

THETA CORRESPONDENCES FOR $\mathrm{GSp}(4)$

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ABSTRACT. We explicitly determine the theta correspondences for GSp_4 and orthogonal similitude groups associated to various quadratic spaces of rank 4 and 6. The results are needed in our proof of the local Langlands correspondence for GSp_4 ([GT1]).

1. Introduction

This is a companion paper to [GT1], in which we proved the local Langlands conjecture for GSp_4 over a non-archimedean local field F of characteristic zero and residue characteristic p . The results of [GT1] depended on a study of the theta correspondences between GSp_4 and the three orthogonal similitude groups in the following diagram:



For example, one of the results needed in [GT1] is that every irreducible representation of $\mathrm{GSp}_4(F)$ participates in theta correspondence with exactly one of $\mathrm{GSO}_{4,0}$ or $\mathrm{GSO}_{3,3}$. This dichotomy follows from a fundamental result of Kudla-Rallis [KR] (which we recall below) for a large class of representations of $\mathrm{GSp}_4(F)$, but to take care of the remaining representations, the results of [KR] needs to be supplemented by the results of this paper. Moreover, the complete determination of the above theta correspondences serves to make the Langlands parametrization for non-supercuspidal representations completely explicit and transparent.

To state a sample result in the introduction, let us note that the orthogonal similitude groups mentioned above are closely related to the groups GL_2 and GL_4 . Indeed, one has:

$$\begin{cases}
 \mathrm{GSO}_{2,2} \cong (\mathrm{GL}_2 \times \mathrm{GL}_2) / \{(z, z^{-1}) : z \in \mathrm{GL}_1\} \\
 \mathrm{GSO}_{4,0} \cong (D^\times \times D^\times) / \{(z, z^{-1}) : z \in \mathrm{GL}_1\} \\
 \mathrm{GSO}_{3,3} \cong (\mathrm{GL}_4 \times \mathrm{GL}_1) / \{(z, z^{-2}) : z \in \mathrm{GL}_1\}
 \end{cases}$$

where D is the quaternion division algebra over F . Via these isomorphisms, an irreducible representation of $\mathrm{GSO}_{2,2}(F)$ (resp. $\mathrm{GSO}_{4,0}(F)$) has the form $\tau_1 \boxtimes \tau_2$ where τ_1 and τ_2 are representations of $\mathrm{GL}_2(F)$ (resp. D^\times) with the same central characters. Similarly, an irreducible representation of $\mathrm{GSO}_{3,3}(F)$ has the form $\Pi \boxtimes \mu$ with Π is a representation of $\mathrm{GL}_4(F)$ with central character $\omega_\Pi = \mu^2$.

For a reductive group G over F , we write $\Pi(G)$ for the set of isomorphism classes of irreducible smooth representations of $G(F)$. Now one of the results we show is the following theorem, which was stated in [GT1, Thm. 5.6].

Theorem 1.1. (i) *The theta correspondence for $\mathrm{GSO}(D) \times \mathrm{GSp}_4$ defines a bijection*

$$\Pi(\mathrm{GSO}_{4,0})/\sim \longleftrightarrow \Pi(\mathrm{GSp}_4)_{ng}^{temp},$$

where \sim is the equivalence relation defined by the action of $\mathrm{GO}_{4,0}(F)$ on $\Pi(\mathrm{GSO}_{4,0})$ and

$$\Pi(\mathrm{GSp}_4)_{ng}^{temp} := \{\text{non-generic essentially tempered representations of } \mathrm{GSp}_4(F)\}.$$

Moreover, the image of the subset of $\tau_1^D \boxtimes \tau_2^D$'s, with $\tau_1^D \neq \tau_2^D$, is precisely the subset of non-generic supercuspidal representations of $\mathrm{GSp}_4(F)$. The other representations in the image are the non-discrete series representations in Table 1, NDS(c).

(ii) *The theta correspondence for $\mathrm{GSO}_{2,2} \times \mathrm{GSp}_4$ defines an injection*

$$\Pi(\mathrm{GSO}_{2,2})/\sim \longrightarrow \Pi(\mathrm{GSp}_4).$$

The image is disjoint from $\Pi(\mathrm{GSp}_4)_{ng}^{temp}$ and consists of:

- (a) *the generic discrete series representations (including supercuspidal ones) whose standard L -factor $L(s, \pi, \text{std})$ has a pole at $s = 0$.*
- (b) *the non-discrete series representations in [Table 1, NDS(b, d, e)].*

Moreover, the images of the representations $\tau_1 \boxtimes \tau_2$'s, with $\tau_1 \neq \tau_2$ discrete series representations of $\mathrm{GL}_2(F)$, are precisely the representations in (a).

(iii) *The theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_{3,3}$ defines an injection*

$$\Pi(\mathrm{GSp}_4) \setminus \Pi(\mathrm{GSp}_4)_{ng}^{temp} \longrightarrow \Pi(\mathrm{GSO}_{3,3}) \subset \Pi(\mathrm{GL}_4) \times \Pi(\mathrm{GL}_1).$$

Moreover, the representations of $\mathrm{GSp}_4(F)$ not accounted for by (i) and (ii) above are

- (a) *the generic discrete series representations π whose standard factor $L(s, \pi, \text{std})$ is holomorphic at $s = 0$. The images of these representations under the above map are precisely the discrete series representations $\Pi \boxtimes \mu$ of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$ such that $L(s, \bigwedge^2 \phi_\Pi \otimes \mu^{-1})$ has a pole at $s = 0$. Here, ϕ_Π is the L -parameter of Π .*
- (b) *the non-discrete series representations in [Table 1, NDS(a)]. The images of these under the above map consists of non-discrete series representations $\Pi \boxtimes \mu$ such that*

$$\phi_\Pi = \rho \oplus \rho \cdot \chi \quad \text{and} \quad \mu = \det \rho \cdot \chi,$$

for an irreducible two dimensional ρ and a character $\chi \neq 1$.

(iv) *If a representation π of $\mathrm{GSp}_4(F)$ participates in theta correspondence with $\mathrm{GSO}(V_2)$, so that*

$$\pi = \theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_2 \boxtimes \tau_1),$$

then π has a nonzero theta lift to $\mathrm{GSO}_{3,3}$. If $\Pi \boxtimes \mu$ is the small theta lift of π to $\mathrm{GSO}_{3,3}(F)$, with Π a representation of $\mathrm{GL}_4(F)$, then

$$\phi_\Pi = \phi_{\tau_1} \oplus \phi_{\tau_2} \quad \text{and} \quad \mu = \omega_\pi = \det \phi_{\tau_1} = \det \phi_{\tau_2}.$$

Indeed, our main results give much more complete and explicit information than the above theorem. In particular, we take note of Thms. 8.1, 8.2 and 8.3, Cors. 12.2 and 12.3, as well as Props. 13.1 and 13.2.

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2. Similitude Theta Correspondences

In this section, we shall describe some basic properties of the theta correspondence for similitude groups. The definitive reference for this subject matter is the paper [Ro1] of B. Roberts. However, the results of [Ro1] are not sufficient for our purposes and need to be somewhat extended.

Consider the dual pair $\mathrm{O}(V) \times \mathrm{Sp}(W)$; for simplicity, we assume that $\dim V$ is even. For each non-trivial additive character ψ , let ω_ψ be the Weil representation for $\mathrm{O}(V) \times \mathrm{Sp}(W)$, which can be described as follows. Fix a Witt decomposition $W = X \oplus Y$ and let $P(Y) = \mathrm{GL}(Y) \cdot N(Y)$ be the parabolic subgroup stabilizing the maximal isotropic subspace Y . Then

$$N(Y) = \{b \in \mathrm{Hom}(X, Y) : b^t = b\},$$

where $b^t \in \mathrm{Hom}(Y^*, X^*) \cong \mathrm{Hom}(X, Y)$. The Weil representation ω_ψ can be realized on $S(X \otimes V)$ and the action of $P(Y) \times \mathrm{O}(V)$ is given by the usual formulas:

$$\begin{cases} \omega_\psi(h)\phi(x) = \phi(h^{-1}x), & \text{for } h \in \mathrm{O}(V); \\ \omega_\psi(a)\phi(x) = \chi_V(\det_Y(a)) \cdot |\det_Y(a)|^{\frac{1}{2} \dim V} \cdot \phi(a^{-1} \cdot x), & \text{for } a \in \mathrm{GL}(Y); \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x \rangle) \cdot \phi(x), & \text{for } b \in N(Y), \end{cases}$$

where χ_V is the quadratic character associated to $\mathrm{disc} V \in F^\times / F^{\times 2}$ and $\langle -, - \rangle$ is the natural symplectic form on $W \otimes V$. To describe the full action of $\mathrm{Sp}(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $\mathrm{O}(V)$ (resp. $\mathrm{Sp}(W)$), the maximal π -isotypic quotient has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation of $\mathrm{Sp}(W)$ (resp. $\mathrm{O}(V)$). We call $\Theta_\psi(\pi)$ the big theta lift of π . It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$; we call it the small theta lift of π . Then it was a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is non-zero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger when the residual characteristic p of F is not 2 and can be checked in many low-rank cases, regardless of the residual characteristic of F . If the Howe conjecture is true in general, our treatment for the rest of the paper can be somewhat simplified. However, because we would like to include the case $p = 2$ in our discussion, we shall refrain from assuming Howe's conjecture in this paper.

With this in mind, we take note of the following result which was shown by Kudla [K] for any residual characteristic p :

Proposition 2.1. (i) *If π is supercuspidal, $\Theta_\psi(\pi) = \theta_\psi(\pi)$ is irreducible or zero.*

(ii) *If $\theta_\psi(\pi_1) = \theta_\psi(\pi_2) \neq 0$ for two supercuspidal representations π_1 and π_2 , then $\pi_1 = \pi_2$.*

One of the main purposes of this section is to extend this result of Kudla to the case of similitude groups.

Let λ_V and λ_W be the similitude factors of $\mathrm{GO}(V)$ and $\mathrm{GSp}(W)$ respectively. We shall consider the group

$$R = \mathrm{GO}(V) \times \mathrm{GSp}(W)^+$$

where $\mathrm{GSp}(W)^+$ is the subgroup of $\mathrm{GSp}(W)$ consisting of elements g such that $\lambda_W(g)$ is in the image of λ_V . In fact, for the cases of interest in this paper (see the next section), λ_V is surjective, in which case $\mathrm{GSp}(W)^+ = \mathrm{GSp}(W)$.

The group R contains the subgroup

$$R_0 = \{(h, g) \in R : \lambda_V(h) \cdot \lambda_W(g) = 1\}.$$

The Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(g, h)\phi = |\lambda_V(h)|^{-\frac{1}{8} \dim V \cdot \dim W} \omega(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(W).$$

Note that this differs from the normalization used in [Ro1]. Observe in particular that the central elements $(t, t^{-1}) \in R_0$ act by the quadratic character $\chi_V(t)^{\frac{\dim W}{2}}$.

Now consider the (compactly) induced representation

$$\Omega = \mathrm{ind}_{R_0}^R \omega_\psi.$$

As a representation of R , Ω depends only on the orbit of ψ under the evident action of $\mathrm{Im} \lambda_V \subset F^\times$. For example, if λ_V is surjective, then Ω is independent of ψ . For any irreducible representation π of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$), the maximal π -isotypic quotient of Ω has the form

$$\pi \otimes \Theta(\pi)$$

where $\Theta(\pi)$ is some smooth representation of $\mathrm{GSp}(W)^+$ (resp. $\mathrm{GO}(V)$). Further, we let $\theta(\pi)$ be the maximal semisimple quotient of $\Theta(\pi)$. Note that though $\Theta(\pi)$ may be reducible, it has a central character $\omega_{\Theta(\pi)}$ given by

$$\omega_{\Theta(\pi)} = \chi_V^{\frac{\dim W}{2}} \cdot \omega_\pi.$$

The extended Howe conjecture for similitudes says that $\theta(\pi)$ is irreducible whenever $\Theta(\pi)$ is non-zero, and the map $\pi \mapsto \theta(\pi)$ is injective on its domain. It was shown by Roberts [Ro1] that this follows from the Howe conjecture for isometry groups, and thus holds if $p \neq 2$.

In any case, we have the following lemma which relates the theta correspondence for isometries and similitudes; the proof is given in [GT1, Lemma 2.2].

Lemma 2.2. (i) *Suppose that π is an irreducible representation of a similitude group and τ is a constituent of the restriction of π to the isometry group. Then $\theta_\psi(\tau) \neq 0$ implies that $\theta(\pi) \neq 0$.*

(ii) *Suppose that*

$$\mathrm{Hom}_R(\Omega, \pi_1 \boxtimes \pi_2) \neq 0.$$

Suppose further that for each constituent τ_1 in the restriction of π_1 to $\mathrm{O}(V)$, $\theta_\psi(\tau_1)$ is irreducible and the map $\tau_1 \mapsto \theta_\psi(\tau_1)$ is injective on the set of irreducible constituents of $\pi_1|_{\mathrm{O}(V)}$. Then there is a uniquely determined bijection

$$f : \{\text{irreducible summands of } \pi_1|_{\mathrm{O}(V)}\} \longrightarrow \{\text{irreducible summands of } \pi_2|_{\mathrm{Sp}(W)}\}$$

such that for any irreducible summand τ_i in the restriction of π_i to the relevant isometry group,

$$\tau_2 = f(\tau_1) \iff \mathrm{Hom}_{\mathrm{O}(V) \times \mathrm{Sp}(W)}(\omega_\psi, \tau_1 \boxtimes \tau_2) \neq 0.$$

One has the analogous statement with the roles of $\mathrm{O}(V)$ and $\mathrm{Sp}(W)$ exchanged.

(iii) If π is a representation of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$) and the restriction of π to the relevant isometry group is $\oplus_i \tau_i$, then as representations of $\mathrm{Sp}(W)$ (resp. $\mathrm{O}(V)$),

$$\Theta(\pi) \cong \bigoplus_i \Theta_\psi(\tau_i).$$

In particular, $\Theta(\pi)$ is admissible of finite length. Moreover, if $\Theta_\psi(\tau_i) = \theta_\psi(\tau_i)$ for each i , then

$$\Theta(\pi) = \theta(\pi).$$

In addition, we have [GT1, Prop. 2.3]:

Proposition 2.3. *Suppose that π is a supercuspidal representation of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$). Then we have:*

- (i) $\Theta(\pi)$ is either zero or is an irreducible representation of $\mathrm{GSp}(W)^+$ (resp. $\mathrm{GO}(V)$).
- (ii) If π' is another supercuspidal representation such that $\Theta(\pi') = \Theta(\pi) \neq 0$, then $\pi' = \pi$.

We now specialize to the cases of interest in this paper. Henceforth, we shall only consider the case when $\dim W = 4$, so that

$$\mathrm{GSp}(W) \cong \mathrm{GSp}_4(F).$$

Moreover, we shall only consider quadratic spaces V with $\dim V = 4$ or 6 . We describe these quadratic spaces in greater detail.

Let D be a (possibly split) quaternion algebra over F and let \mathbb{N}_D be its reduced norm. Then (D, \mathbb{N}_D) is a rank 4 quadratic space. We have an isomorphism

$$\mathrm{GSO}(D, \mathbb{N}_D) \cong (D^\times \times D^\times) / \{(z, z^{-1}) : z \in \mathrm{GL}_1\}$$

via the action of the latter on D given by

$$(\alpha, \beta) \mapsto \alpha x \bar{\beta}.$$

Moreover, an element of $\mathrm{GO}(D, \mathbb{N}_D)$ of determinant -1 is given by the conjugation action $c : x \mapsto \bar{x}$ on D . We have:

$$\mathrm{GSO}(D) \cong \begin{cases} \mathrm{GSO}_{2,2}(F) & \text{if } D \text{ is split;} \\ \mathrm{GSO}_{4,0}(F) & \text{if } D \text{ is non-split.} \end{cases}$$

The similitude character of $\mathrm{GSO}(D)$ is given by

$$\lambda_D : (\alpha, \beta) \mapsto N_D(\alpha \cdot \beta),$$

which is surjective onto F^\times . Since λ_D is surjective, we have $\mathrm{GSp}(W)^+ = \mathrm{GSp}_4(W)$ and the induced Weil representation is a representation of $\mathrm{GO}(D) \times \mathrm{GSp}(W)$. The investigation of the theta correspondence for these dual pairs has been initiated by B. Roberts [Ro2]. In Thm. 8.2 and Thm. 8.1 below, we shall complete the study initiated in [Ro2] by giving an explicit determination of the theta correspondence.

Now consider the rank 6 quadratic space:

$$(V_D, q_D) = (D, \mathbb{N}_D) \oplus \mathbb{H}$$

where \mathbb{H} is the hyperbolic plane. Then one has an isomorphism

$$\mathrm{GSO}(V_D) \cong (\mathrm{GL}_2(D) \times \mathrm{GL}_1) / \{(z \cdot \mathrm{Id}, z^{-2}) : z \in \mathrm{GL}_1\}.$$

To see this, note that the quadratic space V_D can also be described as the space of 2×2 -Hermitian matrices with entries in D , so that a typical element has the form

$$(a, d; x) = \begin{pmatrix} a & x \\ \bar{x} & d \end{pmatrix}, \quad a, d \in F \text{ and } x \in D,$$

equipped with the quadratic form $-\det(a, d; x) = -ad + \mathbb{N}_D(x)$. The action of $\mathrm{GL}_2(D) \times \mathrm{GL}_1$ on this space is given by

$$(g, z)(X) = z \cdot g \cdot X \cdot \bar{g}^\dagger.$$

The similitude factor of $\mathrm{GSO}(V_D)$ is given by

$$\lambda_D(g, z) = N(g) \cdot z^2,$$

where N is the reduced norm on the central simple algebra $M_2(D)$. Thus,

$$\mathrm{SO}(V_D) = \{(g, z) \in \mathrm{GSO}(V_D) : N(g) \cdot z^2 = 1\}.$$

In this paper, we only need to consider V_D when D is split. Thus, we shall suppress D from the notations, so that from now on throughout this paper, V denotes the 6 dimensional split quadratic space, i.e.

$$V = \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{H} \quad \text{and} \quad \mathrm{GSO}(V) = \mathrm{GSO}_{3,3}.$$

Moreover, since λ_V is surjective, we have $\mathrm{GSp}(W)^+ = \mathrm{GSp}(W)$, so that the induced Weil representation Ω is a representation of $R = \mathrm{GSp}(W) \times \mathrm{GO}(V)$. We shall only consider the theta correspondence for $\mathrm{GSp}(W) \times \mathrm{GSO}(V)$. There is no significant loss in restricting to $\mathrm{GSO}(V)$ because of the following lemma:

Lemma 2.4. *Let π (resp. τ) be an irreducible representation of $\mathrm{GSp}(W)$ (resp. $\mathrm{GO}(V)$) and suppose that*

$$\mathrm{Hom}_{\mathrm{GSp}(W) \times \mathrm{GO}(V)}(\Omega, \pi \otimes \tau) \neq 0.$$

Then τ is irreducible as a representation of $\mathrm{GSO}(V)$. If $\nu_0 = \lambda_V^{-3} \cdot \det$ is the unique non-trivial quadratic character of $\mathrm{GO}(V)/\mathrm{GSO}(V)$, then $\tau \otimes \nu_0$ does not participate in the theta correspondence with $\mathrm{GSp}(W)$.

Proof. First note that τ is irreducible when restricted to $\mathrm{GSO}(V)$ if and only if $\tau \otimes \nu_0 \neq \tau$. By a well-known result of Rallis [R, Appendix] (see also [Pr1, §5, Pg. 282]), the lemma holds in the setting of isometry groups. Suppose that $\tau|_{\mathrm{O}(V)} = \oplus_i \tau_i$. Then this result of Rallis implies that τ_i is irreducible when restricted to $\mathrm{SO}(V)$, so that $\tau_i \otimes \nu_0 \neq \tau_i$, and $\tau_i \otimes \nu_0$ does not participate in the theta correspondence with $Sp(W)$. This implies that $\tau \otimes \nu_0 \neq \tau$ and $\tau \otimes \nu_0$ does not participate in the theta correspondence with $\mathrm{GSp}(W)$. \square

Proposition 2.3 and the above lemma imply:

Corollary 2.5. *If π is a supercuspidal representation of $\mathrm{GSp}_4(F)$ and $\theta(\pi)$ is nonzero, then $\theta(\pi)$ is irreducible as a representation of $\mathrm{GSO}(V)$. Moreover, if $\theta(\pi) = \theta(\pi')$ as a representation of $\mathrm{GSO}(V)$ for some other supercuspidal π' , then $\pi = \pi'$.*

If $p \neq 2$, then the above corollary would hold for any irreducible representation π because one knows that Howe conjecture for isometry groups.

A study of the local theta correspondence for $\mathrm{GSp}_4 \times \mathrm{GSO}_{3,3}$ was undertaken in [W] and [GT2]. In Thm 8.3 below, we give a complete determination of this theta correspondence.

3. A Result of Kudla-Rallis

In this section, we recall a fundamental general result of Kudla-Rallis [KR] before specializing it to the cases of interest in this paper.

