CUBIC UNIPOTENT ARTHUR PARAMETERS AND MULTIPLICITIES OF SQUARE INTEGRABLE AUTOMORPHIC FORMS

WEE TECK GAN, NADYA GUREVICH AND DIHUA JIANG

1. **Introduction**

Let G be a connected simple linear algebraic group defined over a number field F. It is a fundamental problem in number theory and the theory of automorphic forms to describe the spectral decomposition of the unitary representation $L^2(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$. By abstract results of functional analysis, such a unitary representation possesses an orthogonal decompostion

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_d(G(F)\backslash G(\mathbb{A})) \oplus L^2_{cont}(G(F)\backslash G(\mathbb{A}))$$

into the direct sum of its discrete spectrum and its continuous spectrum. The theory of Eisenstein series reduces the description of $L^2_{cont}(G(F)\backslash G(\mathbb{A}))$ to that of the discrete spectrum of certain reductive subgroups of G, and thus the basic question is the understanding of the discrete spectrum $L^2_d(G(F)\backslash G(\mathbb{A}))$. The discrete spectrum has a further orthogonal decomposition

$$L^2_d(G(F)\backslash G(\mathbb{A})) = L^2_{cusp} \oplus L^2_{res}$$

where L^2_{cusp} is the subspace of cusp forms, and L^2_{res} is the so-called residual spectrum. Let us write:

$$L^2_{cusp} = \hat{\oplus}_\pi m_{cusp}(\pi) \cdot \pi \quad \text{and} \quad L^2_{res} = \hat{\oplus}_\pi m_{res}(\pi) \cdot \pi.$$

It is known that $m_{cusp}(\pi)$ and $m_{res}(\pi)$ are finite. Two simple-minded questions can now be raised:

- Does there exist π such that $m_{cusp}(\pi) \neq 0$ and $m_{res}(\pi) \neq 0$?
- Can the collection of integers $\{m_{cusp}(\pi)\}$ be unbounded?

Here is a sample of some prior results on these questions:

(i) When $G = \mathrm{PGL}_n$, the results of Jacquet-Shalika [JS] and the multiplicity one theorem imply that the answers are negative for both questions.

(ii) When $G = \mathrm{SL}_2$, it is a recent result of Ramakrishnan [R] that $m_{cusp}(\pi) \leq 1$ for any π.

(iii) For a more general classical group G , it is known that the multiplicities in L^2_d or L^2_{cusp} can be > 1. Examples of such failure of multiplicity one were constructed by Labesse-Langlands [LL] for the inner forms of SL_2, by Blasius [B] for SL_n (with $n \geq 3$) and by Li [L] for quaternionic unitary groups. However, in these examples, the multiplicities are bounded above for the given G and the given number field F.

In view of these results, it is commonly believed that the two phenomena described above do not arise when G is a classical group. In this paper, we show on the contrary that both phenomena do occur when G is the split exceptional group of type G_2.

As an example, for each finite set S of places of F, with $\#S \geq 2$, we construct an irreducible unitary representation π_S of $\mathrm{G}_2(\mathbb{A})$ with

$$\begin{cases} m_{res}(\pi_S) = 1, \\ m_{cusp}(\pi_S) \geq \frac{1}{6}(2^{\#S} + (-1)^{\#S}2) - 1. \end{cases} \tag{1.1}$$

We now briefly explain how one constructs the representation π_S and demonstrates (1.1). Let S_3 be the finite algebraic group over F determined by the symmetric group on 3 letters. Then $\mathrm{S}_3 \times \mathrm{G}_2$ is a dual reductive pair in the disconnected linear algebraic group $H = \mathrm{Spin}_8 \rtimes \mathrm{S}_3$. For each place v of F, the group $H(F_v)$ has a distinguished representation Π_v known as the minimal representation. Restricting the representation Π_v to the subgroup $\mathrm{S}_3(F_v) \times \mathrm{G}_2(F_v)$, we can write:

$$\Pi_v = \bigoplus_{\eta_v \in \widehat{\mathrm{S}_3(F_v)}} \eta_v \otimes \pi_{\eta_v}.$$

