or $\sigma^{\epsilon}_{0,v}$. This means that $[\sigma_0] \sim_{\mathrm{ne}} [\sigma'_0]$ so that $\Phi_0 = \Phi'_0$. Therefore if we define $r(\Phi) = \Phi_0$, then r is well-defined.

For the injectivity, if $r(\Phi) = r(\Phi')$, then for $\sigma \in \Phi$ and $\sigma' \in \Phi'$, one has $\sigma'_v \cong \sigma_v$ or $\sigma'_v \otimes \det$ for almost all v. Since σ_v and σ'_v are unramified with respect to K_v for almost all v, we must have $\sigma'_v \cong \sigma_v$ for almost all v (see Lemma 2.4). Hence we have $\Phi = \Phi'$.

For the surjectivity, we decompose

$$\mathcal{A}_2(\mathrm{O}(V_{2n})) \cong \bigoplus_{\lambda} \sigma_\lambda$$

into a direct sum of irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-modules. Then we may decompose

$$\sigma_\lambda \cong \bigoplus_{\kappa} \sigma_{\lambda,\kappa}$$

into a direct sum of irreducible $\mathcal{H}(\mathrm{SO}(V_{2n}))$-modules. Since $\mathrm{Res}\colon \mathcal{A}_2(\mathrm{O}(V_{2n})) \twoheadrightarrow \mathcal{A}_2(\mathrm{SO}(V_{2n}))$ is a surjective $\mathcal{H}(\mathrm{SO}(V_{2n}))$-homomorphism, any irreducible $\mathcal{H}(\mathrm{SO}(V_{2n}))$-submodule σ_0 of $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ is isomorphic to some $\sigma_{\lambda,\kappa}$ via Res. Then we have $\mathrm{Res}(\sigma_\lambda) \supset \sigma_0$. This shows that If Φ_0 (resp. Φ_λ) is the ϵ-near equivalence class of σ_0 (resp. the near equivalence class of σ_λ), then $r(\Phi_\lambda) = \Phi_0$. □

7.4. **Proof of Theorem 7.1.** In this subsection, we will complete the proof of Theorem 7.1 and show Proposition 7.2.

Recall that $\Psi_2(\mathrm{O}(V_{2n})) = \Psi_2(\mathrm{SO}(V_{2n}))/\sim_\epsilon$. By Propositions 7.6 and 7.9, we obtain a canonical bijection

$$\{\text{near equivalence classes in } \mathcal{A}_2(\mathrm{O}(V_{2n}))\} \longleftrightarrow \Psi_2(\mathrm{O}(V_{2n})).$$

In other words, we obtain a decomposition

$$\mathcal{A}_2(\mathrm{O}(V_{2n})) = \bigoplus_{\Sigma \in \Psi_2(\mathrm{O}(V_{2n}))} \mathcal{A}_{2,\Sigma},$$

where $\mathcal{A}_{2,\Sigma}$ is the direct sum over the near equivalence class corresponding to Σ. Moreover, we have

$$\mathcal{A}_{2,\Sigma} = \bigoplus_{\sigma \in \Pi_\Sigma} m(\sigma)\sigma$$

for some $m(\sigma) \in \mathbb{Z}_{\geq 0}$. We have to show that

$$m(\sigma) = \begin{cases} 1 & \text{if } \iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the restriction map $\mathrm{Res}\colon \mathcal{A}_2(\mathrm{O}(V_{2n})) \to \mathcal{A}_2(\mathrm{SO}(V_{2n}))$. This is an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-equivariant homomorphism.

Lemma 7.10. *Assume Conjectures 6.1 and 6.6. Let σ be an irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-submodule of $\mathcal{A}_2(\mathrm{O}(V_{2n}))$. Then $\mathrm{Res}(\sigma)$ is nonzero and irreducible as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module.*

Proof. It is clear that $\mathrm{Res}(\sigma)$ is nonzero. Decompose $\sigma \cong \otimes_v \sigma_v$. If $\sigma_v \otimes \det \not\cong \sigma_v$ for any v, then σ is irreducible as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module. Hence so is $\mathrm{Res}(\sigma)$.