Let W_n be the $2n$ -dimensional symplectic vector space with associated symplectic group $\mathrm{Sp}(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with trivial discriminant. More precisely, let

$$V_m = \mathbb{H}^m \quad \text{and} \quad V_m^\# = D \oplus \mathbb{H}^{m-2}$$

and denote the orthogonal groups by $\mathrm{O}(V_m)$ and $\mathrm{O}(V_m^\#)$ respectively. For an irreducible representation π of $\mathrm{Sp}(W_n)$, one may consider the theta lifts $\theta_m(\pi)$ and $\theta_m^\#(\pi)$ to $\mathrm{O}(V_m)$ and $\mathrm{O}(V_m^\#)$ respectively (with respect to a fixed non-trivial additive character ψ). Set

$$\begin{cases} m(\pi) = \inf\{m : \theta_m(\pi) \neq 0\}; \\ m^\#(\pi) = \inf\{m : \theta_m^\#(\pi) \neq 0\}. \end{cases}$$

Then Kudla and Rallis [KR, Thms. 3.8 & 3.9] showed:

Theorem 3.1. (i) For any irreducible representation π of $\mathrm{Sp}(W_n)$,

$$m(\pi) + m^\#(\pi) \geq 2n + 2.$$

(ii) If π is a supercuspidal representation of $\mathrm{Sp}(W_n)$, then

$$m(\pi) + m^\#(\pi) = 2n + 2.$$

If we specialize this result to the case $\dim W_n = 4$, we obtain:

Theorem 3.2. (i) Let π be an irreducible supercuspidal representation of $\mathrm{GSp}_4(F)$. Then one has the following two mutually exclusive possibilities:

(A) π participates in the theta correspondence with $\mathrm{GSO}(D) = \mathrm{GSO}_{4,0}(F)$, where D is non-split;

(B) π participates in the theta correspondence with $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$.

One of the purposes of this paper is to extend the dichotomy of this theorem to all irreducible representations of $\mathrm{GSp}_4(F)$. Moreover, our main results make this dichotomy completely explicit. For example, we will see that the supercuspidal representations of Type (A) are precisely those which are non-generic.

On the other hand, one may consider the mirror situation, where one fixes an irreducible representation of $\mathrm{O}(V_m)$ or $\mathrm{O}(V_m^\#)$ and consider its theta lifts $\theta_n(\pi)$ to the tower of symplectic groups $\mathrm{Sp}(W_n)$. Then, with $n(\pi)$ defined in the analogous fashion, one expects that

$$n(\pi) + n(\pi \otimes \det) = 2m.$$

For similitude groups, this implies that

$$n(\pi) + n(\pi \otimes \nu_0) = 2m,$$

where ν_0 is the non-trivial character of $\mathrm{GO}(V_m)/\mathrm{GSO}(V_m)$. When $m = 2$, this expectation has been proved by B. Roberts [Ro2, Thm. 7.8 and Cor. 7.9] as follows:

Theorem 3.3. Let π be an irreducible representation of $\mathrm{GO}(D)$, where D is possibly split. Then

$$n(\pi) + n(\pi \otimes \nu_0) = 4.$$

In particular, if $\pi = \pi \otimes \nu_0$, then $n(\pi) = 2$.

4. Whittaker Modules of Weil Representations

In this section, we describe the Whittaker modules of the Weil representations $\Omega_{W,D}$ and $\Omega_{W,V}$, with $\dim W = 4$. We omit the proofs since they are by-now-standard; see for example [GT2 Prop. 7.4] and [MS, Prop. 4.1].

Proposition 4.1. *Let D be a (possibly split) quaternion algebra over F and consider the Weil representation $\Omega_{D,W}$ of $\mathrm{GO}(D) \times \mathrm{GSp}(W)$. Let U be the unipotent radical of a Borel subgroup of $\mathrm{GSp}(W)$ and let ψ be a generic character of $U(F)$. Similarly, let U_0 be the unipotent radical of a Borel subgroup of $\mathrm{GO}(D)$ when D is split, and let ψ_0 be a generic character of $U_0(F)$. Then we have:*

$$(\Omega_{D,W})_{U,\psi} = 0$$

if D is non-split, and

$$(\Omega_{D,W})_{U,\psi} \cong \mathrm{ind}_{U_0}^{\mathrm{GSO}(D)} \psi_0$$

if D is split.

Corollary 4.2. (i) *If D is non-split, then no generic representation of $\mathrm{GSp}(W)$ participates in theta correspondence with $\mathrm{GO}(D)$.*

(ii) *Let π be an irreducible generic representation of $\mathrm{GO}(V_2) = \mathrm{GSO}_{2,2}(F)$. Then π is generic if and only if $\Theta_{V_2,W}(\pi)$ contains a generic constituent, in which case this generic constituent is unique.*

Proposition 4.3. *Consider the Weil representation $\Omega_{W,V}$ of $\mathrm{GSp}(W) \times \mathrm{GSO}(V)$. Let U be the unipotent radical of a Borel subgroup of $\mathrm{GSp}(W)$ and let ψ be a generic character of $U(F)$. Similarly, let U_0 be the unipotent radical of a Borel subgroup of $\mathrm{GO}(V)$, and let ψ_0 be a generic character of $U_0(F)$. Then we have*

$$(\Omega_{W,V})_{U_0,\psi_0} \cong \mathrm{ind}_U^{\mathrm{GSp}(W)} \psi.$$

Corollary 4.4. *Let π be an irreducible representation of $\mathrm{GSp}(W)$. Then π is generic if and only if $\Theta_{W,V}(\pi)$ contains a generic constituent, in which case this generic constituent is unique.*

Corollaries 4.2(i) and 4.4 imply:

Corollary 4.5. *The dichotomy result of Theorem 3.2 holds for generic representations of $\mathrm{GSp}(W)$.*

5. Representations of $\mathrm{GSp}_4(F)$

In order to state our main results, we need to introduce some notations and recall some results about various principal series representations of $\mathrm{GSp}_4(F)$. In the following, by the term “discrete series” or “tempered” representations of $\mathrm{GSp}(W) = \mathrm{GSp}_4(F)$ or $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$, we mean a representation which is unitarizable discrete series or unitarizable tempered after twisting by a 1-dimensional character.

5.1. Principal series representations. Recall from Section 2 that we have a Witt decomposition $W = Y^* \oplus Y$. Suppose that

$$Y^* = F \cdot e_1 \oplus F \cdot e_2 \quad \text{and} \quad Y = F \cdot f_1 \oplus F \cdot f_2$$

and consider the decomposition $W = F e_1 \oplus W' \oplus F f_1$, where $W' = \langle e_2, f_2 \rangle$. Let $Q(Z) = L(Z) \cdot U(Z)$ be the maximal parabolic stabilizing the line $Z = F \cdot f_1$, so that

$$L(Z) = \mathrm{GL}(Z) \times \mathrm{GSp}(W')$$

and $U(Z)$ is a Heisenberg group:

$$1 \longrightarrow \mathrm{Sym}^2 Z \longrightarrow U(Z) \longrightarrow W' \otimes Z \longrightarrow 1.$$

This is typically called the Klingen parabolic subgroup in the literature. A representation of $L(Z)$ is thus of the form $\chi \boxtimes \tau$ where τ is a representation of $\mathrm{GSp}(W') \cong \mathrm{GL}_2(F)$. We let $I_{Q(Z)}(\chi, \tau)$ be the corresponding parabolically induced representation. If $I_{Q(Z)}(\chi, \tau)$ is a standard module, then it has a unique irreducible quotient (the Langlands quotient), which we shall denote by $J_{Q(Z)}(\chi, \tau)$. The same notation applies to other principal series representations to be introduced later.

The module structure of $I_{Q(Z)}(\chi, \tau)$ is known by Sally-Tadic [ST] and a convenient reference is [RS]. In particular, we note the following:

Lemma 5.1. *(a) Let τ be a supercuspidal representation of $\mathrm{GL}_2(F)$. The induced representation $I_{Q(Z)}(\chi, \tau)$ is reducible iff one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = \chi_0 | - |^{\pm 1}$ and χ_0 is a non-trivial quadratic character such that $\tau \otimes \chi_0 \cong \tau$.

In case (i), the representation $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible tempered representations, exactly one of which is generic. In case (ii), assuming without loss of generality that $\chi = \chi_0 \cdot | - |$, one has a (non-split) short exact sequence:

$$0 \longrightarrow \mathrm{St}(\chi_0, \tau_0) \longrightarrow I_{Q(Z)}(\chi_0 | - |, \tau_0 | - |^{-1/2}) \longrightarrow \mathrm{Sp}(\chi_0, \tau_0) \longrightarrow 0$$

where $\mathrm{St}(\chi_0, \tau_0)$ is a generic discrete series representation and the Langlands quotient $\mathrm{Sp}(\chi_0, \tau_0)$ is non-generic.

(b) If τ is the twisted Steinberg representation of GL_2 , then $I_{Q(Z)}(\chi, \tau)$ is reducible iff one of the following holds:

(i) $\chi = 1$;

(ii) $\chi = | - |^{\pm 2}$

In case (i), $I_{Q(Z)}(1, \mathrm{st}_\chi)$ is the sum of two irreducible tempered representations, exactly one of which is generic. In case (ii), $I_{Q(Z)}(| - |^2, \mathrm{st}_\chi \cdot | - |^{-1})$ has the twisted Steinberg representation $\mathrm{St}_{\mathrm{PGSp}_4} \otimes \chi$ as its unique irreducible submodule.

(c) For general τ , there is a standard intertwining operator

$$I_{Q(Z)}(\chi^{-1}, \tau \otimes \chi) \longrightarrow I_{Q(Z)}(\chi, \tau),$$

which is an isomorphism if $I_{Q(Z)}(\chi, \tau)$ is irreducible. If $I_{Q(Z)}(\chi^{-1}, \tau \otimes \chi)$ is a standard module, then the image of this operator is the unique irreducible submodule of $I_{Q(Z)}(\chi, \tau)$.

Now let $P(Y) = M(Y) \cdot N(Y)$ be the Siegel parabolic subgroup stabilizing Y so that

$$M(Y) = \mathrm{GL}(Y) \times \mathrm{GL}_1$$

and $N(Y) \cong \mathrm{Sym}^2 Y$. A representation of $M(Y)$ is thus of the form $\tau \boxtimes \chi$ with τ a representation of $\mathrm{GL}(Y) \cong \mathrm{GL}_2(F)$ and χ a character of $\mathrm{GL}_1(F)$. We denote the normalized induced representation by $I_{P(Y)}(\tau, \chi)$. As before, the module structure of $I_{P(Y)}(\tau, \chi)$ is completely known [ST] and a convenient reference is [RS]. In particular, we note the following:

Lemma 5.2. (a) Suppose that τ is a supercuspidal representation of $\mathrm{GL}(Y) \cong \mathrm{GL}_2(F)$. Then $I_{P(Y)}(\tau, \mu)$ is reducible iff $\tau = |\cdot|^{-1/2} \cdot \tau_0$ with τ_0 having trivial central character. In this case, one has a non-split short exact sequence:

$$0 \longrightarrow St(\tau_0, \mu_0) \longrightarrow I_{P(Y)}(\tau_0|\cdot|^{-1/2}, \mu_0|\cdot|^{-1/2}) \longrightarrow Sp(\tau_0, \mu_0) \longrightarrow 0$$

where $St(\tau_0, \mu_0)$ is a generic discrete series representation and the Langlands quotient $Sp(\tau_0, \mu_0)$ is non-generic.

(b) Suppose that τ is a twisted Steinberg representation of $\mathrm{GL}(Y)$. Then $I_{P(Y)}(\tau, \mu)$ is reducible iff one of the following holds:

(i) $\tau = st \cdot |\cdot|^{-1/2}$; in this case, $I_{P(Y)}(st|\cdot|^{-1/2}, \mu)$ has a unique irreducible Langlands quotient and a unique irreducible tempered submodule, which is the generic summand of $I_{Q(Z)}(1, st_\mu \cdot |\cdot|^{-1/2})$.

(ii) $\tau = st_\chi \cdot |\cdot|^{-1/2}$, with χ a non-trivial quadratic character. In this case, the representation $I_{P(Y)}(st_\chi|\cdot|^{-1/2}, \mu_0|\cdot|^{-1/2})$ has a unique irreducible Langlands quotient and a unique irreducible submodule which is a generic discrete series representation $St(st_\chi, \mu_0)$. Moreover, $St(st_\chi, \mu_0) \cong St(st_\chi, \chi\mu_0)$.

(iii) $\tau = st \cdot |\cdot|^{-3/2}$; in this case, $I_{P(Y)}(st|\cdot|^{-3/2}, \mu|\cdot|^{-3/2})$ has the twisted Steinberg representation $St_{\mathrm{PGSp}_4} \otimes \mu$ as its unique irreducible submodule.

(c) There is a standard intertwining operator

$$I_{P(Y)}(\tau, \mu) \longrightarrow I_{P(Y)}(\tau^\vee, \omega_\tau \mu),$$

which is an isomorphism if $I_{P(Y)}(\tau, \mu)$ is irreducible. If $I_{P(Y)}(\tau, \mu)$ is a standard module, then the image of this operator is the unique irreducible submodule of $I_{P(Y)}(\tau^\vee, \omega_\tau \mu)$.

Finally, let $B = P(Y) \cap Q(Z) = T \cdot U$ be a Borel subgroup of $\mathrm{GSp}(W)$, so that

$$T \cong (\mathrm{GL}(F \cdot f_1) \times \mathrm{GL}(F \cdot f_2)) \times \mathrm{GL}_1.$$

In particular, for characters χ_1, χ_2 and χ of $\mathrm{GL}_1(F)$, we let $I_B(\chi_1, \chi_2; \chi)$ denote the normalized induced representation. Again, we refer the reader to [RS] for the reducibility points and module structure of $I_B(\chi_1, \chi_2; \chi)$. We simply note here that if χ_1 and χ_2 are unitary, then $I_B(\chi_1, \chi_2; \chi)$ is irreducible.

5.2. Non-Supercuspidal Representations. We can now give a concise enumeration of the non-supercuspidal representations of $\mathrm{GSp}_4(F)$.

5.2.1. Discrete Series Representations. The non-supercuspidal discrete series representations of $\mathrm{GSp}_4(F)$ are precisely the following representations:

- (a) the generalized Steinberg representation $St(\chi, \tau)$ of Lemma 5.1(a)(ii), with τ supercuspidal and χ a non-trivial quadratic character such that $\tau \otimes \chi = \tau$;
- (b) the generalized Steinberg representation $St(\tau, \mu)$ of Lemma 5.2(a) and (b)(ii), so that τ is a discrete series representation of PGL_2 and $\tau \neq st$.
- (c) the twisted Steinberg representation $St_{\mathrm{PGSp}_4} \otimes \chi$ of Lemma 5.1(b)(ii) and Lemma 5.2(b)(iii).

All these representations are generic.

5.2.2. Non-Discrete Series Representations. Now we consider non-discrete series representations. Every tempered representation is a constituent of a twist of an induced representation $I_P(\tau)$ with τ a

unitary discrete series representation. Such an induced representation is irreducible (and thus generic) except in the setting of Lemma 5.1(a)(i) and (b)(i). In this exceptional case, for a discrete series representation τ of $\mathrm{GSp}(W') \cong \mathrm{GL}_2(F)$, we have:

$$I_{Q(Z)}(\mathbf{1}, \tau) = \pi_{gen}(\tau) \oplus \pi_{ng}(\tau)$$

where $\pi_{gen}(\tau)$ is generic and $\pi_{ng}(\tau)$ is non-generic.

Suppose now that π is an irreducible non-tempered representation of $\mathrm{GSp}_4(F)$. By the Langlands classification, there is a unique standard representation $I_P(\sigma)$ (with σ essentially tempered) which has π as its unique irreducible quotient. In fact, it will be more convenient for us to make use of the dual version of Langlands classification, so that π is the unique submodule of $I_P(\sigma)$, for an essentially tempered representation σ whose central character is in the relevant negative Weyl chamber. Note that since the Levi subgroups of GSp_4 are all of GL -type, any essentially tempered representation of a Levi subgroup of $\mathrm{GSp}_4(F)$ is fully induced from a twist of a discrete series representation.

Summarizing the above discussions, we have:

Proposition 5.3. *The non-discrete series representations of $\mathrm{GSp}_4(F)$ fall into the following three disjoint families:*

- (a) $\pi \hookrightarrow I_{Q(Z)}(\chi| - |^{-s}, \tau)$ with χ a unitary character, $s \geq 0$ and τ a discrete series representation of $\mathrm{GSp}(W')$ up to twist. In fact, π is the unique irreducible submodule, except in the exceptional tempered case mentioned above (with χ trivial and $s = 0$).
- (b) $\pi \hookrightarrow I_{P(Y)}(\tau| - |^{-s}, \chi)$ with χ an arbitrary character, $s \geq 0$ and τ a unitary discrete series representation of $\mathrm{GL}(Y)$. In this case, π is the unique irreducible submodule.
- (c) $\pi \hookrightarrow I_B(\chi_1| - |^{-s_1}, \chi_2| - |^{-s_2}; \chi)$ where χ_1 and χ_2 are unitary and $s_1 \geq s_2 \geq 0$. By induction in stages, we see that

$$I_B(\chi_1| - |^{-s_1}, \chi_2| - |^{-s_2}; \chi) = I_{Q(Z)}(\chi_1| - |^{-s_1}, \pi(\chi\chi_2| - |^{-s_2}, \chi)).$$

We may now consider two subcases:

- (i) if $\chi_2| - |^{-s_2} \neq | - |^{-1}$, then $\pi(\chi\chi_2| - |^{-s_2}, \chi)$ is irreducible and we have:

$$\pi \hookrightarrow I_{Q(Z)}(\chi_1| - |^{-s_1}, \pi(\chi\chi_2| - |^{-s_2}, \chi))$$

as the unique irreducible submodule.

- (ii) if $\chi_2| - |^{-s_2} = | - |^{-1}$, then $\pi(\chi\chi_2| - |^{-s_2}, \chi)$ contains $\chi| - |^{-1/2}$ as a unique irreducible submodule and so

$$\pi \hookrightarrow I_{Q(Z)}(\chi_1| - |^{-s_1}, (\chi \circ \det) \cdot |\det|^{-1/2})$$

as the unique irreducible submodule.

6. Representations of $\mathrm{GSO}(D)$

In this section, we set up some notations for the irreducible representations of $\mathrm{GSO}(D)$. In this case, we have

$$V = (D, -\mathbb{N}_D)$$

where D is a quaternion F -algebra (possibly split) with reduced norm \mathbb{N}_D . We have the identification

$$\mathrm{GSO}(V) \cong (D^\times \times D^\times) / \{(z, z^{-1}) \mid z \in F^\times\}$$

via

$$(g_1, g_2) : x \longmapsto g_1 \cdot x \cdot \bar{g}_2.$$

Moreover, the main involution $x \mapsto \bar{x}$ on D gives an order two element c of $O(V)$ with determinant -1 , so that $GO(V) = GSO(V) \rtimes \langle c \rangle$. The conjugation of c on $GSO(V)$ is given by

$$(g_1, g_2) \mapsto (g_2, g_1).$$

Thus, an irreducible representation of $GSO(D)$ is of the form $\tau_1 \boxtimes \tau_2$ for irreducible representations τ_i of D^\times with the same central character $\omega_{\tau_1} = \omega_{\tau_2}$. Moreover, the action of c sends $\tau_1 \boxtimes \tau_2$ to $\tau_2 \boxtimes \tau_1$. If $\tau_1 = \tau_2$, then the representation $\tau_1 \boxtimes \tau_2$ extends to $GO(D)$ in two different ways. If $\tau_1 \neq \tau_2$, then

$$\text{ind}_{GSO(D)}^{GO(D)} \tau_1 \boxtimes \tau_2 = \text{ind}_{GSO(D)}^{GO(D)} \tau_2 \boxtimes \tau_1$$

is an irreducible representation of $GO(D)$. This describes all the irreducible representations of $GO(D)$.

When D is split, the quadratic space D is split and we have a Witt decomposition

$$D = X \oplus X^*$$

for a two dimensional isotropic space X . Let $P(X) = M(X) \cdot N(X)$ be the parabolic subgroup stabilizing X , so that

$$M(X) \cong GL(X) \times GL_1 \quad \text{and} \quad N(X) \cong \wedge^2 X.$$

For an irreducible representation $\tau \boxtimes \chi$ of $GL(X) \times GL_1(F) \cong GL_2(F) \times F^\times$, we let $I_{P(X)}(\tau, \chi)$ denote the normalized induced representation. The following lemma is easy to check.

Lemma 6.1. *Under the identification $GSO(D) \cong (GL_2 \times GL_2)/F^\times$, we have*

$$\pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2) = I_{P(X)}(\tau \cdot \chi_2^{-1}, \chi_2).$$

7. Representations of $GSO(V)$.

Now we need to establish some notations for the representations of $GSO(V) = GSO_{3,3}(F)$. Though we may identify $GSO(V)$ as a quotient of $GL_4(F) \times GL_1(F)$ as in Section 2, it is in fact better not to do so for the purpose of the computation of local theta lifting.

Recall that we have a decomposition

$$V = F \cdot (1, 0) \oplus V_2 \oplus F \cdot (0, 1)$$

where V_2 is the split rank 4 quadratic space. Let $J = F \cdot (1, 0)$ and let $P(J) = M(J) \cdot N(J)$ be the stabilizer of J , so that $M(J) = GL(J) \times GSO(V_2)$. We represent an element of $M(J)$ by (a, α, β) with

$$(\alpha, \beta) \in GSO(V_2) \cong (GL_2 \times GL_2) / \{(z, z^{-1}) : z \in F^\times\}.$$

For a character χ and a representation $\tau_1 \boxtimes \tau_2$ of $GSO(V_2)$, one may consider the normalized induced representation $I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)$.

Under the natural map $GL_4 \times GL_1 \rightarrow GSO(V)$, the inverse image of $P(J)$ is the parabolic $P \times GL_1$, where P is the $(2, 2)$ -parabolic in GL_4 . Indeed, under the natural map $P \times GL_1 \rightarrow P(J)$,

$$\left(\left(\begin{array}{cc} \alpha & \\ & \beta \end{array} \right), z \right) \mapsto (a, \alpha', \beta'),$$

then we have:

$$\begin{cases} \alpha = a \cdot \overline{\alpha'}^{-1} \\ \beta = \beta' \\ z = a^{-1} \cdot \mathbb{N}(\alpha'). \end{cases}$$

From this, one deduces that

$$I_{P(J)}(\tau_1, \tau_2) \boxtimes \mu \cong I_{P(J)}(\omega_1 \mu^{-1}, \tau_1^\vee \mu \boxtimes \tau_2).$$

The following lemma is well-known:

Lemma 7.1. (i) Let τ be a supercuspidal representation of $\mathrm{GL}_2(F)$. Then one has a short exact sequence of representations of $\mathrm{GL}_4(F)$:

$$0 \longrightarrow St(\tau) \longrightarrow I_P(\tau| - |^{1/2}, \tau| - |^{-1/2}) \longrightarrow Sp(\tau) \longrightarrow 0$$

where $St(\tau)$ is a discrete series representation (a generalized Steinberg representation) and $Sp(\tau)$ is the unique Langlands quotient (a generalized Speh representation).