In the beautiful papers [HMS] and [V], Huang-Magaard-Savin (for non-archimedean v) and Vogan (for archimedean v) showed that each π_{η_v} is a non-zero irreducible unitary representation and the π_{η_v}'s are mutually distinct. Moreover, if $\mathbf{1}$ denotes the trivial representation of S_3, then $\pi_{\mathbf{1}}$ is unramified. Consider now the global situation. Let $\eta = \otimes_v \eta_v$ be an irreducible representation of the compact group $\mathrm{S}_3(\mathbb{A})$, so that η_v is the trivial character for almost all v. Set $\pi_\eta = \otimes_v \pi_{\eta_v}$. Using the global theta correspondence furnished by the dual pair $\mathrm{S}_3 \times \mathrm{G}_2$ in H, we show:

Theorem 1.2. *There is a natural embedding of vector spaces*

$$\mathrm{Hom}_{\mathrm{S}_3(\mathbb{A})}(\eta, L^2(\mathrm{S}_3(F)\backslash \mathrm{S}_3(\mathbb{A}))) \hookrightarrow \mathrm{Hom}_{\mathrm{G}_2(\mathbb{A})}(\pi_\eta, L^2_d(\mathrm{G}_2(F)\backslash \mathrm{G}_2(\mathbb{A}))),$$

and thus a natural $\mathrm{G}_2(\mathbb{A})$*-equivariant embedding*

$$\mathrm{Hom}_{\mathrm{S}_3(\mathbb{A})}(\eta, L^2(\mathrm{S}_3(F)\backslash \mathrm{S}_3(\mathbb{A}))) \otimes \pi_\eta \hookrightarrow L^2_d(\mathrm{G}_2(F)\backslash \mathrm{G}_2(\mathbb{A})).$$

The above theorem is a special case of our main result (Theorem 8.2). To obtain the representation π_S of (1.1), with S a given finite set of places of F, let r be the two-dimensional irreducible representation of the finite group S_3 and set:

$$\begin{cases} \eta_S = (\bigotimes_{v \in S} r) \bigotimes (\bigotimes_{v \notin S} \mathbf{1}); \\ \pi_S = \pi_{\eta_S} = (\bigotimes_{v \in S} \pi_r) \bigotimes (\bigotimes_{v \notin S} \pi_{\mathbf{1}}). \end{cases}$$

The theorem implies that

$$m_{res}(\pi_S) + m_{cusp}(\pi_S) \geq m_{\eta_S} := \dim \mathrm{Hom}_{\mathrm{S}_3(\mathbb{A})}(\eta_S, L^2(\mathrm{S}_3(F)\backslash \mathrm{S}_3(\mathbb{A}))).$$

The number m_{η_S} is simply the multiplicity of η_S in the space $L^2(\mathrm{S}_3(F)\backslash \mathrm{S}_3(\mathbb{A}))$ of automorphic forms on S_3. By the Peter-Weyl theorem, we see that

$$m_{\eta_S} = \dim(\eta_S^{\mathrm{S}_3(F)}),$$

the multiplicity of the trivial representation in the restriction of η_S to $\mathrm{S}_3(F)$. A simple character computation now gives:

$$m_{\eta_S} = \frac{1}{6}(2^{\#S} + (-1)^{\#S}2) \quad \text{if } \#S \geq 1.$$

On the other hand, H. Kim [K] and S. Zampera [Z] have determined the residual spectrum L^2_{res} completely. From their results, one sees that $m_{res}(\pi_S) = 1$ if $\#S \neq 1$ and thus (1.1) follows immediately.