We may assume that $\sigma_v \otimes \det \cong \sigma_v$ for some v. This is in the case (B) as in §6.8. We decompose

$$\sigma \cong \bigoplus_{\lambda \in \Lambda} \sigma_\lambda$$

into a direct sum of irreducible $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-modules. Here, Λ is an index set. Then $\mathrm{Res}(\sigma) \cong \oplus_{\lambda \in \Lambda_0} \sigma_\lambda$ for some non-empty subset Λ_0 of Λ. As an $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-module, each σ_λ is a direct sum of two copies of an irreducible $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-module $[\sigma_0]$. Hence $\mathrm{Res}(\sigma) \cong [\sigma_0]^{\oplus 2|\Lambda_0|}$ as $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-modules. By Corollary 7.8 (which we have shown using Conjectures 6.1 and 6.6), we have

$$\dim_\mathbb{C} \mathrm{Hom}_{\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))}([\sigma_0], \mathcal{A}_2(\mathrm{SO}(V_{2n}))) \leq 2.$$

This implies that $|\Lambda_0| = 1$. Hence $\mathrm{Res}(\sigma)$ is irreducible as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module. □

By Arthur's multiplicity formulas for $\mathrm{SO}(V_{2n})$ and $\mathrm{SO}(V_{2n})(\mathbb{A}) \cdot \mathrm{O}(V_{2n})(\mathbb{F})$ (Theorems 6.8 and 6.12), we see that if $\sigma \in \Pi_\Sigma$ occurs in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$, then $\mathrm{Res}(\sigma) \neq 0$ so that $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$. In other words, if $\sigma \in \Pi_\Sigma$ satisfies $\iota_c(\sigma) \circ \Delta \neq \varepsilon_\Sigma$, then σ does not occur in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$, i.e., $m(\sigma) = 0$.

Now we consider the case when $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$.

Proposition 7.11. *Assume Conjectures 6.1 and 6.6. Let $\sigma = \otimes_v \sigma_v \in \Pi_\Sigma$ such that $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$. Then there exists an $\mathcal{H}(\mathrm{O}(V_{2n}))$-subspace $\mathcal{A}_\sigma$ of $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ on which $\mathcal{H}(\mathrm{O}(V_{2n}))$ acts by σ.*

Proof. Let $[\sigma_0] \in \Pi_\Sigma^0$ be an element satisfying $\sigma_{0,v} \subset \sigma_v | \mathrm{SO}(V_{2n})(\mathbb{F}_v)$ for each v. By Arthur's multiplicity formula (Theorem 6.8), we see that $[\sigma_0]$ occurs in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ as an $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-module. We may assume that σ_0 occurs in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ as an $\mathcal{H}(\mathrm{SO}(V_{2n}))$-module. Since $\mathrm{Res}\colon \mathcal{A}_2(\mathrm{O}(V_{2n})) \to \mathcal{A}_2(\mathrm{SO}(V_{2n}))$ is surjective (Proposition 7.5), and $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ is a direct sum of irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-modules, we can find an irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-module $\sigma' = \otimes_v \sigma'_v$ such that σ' occurs in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ and $\mathrm{Res}(\sigma') \supset \sigma_0$. By (the proof of) Proposition 7.9, we see that $\sigma' \in \Pi_\Sigma$ and

$$\sigma' \cong \sigma \otimes \det_S$$

for some finite set S of places of $\mathbb{F}$. Here, $\det_S$ is the determinant for places in S and trivial outside S. We consider 3 cases (A), (B) and (C) as in §6.8 separately.

We consider the case (A). Suppose that $\Sigma \in \Psi_2^\epsilon(\mathrm{O}(V_{2n}))$. Then $[\mathcal{S}_\Sigma : \mathcal{S}_\Sigma^+] = 2$ and $[\mathcal{S}_{\Sigma,v} : \mathcal{S}_{\Sigma,v}^+] = 2$ for any place v of $\mathbb{F}$. By Conjectures 6.1 and 6.6, we see that

$$\iota_c(\sigma') = \iota_c(\sigma) \otimes \left(\prod_{v \in S} \eta_{0,v} \right),$$

where $\eta_{0,v}$ is the non-trivial character of $\mathcal{S}_{\Sigma,v}/\mathcal{S}_{\Sigma,v}^+$. Hence we have

$$\iota_c(\sigma') \circ \Delta = (\iota_c(\sigma) \circ \Delta) \cdot \eta_0^{|S|} = \epsilon_\Sigma \cdot \eta_0^{|S|},$$

where η_0 is the non-trivial character of $\mathcal{S}_\Sigma/\mathcal{S}_\Sigma^+$. Since $\mathrm{Res}(\sigma')$ occurs in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module, by Theorem 6.12, the number of S must be even. Hence if $\varphi' \in \mathcal{A}_2(\mathrm{O}(V_{2n}))$, then

$$\varphi(g) \coloneqq \det_S(g) \cdot \varphi'(g) \in \mathcal{A}_2(\mathrm{O}(V_{2n})).$$

This implies that σ also occurs in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$.