(ii) If $\tau = st_\chi$, then the principal series $I_P(\tau| - |, \tau| - |^{-1})$ has a unique irreducible submodule which is the twisted Steinberg representation $St_\chi := St_{P_{\mathrm{GL}_4}} \otimes \chi$. Moreover, $I_P(\tau| - |^{1/2}, \tau| - |^{-1/2})$ is also reducible, but its constituents are not discrete series representations.

(iii) The situations described in (i) and (ii) are the only cases when $I_P(\tau_1, \tau_2)$ is reducible with τ_i discrete series representations.

We shall need another parabolic of $\mathrm{GSO}(V)$. Let Q be the (1,2,1)-parabolic of GL_4 so that its Levi factor is $\mathrm{GL}_1 \times \mathrm{GL}_2 \times \mathrm{GL}_1$. We let $I_Q(\chi_1, \tau, \chi_2)$ denote the normalized induced representation, with χ_i characters of $\mathrm{GL}_1(F)$ and τ a representation of $\mathrm{GL}_2(F)$. Consider the image of $Q \times \mathrm{GL}_1$ under the natural map $\mathrm{GL}_4 \times \mathrm{GL}_1 \longrightarrow \mathrm{GSO}(V)$. This image is the stabilizer $Q(E)$ of a 2-dimensional isotropic subspace E containing J . Writing

$$V = E \oplus V_1 \oplus E^*$$

where V_1 is a split rank 2 quadratic space, we see that the Levi factor of $Q(E)$ is

$$L(E) = \mathrm{GL}(E) \times \mathrm{GSO}(V_1) \cong \mathrm{GL}_2 \times (\mathrm{GL}_1 \times \mathrm{GL}_1).$$

Thus, for a representation τ of $\mathrm{GL}(E)$ and a character χ of F^\times , we may consider the normalized induced representation $I_{Q(E)}(\tau, \chi \circ \lambda_{V_1})$. Now under the natural map $Q \times \mathrm{GL}_1 \longrightarrow Q(E)$, we have

$$\left(\left(\begin{array}{ccc} t_1 & & \\ & B & \\ & & t_4 \end{array} \right), z \right) \mapsto (t_1 B, (t_1 t_4, \det B)) \in \mathrm{GL}(E) \times \mathrm{GSO}(V_1).$$

From this, one easily calculates that

$$I_{Q(E)}(\tau, \chi \circ \lambda_{V_1}) = I_Q(\omega_\tau \chi, \tau \otimes \chi, \chi) \boxtimes \chi^2 \omega_\tau.$$

Finally, we have the Borel subgroup $B_0 = P \cap Q$ of GL_4 and the principal series $I_B(\chi_1, \chi_2, \chi_3, \chi_4)$.

8. The Main Results.

We are now ready to state our main results concerning the computation of local theta correspondences. But before stating them, let us note that for any quadratic space V , there is an action of $\mathrm{GO}(V)$ on $\Pi(\mathrm{GSO}(V))$, and we denote the class of the equivalence classes defined by this action by

$$\Pi(\mathrm{GSO}(V)) / \sim.$$

Theorem 8.1. Suppose that D is the quaternion division algebra over F . Let $\tau_1^D \boxtimes \tau_2^D$ be an irreducible representation of $\mathrm{GSO}(D)$. Then we have the following:

(i) $\Theta(\tau_1^D \boxtimes \tau_2^D)$ is an irreducible representation of $\mathrm{GSp}_4(F)$.

(ii) If $\tau_1^D = \tau_2^D = \tau_D$, then

$$\Theta(\tau_D \boxtimes \tau_D) = \pi_{ng}(JL(\tau_D)),$$

which is the unique non-generic summand of $I_{Q(Z)}(1, JL(\tau_D))$.

(iii) If $\tau_1^D \neq \tau_2^D$, then $\Theta(\tau_1^D \boxtimes \tau_2^D) = \Theta(\tau_2^D \boxtimes \tau_1^D)$ is a non-generic supercuspidal representation of $\mathrm{GSp}_4(F)$.

(iv) The map

$$\tau_1^D \boxtimes \tau_2^D \mapsto \Theta(\tau_1^D \boxtimes \tau_2^D)$$

defines a bijection between

$$\Pi(\mathrm{GSO}(D))/\sim$$

and

$$\Pi(\mathrm{GSp}_4)_{ng}^{temp} := \{\text{non-generic tempered representations of } \mathrm{GSp}_4\}.$$

Theorem 8.2. *Let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\mathrm{GSO}_{2,2}(F)$ and let $\theta(\tau_1 \boxtimes \tau_2)$ be its small theta lift to $\mathrm{GSp}_4(F)$. Then $\theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_2 \boxtimes \tau_1)$ can be determined as follows.*

(i) If $\tau_1 = \tau_2 = \tau$ is a discrete series representation, then

$$\theta(\tau_1 \boxtimes \tau_2) = \pi_{gen}(\tau),$$

which is the unique generic constituent of $I_{Q(Z)}(1, \tau)$.

(ii) If $\tau_1 \neq \tau_2$ are both supercuspidal, then $\theta(\tau_1 \boxtimes \tau_2)$ is supercuspidal.

(iii) If τ_1 is supercuspidal and $\tau_2 = st_\chi$, then

$$\theta(\tau_1 \boxtimes \tau_2) = St(\tau_1 \otimes \chi^{-1}, \chi).$$

(iv) Suppose that $\tau_1 = st_{\chi_1}$ and $\tau_2 = st_{\chi_2}$ with $\chi_1 \neq \chi_2$, so that $\chi_1^2 = \chi_2^2$. Then

$$\theta(\tau_1 \boxtimes \tau_2) = St(st_{\chi_1/\chi_2}, \chi_2) = St(st_{\chi_2/\chi_1}, \chi_1).$$

(v) Suppose that τ_1 is a discrete series representation and $\tau_2 \hookrightarrow \pi(\chi, \chi')$ with $|\chi/\chi'| = | - |^{-s}$ and $s \geq 0$, so that τ_2 is non-discrete series. Then

$$\theta(\tau_1 \otimes \tau_2) = J_{P(Y)}(\tau_1 \otimes \chi^{-1}, \chi).$$

(vi) Suppose that

$$\tau_1 \hookrightarrow \pi(\chi_1, \chi'_1) \quad \text{and} \quad \tau_2 \hookrightarrow \pi(\chi_2, \chi'_2)$$

with

$$|\chi_i/\chi'_i| = | - |^{-s_i}, \quad s_1 \geq s_2 \geq 0.$$

Then

$$\theta(\tau_1 \boxtimes \tau_2) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

In particular, the map $\tau_1 \boxtimes \tau_2 \mapsto \theta(\tau_1 \boxtimes \tau_2)$ defines an injection

$$\Pi(\mathrm{GSO}_{2,2})/\sim \hookrightarrow \Pi(\mathrm{GSp}_4).$$

Theorem 8.3. *Let π be an irreducible representation of $\mathrm{GSp}_4(F)$ and consider its theta lift to $\mathrm{GSO}(V)$.*

(i) The representation $\theta(\pi)$ is nonzero iff $\pi \notin \Pi(\mathrm{GSp}_4)_{ng}^{temp}$, in which case it is irreducible.

(ii) The map $\pi \mapsto \theta(\pi)$ defines an injective map from $\Pi(\mathrm{GSp}_4) \setminus \Pi(\mathrm{GSp}_4)_{ng}^{temp}$ into the set of irreducible representations of $\mathrm{GSO}(V)$.

Moreover, for $\pi \notin \Pi(\mathrm{GSp}_4)_{ng}^{temp}$, $\theta(\pi)$ can be described in terms of π as follows.

(iii) **(Supercuspidal)** If π is supercuspidal, then $\theta(\pi)$ is supercuspidal unless π has a nonzero theta lift to $\mathrm{GSO}_{2,2}(F)$. If π is the theta lift of $\tau_1 \boxtimes \tau_2$ from $\mathrm{GSO}_{2,2}(F)$, then

$$\theta(\pi) = I_P(\tau_1, \tau_2) \boxtimes \omega_{\tau_1}.$$

(iv) **(Generalized Steinberg)** Suppose that $\pi = St(\chi, \tau)$ as in 5.2.1(a). Then

$$\theta(St(\chi, \tau)) = St(\tau) \boxtimes \omega_{\tau\chi}.$$

On the other hand, if $\pi = St(\tau, \mu)$ as in 5.2.1(b), then

$$\theta(St(\tau, \mu)) = I_P(\tau \otimes \mu, st \otimes \mu) \boxtimes \mu^2.$$

(v) **(Twisted Steinberg)** If $\pi = St_{\mathrm{PGSp}_4} \otimes \chi$ is a twisted Steinberg representation, then

$$\theta(St_{\mathrm{PGSp}_4} \otimes \chi) = St_{\chi} \boxtimes \chi^2.$$

(vi) **(Non-discrete series)** We consider the different cases according to 5.2.2:

(a) Suppose that $\pi \hookrightarrow I_{Q(Z)}(\chi, \tau)$ as in 5.2.2(a), so that $|\chi| = | - |^{-s}$ with $s \geq 0$. Then

$$\theta(\pi) = J_P(\tau, \tau \cdot \chi) \boxtimes \omega_{\tau\chi}.$$

(b) Suppose that $\pi \hookrightarrow I_{P(Y)}(\tau, \chi)$ as in 5.2.2(b), so that $|\omega_{\tau}| = | - |^{-2s}$ with $s \geq 0$. Then

$$\theta(\pi) = J_Q(1, \tau, \omega_{\tau}) \cdot \chi \boxtimes \chi^2 \omega_{\tau}.$$

(c) Suppose that

$$\pi \hookrightarrow I_B(\chi_1, \chi_2; \chi)$$

where $|\chi_1| = | - |^{-s_1}$ and $|\chi_2| = | - |^{-s_2}$ with $s_1 \geq s_2 \geq 0$. Then

$$\theta(\pi) = \chi \cdot J_{B_0}(1, \chi_2, \chi_1, \chi_1\chi_2) \boxtimes \chi^2 \chi_1 \chi_2.$$

Remarks: In Section 14, we display the results of the above three theorems in the form of three tables. The results there are given in terms of the usual Langlands classification which describes π as a unique irreducible quotient of a standard module, as opposed to describing π as a unique submodule as we have done in the theorems.

9. Proof of Theorem 8.1

We begin with the proof of Thm. 8.1.

(i) Let $\pi = \tau_1^D \boxtimes \tau_2^D$ be an irreducible representation of $\mathrm{GSO}(D)$ which is contained in a (unique) irreducible representation $\tilde{\pi}$ of $\mathrm{GO}(D)$. Then by Prop. 2.3, $\Theta(\tilde{\pi})$ is either zero or an irreducible representation of $\mathrm{GSp}_4(F)$. By Thm. 3.3, at least one of $\Theta(\tilde{\pi})$ or $\Theta(\tilde{\pi} \otimes \nu_0)$ is nonzero. Thus we conclude that

$$\Theta(\tau_1^D \boxtimes \tau_2^D) = \Theta(\tau_2^D \boxtimes \tau_1^D)$$

is nonzero and irreducible. This proves (i). Moreover, Prop. 2.3 implies that one has an injection

$$\Pi(\mathrm{GSO}(D))/ \sim \hookrightarrow \Pi(\mathrm{GSp}(W)).$$

(ii) Now suppose that $\tau_1^D = \tau_2^D = \tau_D$. Then it is well-known that $\pi = \tau_D \boxtimes \tau_D$ participates in the theta correspondence with $\mathrm{GSp}(W') = \mathrm{GSp}_2 = \mathrm{GL}_2$. Indeed, one has

$$\Theta_{D, W_1}(\pi) = JL(\tau_D).$$

On the other hand, by Theorem A.2 in the appendix, one has an $\mathrm{GSO}(D) \times Q(Z)$ -equivariant surjection

$$R_{Q(Z)}(\Omega_{D, W}) \twoheadrightarrow \Omega_{D, W'}.$$

By Frobenius reciprocity, one has a nonzero $\mathrm{GSO}(D) \times \mathrm{GSp}(W)$ -equivariant map

$$\Omega_{D, W'} \longrightarrow \pi \boxtimes I_{Q(Z)}(1, JL(\tau_D)).$$

In view of (i), we see that $\Theta(\pi)$ is either equal to $\pi_{gen}(JL(\tau_D))$ or $\pi_{ng}(JL(\tau_D))$. By Cor. 4.2(i), we see that the former possibility cannot occur, so that

$$\Theta(\tau_D \boxtimes \tau_D) = \pi_{ng}(JL(\tau_D)).$$

We have thus shown (ii). Moreover, from our enumeration of the non-supercuspidal representations of $\mathrm{GSp}(W)$ given in §5.2, we see that the representations of $\mathrm{GSp}(W)$ obtained in this way are precisely the essentially tempered non-generic non-supercuspidal representations.

(iii) On the other hand, suppose that $\tau_1^D \neq \tau_2^D$. Then by Thm. 3.3, $\Theta(\tau_1^D \boxtimes \tau_2^D)$ is supercuspidal. Cor. 4.2(i) implies that it is non-generic. This proves (iii).

(iv) By the above, we see that the map Θ gives an injection

$$\Pi(\mathrm{GSO}(D))/\sim \hookrightarrow \Pi(\mathrm{GSp}(W))_{ng}^{temp}.$$

Moreover, as we remarked in the proof of (ii), any non-supercuspidal representation in $\Pi(\mathrm{GSp}(W))_{ng}^{temp}$ lies in the image. Thus, to prove (iv), we need to show that every non-generic supercuspidal representation lies in the image.

Suppose then that π is a non-generic supercuspidal representation of $\mathrm{GSp}(W)$ which does not participate in the theta correspondence with $\mathrm{GSO}(D)$. By Thm. 3.2, π must participate in the theta correspondence with $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$. If this is the first occurrence of π in the tower of split orthogonal similitude groups, then $\Theta_{W, D}(\pi)$ is a supercuspidal representation of $\mathrm{GSO}_{3,3}(F)$ which is necessarily generic. By Cor. 4.4, this implies that π is itself generic, which is a contradiction. Thus, π must participate in the theta correspondence with $\mathrm{GSO}(V_2) = \mathrm{GSO}_{2,2}(F)$, which is the lower step of the tower. Moreover,

$$\sigma = \Theta_{W, V_2}(\pi)$$

is then a supercuspidal representation of $\mathrm{GO}(V_2)$, since no supercuspidal representations of $\mathrm{GSp}(W)$ participate in the theta correspondence with $\mathrm{GO}(V_1) = \mathrm{GO}_{1,1}(F)$. Since σ is necessarily generic, we see by Cor. 4.2(ii) that $\pi = \Theta_{V_2, W}(\sigma)$ must also be generic. This contradiction completes the proof of (iv).

Thm. 8.1 is proved.

10. Proof of Theorem 8.2

We now give the proof of Theorem 8.2. The key step is the computation of the normalized Jacquet modules of the induced Weil representation $\Omega_{V_2, W}$, where V_2 is the split four dimensional quadratic space. Before coming to this computation, we first introduce some more notations.

Recall that $V_2 = X \oplus X^*$, where X is a two dimensional isotropic space. We can write

$$X = F \cdot u_1 \oplus F \cdot u_2 \quad \text{and} \quad X^* = F \cdot v_1 \oplus F \cdot v_2$$

with $(u_i, v_j) = \delta_{ij}$. Let $P(X)$ be the parabolic subgroup of $\mathrm{GSO}(V_2)$ stabilizing X with Levi factor

$$M(X) \cong \mathrm{GL}(X) \times \mathrm{GL}_1.$$

Let $J = F \cdot u_1$ be the isotropic line spanned by u_1 in X and let $B(J)$ be the stabilizer of J in $M(X)$; it is also the stabilizer of the isotropic line spanned by v_2 in X^* . With respect to the basis $\{u_1, u_2\}$ of X , $B(J)$ is the group of upper triangular matrices in $M(X) \cong \mathrm{GL}(X) \times \mathrm{GL}_1$. We write

$$(t(a, b), \lambda) = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \lambda \right) \in B(J) \subset M(X).$$

Similarly, recall that $W = Y^* \oplus Y$,

$$Y^* = F \cdot e_1 \oplus F \cdot e_2 \quad \text{and} \quad Y = F \cdot f_1 \oplus F \cdot f_2$$

with $\langle e_i, f_j \rangle = \delta_{ij}$. The stabilizer of Y in $\mathrm{GSp}(W)$ is the Siegel parabolic subgroup $P(Y)$ with Levi factor

$$M(Y) \cong \mathrm{GL}(Y) \times \mathrm{GL}_1$$

and the stabilizer of $Z = F \cdot f_1$ in $\mathrm{GSp}(W)$ is the Klingen parabolic subgroup $Q(Z)$ with Levi factor

$$L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$$

where $W' = F \cdot e_2 \oplus F \cdot f_2$.

10.1. Jacquet modules. With the above notations, we can now compute the Jacquet module of $\Omega_{V_2, W}$ relative to $P(X)$. This follows the lines of [K] and we give the detailed computation in the appendix.

Proposition 10.1. *The normalized Jacquet module $R_{P(X)}(\Omega_{V_2, W})$ of $\Omega_{V_2, W}$ along $P(X)$ has a natural three step filtration as an $M(X) \times \mathrm{GSp}(W)$ -module whose successive quotients are described as follows:*

(1) *The top quotient is*

$$C \cong S(F^\times).$$

Here the action of $(m, \lambda) \in M(X) \cong \mathrm{GL}(X) \times \mathrm{GL}_1$ on $S(F^\times)$ is given by

$$((m, \lambda) \cdot f)(t) = |\det_X(m)|^{3/2} \cdot |\lambda|^{-3/2} \cdot f(\lambda \cdot t).$$

(2) *The middle subquotient is*

$$B \cong I_{B(J) \times Q(Z)}(S(F^\times) \otimes S(F^\times \cdot v_2 \otimes f_1)).$$

Here the action of $(t(a, b), \lambda) \in B(J)$ on $S(F^\times) \otimes S(F^\times \cdot v_2 \otimes f_1)$ is given by

$$((t(a, b), \lambda) \cdot f)(t, x) = |a| \cdot |\lambda|^{-3/2} \cdot f(\lambda \cdot t, b \cdot x),$$

whereas the action of $(\alpha, g) \in L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$ is given by

$$((\alpha, g) \cdot f)(t, x) = |\nu_{W'}(g)|^{-1} \cdot f(\nu_{W'}(g) \cdot t, \alpha^{-1} \cdot \nu_{W'}(g) \cdot x).$$

(3) *Finally, the submodule is*

$$A \cong I_{P(Y)}(S(F^\times) \otimes S(\mathrm{Isom}(X, Y)))$$

where $\mathrm{Isom}(X, Y)$ is the set of isomorphisms from X to Y as vector spaces (which is a torsor for $\mathrm{GL}(X)$ as well as for $\mathrm{GL}(Y)$). Here the action of $(m, \lambda) \in M(X) \cong \mathrm{GL}(X) \times \mathrm{GL}_1$ on $S(F^\times) \otimes S(\mathrm{Isom}(X, Y))$ is given by

$$((m, \lambda) \cdot f)(t, h) = f(\lambda \cdot t, h \circ m),$$

whereas the action of $(m', \lambda') \in M(Y) \cong \mathrm{GL}(Y) \times \mathrm{GL}_1$ is given by

$$((m', \lambda') \cdot f)(t, h) = f(\lambda' \cdot t, \lambda' \cdot m'^{-1} \circ h).$$

Proof. This is Theorem A.1, with $m = 4$, $n = 2$ and $t = 2$, and with Remark A.5 taken into account. \square

Corollary 10.2. *Let $\sigma = \pi(\chi_1, \chi_2) \boxtimes \tau$ be a representation of $\mathrm{GSO}(V_2) \cong (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))/F^\times$ such that τ is irreducible but $\pi(\chi_1, \chi_2)$ may be reducible, so that $\omega_\tau = \chi_1\chi_2$ and*

$$\sigma = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2).$$

Then

$$\mathrm{Hom}_{\mathrm{GSO}(V_2)}(\Omega, \sigma) = \mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee \cdot \chi_1 \boxtimes \chi_2).$$

(1) *If $\chi_1/\chi_2 \neq |-\|^3$, then*

$$\mathrm{Hom}_{M(X)}(C, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = 0.$$

(2) *If $R_B(\tau)$ does not have $\chi_1|-\|^{-1} \boxtimes \eta$ as a subquotient for any character η , then*

$$\mathrm{Hom}_{M(X)}(B, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = 0.$$

(3) *If the conditions in (i) and (ii) hold, then*

$$\mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee \cdot \chi_1 \boxtimes \chi_2) \subset \mathrm{Hom}_{M(X)}(A, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = I_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1)^*,$$

where $*$ indicates the full linear dual.

Proof. The first equality is just the Frobenius reciprocity.

(1) This follows from the fact that the center of $\mathrm{GL}(X)$ acts by $|\det_X|^3$ on C and by the character χ_1/χ_2 on $\tau^\vee \cdot \chi_1 \boxtimes \chi_2$.