The occurrence of π_S in $L^2_d(\mathrm{G}_2(F)\backslash \mathrm{G}_2(\mathbb{A}))$ with multiplicity $\geq m_{\eta_S}$ is predicted by the conjecture of Arthur on the discrete spectrum (cf. [A1] and [A2]), as we explain in Section 2 which serves as an extended introduction. We shall see that to every étale cubic F-algebra E, one can naturally associate an Arthur parameter ψ_E for the group G_2 and our main result (Theorem 8.2) gives a construction of what one might hope is the global Arthur packet associated to ψ_E. Theorem 1.2 above is simply a special case of our main result, for the case $E = F \times F \times F$.

We now give an outline of the remainder of the paper. Section 3 introduces the relevant groups involved in this paper and Sections 4 and 5 are devoted to the properties of the local and global minimal representations respectively. There is nothing very illuminating here, but they are quite necessary and the proofs of various technical results there are deferred to the appendix at the end of the paper. The exception is the important Proposition 5.5, which is the main result of these preliminary sections. After recalling the local results of [HMS] and [V] in Section 6, we summarize in Section 7 the results of [K] and [Z] concerning the residual spectrum and give an interpretation of their results in terms of Arthur parameters. After this long preparation, the proof of our main global result (Theorem 8.2) is given in Section 8.

The cuspidal representations constructed in the main theorem are non-tempered and have some very special properties. For one thing, these cusp forms are distinguished, in the sense that they have only one orbit of non-vanishing generic Fourier coefficients along the Heisenberg parabolic subgroup of G_2. Further, the cuspidal representations they afford are CAP with respect to the Borel subgroup (respectively, the non-Heisenberg maximal parabolic subgroup) of G_2 if E is Galois (respectively, non-Galois) over F. Here, following Piatetski-Shapiro [PS], we say that a cuspidal representation $\pi = \otimes_v \pi_v$ is CAP (cuspidal associated to parabolic) with respect to a parabolic $P = M{\cdot}N$ if there is a cuspidal representation $\sigma = \otimes_v \sigma_v$ of the Levi subgroup M such that π_v is equivalent to a subquotient of $Ind_{P(F_v)}^{G(F_v)}\sigma_v$ for almost all v. In [RS], Rallis and Schiffmann constructed CAP representations of G_2 with respect to the Borel subgroup and the Heisenberg parabolic. When E is non-Galois, the cuspidal representations constructed here seems to be the first examples of CAP representations with respect to the non-Heisenberg maximal parabolic. It will be very interesting to see if these special properties characterize the cusp forms constructed in this paper.

General Notations: If G is a connected reductive group over a number field F, we shall let $\mathcal{A}(G)$ denote the space of automorphic forms on G. This possesses a decomposition $\mathcal{A}(G) = \mathcal{A}(G)_{cusp} \oplus \mathcal{A}(G)_{Eis}$, where $\mathcal{A}(G)_{cusp}$ is the subspace of cusp forms and $\mathcal{A}(G)_{Eis}$ is the subspace spanned by Eisenstein series. The definition of automorphic forms used here differs from the usual definition (as given in [BJ]) in that we do not require an automorphic form to be K_∞-finite, where K_∞ is a maximal compact subgroup of $G(F_\infty)$. In particular, $\mathcal{A}(G)$ is a representation of the group $G(\mathbb{A})$. Similarly, instead of working with $(\mathfrak{g}_\infty, K_\infty)$-modules, we shall work with their canonical Casselman-Wallach globalizations (cf. [C] and [W]), which are representations of $G(F_\infty)$. For a smooth representation V of $G(F_\infty)$, we write V_K for the associated $(\mathfrak{g}_\infty, K_\infty)$-module.

2. **Cubic Unipotent Arthur Parameters**

In this section, we explain how the two questions raised in the introduction can be viewed in the framework of Arthur's conjecture on the discrete spectrum.