We consider the case (B). Suppose that $\Sigma \notin \Psi_2^\epsilon(\mathrm{O}(V_{2n}))$ and $\sigma_0^\epsilon \not\cong \sigma_0$. Then $\sigma_v \otimes \det_v \cong \sigma_v$ for some v. Hence we can take S such that $|S|$ is even, and we see that σ also occurs in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$.

We consider the case (C). Suppose that $\Sigma \notin \Psi_2^\epsilon(\mathrm{O}(V_{2n}))$ and $\sigma_0^\epsilon \cong \sigma_0$. Then there are two extension of σ_0 to irreducible $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-modules, and both of them occur in $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$. Note that $\mathrm{Res}(\sigma') \cong \mathrm{Res}(\sigma' \otimes \det_S)$ as $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-modules if and only if $|S|$ is even. This implies that $\sigma \cong \sigma' \otimes \det_S$ occurs in $\mathcal{A}_2(\mathrm{O}(V_{2n}))$ for any S. This completes the proof. □

Under assuming Conjectures 6.1 and 6.6, by a similar argument to Corollary 7.8, we have

$$m(\sigma) = \dim_\mathbb{C} \mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))).$$

Hence Proposition 7.11 says that if $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$, then $m(\sigma) > 0$. To prove $m(\sigma) = 1$ for such σ, it suffices to show Proposition 7.2.

Proof of Proposition 7.2. Let σ be an irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-module. We have to show that

$$\dim_\mathbb{C} \mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \leq 1.$$

First we assume that σ remains irreducible as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module. Then by Lemma 7.10, the map

$$\mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \to \mathrm{Hom}_{(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)}(\sigma, \mathcal{A}_2(\mathrm{SO}(V_{2n}))), \ f \mapsto \mathrm{Res} \circ f$$

is injective. Hence by Corollary 7.8, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \leq \dim_{\mathbb{C}} \mathrm{Hom}_{(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)}(\sigma, \mathcal{A}_2(\mathrm{SO}(V_{2n}))) \leq 1.$$

Now suppose for the sake of contradiction that σ is an irreducible $\mathcal{H}(\mathrm{O}(V_{2n}))$-module such that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \geq 2.$$

We know that as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module, σ is a multiplicity-free sum

$$\sigma = \bigoplus_{\lambda} \sigma_\lambda$$

of irreducible $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-modules σ_λ. For a fixed nonzero f_1 in the above Hom space, we have shown that $\mathrm{Res}(f_1(\sigma))$ is irreducible (Lemma 7.10), so say $\mathrm{Res}(f_1(\sigma)) = \sigma_{\lambda_1}$. Consider the natural map

$$\mathrm{Hom}_{\mathcal{H}(\mathrm{O}(V_{2n}))}(\sigma, \mathcal{A}_2(\mathrm{O}(V_{2n}))) \to \mathrm{Hom}_{(\mathcal{H}(\mathrm{SO}(V_{2n})),\epsilon)}(\sigma_{\lambda_1}, \mathcal{A}_2(\mathrm{SO}(V_{2n})))$$

given by

$$f \mapsto \mathrm{Res} \circ f \circ \iota_1,$$

where $\iota_1 \colon \sigma_{\lambda_1} \hookrightarrow \sigma$ is a fixed inclusion. We have shown that the right hand side has dimension one (Corollary 7.8), so this map has nonzero kernel, i.e., there is f_2 in the left hand side such that $0 \neq \mathrm{Res}(f_2(\sigma)) = \sigma_{\lambda_2}$, with $\lambda_1 \neq \lambda_2$. This shows that $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ contains $\sigma_{\lambda_1} \oplus \sigma_{\lambda_2}$ as an $(\mathcal{H}(\mathrm{SO}(V_{2n})), \epsilon)$-module, which implies that $\mathcal{A}_2(\mathrm{SO}(V_{2n}))$ contains $4[\sigma_0]$ as an $\mathcal{H}^\epsilon(\mathrm{SO}(V_{2n}))$-module. This contradicts Corollary 7.8 (1). This completes the proof. □

By Propositions 7.2 and 7.11, we see that if $\iota_c(\sigma) \circ \Delta = \varepsilon_\Sigma$, then $m(\sigma) > 0$. This completes the proof of Theorem 7.1.

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