(2) We have:

$$\begin{aligned} & \mathrm{Hom}_{M(X)}(B, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) \\ &= \mathrm{Hom}_{M(X)}(I_{B(J) \times Q(Z)}(S(F^\times) \otimes S(F^\times \cdot v_2 \otimes f_1)), \tau^\vee \cdot \chi_1 \boxtimes \chi_2) \\ &= \mathrm{Hom}_{B(J)}(S(F^\times) \otimes S(F^\times \cdot v_2 \otimes f_1), R_{\overline{B}}(\tau^\vee \cdot \chi_1) \boxtimes \chi_2), \end{aligned}$$

where $R_{\overline{B}}$ denotes the normalized Jacquet module relative to the opposite Borel subgroup \overline{B} . From the action of $B(J)$ described in Proposition 10.1(2), one sees that the element $t(a, 1) \in B(J)$ acts on $S(F^\times) \otimes S(F^\times \cdot v_2 \otimes f_1)$ by the character $|a|$. It follows that if the last Hom space is nonzero, $R_{\overline{B}}(\tau^\vee \cdot \chi_1)$ must contain $|-\| \boxtimes \eta^{-1}$ as a subquotient, for some character η . This is equivalent to $R_B(\tau)$ having $\chi_1|-\|^{-1} \boxtimes \eta$ as a subquotient. This proves (2).

(3) By (1) and (2), one obtains the inclusion in (3). Now let $(\tau^\vee \cdot \chi_1 \boxtimes \chi_2) \otimes \Pi$ be the maximal $\tau^\vee \cdot \chi_1 \boxtimes \chi_2$ -isotypic quotient of A . Then

$$\mathrm{Hom}_{M(X)}(A, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = \mathrm{Hom}_{M(X)}((\tau^\vee \cdot \chi_1 \boxtimes \chi_2) \otimes \Pi, \tau^\vee \cdot \chi_1 \boxtimes \chi_2) = \mathrm{Hom}_{\mathbb{C}}(\Pi, \mathbb{C}) = \Pi^*.$$

But by [GG, Lemma 9.4], Π is of the form $I_{P(Y)}(\pi_0)$, where π_0 is such that $(\tau^\vee \cdot \chi_1 \boxtimes \chi_2) \otimes \pi_0$ is the maximal $\tau^\vee \cdot \chi_1 \boxtimes \chi_2$ -isotypic quotient of $S(F^\times) \otimes S(\mathrm{Isom}(X, Y))$ which we view as a representation of $\mathrm{GL}(X) \times \mathrm{GL}(Y)$. Now by [MVW, Lemma, Pg. 59], with the actions of GL_1 taken into account, one can see that $\pi_0 = \tau \cdot \chi_1^{-1} \boxtimes \chi_1$. This completes the proof. \square

10.2. Proof of Theorem 8.2. We are now ready to give the proof of Theorem 8.2. In the following, we shall repeatedly use the following simple fact:

- if σ is an irreducible representation of $\mathrm{GSO}(V_2)$, then

$$\Theta(\sigma)^* \cong \mathrm{Hom}_{\mathrm{GSO}(V_2)}(\Omega_{V_2, W}, \sigma).$$

Let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\mathrm{GSO}(V_2) \cong (\mathrm{GL}_2(F) \times \mathrm{GL}_2(F))/F^\times$. As in the proof of Thm. 8.1(i), it follows from Thm. 3.3 that

$$\Theta(\tau_1 \boxtimes \tau_2) = \Theta(\tau_2 \boxtimes \tau_1) \neq 0.$$

We now consider the various cases in Theorem 8.2 in turn.

Supercuspidal representations.

Suppose that $\tau_1 \boxtimes \tau_2$ is supercuspidal. Then one knows that $\Theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_1 \boxtimes \tau_2)$ is non-zero and irreducible. Moreover, if $\tau_1 \neq \tau_2$, then the theta lift of $\tau_1 \boxtimes \tau_2$ to $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ is zero and hence $\theta(\tau_1 \boxtimes \tau_2)$ is supercuspidal.

On the other hand, if $\tau_1 = \tau_2 = \tau$, then $\tau \boxtimes \tau$ participates in the theta correspondence with $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ and its big theta lift to GL_2 is τ . As in the proof of Thm. 8.1(ii), there is a $\mathrm{GSO}(V_2) \times L(Z)$ -equivariant surjective map

$$R_{Q(Z)}(\Omega_{V_2, W}) \longrightarrow \Omega_{V_2, W'}.$$

By Frobenius reciprocity, one has a non-zero $\mathrm{GSO}(V_2) \times \mathrm{GSp}(W)$ -equivariant map

$$\Omega_{V_2, W} \longrightarrow (\tau \boxtimes \tau) \boxtimes I_{Q(Z)}(1, \tau).$$

Thus, we see that

$$\Theta(\tau \boxtimes \tau) \hookrightarrow I_{Q(Z)}(1, \tau).$$

We know that $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible constituents with a unique generic constituent $\pi_{gen}(\tau)$. It follows from Cor. 4.2(ii) that

$$\Theta(\tau \boxtimes \tau) = \pi_{gen}(\tau).$$

Discrete series representations.

Suppose that $\sigma = \mathrm{st}_\chi \boxtimes \tau$ where st_χ is a twisted Steinberg representation and τ is a discrete series representation so that $\omega_\tau = \chi^2$. Note that τ is either supercuspidal or equal to st_μ . Then

$$\sigma \hookrightarrow \pi(\chi| - |^{1/2}, \chi| - |^{-1/2}) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi| - |^{1/2}, \chi| - |^{-1/2}).$$

We would like to apply Corollary 10.2 (3) and so we need to verify that the conditions in Corollary 10.2 (1) and (2) hold. The condition in Corollary 10.2 (1) obviously holds, and that in Corollary 10.2 (2) holds when τ is supercuspidal. If $\tau = \mathrm{st}_\mu$ is a twisted Steinberg representation (so that $\chi^2 = \mu^2$), then

$$R_B(\tau) = \mu| - |^{1/2} \boxtimes \mu| - |^{-1/2} \neq \chi| - |^{-1/2} \boxtimes \eta$$

for any character η . Hence the condition in Corollary 10.2 (2) also holds when τ is a twisted Steinberg representation. In particular, we conclude by Corollary 10.2 (3) that

$$I_{P(Y)}(\tau \cdot \chi^{-1}| - |^{-1/2}, \chi| - |^{1/2}) \twoheadrightarrow \Theta(\sigma).$$

By Lemma 5.2, the above induced representation is multiplicity free and of length two with a unique irreducible quotient, so that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma)$ is irreducible. Moreover,

$$\theta(\sigma) = \begin{cases} \mathrm{St}(\tau \cdot \chi^{-1}, \chi) & \text{if } \tau \neq \mathrm{st}_\chi; \\ \pi_{gen}(\tau) & \text{if } \tau = \mathrm{st}_\chi. \end{cases}$$

Non-discrete series representations I.

Suppose that

$$\sigma \hookrightarrow \pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee \cdot \chi_1, \chi_2)$$

where τ is a discrete series representation with $\omega_\tau = \chi_1 \chi_2$ and

$$|\chi_1 / \chi_2| = | - |^{-s_0} \quad \text{and} \quad s_0 \geq 0.$$

Again, we would like to apply Corollary 10.2 (3) and so we need to verify the conditions there. As before, the only issue is the condition in Corollary 10.2 (2) when $\tau = \mathrm{st}_\chi$ is a twisted Steinberg representation, in which case

$$R_B(\tau) = \chi| - |^{1/2} \boxtimes \chi| - |^{-1/2}$$

and we need to show that this is different from $\chi_1| - |^{-1} \boxtimes \eta$ for any character η . In other words, we need to show that

$$\chi/\chi_1 \neq | - |^{-3/2}.$$

But observe that

$$|\chi|^2 = |\chi_1\chi_2| = |\chi_1|^2 \cdot |\chi_2/\chi_1| = |\chi_1|^2 \cdot | - |^{s_0},$$

so that

$$|\chi/\chi_1| = | - |^{s_0/2} \neq | - |^{-3/2}.$$

This verifies that the conditions in Corollary 10.2 (1) and (2) hold, so that we conclude that

$$I_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1) \twoheadrightarrow \Theta(\sigma).$$

Since the above induced representation is multiplicity free with a unique irreducible quotient, we conclude that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma) = J_{P(Y)}(\tau \cdot \chi_1^{-1}, \chi_1)$ is irreducible.

Non-discrete series representations II.

Finally, we consider the case where

$$\sigma \hookrightarrow \pi(\chi_1, \chi'_1) \boxtimes \pi(\chi_2, \chi'_2)$$

with $\chi_1\chi'_1 = \chi_2\chi'_2$ and

$$|\chi_i/\chi'_i| = | - |^{-s_i} \quad \text{and} \quad s_1 \geq s_2 \geq 0.$$

We consider two subcases:

- (a) $\chi_2/\chi'_2 \neq | - |^{-1}$; in this case $\pi(\chi_2, \chi'_2) = \pi(\chi'_2, \chi_2)$ is irreducible and

$$\sigma \hookrightarrow I_{P(X)}(\pi(\chi_2, \chi'_2)^\vee \cdot \chi_1, \chi'_1).$$

Again, to apply Corollary 10.2 (3), we need to verify the conditions there, and in particular the condition in Corollary 10.2 (2). We have

$$R_B(\pi(\chi_2, \chi'_2)) = (\chi_2 \boxtimes \chi'_2) \oplus (\chi'_2 \boxtimes \chi_2)$$

up to semisimplification and so we need to verify that

$$\chi_2 \neq \chi_1| - |^{-1} \quad \text{and} \quad \chi'_2 \neq \chi_1| - |^{-1}.$$

To see these, we argue by contradiction. If $\chi_2 = \chi_1| - |^{-1}$, then $\chi'_2 = \chi'_1| - |$, so that

$$| - |^{-s_2} = |\chi_2/\chi'_2| = |\chi_1/\chi'_1| \cdot | - |^{-2} = | - |^{-s_1-2}.$$

This would give $s_2 = s_1 + 2$ which contradicts $s_1 \geq s_2$. On the other hand, if $\chi'_2 = \chi_1| - |^{-1}$, then $\chi_2 = \chi'_1| - |$, so that

$$| - |^{s_2} = |\chi'_2/\chi_2| = |\chi_1/\chi'_1| \cdot | - |^{-2} = | - |^{-s_1-2}.$$

This would give $s_2 = -s_1 - 2 < 0$, which is a contradiction. Thus, we may apply Corollary 10.2 (3) to conclude that

$$I_{P(Y)}(\pi(\chi'_2, \chi_2) \cdot \chi_1^{-1}, \chi_1) = I_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow \Theta(\sigma).$$

This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

- (b) $\chi_2/\chi'_2 = |\cdot|^{-1}$; in this case, $\pi(\chi_2, \chi'_2)$ is reducible and has the one dimensional representation $\chi_2|\cdot|^{-1/2}$ as its unique irreducible submodule. Then

$$\sigma \hookrightarrow \pi(\chi_1, \chi'_1) \boxtimes \chi_2|\cdot|^{-1/2} = I_{P(X)}(\chi_1\chi_2^{-1}|\cdot|^{-1/2}, \chi'_1).$$

Applying Corollary 10.2 (3) (we leave the verification of the conditions there to the reader), we conclude that

$$I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{-1/2}, \chi_1) \twoheadrightarrow \Theta(\sigma).$$

Observe that

$$I_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{-1/2}, \chi_1)$$

and the former induced representation is a standard module. This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1).$$

This completes the proof of Theorem 8.2.

11. Proof of Theorem 8.3

In this section, we give the proof of Theorem 8.3.

11.1. Jacquet Modules. The key step is the computation of the normalized Jacquet modules of the induced Weil representation Ω with respect to $Q(Z)$, $P(Y)$ and $P(J)$. This is carried out in the appendix, following the lines of [K].

Proposition 11.1. *Let $R_{P(J)}(\Omega)$ denote the normalized Jacquet module of Ω along $P(J)$. Then we have a short exact sequence of $M(J) \times \mathrm{GSp}(W)$ -modules:*

$$0 \longrightarrow A \longrightarrow R_{P(A)}(\Omega) \longrightarrow B \longrightarrow 0.$$

Here,

$$B \cong \Omega_{W, V_2},$$

where Ω_{W, V_2} is the induced Weil representation for $\mathrm{GSp}(W) \times \mathrm{GSO}(V_2)$, and

$$A \cong I_{Q(Z)}(S(F^\times) \otimes \Omega_{W', V_2}),$$

where the action of $(GL(J) \times \mathrm{GSO}(V_2)) \times (GL(Z) \times \mathrm{GSp}(W'))$ on $S(F^\times)$ is given by:

$$((a, h), (b, g)) \cdot f(x) = f(b^{-1} \cdot x \cdot a \cdot \lambda_{W'}(g)),$$

and Ω_{W', V_2} denotes the induced Weil representation of $\mathrm{GSp}(W') \times \mathrm{GSO}(V_2)$.

Proof. This is Theorem A.1 with $m = 6$, $n = 2$ and $t = 1$, and with Remark A.5 taken into account. \square

Proposition 11.2. *Let $R_{Q(Z)}(\Omega)$ denote the normalized Jacquet module of Ω along $Q(Z)$. Then we have a short exact sequence of $\mathrm{GSO}(V) \times L(Z)$ -modules*

$$0 \longrightarrow A' \longrightarrow R_{Q(Z)}(\Omega_D) \longrightarrow B' \longrightarrow 0.$$

Here,

$$B' \cong |\det_Z| \cdot |\lambda_W|^{-1/2} \boxtimes \Omega_{W', V}$$

where $\Omega_{W',V}$ is the induced Weil representation of $\mathrm{GSp}(W') \times \mathrm{GSO}(V)$ and

$$A' \cong I_{P(J)}(S(F^\times) \otimes \Omega_{W',V_2}),$$

where the action of $(\mathrm{GL}(J) \times \mathrm{GSO}(V_2)) \times (\mathrm{GL}(Z) \times \mathrm{GSp}(W'))$ on $S(F^\times)$ is given by

$$((a, h), (b, g)) \cdot f(x) = f(a^{-1} \cdot \lambda_{W'}(g)^{-1} \cdot x \cdot b),$$

and Ω_{W',V_2} is the induced Weil representation of $\mathrm{GSp}(W') \times \mathrm{GSO}(V_2)$.

Proof. This is Theorem A.2, with $m = 6$, $n = 2$ and $k = 1$ and with Remark A.5 taken into account. \square

Proposition 11.3. *Let $R_{P(Y)}(\Omega)$ denote the normalized Jacquet module of Ω along $P(Y)$. Then as a representation of $M(Y) \times \mathrm{GSO}(V)$, $R_{P(Y)}(\Omega)$ has a 3-step filtration whose successive quotients are given as follows:*

(i) *the top piece of the filtration is:*

$$A'' = S(F^\times) \otimes |\det_Y|^{3/2} \cdot |\lambda_W|^{-3/2},$$

where $(a, \lambda, h) \in \mathrm{GL}(Y) \times \mathrm{GL}_1 \times \mathrm{GSO}(V)$ acts on $S(F^\times)$ by

$$(a, \lambda, h)\phi(t) = \phi(t \cdot \lambda \cdot \lambda(h)).$$

(ii) *the second piece in the filtration is:*

$$B'' = I_{B \times P(J)}(S(F^\times \times F^\times))$$

where the action of the diagonal torus in B on $S(F^\times \times F^\times)$ is given by

$$\left(\begin{pmatrix} a & \\ & d \end{pmatrix} \cdot \phi \right) (\lambda, t) = |a| \cdot \phi(\lambda, td).$$

(iii) *the bottom piece of the filtration is:*

$$C'' = I_{Q(E)}(S(F^\times) \otimes S(\mathrm{GL}_2)),$$

where the action of $(\mathrm{GL}(Y) \times \mathrm{GL}_1) \times (\mathrm{GL}(E) \times \mathrm{GSO}(V_1))$ on $S(F^\times) \otimes S(\mathrm{GL}_2)$ is given by:

$$(a, \lambda; b, h)\phi(t, g) = \phi(t \cdot \lambda \cdot \lambda_{V_1}(h), b^{-1}ga \cdot \lambda_{V_1}(h)).$$

Proof. This is Theorem A.2, with $m = 6$, $n = 2$ and $k = 2$, and with Remark A.5 taken into account. \square

11.2. Consequences. Applying Frobenius reciprocity as well as Props. 11.1, 11.2 and 11.3, we obtain the following 3 propositions as consequences. Since the proofs of these 3 propositions are similar, we shall only give the proof of Prop. 11.5.

Proposition 11.4. *Assume that $\chi \neq | - |$. Then as a representation of $\mathrm{GSO}(V)$,*

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega, I_{Q(Z)}(\chi, \tau)) = I_{P(J)}(\chi^{-1}, (\tau \cdot \chi) \boxtimes (\tau \cdot \chi))^* \quad (\text{full linear dual}).$$

Proposition 11.5. *Suppose that τ is a discrete series representation of $\mathrm{GL}(Y)$ and $\omega_\tau \neq | - |^3$. Then as a representation of $\mathrm{GSO}(V)$,*

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega, I_{P(Y)}(\tau, \chi)) \hookrightarrow I_{Q(E)}(\tau^\vee, (\chi\omega_\tau) \circ \lambda_{V_1})^*.$$

Further, if τ is supercuspidal, then

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega, I_{P(Y)}(\tau, \chi)) = I_{Q(E)}(\tau^\vee, (\chi\omega_\tau) \circ \lambda_{V_1})^*.$$

Proposition 11.6. (i) *Consider the space*

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2))$$

as a representation of $\mathrm{GSp}(W)$. Then we have:

(a) *If $\chi \neq 1$, then*

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)) = 0$$

unless

$$\tau_1 = \tau_2 = \tau,$$

in which case

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi, \tau \boxtimes \tau)) = I_{Q(Z)}(\chi^{-1}, \tau \otimes \chi)^*.$$

(b) *If $\chi = 1$ but $\tau_1 \neq \tau_2$, then*

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)) = \Theta_{W, V_2}(\tau_1 \boxtimes \tau_2)^*,$$

where $\Theta_{W, V_2}(\tau_1 \boxtimes \tau_2)$ denotes the big theta lift of $\tau_1 \boxtimes \tau_2$ from $\mathrm{GSO}(V_2)$ to $\mathrm{GSp}(W)$.

(c) *If $\chi = 1$ and $\tau_1 = \tau_2 = \tau$, then we have an exact sequence:*

$$0 \longrightarrow \Theta_{W, V_2}(\tau \boxtimes \tau)^* \longrightarrow \mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi, \tau \boxtimes \tau)) \longrightarrow (I_{Q(Z)}(1, \tau))^*$$

Proof of Prop. 11.5. By Frobenius reciprocity, we have:

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega, I_{P(Y)}(\tau, \chi)) = \mathrm{Hom}_{M(Y)}(R_{P(Y)}(\Omega), \tau \boxtimes \chi).$$

The 3-step filtration of $R_{P(Y)}(\Omega)$ thus induces one on this Hom space.

For τ as in the proposition, we see that

$$\mathrm{Hom}_{M(Y)}(A'', \tau \boxtimes \chi) = 0.$$

This is because the center of $\mathrm{GL}(Y)$ acts by the character $| - |^3$ on A'' and by ω_τ on $\tau \boxtimes \chi$, and by our assumption, these two characters are different.

We claim now that

$$\mathrm{Hom}_{M(Y)}(B'', \tau \boxtimes \chi) = 0$$

as well. This is clear if τ is supercuspidal. On the other hand, suppose that $\tau = st_\mu$ is a twisted Steinberg representation. If $\text{Hom}_{M(Y)}(B'', \tau \boxtimes \chi) \neq 0$, then one deduces that

$$\text{Hom}_{GL(Y)}(I_B(|-| \boxtimes V), st_\mu) \neq 0,$$

where $|-| \boxtimes V$ is a representation of the diagonal torus $GL_1 \times GL_1$. By Frobenius reciprocity, this implies that

$$\text{Hom}_{GL_1 \times GL_1}(R_B(st_{\mu^{-1}}), |-|^{-1} \boxtimes V^\vee) \neq 0.$$

But $R_B(st_{\mu^{-1}}) = |-|^{1/2} \mu^{-1} \boxtimes |-|^{-1/2} \mu^{-1}$. This would imply that $\mu = |-|^{3/2}$. However, this is ruled out by the assumption of the proposition.

Hence, we have shown that

$$\text{Hom}_{\text{GSp}(W)}(\Omega, I_{P(Y)}(\tau, \chi)) \hookrightarrow \text{Hom}_{M(Y)}(C'', \tau \boxtimes \chi).$$

Now by arguing in the same way as the proof of Cor. 10.2(3)

$$\text{Hom}_{M(Y)}(C'', \tau \boxtimes \chi) = I_{Q(E)}(\tau^\vee, (\chi\omega_\tau) \circ \lambda_{V_1})^*.$$

Suppose further that τ is supercuspidal. Since the representations A'' and B'' do not contain any supercuspidal constituents and hence belong to different Bernstein components of GSp_4 , one has

$$\text{Hom}_{\text{GSp}(W)}(\Omega, I_{P(Y)}(\tau, \chi)) = \text{Hom}_{M(Y)}(C'', \tau \boxtimes \chi).$$

The proposition is proved. \square

11.3. Proof of Thm. 8.3. Now we can prove Thm. 8.3. In the following, we shall repeatedly use the following two simple facts:

- (a) if π is an irreducible representation of $\text{GSp}(W)$, then

$$\Theta(\pi)^* \cong \text{Hom}_{\text{GSp}(W)}(\Omega, \pi).$$

- (b) if Π is an irreducible representation of $\text{GSO}(V)$ such that

$$\Pi^\vee \hookrightarrow \text{Hom}_{\text{GSp}(W)}(\Omega, \Sigma),$$

where Σ is not necessarily irreducible (typically, Σ is a principal series representation), then there is a nonzero equivariant map

$$\Omega \longrightarrow \Pi \boxtimes \Sigma.$$

In particular, $\Theta(\Pi) \neq 0$. The analogous result with the roles of $\text{GSp}(W)$ and $\text{GSO}(V)$ exchanged also holds.

By [KR, Thm. 3.8] (i.e. Thm. 3.1(i)), we know that if $\pi \in \Pi(\text{GSp}_4)_{ng}^{temp}$, then $\Theta(\pi) = 0$. Parts (i) and (ii) of the theorem will follow if we can determine $\theta(\pi)$ for $\pi \notin \Pi(\text{GSp}_4)_{ng}^{temp}$, i.e. if we can demonstrate parts (iii), (iv), (v) and (vi). We consider the different cases separately.