Let F be a number field and let L_F be the conjectural Langlands group of F. Let G be the split exceptional group over F of type G_2, so that the dual group of G is the complex Lie group $\mathrm{G}_2(\mathbb{C})$. To each étale cubic algebra E over F, we shall first attach a cubic unipotent Arthur parameter

$$\psi_E : L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C}).$$

The parameter ψ_E will factor through the Galois group of a finite extension of F (which is a quotient of L_F), and thus we shall simply construct a map $\psi_E : \mathrm{Gal}(\overline{F}/F) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$.

To begin, observe that there is a natural bijection between the set of isomorphism classes of étale cubic algebras E over F and the set of conjugacy classes of group homomorphisms

$$\rho_E : \mathrm{Gal}(\overline{F}/F) \to S_3.$$

Here S_3 is the symmetric group on 3 letters and is the automorphism group of the split cubic algebra $F \times F \times F$. Henceforth, we fix the étale cubic algebra E. Via ρ_E, we have an action of $\mathrm{Gal}(\overline{F}/F)$ on S_3 which allows one to define a twisted form S_E of the finite (constant) group scheme S_3. For any commutative F-algebra K, we have

$$S_E(K) = \mathrm{Aut}_K(E \otimes K).$$

Now let $\iota : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$ be a morphism furnished by the Jacobson-Morozov theorem, such that

$$\iota \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a subregular unipotent element of $\mathrm{G}_2(\mathbb{C})$. Note that this morphism is not injective; it factors through the quotient $SO_3(\mathbb{C})$ of $\mathrm{SL}_2(\mathbb{C})$. It is known that the centralizer of $\iota(\mathrm{SL}_2(\mathbb{C}))$ is isomorphic to the finite group S_3, so that we have a subgroup

$$S_3 \times SO_3(\mathbb{C}) \subset \mathrm{G}_2(\mathbb{C}).$$

We now set

$$\psi_E = \rho_E \times \iota : \mathrm{Gal}(\overline{F}/F) \times \mathrm{SL}_2(\mathbb{C}) \to S_3 \times SO_3(\mathbb{C}) \subset \mathrm{G}_2(\mathbb{C}).$$

This is the desired (global) cubic unipotent Arthur parameter. Observe that the centralizer in $\mathrm{G}_2(\mathbb{C})$ of the image of ψ_E is a subgroup of S_3 isomorphic to $S_E(F)$.

For each place v of F, we have a canonical conjugacy class of embeddings $\mathrm{Gal}(\overline{F_v}/F_v) \hookrightarrow \mathrm{Gal}(\overline{F}/F)$ and thus we obtain from ψ_E a local Arthur parameter

$$\psi_{E,v} : \mathrm{Gal}(\overline{F}_v/F_v) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C}).$$

The centralizer of the image of ψ_{E_v} is a subgroup of S_3 isomorphic to $S_E(F_v)$.

Remarks: According to Arthur [A2], a parameter ψ is called "unipotent" if it is trivial when restricted to L_F. On the other hand, Moeglin [M] called a parameter ψ "quadratic unipotent" if its restriction to L_F has image equal to a product of $\mathbb{Z}/2\mathbb{Z}$'s, so that $\psi|_{L_F}$ factors through the Galois group of the compositum of finitely many quadratic extensions of F. We call the parameter ψ_E "cubic unipotent" since it factors through the automorphism group of the étale cubic algebra E.

We can now state Arthur's conjecture for the parameter ψ_E.

Arthur's conjecture:

- For each place v and to each irreducible representation η_v of the finite group $S_E(F_v)$, we can attach a unitarizable admissible (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. The set
$$A_{E,v} = \{\pi_{\eta_v} : \eta_v \in \widehat{S_E(F_v)}\}$$
is called the **local Arthur packet** determined by $E_v = E \otimes_F F_v$. It is also required that, for almost all v, π_{η_v} is irreducible and unramified if η_v is the trivial character. Further, one expects that the distribution
$$\Delta_{E,v} = \sum_{\eta_v} \dim(\eta_v)\,\mathrm{Tr}(\pi_{\eta_v})$$
is a stable distribution on $G(F_v)$ and that certain identities involving transfer of distributions to endoscopic groups of $G(F_v)$ should hold. We will not go into these important details here; the reader can consult [A2].
- If $\eta = \otimes_v \eta_v$ is an irreducible representation of $S_E(\mathbb{A})$ (so that η_v is the trivial representation for almost all v), let
$$\pi_\eta = \otimes_v \pi_{\eta_v}$$
and
$$m_\eta := \dim(\eta^{S_E(F)}) = \dim \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A}))).$$
Then there is a $G(\mathbb{A})$-equivariant embedding
$$m_\eta \pi_\eta \hookrightarrow L^2_d(G(F)\backslash G(\mathbb{A})).$$