Supercuspidal Representations

If π is a supercuspidal representation of $\text{GSp}_4(F)$ and $\pi \notin \Pi(\text{GSp}_4)_{ng}^{temp}$, then π is generic and we know that $\Theta(\pi)$ is a nonzero irreducible representation of $\text{GSO}(V)$. It is supercuspidal unless the theta lift of π to $\text{GSO}_{2,2}(F)$ is nonzero, in which case its theta lift to $\text{GSO}_{2,2}(F)$ is also supercuspidal. Suppose that

$$\pi = \Theta(\tau_1 \boxtimes \tau_2) = \Theta(\tau_2 \boxtimes \tau_1),$$

so that τ_1 and τ_2 are supercuspidal representations of $\mathrm{GL}_2(F)$ with the same central character. Then it follows by Prop. 11.1 that

$$\mathrm{Hom}_{\mathrm{GSp}(W) \times \mathrm{GSO}(V)}(\Omega, \pi \boxtimes I_{P(J)}(1, \tau_1 \boxtimes \tau_2)) \neq 0.$$

This shows that

$$\Theta(\pi) = I_{P(J)}(1, \tau_1 \boxtimes \tau_2) = I_P(\tau_1, \tau_2) \boxtimes \omega_{\tau_1}.$$

This proves Thm. 8.3(iii).

The Generalized Steinberg Representation $St(\chi, \tau)$

Now we consider the first family of generalized Steinberg representations, so that $\pi = St(\chi, \tau)$ is as in 5.2.1(a) with χ a non-trivial quadratic character and τ supercuspidal so that $\tau \otimes \chi = \tau$. Recall that we need to show:

$$\theta(St(\chi, \tau)) = St(\tau) \boxtimes \omega_{\tau\chi}.$$

Since

$$St(\tau) \boxtimes \omega_{\tau\chi} \hookrightarrow I_{P(J)}(\chi| - |, (\tau \cdot \chi| - |^{-1/2}) \boxtimes \tau| - |^{-1/2}),$$

we deduce by the fact (a) above and Prop. 11.6(i)(a) that

$$\Theta(St(\tau) \boxtimes \omega_{\tau\chi})^* \hookrightarrow \mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi| - |, (\tau \cdot \chi| - |^{-1/2}) \boxtimes \tau| - |^{-1/2})),$$

which vanishes unless $\tau \otimes \chi \cong \tau$, in which case one has:

$$I_{Q(Z)}(\chi^{-1}| - |^{-1}, \tau| - |^{1/2}) \rightarrow \Theta(St(\tau) \boxtimes \omega_{\tau\chi}).$$

Recall that the above induced representation has $St(\chi, \tau)$ as its unique irreducible quotient (since $\chi \neq 1$). From this, we conclude that:

- $\theta(St(\tau) \boxtimes \omega_{\tau\chi}) \subseteq St(\chi, \tau)$;
- $\theta(St(\chi, \tau)) \neq 0$.

On the other hand, since $\chi \neq 1$, one may apply Prop. 11.4 to $I_{Q(Z)}(\chi| - |, \tau| - |^{-1/2})$ and arguing as above, one obtains:

- $\theta(St(\chi, \tau)) \subseteq St(\tau) \boxtimes \omega_{\tau\chi}$;
- $\theta(St(\tau) \boxtimes \omega_{\tau\chi}) \neq 0$.

Hence, we have shown that

$$\begin{cases} \theta(St(\tau) \boxtimes \omega_{\tau\chi}) = St(\chi, \tau); \\ \theta(St(\chi, \tau)) = St(\tau) \boxtimes \omega_{\tau\chi}. \end{cases}$$

The Generalized Steinberg Representation $St(\tau, \mu)$

Now we consider the second family of generalized Steinberg representations, so that $\pi = St(\tau, \mu)$ as in 5.2.1(b), with $\tau \neq st$ a discrete series representation of $\mathrm{PGL}(Y)$. Recall that we need to show:

$$\theta(St(\tau, \mu)) = I_P(\tau \otimes \mu, st \otimes \mu) \boxtimes \mu^2.$$

Since

$$St(\tau, \mu) \hookrightarrow I_{P(Y)}(\tau| - |^{1/2}, \mu| - |^{-1/2}),$$

Prop. 11.5 implies that

$$I_{Q(E)}(\tau | - |^{-1/2}, (\mu | - |^{1/2}) \circ \lambda_{V_1}) \twoheadrightarrow \Theta(St(\tau, \mu))^*.$$

Now note that as a representation of $GL_4(F) \times GL_1(F)$,

$$I_{Q(E)}(\tau | - |^{-1/2}, (\mu | - |^{1/2}) \circ \lambda_{V_1}) = \mu \cdot I_Q(| - |^{-1/2}, \tau, | - |^{1/2}) \boxtimes \mu^2,$$

and the latter representation has a unique irreducible quotient isomorphic to $\mu \cdot I_P(\tau, st) \boxtimes \mu^2$. Hence, we have shown that

$$\theta(St(\tau, \mu)) \subseteq I_P(\tau \otimes \mu, st \otimes \mu) \boxtimes \mu^2.$$

On the other hand, by Thm. 8.2, one knows that the representation $St(\tau, \mu)$ participates in the theta correspondence with $GSO(V_2)$: it is the theta lift of $(\tau \otimes \mu) \boxtimes (st \otimes \mu)$. Hence it follows by Prop. 11.1 that $\Theta(St(\tau, \mu)) \neq 0$ and so

$$\theta(St(\tau, \mu)) = I_P(\tau \otimes \mu, st \otimes \mu) \boxtimes \mu^2,$$

as desired.

Twisted Steinberg Representations

Now consider the twisted Steinberg representation $St_{\text{PGSp}_4} \otimes \chi$. Since

$$St_{\text{PGSp}_4} \otimes \chi \hookrightarrow I_{Q(Z)}(| - |^2, st_\chi | - |^{-1}),$$

and

$$St_\chi \boxtimes \chi^2 \hookrightarrow I_{P(J)}(| - |^2, st_\chi | - |^{-1} \boxtimes st_\chi | - |^{-1}),$$

we may apply Props. 11.4 and 11.6 to conclude that

$$\theta(St_\chi \boxtimes \chi^2) = St_{\text{PGSp}_4} \otimes \chi$$

and

$$\theta(St_{\text{PGSp}_4} \otimes \chi) = St_\chi \boxtimes \chi^2.$$

Since the arguments are similar to the above, we omit the details.

Non-Discrete Series Representations

Finally, we come to the non-discrete series representations in part (vi) of Thm. 8.3. We will consider the three cases (a), (b) and (c) separately.

(a) Suppose that

$$\pi \hookrightarrow I_{Q(Z)}(\chi, \tau)$$

as in 5.2.2(a), so that $|\chi| = | - |^{-s}$ with $s \geq 0$. Recall that we need to show:

$$\theta(\pi) = J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi.$$

By Prop. 11.4, we deduce that $\Theta(\pi)$ is a quotient of

$$I_{P(J)}(\chi^{-1}, (\tau \cdot \chi) \boxtimes (\tau \cdot \chi)) = I_P(\tau, \tau \cdot \chi) \boxtimes (\omega_\tau \cdot \chi).$$

But this induced representation has a unique irreducible quotient $J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi$, since it is a standard module. This shows that

- $\theta(\pi) \subseteq J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi$;
- $\theta(J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi) \neq 0$.

On the other hand, since

$$J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi \hookrightarrow I_P(\tau \cdot \chi, \tau) \boxtimes \omega_\tau \chi \cong I_{P(J)}(\chi, \tau \boxtimes \tau),$$

we may apply Prop. 11.6(a) and (c) to conclude that

- $\theta(J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi) \subseteq \pi$;
- $\Theta(\pi) \neq 0$.

From this, we conclude that

$$\theta(\pi) = J_P(\tau, \tau \cdot \chi) \boxtimes \omega_\tau \chi,$$

as desired.

(b) Suppose now that

$$\pi \hookrightarrow I_{P(Y)}(\tau, \chi)$$

as in 5.2.2(b), so that τ is a twist of a discrete series representation with $|\omega_\tau| = | - |^{-2s}$ with $s \geq 0$. Applying Prop. 11.5, one deduces that $\Theta(\pi)$ is a quotient of

$$I_{Q(E)}(\tau^\vee, (\omega_\tau \chi) \circ \lambda_{V_1}) \cong I_Q(1, \tau, \omega_\tau) \cdot \chi \boxtimes \chi^2 \omega_\tau.$$

This induced representation has a unique irreducible quotient $J_Q(1, \tau, \omega_\tau) \cdot \chi \boxtimes \chi^2 \omega_\tau$. Thus, one sees that

$$\theta(\pi) \subseteq J_Q(1, \tau, \omega_\tau) \cdot \chi \boxtimes \chi^2 \omega_\tau.$$

On the other hand, since π participates in the theta correspondence with $\mathrm{GSO}(V_2)$ by Thm. 8.2, one knows that $\theta(\pi) \neq 0$. Hence, one has

$$\theta(\pi) = J_Q(1, \tau, \omega_\tau) \cdot \chi \boxtimes \chi^2 \omega_\tau,$$

as desired.

(c) Suppose first that

$$\pi \hookrightarrow I_B(\chi_1, \chi_2; \chi)$$

as in 5.2.2(c)(i), so that $\chi_2 \neq | - |^{-1}$. Thus, $|\chi_1| = | - |^{-s_1}$ and $|\chi_2| = | - |^{-s_2}$ with $s_1 \geq s_2 \geq 0$, but $\chi_2 \neq | - |^{-1}$. In this case, we have

$$\pi \hookrightarrow I_{Q(Z)}(\chi_1, \pi(\chi \chi_2, \chi))$$

as the unique irreducible submodule. Applying Prop. 11.4, one deduces that as a representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$, $\Theta(\pi)$ is a quotient of

$$\begin{aligned} & I_{P(J)}(\chi_1^{-1}, \pi(\chi \chi_1 \chi_2, \chi \chi_1) \boxtimes \pi(\chi \chi_1 \chi_2, \chi \chi_1)) \\ &= \chi \cdot I_P(\pi(\chi_2, 1), \pi(\chi_1 \chi_2, \chi_1)) \boxtimes \chi^2 \chi_1 \chi_2 \\ &= \chi \cdot I_{B_0}(1, \chi_2, \chi_1, \chi_1 \chi_2) \boxtimes \chi^2 \chi_1 \chi_2. \end{aligned}$$

This induced representation has a unique irreducible quotient

$$\Pi = \chi \cdot J_{B_0}(1, \chi_2, \chi_1, \chi_1 \chi_2) \boxtimes \chi^2 \chi_1 \chi_2.$$

Hence, we have:

$$\theta(\pi) \subseteq \Pi \quad \text{and} \quad \Theta(\Pi) \neq 0.$$

In fact, if $I_B(\chi_1, \chi_2; \chi)$ is irreducible, so that it is equal to π , then we would have

$$\Theta(\pi) = I_{B_0}(1, \chi_2, \chi_1, \chi_1 \chi_2) \boxtimes \chi^2 \chi_1 \chi_2.$$

This is the case, for example, when χ_1 (and hence χ_2) is unitary. In that case, we have the desired identity $\theta(\pi) = \Pi$.

To prove the desired identity in general, we may thus assume that $\chi_1 \neq 1$. Then one may apply Prop. 11.6(a) to the representation

$$I_{P(J)}(\chi_1, \pi(\chi\chi_2, \chi) \boxtimes \pi(\chi\chi_2, \chi))$$

which contains Π as its unique irreducible submodule. By Prop. 11.6(a), one has

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega, I_{P(J)}(\chi_1, \pi(\chi\chi_2, \chi) \boxtimes \pi(\chi\chi_2, \chi))) = I_{Q(Z)}(\chi_1^{-1}, \pi(\chi\chi_1\chi_2, \chi\chi_1) \boxtimes \pi(\chi\chi_1\chi_2, \chi\chi_1))^*.$$

It follows from this that

$$\theta(\Pi) \subseteq \pi \quad \text{and} \quad \Theta(\pi) \neq 0.$$

Hence we have the desired equality $\theta(\pi) = \Pi$ in general.

Finally, we need to treat the case when

$$\pi \hookrightarrow I_{Q(Z)}(\chi_1, \chi \cdot | - |^{-1/2})$$

as the unique irreducible submodule, as given in 5.2.2(c)(ii), so that $|\chi_1| = | - |^{-s_1}$ with $s_1 \geq 1$. Application of Prop. 11.4 shows that $\Theta(\pi)$ is a quotient of

$$I_{P(J)}(\chi_1^{-1}, \chi_1\chi | - |^{-1/2} \boxtimes \chi_1\chi | - |^{-1/2}) = \chi | - |^{-1/2} \cdot I_P(1, \chi_1) \boxtimes \chi_1\chi^2 | - |^{-1}.$$

But $I_P(1, \chi_1)$ is a quotient of $I_{B_0}(| - |^{1/2}, | - |^{-1/2}, \chi_1 | - |^{1/2}, \chi_1 | - |^{-1/2})$ which is a standard module. This shows that

$$\theta(\pi) \subset \Pi := \chi | - |^{-1/2} J_{B_0}(| - |^{1/2}, | - |^{-1/2}, \chi_1 | - |^{1/2}, \chi_1 | - |^{-1/2}) \boxtimes \chi_1\chi^2 | - |^{-1}.$$

On the other hand, since

$$\Pi \hookrightarrow I_{P(J)}(\chi_1, \chi | - |^{-1/2} \boxtimes \chi | - |^{-1/2})$$

and $\chi_1 \neq 1$, one may apply Prop. 11.6(a) to conclude that $\theta(\pi) = \Pi$. This completes the proof of Thm. 8.3. \square

12. Some Corollaries

We note some corollaries of our explicit determination of theta correspondences.

The following is an immediate consequence of Thms. 8.1, 8.2 and 8.3.

Corollary 12.1. *The dichotomy statement of Theorem 3.2 holds for all irreducible representations of $\mathrm{GSp}_4(F)$.*

The following result was stated in [GT1, Thm. 5.6(iv)]:

Corollary 12.2. *Let π be an irreducible representation of $\mathrm{GSp}_4(F)$ with central character μ and suppose that π participates in the theta correspondence with $\mathrm{GSO}(V_2)$, so that*

$$\pi = \theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_2 \boxtimes \tau_1).$$

Let $\Pi \boxtimes \mu$ be the small theta lift of π to $\mathrm{GSO}(V)$, with Π a representation of $\mathrm{GL}_4(F)$. Then we have the following equality of L-parameters for $\mathrm{GL}_4 \times \mathrm{GL}_1$:

$$\phi_\Pi \times \mu = (\phi_{\tau_1} \oplus \phi_{\tau_2}) \times \mu.$$

Proof. Suppose first that π is a discrete series representation so that τ_i is also discrete series. By Thm. 8.3, we know that $\theta(\pi)$ is irreducible. By Prop. 11.1 and Frobenius reciprocity, we see that there is a nonzero map

$$\Omega \longrightarrow \pi \boxtimes I_P(\tau_1, \tau_2).$$

Since $I_P(\tau_1, \tau_2)$ is irreducible, we see that $\Pi = I_P(\tau_1, \tau_2)$ and we have the desired equality of L -parameters.

If π is not a discrete series representation, then π is of the type occurring in Thm. 8.2(v) or (vi). On the other hand, we can determine Π from Thm. 8.3(vi)(b) or (c). Let us illustrate this for the case when π is as in Thm. 8.2(vi), so that

$$\pi = \theta(\tau_1 \boxtimes \tau_2) = J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1),$$

and $\tau_i \hookrightarrow \pi(\chi_i, \chi'_i)$ is non-discrete series. Then

$$\phi_{\tau_1} \oplus \phi_{\tau_2} = \chi_1 \oplus \chi'_1 \oplus \chi_2 \oplus \chi'_2.$$

On the other hand, since

$$J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1) \hookrightarrow I_B(\chi_1/\chi'_2, \chi_1/\chi_2; \chi'_1),$$

it follows by Thm. 8.3(vi)(c) that

$$\Pi = J_B(\chi'_1, \chi'_2, \chi_2, \chi_1)$$

and so

$$\phi_{\Pi} = \chi_1 \oplus \chi'_1 \oplus \chi_2 \oplus \chi'_2$$

as well. This proves the corollary. \square

Now we consider the theta lifts of unramified representations. The dual group of GSp_4 is $\mathrm{GSp}_4(\mathbb{C})$ while that of $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}$ is a subgroup of $\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. There is a natural embedding of dual groups

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GSO}(V)^\vee \subset \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}),$$

where the first projection is given by the tautological embedding and the second projection is given by the similitude character. The following corollary gives the lifting of unramified representations in terms of their Satake parameters. It was stated in [GT1, Prop. 3.4].

Corollary 12.3. *Let $\pi = \pi(s)$ be an unramified representation of $\mathrm{GSp}(W)$ corresponding to the semisimple class $s \in \mathrm{GSp}_4(\mathbb{C})$. Then $\theta(\pi(s))$ is the unramified representation of $\mathrm{GSO}(V)$ corresponding to the semisimple class $\iota(s) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.*

Proof. If $\pi(s) \hookrightarrow I_B(\chi_1, \chi_2; \chi)$ and we set $\chi_i(\varpi) = t_i$ and $\chi(\varpi) = \nu$, then

$$s = \begin{pmatrix} \nu t_1 t_2 & & & \\ & \nu t_1 & & \\ & & \nu t_2 & \\ & & & \nu \end{pmatrix}.$$

The unramified representation of $\mathrm{GL}_4(F) \times \mathrm{GL}_1(F)$ with Satake parameter $\iota(s)$ is the unramified constituent of

$$\chi \cdot I_B(1, \chi_2, \chi_1, \chi_1 \chi_2) \boxtimes \chi^2 \chi_1 \chi_2.$$

The corollary follows by Thm. 8.3(vi)(c). \square

13. L -parameters and Genericity.

In [GT1], using Theorem 1.1 in the introduction, the local Langlands correspondence for GSp_4 was obtained from that for GL_2 and GL_4 . Given an irreducible representation π of $\mathrm{GSp}_4(F)$, we briefly recall how one obtains its L -parameter:

$$\phi_\pi : WD_F = W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

where WD_F (resp. W_F) denotes the Weil-Deligne (resp. Weil) group of F .

Firstly, we note that in [GT1, Lemma 6.1], it was shown that the embedding

$$\iota : \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

induces an injection

$$\Phi(\mathrm{GSp}_4) \hookrightarrow \Phi(\mathrm{GL}_4) \times \Phi(\mathrm{GL}_1)$$

where $\Pi(G)$ denotes the set of L -parameters of G . In particular, ϕ_π can be specified by describing it as a 4-dimensional representation of WD_F and giving its similitude character $\mathrm{sim} \phi_\pi$.

The following describes the construction of ϕ_π :

- (a) Suppose that π participates in the theta correspondence with $\mathrm{GSO}(D)$, where D is possibly split. Then we have:

$$\pi = \theta_{D,W}(\tau_1^D \boxtimes \tau_2^D),$$

where τ_1^D and τ_2^D have the same central characters. Let ϕ_i denote the L -parameter of the Jacquet-Langlands lift of τ_i^D to $\mathrm{GL}_2(F)$. Then one sets:

$$\phi_\pi = \phi_1 \oplus \phi_2 \quad \text{with } \mathrm{sim} \phi_\pi = \det \phi_1 = \det \phi_2.$$

- (b) Suppose that π participates in the theta correspondence with $\mathrm{GSO}(V) = \mathrm{GSO}_{3,3}(F)$. Then we have

$$\theta_{W,V}(\pi) = \Pi \boxtimes \mu$$

for an irreducible representation Π of $\mathrm{GL}_4(F)$ and $\mu = \omega_\pi$ is such that $\omega_\Pi = \mu^2$. One then sets:

$$\phi_\pi = \phi_\Pi \quad \text{with } \mathrm{sim} \phi_\pi = \mu.$$

Now by our explicit determination of local theta correspondence, we can explicitly write down the L -parameter of any non-supercuspidal representation. This is given in the following proposition.

Proposition 13.1. *Let S_n denote the n -dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. The L -parameter of a non-supercuspidal representation π of GSp_4 can be given as follows.*

- (i) If $\pi = \mathrm{St}(\chi, \tau)$, then

$$\phi_\pi = \phi_\tau \boxtimes S_2 : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GO}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

with similitude factor $\omega_\tau \cdot \chi$, so that the composite

$$W_F \longrightarrow \mathrm{GO}_2(\mathbb{C}) \longrightarrow \mathrm{GO}_2(\mathbb{C}) / \mathrm{GSO}_2(\mathbb{C}) \cong \{\pm 1\}$$

is the quadratic character χ .

- (ii) If $\pi = \mathrm{St}(\tau, \mu)$, then

$$\phi_\pi = \mu \cdot \phi_\tau \oplus (\mu \boxtimes S_2) : WD_F \longrightarrow (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

with $\mathrm{sim} \phi_\pi = \mu^2$.

(iii) If $\pi = St_{\mathrm{PGSp}_4} \otimes \chi$ is a twisted Steinberg representation, then

$$\phi_\pi = \chi \boxtimes S_4 : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

with $\mathrm{sim} \phi_\pi = \chi^2$.

(iv) If $\pi \hookrightarrow I_{Q(Z)}(\chi, \tau)$ as in Thm. 8.3(vi)(a), then

$$\phi_\pi = \phi_\tau \oplus \phi_\tau \cdot \chi : WD_F \longrightarrow M(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

where $M(\mathbb{C})$ is the Levi subgroup of the Siegel parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$, and $\mathrm{sim} \phi_\pi = \chi \cdot \omega_\tau$.