It is not difficult to see that the collection of numbers $\{m_\eta : \eta \in \widehat{S_E(\mathbb{A})}\}$ is unbounded. For example, when $E = F \times F \times F$, $S_E = \mathrm{S}_3$ and we have seen in the introduction that for a non-empty finite set S of places of F, the representation η_S of $S_E(\mathbb{A})$ satisfies

$$m_{\eta_S} = \frac{1}{6}(2^{\#S} + (-1)^{\#S}2)$$

so that $m_{\eta_S} \to \infty$ as $\#S \to \infty$. Hence, Arthur's conjecture predicts that the multiplicities of representations occurring in the discrete spectrum are unbounded when $G = \mathrm{G}_2$. As the reader will undoubtedly realize by now, the key reason for this is the fact that the centralizer of the image of ψ_{E_v} is non-abelian (equal to S_3) for infinitely many places v. This also (partially) explains why the two phenomena of the introduction do not arise in the case of classical groups: the component group of the centralizer of an Arthur parameter is always abelian there.

Despite the above, Arthur's conjecture is not sufficient to imply the two phenomena highlighted in the introduction, since the conjecture does not distinguish between the cuspidal and residual spectrum (cf. the paper [M2] of Moeglin which addresses this issue conjecturally, at least for classical groups). However, the theory of Eisenstein series furnishes a systematic procedure for constructing the residual spectrum: L^2_{res} can be spanned by residues of Eisenstein series and can in principle be described explicitly as an abstract representation. As we mentioned earlier, for $G = \mathrm{G}_2$, this has actually been carried out by H. Kim [K] and S. Zampera [Z]. Parts of the results of [K] and [Z] were established under the assumption that the symmetric cube L-function of GL_2 is entire for non-monomial cuspidal representations. This assumption has since been removed by Kim-Shahidi [KS]. As we mentioned in the introduction, a consequence of [K] and [Z] is that $m_{res}(\pi_{\eta_S}) = 1$ if $\#S \neq 1$. Coupled with this, Arthur's conjecture gives affirmative answers to the two initial questions.

It remains then to verify Arthur's conjectures for the parameter ψ_E. The results of Huang-Magaard-Savin [HMS, Prop. 6.1, Pg. 76] and Vogan [V, Thm. 18.10, Pg. 788] can be viewed as a natural construction of what one might hope is the local Arthur packet $A_{E,v}$. Using the local packets furnished by [HMS] and [V], our main result (Theorem 8.2) can be viewed as a natural construction of what one might hope is the global Arthur packet associated to the parameter ψ_E.

3. **Groups**

In this section, we set up some notations and introduce the basic objects of interest. Let F be any field of characteristic zero (for simplicity) and let E be an étale cubic F-algebra, with norm map Nm. More explicitly, E is one of the following:

$$E = \begin{cases} F \times F \times F, \\ F \times K, \text{ with } K \text{ a quadratic extension of } F, \\ \text{a Galois field extension of } F, \\ \text{a non-Galois field extension of } F. \end{cases} \tag{3.1}$$

We shall say that E is Galois in the first three cases, and non-Galois otherwise. In any case, E corresponds to a conjugacy class of group homomorphisms

$$\rho_E : \mathrm{Gal}(\overline{F}/F) \to S_3.$$

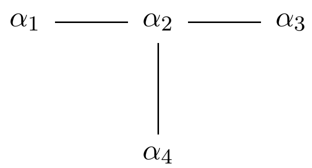
This determines a twisted form S_E of the finite group scheme S_3, characterized by

$$S_E(K) = \mathrm{Aut}_K(E \otimes K) \quad \text{for any } F\text{-algebra } K.$$

It also determines a simply-connected quasi-split group G_E of type D_4. Let us fix a Chevalley-Steinberg system of épinglage [BT, 4.1.3, Pg. 78]

$$\{T_E, B_E, x_\alpha : \mathbb{G}_a \to (G_E)_\alpha, \alpha \in \Phi\}$$

where $T_E \subset B_E$ is a maximal torus contained in a Borel subgroup (both defined over F) and Φ denotes the roots of $G_E \otimes \overline{F}$ with respect to $T_E \otimes \overline{F}$. We shall label the simple roots of $G_E \otimes \overline{F}$ according to the following diagram.



The automorphism group $\mathrm{Aut}(G_E)$ of G_E is a disconnected linear algebraic group and there is a split exact sequence:

$$1 \longrightarrow G_E^{ad} \longrightarrow \mathrm{Aut}(G_E) \longrightarrow S_E \longrightarrow 1.$$

The choice of épinglage defines a splitting of this exact sequence [Sp] and thus we can regard S_E as a subgroup of $\mathrm{Aut}(G_E)$. This allows one to define the semi-direct product $H_E = G_E \rtimes S_E$. The centralizer of S_E in H_E is easily seen to be a subgroup $G \subset G_E$, isomorphic to the split exceptional group of type G_2. This gives a dual reductive pair

$$S_E \times G \hookrightarrow H_E.$$

Moreover, $T = T_E \cap G$ is a maximal split torus of G, contained in the Borel subgroup $B = B_E \cap G$. If Φ_G denotes the roots of G with respect to T, with short simple root α and long simple root β, then there is a natural restriction map $\Phi \to \Phi_G$. Under this map,

$$\begin{cases} \alpha_2 \mapsto \beta, \\ \alpha_i \mapsto \alpha, \ \text{for } i = 1, 3, 4. \end{cases}$$

We now describe various parabolic subgroups of G_E and G. Let $P_E = M_E \cdot N_E$ be the (standard) Heisenberg parabolic subgroup of G_E. Its Levi subgroup is

$$M_E \cong \{g \in \mathrm{Res}_{E/F}\,\mathrm{GL}_2 : det(g) \in \mathbb{G}_m\} \tag{3.2}$$

and is generated by the simple roots α_1, α_3 and α_4. We fix the isomorphism (3.2) so that the modulus character of P_E is given by

$$\delta_{P_E} = det^5.$$

The unipotent radical N_E is a 9-dimensional Heisenberg group, and thus $\mathrm{Hom}(N_E, \mathbb{G}_a)$ is an 8-dimensional affine space on which M_E naturally acts. We let Ω_E denote the minimal non-trivial M_E-orbit in $\mathrm{Hom}(N_E, \mathbb{G}_a)$.

Let $P = G \cap P_E$. Then $P = M \cdot N$ is the (standard) Heisenberg parabolic subgroup of G. Its Levi subgroup is $M \cong \mathrm{GL}_2$, and we pick the isomorphism so that the inclusion $M \hookrightarrow M_E$

is the natural inclusion $\mathrm{GL}_2 \hookrightarrow \{g \in \mathrm{Res}_{E/F}\mathrm{GL}_2 : det(g) \in \mathbb{G}_m\}$. The modulus character of P is thus