(v) If $\pi \hookrightarrow I_{P(Y)}(\tau, \chi)$ as in Thm. 8.3(vi)(b), then

$$\phi_\pi = \chi \oplus \chi \cdot \phi_\tau \oplus \chi \omega_\tau : WD_F \longrightarrow L(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$$

where $L(\mathbb{C})$ is the Levi subgroup of the Klingen (or Heisenberg) parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$, and $\mathrm{sim} \phi_\pi = \chi^2 \cdot \omega_\tau$.

(vi) If $\pi \hookrightarrow I_B(\chi_1, \chi_2; \chi)$ as in Thm. 8.3(vi)(c), then

$$\phi_\pi = \chi \chi_1 \chi_2 \oplus \chi \oplus \chi \chi_1 \oplus \chi \chi_2 : WD_F \longrightarrow T(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C}),$$

where $T(\mathbb{C})$ is the diagonal maximal torus of $\mathrm{GSp}_4(\mathbb{C})$ and $\mathrm{sim} \phi_\pi = \chi^2 \chi_1 \chi_2$.

In particular, we see that ϕ_π is a discrete series parameter if and only if π is a discrete series representation. Moreover, the map $\pi \mapsto \phi_\pi$ defines a bijective map

$$\Pi(\mathrm{GSp}_4)_{\mathrm{NDS}} \setminus \Pi(\mathrm{GSp}_4)_{\mathrm{ng}}^{\mathrm{temp}} \longrightarrow \Phi(\mathrm{GSp}_4)_{\mathrm{NDS}}$$

where the subscript NDS on both sides stand for ‘‘non-discrete series’’.

The reader can easily verify that the above L -parameters agree with the prescription given in [RS, §A.5].

Finally, the following proposition was used in the proof of [GT1, Main Theorem (vii)], which relates the genericity of π with its adjoint L -factor. A proof of this was also given in [AS], but our verification is more concise.

Proposition 13.2. *For the L -parameters ϕ described in Prop. 13.1, the adjoint L -factor $L(s, Ad \circ \phi)$ is holomorphic at $s = 1$ if and only if the L -packet L_ϕ contains a generic representation.*

Proof. This is a simple calculation using Prop. 13.1 and the knowledge of reducibility points of various principal series. But let us make a few simple observations:

- (a) If $\phi : WD_F \longrightarrow \mathrm{GSp}_4(\mathbb{C})$ is an L -parameter, then $Ad \circ \phi = \mathrm{Sym}^2 \phi \otimes \mathrm{sim}(\phi)^{-1}$;
- (b) If $\rho \boxtimes S_r$ is a representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$, then $L(s, \rho \boxtimes S_r) = L(s + (r - 1)/2, \rho)$. Thus $L(s, \rho \boxtimes S_r)$ has a pole at $s = 1$ if and only if ρ contains the unramified character $|\cdot|^{-(r+1)/2}$ (regarded as a character of W_F) as a constituent.
- (c) In the context of Prop. 13.1(iv), (v) and (vi) (but with $\chi \neq 1$ in case (iv)), the L -packet for ϕ_π is a singleton set containing only π . Moreover, π is generic precisely when the relevant principal series containing π as the unique irreducible submodule is irreducible. This is because the standard module conjecture holds for GSp_4 .

Now we consider each case of Prop. 13.1 in turn.

(i) If $\pi = St(\chi, \tau)$, which is generic, then by (a),

$$Ad \circ \phi_\pi = (\chi \cdot Ad(\phi_\tau) \boxtimes S_3) \oplus (\chi \boxtimes S_1)$$

so that

$$L(s, Ad \circ \phi_\pi) = L(s+1, Ad(\phi_\tau) \times \chi) \cdot L(s, \chi).$$

Since the only 1-dimensional characters in $Ad(\phi_\tau)$ are precisely those quadratic χ_K such that $\tau \otimes \chi_K = \tau$, we see that $L(s, Ad \circ \phi_\pi)$ is holomorphic at $s = 1$ by (b).

(ii) If $\pi = St(\tau, \mu)$, which is generic, then by (a),

$$Ad \circ \phi_\pi = Ad(\phi_\tau) \oplus (1 \boxtimes S_3) \oplus (\phi_\tau \boxtimes S_2).$$

If τ is supercuspidal, then as in (i), the only characters contained in $Ad(\phi_\tau)$ are quadratic. It follows by (b) that the adjoint L -factor is holomorphic at $s = 1$. On the other hand, if $\tau = st_\chi$ is a twisted Steinberg representation with χ a non-trivial quadratic character, then

$$Ad \circ \phi_\pi = 2 \cdot (1 \boxtimes S_3) \oplus (\chi \boxtimes S_3) \oplus (\chi \boxtimes S_1).$$

It follows from (b) that the adjoint L -factor is holomorphic at $s = 1$.

(iii) If $\pi = St_{PGSp_4} \otimes \chi$, which is generic, then by (a),

$$Ad \circ \phi_\pi = (1 \boxtimes S_3) \oplus (1 \boxtimes S_7),$$

so that $L(s, Ad \circ \phi_\pi) = \zeta(s+1) \cdot \zeta(s+3)$, which is clearly holomorphic at $s = 1$.

(iv) If $\pi \hookrightarrow I_{Q(Z)}(\chi, \tau)$, where $|\chi| = | - |^{-s_0}$ with $s_0 \geq 0$, then

$$Ad \circ \phi_\pi = \chi \cdot Ad(\phi_\tau) \oplus \chi^{-1} \cdot Ad(\phi_\tau) \oplus (\phi_\tau \otimes \phi_\tau^\vee).$$

If τ is supercuspidal, then it follows from (b) that the adjoint L -factor is non-holomorphic at $s = 1$ if and only if there is a quadratic character χ_0 such that

$$\tau \otimes \chi_0 = \tau \quad \text{and} \quad \chi \cdot \chi_0 = | - |^{-1}.$$

Similarly, when $\tau = st_\mu$ is a twisted Steinberg representation, then

$$Ad \circ \phi_\pi = ((\chi \oplus 1 \oplus \chi^{-1}) \boxtimes S_3) \oplus (1 \boxtimes S_1).$$

By (b), it follows that the adjoint L -factor is non-holomorphic at $s = 1$ precisely when

$$\chi = | - |^{-2}.$$

Comparing this with Lemma 5.1 and taking note of (c), we see that when $\chi \neq 1$, π is non-generic iff $I_{Q(Z)}(\chi, \tau)$ is reducible iff the adjoint L -factor is holomorphic at $s = 1$. If $\chi = 1$, then $I_{Q(Z)}(1, \tau)$ is the sum of two representations which form an L -packet. This L -packet thus contains a generic element and the adjoint L -factor is indeed holomorphic at $s = 1$.

(v) If $\pi \hookrightarrow I_{P(Y)}(\tau, \chi)$, where $|\omega_\tau| = | - |^{-s_0}$ with $s_0 \geq 0$, then by (a),

$$Ad \circ \phi_\pi = \omega_\tau \oplus \omega_\tau^{-1} \oplus \phi_\tau \oplus \phi_\tau^{-1} \oplus Ad(\phi_\tau) \oplus 1.$$

If τ is supercuspidal, it follows by (b) that the adjoint L -factor is non-holomorphic at $s = 1$ precisely when

$$\omega_\tau = | - |^{-1}.$$

Similarly, when $\tau = st_\mu$ is a twisted Steinberg representation, then

$$Ad \circ \phi = ((\mu^2 \oplus \mu^{-2} \oplus 1) \boxtimes S_1) \oplus (\mu \oplus \mu^{-1}) \boxtimes S_2 \oplus (1 \boxtimes S_3).$$

By (b), it follows that the adjoint L -factor is non-holomorphic at $s = 1$ iff

$$\mu^2 = | - |^{-1} \quad \text{or} \quad \mu = | - |^{-3/2}.$$

In view of (c), a comparison with Lemma 5.2 gives the desired result.

(vi) If $\pi \hookrightarrow I_B(\chi_1, \chi_2; \chi)$, then it follows by (a) that $Ad \circ \phi_\pi$ is the direct sum of the following 1-dimensional characters:

$$\chi_1^{\pm 1}, \chi_2^{\pm 1}, (\chi_1 \chi_2)^{\pm 1}, (\chi_1 / \chi_2)^{\pm 1}, 1 \text{ (with multiplicity 2)}.$$

By (b), the adjoint L -factor is non-holomorphic at $s = 1$ precisely when one of the above character is equal to $| - |^{-1}$. By [ST] (see also [RS, Pg. 37]), this is precisely when the induced representation $I_B(\chi_1, \chi_2; \chi)$ is reducible so that π is non-generic. This proves the desired result. □

14. Tables of Explicit Local Theta Correspondence.

In this section, we display the results of local theta correspondences in the form of tables. Note however that, unlike Thms. 8.1, 8.2 and 8.3, we have described the representation π in terms of the usual Langlands classification, so that π is the unique irreducible quotient of a standard module. So, for example, $J(\pi(\chi_1, \chi_2))$ stands for the unique irreducible quotient of the principal series representation $\pi(\chi_1, \chi_2)$ of $\mathrm{GL}_2(F)$.

TABLE 1. Explicit theta lifts from GSp_4

		π	$\theta_{(3,3)}(\pi)$	$\theta_{(2,2)}(\pi)$	$\theta_{(4,0)}(\pi)$	
S.C.	a	not a lift from $\mathrm{GO}_{2,2}$ or $\mathrm{GO}_{4,0}$	S.C.	0	0	
	b	$\theta(\tau_1 \boxtimes \tau_2)$, $\tau_1 \neq \tau_2$, both S.C.	$I_P(\tau_1, \tau_2) \boxtimes \omega_{\tau_1}$	$\tau_1 \boxtimes \tau_2$	0	
	c	$\theta(\tau_1^D \boxtimes \tau_2^D)$, $\tau_1^D \neq \tau_2^D$	0	0	$\tau_1^D \boxtimes \tau_2^D$	
D.S.	a	$St(\chi, \tau)$	$St(\tau) \boxtimes \omega_{\tau\chi}$	0	0	
	b	$St(\tau, \mu)$	$I_P(\tau \cdot \mu, st \cdot \mu) \boxtimes \mu^2$	$(\tau \cdot \mu) \boxtimes st_{\mu}$	0	
	c	$St_{\mathrm{PGSp}_4} \otimes \chi$	$St_{\mathrm{PGL}_4} \boxtimes \chi^2$	0	0	
N.D.S.	a	$J_{Q(Z)}(\chi, \tau)$, $\chi \neq 1$	$J_P(\tau \cdot \chi, \tau) \boxtimes \omega_{\tau\chi}$	0	0	
	b	$\pi \leftrightarrow$	$\pi = \pi_{gen}(\tau)$	$J_P(\tau, \tau) \boxtimes \omega_{\tau}$	$\tau \boxtimes \tau$	0
	c	$I_{Q(Z)}(1, \tau)$	$\pi = \pi_{ng}(\tau)$	0	0	$\tau_D \boxtimes \tau_D$
	d	$J_{P(Y)}(\tau, \chi)$	$J_Q(\omega_{\tau}, \tau, 1) \cdot \chi \boxtimes \chi^2 \omega_{\tau}$	$(\tau \cdot \chi) \boxtimes J(\pi(\omega_{\tau\chi}, \chi))$	0	
	e	$J_B(\chi_1, \chi_2; \chi)$	$\chi \cdot J_{B_0}(\chi_1 \chi_2, \chi_1, \chi_2, 1) \boxtimes \chi^2 \chi_1 \chi_2$	$J(\pi(\chi \chi_1, \chi \chi_2)) \boxtimes J(\pi(\chi \chi_1 \chi_2, \chi))$	0	

TABLE 2. Explicit theta lifts from $\mathrm{GSO}_{2,2}$ to GSp_4

	$\tau_1 \boxtimes \tau_2$	$\theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_2 \boxtimes \tau_1)$
a	$\tau_1 = \tau_2 = \tau = \mathrm{D.S.}$	$\pi_{gen}(\tau)$
b	$\tau_1 \neq \tau_2$ both S.C.	S.C.
c	$\tau_1 = \mathrm{S.C.}, \tau_2 = st_{\chi}$	$St(\tau_1 \otimes \chi^{-1}, \chi)$
d	$\tau_1 = st_{\chi_1}, \tau_2 = st_{\chi_2}, \chi_1 \neq \chi_2$	$St(st_{\chi_1/\chi_2}, \chi_2) = St(st_{\chi_2/\chi_1}, \chi_1)$
e	$\tau_1 = \mathrm{D.S.}, \tau_2 = J(\pi(\chi', \chi))$	$J_{P(Y)}(\tau_1 \otimes \chi^{-1}, \chi)$
f	$\tau_1 = J(\pi(\chi'_1, \chi_1)), \tau_2 = J(\pi(\chi'_2, \chi_2))$	$J_B(\chi'_2/\chi_1, \chi_2/\chi_1; \chi_1)$

TABLE 3. Explicit theta lifts from $\mathrm{GSO}_{4,0}$ to GSp_4

	$\tau_1^D \boxtimes \tau_2^D$	$\Theta(\tau_1^D \boxtimes \tau_2^D) = \theta(\tau_2^D \boxtimes \tau_1^D)$
a	$\tau_1^D = \tau_2^D = \tau_D$	$\pi_{ng}(JL(\tau_D))$
b	$\tau_1^D \neq \tau_2^D$	non-generic, S.C.

APPENDIX A. **Jacquet modules of the Weil representation**

In this appendix, we give a detailed computation of the Jacquet module of the (induced) Weil representation. We deal not only with the dual pairs for the small ranks we worked with but for all ranks. Hence throughout this appendix, $(V_m, (-, -))$ will denote a symmetric bilinear space with $\dim V_m = m$ even and $(W_n, \langle -, - \rangle)$ a symplectic space with $\dim W_n = 2n$. And χ_V denotes the character of V_m as usual. We fix a polarization

$$V_m = X_r + V_{an} + X_r^*$$

of V_m where X_{an} is anisotropic and $X_r + X_r^* = \mathbb{H}^r$. We let $\{v_1, \dots, v_r\}$ (resp. $\{v_1^*, \dots, v_r^*\}$) be a basis of X_r (resp. X_r^*) with $(v_i, v_j^*) = \delta_{ij}$. Also we fix a polarization

$$W_n = Y_n \oplus Y_n^*$$

of W_n and we let $\{e_1, \dots, e_n\}$ (resp. $\{e_1^*, \dots, e_n^*\}$) be a basis of Y_n (resp. Y_n^*) with $\langle e_i, e_j^* \rangle = \delta_{ij}$.

Let

$$R_{m,n} = R = \mathrm{GO}(V_m) \times \mathrm{GSp}(W_n)^+$$

and $\omega_{m,n} = \omega_{V_m W_n}$ be the Weil representation of

$$R_0 = \{(h, g) \in R_{m,n} : \lambda_V(h) \cdot \lambda_W(g) = 1\}.$$

Note that in this appendix, we consider various subspaces of V_m and W_n and their similitude groups. For the similitude characters of those similitude groups, we always use the same symbols λ_V and λ_W because this will not produce any confusion. Also let

$$\Omega_{m,n} = \Omega_{V_m, W_n} = \mathrm{ind}_{R_0}^R \omega_{m,n}$$

be the induced Weil representation of $\mathrm{GO}(V_m) \times \mathrm{GSp}(W_n)$.

Now let

$$X_t = \mathrm{span}\{v_1, \dots, v_t\} \text{ and } X_t^* = \mathrm{span}\{v_1^*, \dots, v_t^*\}$$

and write

$$V_m = X_t + V_{m_0} + X_t^*$$

so that $V_{m_0} = V_{an} + \mathbb{H}^{r-t}$ and

$$\dim V_{m_0} = m_0 = m - 2t.$$

Let $P(X_t)$ be the parabolic subgroup that stabilizes X_t . Then we write

$$P(X_t) = M(X_t)N(X_t)$$

where $M(X_t) \cong \mathrm{GL}(X_t) \times \mathrm{GO}(V_{m_0})$ is the Levi part and $N(X_t)$ is the unipotent part. We use $[a, h]$ to denote an element in $M(X_t)$ where $(a, h) \in \mathrm{GL}(X_t) \times \mathrm{GO}(V_{m_0})$.

Next let

$$Y_k = \mathrm{span}\{e_1, \dots, e_k\} \text{ and } Y_k = \mathrm{span}\{e_1^*, \dots, e_k^*\}$$

and we write

$$W_n = Y_k + W_{n_0} + Y_k^*$$

so that

$$\dim W_{n_0} = 2n_0 = 2(n - k).$$

Let $Q(Y_k) = M(Y_k)N(Y_k)$ be the parabolic that stabilizes Y_k , so that $M(Y_k) \cong \mathrm{GL}(Y_k) \times \mathrm{GSp}(W_{n_0})$. We use $[b, g]$ to denote an element in $M(Y_k)$ where $(b, g) \in \mathrm{GL}(Y_k) \times \mathrm{GO}(W_{n_0})$.

Then we compute the Jacquet modules $R_{P(X_t)}(\Omega_{m,n})$ and $R_{Q(Y_k)}(\Omega_{m,n})$ of $\Omega_{m,n}$ along the parabolic subgroups $P(X_t)$ and $Q(Y_k)$, respectively. For the former, assuming $t \geq k$, let

$$X_{t-k} = \mathrm{span}\{v_1, \dots, v_{t-k}\} \text{ and } X_{t-k}^* = \mathrm{span}\{v_1^*, \dots, v_{t-k}^*\}$$

and $P(X_{t-k}, X_t)$ be the parabolic subgroup of $M(X_t)$ that preserves the flag

$$0 \subseteq X_{t-k} \subseteq X_t \subseteq V_m,$$

so that

$$P(X_{t-k}, X_t) \cong R(X_{t-k}, X_t) \times \mathrm{GO}(V_{m_0}),$$

where $R(X_{t-k}, X_t)$ is the parabolic subgroup of $\mathrm{GL}(X_t)$ that preserves $0 \subseteq X_{t-k} \subseteq X_t$, i.e.

$$R(X_{t-k}, X_t) = \left\{ a = \begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix} : a_1 \in \mathrm{GL}(X_{t-k}), a_2 \in \mathrm{GL}(X'_k) \right\},$$

and

$$X'_k = \mathrm{span}\{v_{t-k+1}, \dots, v_t\} \text{ and } X'_k{}^* = \mathrm{span}\{v_{t-k+1}^*, \dots, v_t^*\}.$$

For the latter, assuming $k \geq t$, let

$$Y_{k-t} = \mathrm{span}\{e_1, \dots, e_{k-t}\} \text{ and } Y_{k-t}^* = \mathrm{span}\{e_1^*, \dots, e_{k-t}^*\}$$

and $Q(Y_{k-t}, Y_k)$ be the parabolic subgroup of $M(Y_k)$ that preserves the flag

$$0 \subseteq Y_{k-t} \subseteq Y_k \subseteq W_n,$$

so that

$$Q(Y_{k-t}, Y_k) \cong R(Y_{k-t}, Y_k) \times \mathrm{GSp}(W_{n_0}),$$

where $R(Y_{k-t}, Y_k)$ is the parabolic subgroup of $\mathrm{GL}(Y_k)$ that preserves $0 \subseteq Y_{k-t} \subseteq Y_k$, i.e.

$$R(Y_{k-t}, Y_k) = \left\{ b = \begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix} : b_1 \in \mathrm{GL}(Y_{k-t}), b_2 \in \mathrm{GL}(Y'_t) \right\},$$

and

$$Y'_t = \mathrm{span}\{e_{k-t+1}, \dots, e_k\} \text{ and } Y'_k{}^* = \mathrm{span}\{e_{k-t+1}^*, \dots, e_k^*\}.$$

Then we have

Theorem A.1. *The normalized Jacquet module $R_{P(X_t)}(\Omega_{m,n})$ of the Weil representation $\Omega_{m,n}$ along the parabolic $P(X_t)$ has an $M(X_t) \times \mathrm{GSp}(W_n)$ invariant filtration*

$$\{0\} \subseteq J^{(\min\{t,n\})} \subseteq \dots \subseteq J^{(1)} \subseteq J^{(0)} = R_{P(X_t)}(\Omega_{m,n})$$

with the successive quotient

$$J^k := J^k / J^{k+1} \cong \mathrm{Ind}_{P(X_{t-k}, X_t) \times Q(Y_k)^+}^{M(X_t) \times \mathrm{GSp}(W_n)^+} (S(\mathrm{Isom}(X'_k, Y_k)) \otimes \Omega_{m_0, n-k}),$$

where $\Omega_{m_0, n-k}$ is the Weil representation of the group $\mathrm{GO}(V_{m_0}) \times \mathrm{GSp}(W_{n-k})$, and the group $P(X_{t-k}, X_t) \times Q(Y_k)^+$ acts on $S(\mathrm{Isom}(X'_k, Y_k))$ as follows: Let $\varphi(A) \in S(\mathrm{Isom}(X'_k, Y_k))$. Then each element $\left[\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h \right] \in P(X_{t-k}, X_t)$ acts as

$$\left[\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h \right] \cdot \varphi(A) = \chi_V(\det a_2) |\lambda_V(h)|^{e_0} |\det a_1|^{e_1} \varphi(Aa_2)$$

where

$$e_0 = -\frac{1}{4}(m-2t)k - \frac{1}{2}tn + \frac{1}{4}mt - \frac{1}{4}t(t+1)$$

$$e_1 = n - \frac{1}{2}m + \frac{1}{2}t - \frac{1}{2}(k-1), \quad (\text{for } k < t)$$

and each element $[b, g]n \in Q(Y_k)$ acts as

$$[b, g]n \cdot \varphi(A) = |\lambda_W(g)|^{f_0} \varphi(\lambda_W(g)b^{-1}A),$$

where

$$f_0 = -\frac{1}{2}kn + \frac{1}{4}k(k-1).$$

Note that the induction is normalized.

Theorem A.2. *The normalized Jacquet module $R_{Q(Y_k)}(\Omega_{m,n})$ of the Weil representation $\Omega_{m,n}$ along the parabolic $Q(Y_k)$ has an $\mathrm{GO}(V_m) \times M(Y_k)^+$ invariant filtration*

$$\{0\} \subseteq J^{(\min\{k,r\})} \subseteq \dots \subseteq J^{(1)} \subseteq J^{(0)} = R_{Q(Y_k)}(\Omega_{m,n})$$

with the successive quotient

$$J^t := J^t / J^{t+1} \cong \mathrm{Ind}_{P(X_t) \times Q(Y_{k-t}, Y_k)^+}^{\mathrm{GO}(V_m) \times M(Y_k)^+} (S(\mathrm{Isom}(Y'_t, X_t)) \otimes \Omega_{m-2t, n_0}),$$

where Ω_{m-2t, n_0} is the Weil representation of the group $\mathrm{GO}(V_{m-2t}) \times \mathrm{GSp}(W_{n_0})$, and the group $P(X_t) \times Q(Y_{k-t}, Y_k)^+$ acts on $S(\mathrm{Isom}(Y'_t, X_t))$ as follows: Let $\varphi(A) \in S(\mathrm{Isom}(Y'_t, X_t))$. Then each element $[\begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix}, g] \in Q(Y_{k-t}, Y_k)^+$ acts as

$$[\begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix}, g] \cdot \varphi(A) = \chi_V(\det b_1) \chi_V(\det b_2) |\lambda_W(g)|^{e_0} |\det b_1|^{e_1} \varphi(Ab_2)$$

where

$$\begin{aligned} e_0 &= -\frac{1}{2}t(n-k) - \frac{1}{4}mk + \frac{1}{2}kn - \frac{1}{4}k(k-1) \\ e_1 &= \frac{1}{2}m - n + \frac{1}{2}k - \frac{1}{2}(t+1), \quad (\text{for } t < k) \end{aligned}$$

and each element $[a, h]n \in P(X_t)$ acts as

$$[a, h]n \cdot \varphi(A) = |\lambda_V(h)|^{f_0} \varphi(\lambda_V(h)a^{-1}A),$$

where

$$f_0 = -\frac{1}{4}mt + \frac{1}{4}t(t+1).$$

Note that the induction is normalized.

Remark A.3. *If m_0 or $n-k$ in $\Omega_{m_0, n-k}$ (resp. $m-2k$ or n_0 in Ω_{m-2k, n_0}) is zero, then $\Omega_{m_0, n-k}$ (resp. Ω_{m-2k, n_0}) is the induced representation of the trivial representation. For example if $m_0 = 0$, then $\Omega_{m_0, n-k}$ is realized in the space $S(F^\times)$ where $([1, \lambda], [1, g])$ acts as $([1, \lambda], [1, g]) \cdot f(x) = (x \cdot \lambda \cdot \lambda_W(g))$ for $\lambda \in \mathrm{GL}_1$, $g \in \mathrm{GSp}_{n-k}$ and $f \in S(F^\times)$.*

Remark A.4. *Note that the roles of k and t are switched in Theorems A.1 and A.2.*

Remark A.5. *For an induced Weil representation $\Omega_{W,V}$, and any character χ , one has*

$$\Omega_{V,W} \otimes ((\chi \circ \lambda_W) \boxtimes (\chi \circ \lambda_V)) \cong \Omega_{W,V}.$$

Thus, in Theorems A.1 and A.2, one could replace the pair (e_0, f_0) by $(e_0 - f_0, 0)$. Now observe that when $k = t$ in Theorems A.1 and A.2, one has $e_0 = f_0$. Thus, for $k = t$, one could simply take $e_0 = f_0 = 0$.

Remark A.6. *One can obtain the analogous theorems for the isometry case simply by replacing the induced Weil representation $\Omega_{m_0, n-k}$ (or Ω_{m-2t, n_0}) by the Weil representation $\omega_{m_0, n-k}$ (or ω_{m-2t, n_0}) and disregarding the similitude factors. For example, the Jacquet module $R_{P(X_t)}(\omega_{m,n})$ along the parabolic $P(X_t)$ of $\mathrm{O}(V_m)$ has the analogous filtration where each successive quotient has the inducing data of the form $S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_{m_0, n-k}$ where the action of the relevant subgroup of $\mathrm{O}(V_m) \times \mathrm{Sp}(W_n)$ on $S(\mathrm{Isom}(X'_k, Y_k))$ is simply the restriction of the similitude case. And the similar statement holds for $R_{Q(Y_k)}(\omega_{m,n})$.*

The rest of the appendix is devoted to the proof of those theorems. The proof in the context of the isometry groups appears in Kudla ([K]). Our proof closely follows Kudla's computation, though we give more details. Also in [K], Kudla considers the Jacquet module along the parabolic of the symplectic group, but we do the other way around. Namely we consider the Jacquet module $R_{P(X_t)}(\Omega_{m,n})$ along the parabolic $P(X_t)$ of $\mathrm{GO}(V_m)$ i.e. Theorem A.1, and leave the other case to the reader.

First recall that

$$\mathbb{W} := V_m \otimes W_n$$

is equipped with the obvious symplectic structure $\langle\langle -, - \rangle\rangle$ and for each polarization $\mathbb{W} = \mathbb{Y} + \mathbb{Y}^*$, the Weil representation $\omega_{m,n}$ can be realized in the space $S(\mathbb{Y})$ of Schwartz functions on \mathbb{Y} called the Schrodinger model of $\omega_{m,n}$ with respect to the polarization. To compute the Jacquet module of the Weil representation $\omega_{m,n}$, one needs to consider the Schrodinger models for various polarizations. First, consider the Schrodinger model with respect to the polarization

$$\mathbb{W} = V_m \otimes Y_n^* + V_m \otimes Y_n$$

so that $\omega_{m,n}$ is realized in the space $S(V_m \otimes Y_n^*)$. Then each $(h, g) \in R_0$ acts as

$$(h, g)\varphi(x) = |\lambda_W(g)|^{\frac{1}{4}mn} \omega_{m,n}(1, g_1) \cdot \varphi(h^{-1}x)$$

where

$$g_1 = g \begin{pmatrix} \lambda_W(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and $\omega_{m,n}(1, g_1)$ is the action of the Weil representation for the usual isometry group.

Next let $t \leq r$ and m_0 be such that $2t + m_0 = m$. Consider

$$V_m = X_t + V_{m_0} + X_t^*$$

where X_t , V_{m_0} and X_t^* are as above. Then we compute the Jacquet module of $\Omega_{m,n}$ along the parabolic subgroup $P(X_t) = M(X_t)N(X_t)$ of $\mathrm{GO}(V_m)$. Let $N(X_t)_0$ be the center of $N(X_t)$, so that it fits in the exact sequence

$$1 \longrightarrow N(X_t)_0 \longrightarrow N(X_t) \longrightarrow \mathrm{Hom}(V_{m_0}, X_t) \longrightarrow 1.$$

Indeed in terms of the obvious matrix realization of $\mathrm{GO}(V_m)$, $N(X_t)$ is written as

$$N(X_t) = \left\{ n(c, d) = \begin{pmatrix} 1 & c & d - \frac{1}{2}c \circ c^* \\ & 1 & -c^* \\ & & 1 \end{pmatrix} : c \in \mathrm{Hom}(V_{m_0}, X_t), d \in \mathrm{Hom}(X_t^*, X_t) \right\},$$

where c^* is the adjoint of c , i.e. $c^* \in \mathrm{Hom}(X_t^*, V_{m_0})$ such that

$$(cv, u) = (v, c^*u) \quad \text{for } v \in V_{m_0}, u \in V_{m_0},$$

and d is such that $(dx_1, x_2) = (x_1, -dx_2)$ for all $x_1, x_2 \in X_t^*$. Note that if we make the identification $\mathrm{Hom}(X_t^*, X_t) = \mathrm{GL}_t$ with respect to the above chosen bases of X_t and X_t^* , we have ${}^t d = -d$. Note that

$$N(X_t)_0 = \{n(c, d) : c = 0\} = \left\{ \begin{pmatrix} 1 & d \\ & 1 & \\ & & 1 \end{pmatrix} : {}^t d = -d \right\}.$$

Now we consider the polarization

$$\mathbb{W} = \mathbb{Y}^* + \mathbb{Y} = (W_n \otimes X_t^* + V_{m_0} \otimes Y_n^*) + (W_n \otimes X_t + V_{m_0} \otimes Y_n)$$

to realize the Weil representation $\omega_{m,n}$ of $R_0 \subset \mathrm{GO}(V_m) \times \mathrm{GSp}(W_n)^+$. So the space of $\omega_{m,n}$ is

$$S(\mathbb{Y}^*) = S(W_n \otimes X_t^* + V_{m_0} \otimes Y_n^*) \cong S(W_n \otimes X_t^*) \otimes S(V_{m_0} \otimes Y_n^*).$$

Now for any subgroup H of R , let us define

$$R_0(H) := R_0 \cap H.$$

Write

$$W_n \otimes X_t^* = Y_n \otimes X_t^* + Y_n^* \otimes X_t^*$$

and denote each element $w \in W_n \otimes X_t^*$ as

$$w = y + y^* \in Y_n \otimes X_t^* + Y_n^* \otimes X_t^*$$

where $y \in Y_n \otimes X_t^*$ and $y^* \in Y_n^* \otimes X_t^*$. Then the action of R_0 is described as follows: Let $([a, h], g) \in R_0(M(X_t) \times \mathrm{GSp}(W_n)^+)$ where $(a, h) \in \mathrm{GL}(X_t) \times \mathrm{GO}(V_{m_0})$. Then for $\phi_1(y + y^*) \otimes \phi_2(x) \in S(W_n \otimes X_t^*) \otimes S(V_{m_0} \otimes Y_n^*)$,

$$\begin{aligned} & ([a, h], g) \cdot \phi_1(y + y^*) \otimes \phi_2(x) \\ &= |\det a|^n |\lambda_V(h)|^{-\frac{1}{2}tn} \phi_1((g_1^{-1} \otimes a^*)(y + \lambda_V(h)^{-1}y^*)) \otimes \omega_{m_0, n}(h, g)\phi_2(x) \end{aligned}$$

where $x \in V_{m_0} \otimes Y_n^*$, and $a^* \in \mathrm{GL}(X_t^*)$ is such that $(ax_1, x_2) = (x_1, a^*x_2)$ for all $x_1 \in X_t$ and $x_2 \in X_t^*$. Note that $\omega_{m_0, n}$ is the Weil representation for the pair $(\mathrm{GO}(V_{m_0}), \mathrm{GSp}(W_n)^+)$. Also the action of $N(X_t)$ is described as follows: Let $\phi(y + y^* + x) \in S(\mathbb{Y}^*) = S(W_n \otimes X_t^* + V_{m_0} \otimes Y_n^*)$. Then

$$n(c, d) \cdot \phi(y + x + y^*) = \psi(\langle\langle y, dy^* \rangle\rangle) \rho(-c^*(y + y^*)) \phi(y + x + y^*)$$

where ρ is the action of the Heisenberg group $H(\mathbb{W})$ in $S(\mathbb{Y}^*)$. Those actions can be shown by looking at how R_0 acts on the Weil representation realized in the space

$$S(V_n \otimes Y_n^*) = S(Y_n^* \otimes X_t + V_{m_0} \otimes Y_n^* + Y_n^* \otimes X_t^*)$$

and the partial Fourier transform

$$\mathcal{F} : S(Y_n^* \otimes X_t + V_{m_0} \otimes Y_n^* + Y_n^* \otimes X_t^*) \rightarrow S(\mathbb{Y}^*) = S(Y_n \otimes X_t^* + V_{m_0} \otimes Y_n^* + Y_n^* \otimes X_t^*)$$

given by

$$\mathcal{F}(\varphi)(y + x + y^*) = \int_{Y_n^* \otimes X_t} \psi(\langle\langle y, z \rangle\rangle) \varphi \begin{pmatrix} z \\ x \\ y^* \end{pmatrix} dz,$$

where $y + x + y^* \in Y_n \otimes X_t^* + V_{m_0} \otimes Y_n^* + Y_n^* \otimes X_t^*$ and $z \in Y_n^* \otimes X_t$. For example, let $[a, 1] \in M(X_t)$. Then

$$\begin{aligned} [a, 1] \cdot \mathcal{F}(\varphi)(y + x + y^*) &= \mathcal{F}([a, 1] \cdot \varphi)(y + x + y^*) \\ &= \int_{Y_n^* \otimes X_t} \psi(\langle\langle y, z \rangle\rangle) [a, 1] \cdot \varphi \begin{pmatrix} z \\ x \\ y^* \end{pmatrix} dz \\ &= \int_{Y_n^* \otimes X_t} \psi(\langle\langle y, z \rangle\rangle) \varphi \begin{pmatrix} a^{-1}z \\ x \\ a^*y^* \end{pmatrix} dz \\ &= |\det a|^n \int_{Y_n^* \otimes X_t} \psi(\langle\langle y, az \rangle\rangle) \varphi \begin{pmatrix} z \\ x \\ a^*y^* \end{pmatrix} dz \\ &= |\det a|^n \int_{Y_n^* \otimes X_t} \psi(\langle\langle a^*y, z \rangle\rangle) \varphi \begin{pmatrix} z \\ x \\ a^*y^* \end{pmatrix} dz \\ &= |\det a|^n \mathcal{F}(\varphi)(a^*y + x + a^*y^*). \end{aligned}$$

In particular, the group $N(X_t)_0$ simply acts as multiplication by $\psi(\langle\langle y, dy^* \rangle\rangle)$. It is easy to see that $\langle\langle y, dy^* \rangle\rangle = \frac{1}{2} \langle\langle w, dw \rangle\rangle$, where $w = y + y^* \in W_n \otimes X_t^*$. Now let

$$W_0 = \{w \in W_n \otimes X_t^* : \langle\langle w, dw \rangle\rangle = 0 \text{ for all } d = -{}^t d\}.$$

If we write $w = \sum_{i=1}^t w_i \otimes v_i^*$, we have $\langle\langle w, dw \rangle\rangle = \sum_{i \neq j} \langle w_i, w_j \rangle \langle v_i^*, dv_j^* \rangle$. Since for each pair i and j with $i \neq j$ one can choose d so that $\sum_{i \neq j} \langle w_i, w_j \rangle \langle v_i^*, dv_j^* \rangle = 2\langle w_i, w_j \rangle$, we have $W_0 = \{w \in W_n \otimes X_t^* : \langle w_i, w_j \rangle = 0\}$. Or by identifying $W_n \otimes X_t^*$ with $\text{Hom}(X_t, W_n)$ in the obvious way, one can see that

$$W_0 = \{\phi \in \text{Hom}(X_t, W_n) : \phi^*(\langle -, - \rangle) = 0\},$$

where $\phi^*(\langle -, - \rangle)$ is the pullback of the symplectic form $\langle -, - \rangle$ on W_n to X_t via ϕ . Then it is clear that the restriction map

$$S(W_n \otimes X_t^*) \otimes S(V_{m_0} \otimes Y_n^*) \rightarrow S(W_0) \otimes S(V_{m_0} \otimes Y_n^*)$$

induces an isomorphism

$$(\omega_{m,n})_{N(X_t)_0} \cong S(W_0) \otimes S(V_{m_0} \otimes Y_n^*).$$

We can decompose the space W_0 as

$$W_0 = \prod_{k=0}^{\min\{t,n\}} W_{0,k}$$

where

$$W_{0,k} = \{\phi \in W_0 : \dim \text{Im} \phi = k\}.$$

Define $T^{(k)} \subseteq S(W_0) \otimes S(V_{m_0} \otimes Y_n^*)$ by

$$T^{(k)} = \{\varphi \in S(W_0) : \varphi|_{W_{0,i}} = 0 \text{ for all } i < k\} \otimes S(V_{m_0} \otimes Y_n^*).$$

Clearly $T^{(k+1)} \subseteq T^{(k)}$. One can see that each $T^{(k)}$ is invariant under $R_0(P(X_t) \times \text{GSp}(W_n)^+)$, and moreover we have a short exact sequence of $R_0(P(X_t) \times \text{GSp}(W_n)^+)$ modules

$$0 \rightarrow T^{(k+1)} \rightarrow T^{(k)} \rightarrow S(W_{0,k}) \otimes S(V_{m_0} \otimes Y_n^*) \rightarrow 0,$$

where the map $T^{(k)} \rightarrow S(W_{0,k}) \otimes S(V_{m_0} \otimes Y_n^*)$ is the obvious restriction map. Hence we have an $R_0(P(X_t) \times \text{GSp}(W_n)^+)$ invariant filtration

$$\{0\} \subseteq T^{(\min\{t,n\})} \subseteq \dots \subseteq T^{(1)} \subseteq T^{(0)} = S(W_0) \otimes S(V_{m_0} \otimes Y_n^*),$$

where the successive quotient is given by

$$T^k := T^{(k)} / T^{(k+1)} = S(W_{0,k}) \otimes S(V_{m_0} \otimes Y_n^*).$$

Then we have

Lemma A.7. *There is an isomorphism of $P(X_t) \times \text{GSp}(W_n)^+$ modules*

$$(\Omega_{m,n})_{N(X_t)_0} = (\text{ind}_{R_0}^R \omega_{m,n})_{N(X_t)_0} \xrightarrow{\sim} \text{ind}_{R_0(P(X_t) \times \text{GSp}(W_n)^+)}^{P(X_t) \times \text{GSp}(W_n)^+} S(W_0) \otimes S(V_{m_0} \otimes Y_n^*).$$

Moreover the above filtration induces a $P(X_t) \times \text{GSp}(W_n)^+$ invariant filtration

$$\{0\} \subseteq \tilde{T}^{(\min\{t,n\})} \subseteq \dots \subseteq \tilde{T}^{(1)} \subseteq \tilde{T}^{(0)} = S(W_0) \otimes S(V_{m_0} \otimes Y_n^*),$$

where

$$\tilde{T}^{(k)} = \text{ind}_{R_0(P(X_t) \times \text{GSp}(W_n)^+)}^{P(X_t) \times \text{GSp}(W_n)^+} T^{(k)}$$

and the successive quotient is given by

$$\tilde{T}^k := \tilde{T}^{(k)} / \tilde{T}^{(k+1)} \cong \text{ind}_{R_0(P(X_t) \times \text{GSp}(W_n)^+)}^{P(X_t) \times \text{GSp}(W_n)^+} S(W_{0,k}) \otimes S(V_{m_0} \otimes Y_n^*).$$

Proof. This follows because an induction is an exact functor. \square

Now we will describe the representation \tilde{T}^k of $P(X_t) \times \mathrm{GSp}(W_n)^+$ in terms of a certain induced representation, and then compute $\tilde{T}_{N(X_t)}^k$, which gives the desired filtration of the Jacquet module of the Weil representation $\Omega_{m_0, n}$. For that purpose, we need to describe the representation $T^k = S(W_{0, k}) \otimes S(V_{m_0} \otimes Y_n^*)$ of $R_0(P(X_t) \times \mathrm{GSp}(W_n)^+)$ in terms of an induced representation. For this, let us write

$$R_0(P(X_t) \times \mathrm{GSp}(W_n)^+) = R_0(M(X_t) \times \mathrm{GSp}(W_n)^+) \ltimes N(X_t)$$

and fix $w_0 \in W_{0, k}$ given by

$$w_0 = e_1 \otimes v_{t-k+1}^* + e_2 \otimes v_{t-k+2}^* + \cdots + e_k \otimes v_t^*.$$

Let

$$H = \{([a, h], g) \in R_0(M(X_t) \times \mathrm{GSp}(W_n)^+) : (g_1^{-1} \otimes a^*)w_0 = w_0\}.$$

Recalling

$$Y_k = \mathrm{span}\{e_1, \dots, e_k\},$$

we have

$$H \subseteq R_0(M(X_t) \times Q(Y_k)^+)$$

where $Q(Y_k)^+ \subseteq \mathrm{GSp}(W_n)^+$ is the maximal parabolic that preserves the flag $0 \subseteq Y_k \subseteq W_n$. Recall we denote each element $q \in Q(Y_k)^+$ by

$$q = [b, g]n, \quad [b, g] \in M(Y_k) \cong \mathrm{GL}(Y_k) \times \mathrm{GSp}(W_{n-k})^+$$

where $b \in \mathrm{GL}(Y_k)$, $g \in \mathrm{GSp}(W_{n-k})^+$ and n is in the unipotent radical of $Q(Y_k)$. So each element in H can be denoted by $([a, h], [b, g]n)$ where $[a, h] \in M(X_t)$ and $[b, g]n \in Q(Y_k)$. Then we define a representation

$$(\tau^k, S(V_{m_0} \otimes Y_n^*))$$

of $H \ltimes N(X_t)$ on the space $S(V_{m_0} \otimes Y_n^*)$ as follows: For $([a, h], q) \in H$, we define

$$\tau^k([a, h], q) = \xi(\det a) |\lambda_V(h)|^{-\frac{1}{2}tn} \omega_{m_0, n}(h, q),$$

where $\omega_{m_0, n}$ is the Weil representation of the pair $(\mathrm{GO}(V_{m_0}), \mathrm{GSp}(W_n)^+)$ and

$$\xi(\det a) = |\det a|^n.$$

Also for $n(c, d) \in N(X_t)$,

$$\tau^k(n(c, d)) = \rho_0(-c^*w_0)$$

where ρ_0 is the action of the Heisenberg group $H(V_{m_0} \otimes W_n)$ on $S(V_{m_0} \otimes Y_n^*)$. Note that $-c^*w_0 \in V_{m_0} \otimes W_n$ and hence the action of $\rho_0(-c^*w_0)$ on $S(V_{m_0} \otimes Y_n^*)$ makes sense. Then we have

Lemma A.8. *There is an isomorphism*

$$T^k \cong \mathrm{ind}_{H \ltimes N(X_t)}^{R_0(M(X_t) \times \mathrm{GSp}(W_n)^+) \ltimes N(X_t)} \tau^k$$

of $R_0(M(X_t) \times \mathrm{GSp}(W_n)^+) \ltimes N(X_t)$ -modules.

Proof. The proof is almost identical to Lemma 5.2 of [K]. □

We would like to compute $T_{N(X_t)}^k$. But as in [K, Lemma 5.3], we have

$$T_{N(X_t)}^k \cong (\mathrm{ind}_{H \ltimes N(X_t)}^{R_0(M(X_t) \times \mathrm{GSp}(W_n)^+) \ltimes N(X_t)} \tau^k)_{N(X_t)} \cong \mathrm{ind}_H^{R_0(M(X_t) \times \mathrm{GSp}(W_n)^+)} (\tau_{N(X_t)}^k).$$

Hence we first compute $\tau_{N(X_t)}^k$. For this purpose, let us write

$$Y_n^* = Y_k^* + Y_{n-k}^*$$

where $Y_k^* = \mathrm{span}\{e_1^*, \dots, e_k^*\}$ and $Y_{n-k}^* = \mathrm{span}\{e_{k+1}^*, \dots, e_n^*\}$, and consider the polarization

$$V_{m_0} \otimes W_n = (V_{m_0} \otimes Y_n^*) + (V_{m_0} \otimes Y_n).$$

Then we can write the Schwartz space as

$$S(V_{m_0} \otimes Y_n^*) = S((V_{m_0} \otimes Y_k^*) + (V_{m_0} \otimes Y'_{n-k}{}^*)) \cong S(V_{m_0} \otimes Y_k^*) \otimes S(V_{m_0} \otimes Y'_{n-k}{}^*),$$

where $S(V_{m_0} \otimes Y'_{n-k}{}^*)$ provides a model of the Weil representation $\omega_{m_0, n-k}$ for the pair $(\mathrm{GO}(V_{m_0}), \mathrm{GSp}(W_{n-k})^+)$. Note that for $\varphi_1(x_1) \otimes \varphi_2(x_2) \in S(V_{m_0} \otimes Y_k^*) \otimes S(V_{m_0} \otimes Y'_{n-k}{}^*)$, each $([1, h], [b, g]) \in R_0(M(X_t) \times M(Y_k))$ acts as

$$\begin{aligned} & \omega_{m_0, n}([1, h], [b, g]) \cdot \varphi_1(x_1) \otimes \varphi_2(x_2) \\ &= \xi'(\det(\lambda_W(g)^{-1}b)) |\lambda_V(h)|^{-\frac{1}{4}m_0k} \varphi_1((h^{-1} \otimes \lambda_W(g)^{-1}b^*)x_1) \otimes \omega_{m_0, n-k}(h, g)\varphi_2(x_2), \end{aligned}$$

where

$$\xi'(x) = \chi_V(x) |x|^{\frac{m_0}{2}} \quad \text{for } x \in F^\times.$$

Then let us define a representation

$$(\omega_0, S(V_{m_0} \otimes Y'_{n-k}{}^*))$$

of H by

$$\omega_0([a, h], [b, g]n)\varphi = \xi(\det a)\xi'(\det(\lambda_W(g)^{-1}b))\eta(\lambda_V(h))\omega_{m_0, n-k}(h, g)\varphi$$

for $([a, h], [b, g]n) \in H \subseteq R_0(M(X_t) \times Q(Y_k)^+)$ and $\varphi \in S(V_{m_0} \otimes Y'_{n-k}{}^*)$, where

$$\eta(\lambda_V(h)) = |\lambda_V(h)|^{-\frac{1}{2}tn - \frac{1}{4}m_0k}.$$

Then

Lemma A.9. *There is a natural isomorphism of H -modules*

$$\tau^k_{N(X_t)} \cong \omega_0$$

induced by the surjection

$$S((V_{m_0} \otimes Y_k^*) + (V_{m_0} \otimes Y'_{n-k}{}^*)) \twoheadrightarrow S(V_{m_0} \otimes Y'_{n-k}{}^*)$$

defined by $\varphi(x_1 + x_2) \mapsto \varphi(0 + x_2)$ where $x_1 \in V_{m_0} \otimes Y_k^*$ and $x_2 \in V_{m_0} \otimes Y'_{n-k}{}^*$.

Proof. Since $-c^*x_0 \in V_{m_0} \otimes Y_k$, we have

$$\tau^k(-c^*x_0)\varphi(x_1 + x_2) = \rho_0(-c^*x_0)\varphi(x_1 + x_2) = \psi(\langle\langle x_1, -c^*x_0 \rangle\rangle)\varphi(x_1 + x_2).$$

By looking at this action, one can easily see that the lemma follows. \square

Thus we have proven

$$T^k_{N(X_t)} \cong \mathrm{ind}_H^{R_0(M(X_t) \times \mathrm{GSp}(W_n)^+)} \omega_0.$$

Now let

$$\mu_{tk} := \mathrm{ind}_H^{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)} \omega_0$$

where

$$X_{t-k} = \mathrm{span}\{v_1, \dots, v_{t-k}\}$$

and $P(X_{t-k}, X_t)$ is the parabolic subgroup of $M(X_t)$ that preserves the flag

$$\{0\} \subseteq X_{t-k} \subseteq X_t \subseteq V_m.$$

Note that

$$P(X_{t-k}, X_t) \cong R(X_{t-k}, X_t) \times \mathrm{GO}(V_{m_0}),$$

where $R(X_{t-k}, X_t)$ is the parabolic subgroup of $\mathrm{GL}(X_t)$ that preserves $0 \subseteq X_{t-k} \subseteq X_t$, i.e.

$$R(X_{t-k}, X_t) = \left\{ a = \begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix} : a_1 \in \mathrm{GL}(X_{t-k}), a_2 \in \mathrm{GL}(X'_k) \right\},$$

where

$$X'_k = \mathrm{span}\{v_{t-k+1}, \dots, v_n\}.$$

Then we have

$$T^k_{N(X_t)} \cong \mathrm{ind}_{R_0(M(X_{t-k}, X_t) \times Q(Y_k)^+)}^{R_0(M(X_t) \times \mathrm{GSp}(W_n)^+)} \mu_{tk}.$$

In what follows, we realize μ_{tk} in a more concrete space. For this, let $\mathrm{Isom}(X'_k, Y_k)$ be the space of isomorphisms from X'_k to Y_k as vector spaces. Note $\mathrm{GL}(X'_k) \times \mathrm{GL}(Y_k)$ acts on this space in the obvious way. One can also see that

$$H = \left\{ \left(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h \right), [b, g]n \in P(X_{t-k}, X_t) \times Q(Y_k) : (\lambda_W(g)b^{-1} \otimes a_2^*) \cdot w_0 = w_0 \right\}.$$

Notice that if $I \in \mathrm{Isom}(X'_k, Y_k)$ is such that $I(v_{t-k+i}) = e_i$, then

$$(\lambda_W(g)b^{-1} \otimes a_2^*) \cdot w_0 = w_0 \iff b = I\lambda_W(g)a_2I^{-1}.$$

Also notice that the set

$$\left\{ \left(\begin{pmatrix} 1 & * \\ & a_2 \end{pmatrix}, 1 \right), [1, 1] \in P(X_{t-k}, X_t) \times Q(Y_k) : a_2 \in \mathrm{GL}(X'_k) \right\}$$

is a set of representatives of

$$H \backslash P(X_{t-k}, X_t) \times Q(Y_k).$$

Then we have

Lemma A.10. *There is an isomorphism*

$$\mu_{tk} = \mathrm{ind}_H^{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)} \omega_0 \cong S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_0$$

of $R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)$ -modules, where each $(\left(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h \right), [b, g]n)$ acts in the following way:

$$\begin{aligned} & \left(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h \right), [b, g]n \cdot \varphi(A) \\ &= \xi(\det a_2)\xi'(\det a_2)\xi(\det a_1)\eta(\lambda_V(h))\omega_{m_0, n-k}(h, g)\varphi(\lambda_W(g)b^{-1}Aa_2), \end{aligned}$$

where $A \in \mathrm{Isom}(X'_k, Y_k)$, and we identify each element $\varphi \in S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_0$ with

$$\varphi : \mathrm{Isom}(X'_k, Y_k) \rightarrow \omega_0.$$

Proof. Define

$$\alpha : \mathrm{ind}_H^{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)} \omega_0 \rightarrow S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_0$$

by

$$\alpha(F)(A) = F([1, 1], [IA^{-1}, 1]) \in \omega_0,$$

where $F \in \mathrm{ind}_H^{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)} \omega_0$ and $([1, 1], [IA^{-1}, 1]) \in P(X_{t-k}, X_t) \times Q(Y_k)$, and also define

$$\beta : S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_0 \rightarrow \mathrm{ind}_H^{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)} \omega_0$$

by

$$\beta(\varphi)\left(\left(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h\right), [b, g]n\right) = \omega_0\left(\left(\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h\right), [I\lambda_W(g)a_2I^{-1}, g]n\right)\varphi(\lambda_W(g)b^{-1}Ia_2),$$

where we identify each element $\varphi \in S(\mathrm{Isom}(X'_k, Y_k)) \otimes \omega_0$ with $\varphi : \mathrm{Isom}(X'_k, Y_k) \rightarrow \omega_0$. Then it is straightforward to verify that α and β are inverses to each other.

Also one can see that this map indeed intertwines the actions of $R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)$ by considering

$$\begin{aligned}
& \alpha(\mu_{tk}([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [b, g]n)F)(A) \\
&= \mu_{tk}([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [b, g]n)F([1, 1], [IA^{-1}, 1]) \\
&= F([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [IA^{-1}b, g]n) \\
&= \omega_0([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [I\lambda_W(g)a_2I^{-1}, g]n)F([1, 1], [I\lambda_W(g)^{-1}a_2^{-1}I^{-1}A^{-1}b, 1]) \\
&= \omega_0([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [I\lambda_W(g)a_2I^{-1}, g]n)\alpha(F)(\lambda_W(g)b^{-1}Aa_2) \\
&= \xi(\det a_2)\xi'(\det(\lambda_W(g)^{-1}I\lambda_W(g)a_2I^{-1}))\xi(\det a_1)\eta(\lambda_V(h))\omega_{m_0, n-k}(h, g)\alpha(F)(\lambda_W(g)b^{-1}Aa_2) \\
&= \xi(\det a_2)\xi'(\det a_2)\xi(\det a_1)\eta(\lambda_V(h))\omega_{m_0, n-k}(h, g)\alpha(F)(\lambda_W(g)b^{-1}Aa_2).
\end{aligned}$$

□

This lemma gives

$$T_{N(X_t)}^k \cong \text{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{R_0(M(X_t) \times \text{GSp}(W_n)^+)} \mu_{tk}.$$

Recall that we have been trying to compute $\tilde{T}_{N(X_t)}^k \cong (\text{ind}_{R_0(P(X_t) \times \text{GSp}(W_n)^+)}^{P(X_t) \times \text{GSp}(W_n)^+} T^k)_{N(X_t)}$. But note that

$$\begin{aligned}
\tilde{T}_{N(X_t)}^k &\cong (\text{ind}_{R_0(P(X_t) \times \text{GSp}(W_n)^+)}^{P(X_t) \times \text{GSp}(W_n)^+} T^k)_{N(X_t)} \\
&\cong (\text{ind}_{R_0(M(X_t) \times \text{GSp}(W_n)^+) \times N(X_t)}^{(M(X_t) \times \text{GSp}(W_n)^+) \times N(X_t)} T^k)_{N(X_t)} \\
&\cong \text{ind}_{R_0(M(X_t) \times \text{GSp}(W_n)^+)}^{M(X_t) \times \text{GSp}(W_n)^+} (T_{N(X_t)}^k),
\end{aligned}$$

where the last isomorphism can be proven in the same way as [K, Lemma 5.3]. Therefore by inducing in stages, we obtain

$$\begin{aligned}
\tilde{T}_{N(X_t)}^k &\cong \text{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{M(X_t) \times \text{GSp}(W_n)^+} \mu_{tk} \\
&\cong \text{ind}_{P(X_{t-k}, X_t) \times Q(Y_k)^+}^{M(X_t) \times \text{GSp}(W_n)^+} \left(\text{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{P(X_{t-k}, X_t) \times Q(Y_k)^+} \mu_{tk} \right).
\end{aligned}$$

Now let $(\sigma_{tk}, S(\text{Isom}(X'_k, Y_k)))$ be the representation of $R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)$ defined by

$$\sigma_{tk}([\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h], [b, g]n)\varphi(A) = \xi(\det a_2)\xi'(\det a_2)\xi(\det a_1)\eta(\lambda_V(h))\varphi(\lambda_W(g)b^{-1}Aa_2)$$

so that

$$\mu_{tk} = \sigma_{tk} \otimes \omega_{m_0, n-k}.$$

Remark A.11. *At this point, one can derive the isometry version of the theorem simply by restricting μ_{tk} to the corresponding isometry group, and writing down the induction in the normalized form.*

We need to extend σ_{tk} to a representation of $P(X_{t-k}, X_t) \times Q(Y_k)^+$. Namely define the representation

$$(\tilde{\sigma}_{tk}, S(\text{Isom}(X'_k, Y_k)))$$

of $P(X_{t-k}, X_t) \times Q(Y_k)^+$ by

$$\begin{aligned} \tilde{\sigma}_{tk}([a, h], [b, g]n)\varphi(A) &= \eta(\lambda_V(h))\sigma_{tk}([a, 1], [\lambda_W(g)^{-1}b, 1])\varphi(A) \\ &= \xi(\det a_2)\xi'(\det a_2)\xi(\det a_1)\eta(\lambda_V(h))\varphi(\lambda_W(g)b^{-1}Aa_2), \end{aligned}$$

where

$$a = \begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}.$$

Then we have

Lemma A.12. *There is a $P(X_{t-k}, X_t) \times Q(Y_k)^+$ -isomorphism*

$$\mathrm{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{P(X_{t-k}, X_t) \times Q(Y_k)^+} \sigma_{tk} \otimes \omega_{m_0, n-k} \cong \tilde{\sigma}_{tk} \otimes \Omega_{m_0, n-k},$$

where $\Omega_{m_0, n-k}$ is the induced Weil representation of the pair $(\mathrm{GO}(V_{m_0}), \mathrm{GSp}(W_{n-k}))$.

Proof. Define

$$\alpha : \mathrm{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{P(X_{t-k}, X_t) \times Q(Y_k)^+} \sigma_{tk} \otimes \omega_{m_0, n-k} \rightarrow \tilde{\sigma}_{tk} \otimes \Omega_{m_0, n-k}$$

by

$$\alpha(F)(A)(h', g') = \eta(\lambda_V(h'))^{-1}F([1, h'], [\lambda_W(g'), g'])(A)$$

for $A \in \mathrm{Isom}(X'_k, Y_k)$ and $(h', g') \in \mathrm{GO}(V_{m_0}) \times \mathrm{GSp}(W_{n-k})$. One can verify that indeed $\alpha(F)(A) \in \Omega_{m_0, n-k}$. Here note that we identify an element $\alpha(F) \in \mathrm{Isom}(X'_k, Y_k) \otimes \Omega_{m_0, n-k}$ with a map

$$\alpha(F) : S(\mathrm{Isom}(X'_k, Y_k)) \rightarrow \Omega_{m_0, n-k}$$

and so $\alpha(F)(A)(h', g') \in \omega_{m_0, n-k}$. Also define

$$\beta : \tilde{\sigma}_{tk} \otimes \Omega_{m_0, n-k} \rightarrow \mathrm{ind}_{R_0(P(X_{t-k}, X_t) \times Q(Y_k)^+)}^{P(X_{t-k}, X_t) \times Q(Y_k)^+} \sigma_{tk} \otimes \omega_{m_0, n-k}$$

by

$$\beta(\Phi)([a, h], [b, g]n)(A) = \eta(\lambda_V(h))\sigma_{tk}([a, 1], [\lambda_W(g)^{-1}b, 1])\Phi(A)(h, g),$$

for $A \in \mathrm{Isom}(X'_k, Y_k)$ and $([a, h], [b, g]) \in P(X_{t-k}, X_t) \times Q(Y_k)^+$. One can verify that $\beta(\Phi)$ is indeed in the induced space. Then by direction computation, one can see that α and β are inverses to each other.

To see that α is intertwining, consider

$$\begin{aligned} &\alpha([a, h], [b, g]n \cdot F)(A)(h', g') \\ &= \eta(\lambda_V(h'))^{-1}([a, h], [b, g]n \cdot F)([1, h'], [\lambda_W(g'), g'])(A) \\ &= \eta(\lambda_V(h'))^{-1}F([a, h'h], [\lambda_W(g')b, g'g]n)(A) \\ &= \eta(\lambda_V(h'))^{-1}\sigma_{tk}([a, 1], [\lambda_W(g)^{-1}b, 1])\eta(\lambda_V(h'h))\eta(\lambda_V(h'h))^{-1}F([1, h'h], [\lambda_W(g'g), g'g])(A) \\ &= \eta(\lambda_V(h))\sigma_{tk}([a, 1], [\lambda_W(g)^{-1}b, 1])\alpha(F)(A)(h'h, g'g) \\ &= \tilde{\sigma}_{tk}([a, h], [b, g]n)\alpha(F)(A)(h'h, g'g) \\ &= \tilde{\sigma}_{tk}([a, h], [b, g]n)\Omega_{m_0, n-k}(h, g)\alpha(F)(A)(h', g'). \end{aligned}$$

□

Note that the induction ind has not been normalized. To express it in the normalized way, recall the modular characters for the parabolic subgroups involved:

$$\begin{aligned} \delta_{R(X_{t-k}, X_t)}\left(\left[\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}\right]\right) &= |\det a_1|^k |\det a_2|^{-(t-k)} \\ \delta_{P(X_t)}([a, h]n) &= |\det a|^{m-t-1} |\lambda_{V_{m_0}}(h)|^{-\frac{m-t}{2} + \frac{1}{2}t(t+1)} \\ \delta_{Q(Y_k)}([b, g]n) &= |\det b|^{2n-k+1} |\lambda_W(g)|^{-kn + \frac{1}{2}k(k-1)}. \end{aligned}$$

Hence we obtain each quotient of the filtration of the normalized Jacquet module as

$$J^k = \text{Ind}_{P(X_{t-k}, X_t) \times Q(Y_k)^+}^{M(X_t) \times \text{GSp}(W_n)^+} (\tilde{\sigma}_{tk} \otimes \Omega_{m_0, n-k}) (\delta_{R(X_{t-k}, X_t)} \delta_{P(X_t)} \delta_{Q(Y_k)})^{-\frac{1}{2}}$$

where the induction is also normalized. By writing down the characters involved explicitly, we see that

$$J^k = \text{Ind}_{P(X_{t-k}, X_t) \times Q(Y_k)^+}^{M(X_t) \times \text{GSp}(W_n)^+} (S(\text{Isom}(X'_k, Y_k)) \otimes \Omega_{m_0, n-k}),$$

where the group $P(X_{t-k}, X_t) \times Q(Y_k)^+$ acts on $S(\text{Isom}(X'_k, Y_k))$ as follows: Let $\varphi(A) \in S(\text{Isom}(X'_k, Y_k))$.

Then each element $[\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h] \in P(X_{t-k}, X_t)$ acts as

$$[\begin{pmatrix} a_1 & * \\ & a_2 \end{pmatrix}, h] \cdot \varphi(A) = \chi_V(\det a_2) |\lambda_V(h)|^{e_0} |\det a_1|^{e_1} |\det a_2|^{e_2} \varphi(Aa_2)$$

where

$$\begin{aligned} e_0 &= -\frac{1}{4}(m-2t)k - \frac{1}{2}tn + \frac{1}{4}mt - \frac{1}{4}t(t+1) \\ e_1 &= n - \frac{1}{2}m + \frac{1}{2}t - \frac{1}{2}(k-1) \\ e_2 &= n - \frac{1}{2}(k-1), \end{aligned}$$

and each element $[b, g]n \in Q(Y_k)$ acts as

$$[b, g]n \cdot \varphi(A) = |\lambda_W(g)|^{f'_0} |\det b|^{f_1} \varphi(\lambda_W(g)b^{-1}A),$$

where

$$\begin{aligned} f'_0 &= \frac{1}{2}kn - \frac{1}{4}k(k-1) \\ f_1 &= -e_2 = -n + \frac{1}{2}(k-1). \end{aligned}$$

This is essentially the similitude analogue of the formula obtained by Kudla ([K]). However one can simplify it further by ‘‘absorbing away’’ the characters $|\det a_2|^{e_2}$ and $|\det b|^{f_1}$ in the regular representation realized in $S(\text{Isom}(X'_k, Y_k))$ by using the following lemma.

Lemma A.13. *Let χ be a character and σ the representation of the group of elements of the form $([\begin{pmatrix} 1 & * \\ & a_2 \end{pmatrix}, 1], [b, g]) \in P(X_{t-k}, X_t) \times Q(Y_k)$ realized on the space $S(\text{Isom}(X'_k, Y_k))$ defined by*

$$\sigma([\begin{pmatrix} 1 & * \\ & a_2 \end{pmatrix}, 1], [b, g])\varphi(A) = \chi(\det a_2)\chi(\det b)^{-1}\varphi(\lambda_W(g)b^{-1}Aa_2).$$

Then σ is equivalent to the representation σ' realized on the same space $S(\text{Isom}(X'_k, Y_k))$ defined by

$$\sigma'([\begin{pmatrix} 1 & * \\ & a_2 \end{pmatrix}, 1], [b, g])\varphi(A) = \chi(\lambda_W(g))^{-k}\varphi(\lambda_W(g)b^{-1}Aa_2).$$

Proof. Define a map $S(\text{Isom}(X'_k, Y_k)) \rightarrow S(\text{Isom}(X'_k, Y_k))$ by $\varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi}$ is defined as

$$\tilde{\varphi}(A) = \chi(\det A)\varphi(A).$$

One can see that this map is an intertwining map from σ to σ' . □

By taking $\chi = |\cdot|^{e_2}$ in this lemma, one can see that the exponents e_2 and f_1 can be absorbed away, and the similitude factor $\lambda_W(g)$ acts by the character $|\cdot|^{f_0}$ where $f_0 = f'_0 - ke_2$. The theorem follows.

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