$$\delta_P = det^3.$$

The unipotent radical N is a 5-dimensional Heisenberg group, and thus $\mathrm{Hom}(N, \mathbb{G}_a)$ is a 4-dimensional affine space on which M naturally acts. It is well-known that this representation of M is isomorphic to $Sym^3(F^2)^* \otimes det$, and that the $M(F)$-orbits on $\mathrm{Hom}(N(F), F)$ are naturally parametrized by isomorphism classes of cubic F-algebras [HMS]. Moreover, the generic orbits (i.e. those of dimension 4) correspond to those cubic algebras which are étale. Given a cubic F-algebra E', we shall denote by $\mathcal{O}_{E'}$ the $M(F)$-orbit determined by E'. If E' is étale and $x \in \mathcal{O}_{E'}$, then the stabilizer $M_x(F)$ of x in $M(F)$ is a finite subgroup isomorphic to $S_{E'}(F)$. The inclusion $N \hookrightarrow N_E$ now induces a natural projection

$$\tau_P : \mathrm{Hom}(N_E, \mathbb{G}_a) \to \mathrm{Hom}(N, \mathbb{G}_a)$$

and for $x \in \mathrm{Hom}(N(F), F)$, we let

$$\Omega_E(x) = \{y \in \Omega_E(F) : \tau_P(y) = x\}.$$

Now we have the following lemma whose proof is essentially contained in [HMS]:

Lemma 3.3. *Let $x \in \mathcal{O}_{E'}$ be an element of a generic orbit.*

(i) If $E' \neq E$, then $\Omega_E(x)$ is empty.

(ii) If $E' = E$, then $M_x(F) \times S_E(F)$ acts naturally on $\Omega_E(x)$ with each group acting simply transitively.

(iii) $\Omega_E(0)$ is empty.

Now let $Q_E = L_E \cdot U_E$ be the standard parabolic subgroup of G_E whose Levi subgroup L_E is generated by the simple root α_2. Then

$$L_E \cong (\mathrm{SL}_2 \times \mathrm{Res}_{E/F}\mathbb{G}_m)/\Delta\mu_2$$

and the isomorphism can be chosen so that the modulus character is $\delta_{Q_E} = Nm^6$. The unipotent radical U_E is a 3-step unipotent group and $\mathrm{Hom}(U_E, \mathbb{G}_a)$ is a 6-dimensional affine space on which L_E acts. This representation of L_E is isomorphic to $(F^2 \otimes E)^*$ and we let ω_E denote the set of non-zero $v \otimes x$ such that $x \in E^*$ has rank 1.

Let $Q = G \cap Q_E$. Then $Q = L \cdot U$ is the non-Heisenberg maximal parabolic subgroup of G. Its Levi subgroup is $L \cong \mathrm{GL}_2$, and this isomorphism is chosen so that the embedding $L \hookrightarrow L_E$ is the natural one

$$\mathrm{GL}_2 \cong (\mathrm{SL}_2 \times \mathbb{G}_m)/\Delta\mu_2 \hookrightarrow (\mathrm{SL}_2 \times \mathrm{Res}_{E/F}\mathbb{G}_m)/\Delta\mu_2.$$

The modulus character of Q is thus given by

$$\delta_Q = det^5.$$

Note that L and L_E have the same derived group SL_2. The unipotent radical is a 3-step unipotent group, and the inclusion $U \hookrightarrow U_E$ induces a natural projection

$$\tau_Q : \mathrm{Hom}(U_E, \mathbb{G}_a) \to \mathrm{Hom}(U, \mathbb{G}_a).$$

For $x \in \mathrm{Hom}(U(F), F)$, let

$$\omega_E(x) = \{y \in \omega_E(F) : \tau_Q(y) = x\}.$$

Now we have the following simple lemma:

Lemma 3.4. *$\omega_E(0)$ is empty.*

We conclude this section with a description of the character group $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$. This depends on the type of E as described in (3.1). When $E = F \times F \times F$,

$$\mathrm{Hom}_F(L_E, \mathbb{G}_m) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_3 \oplus \mathbb{Z}\chi_4$$

where, for $(g, a, b, c) \in \mathrm{SL}_2 \times \mathbb{G}_m^3$,

$$\begin{cases} \chi_1 : (g, a, b, c) \mapsto bc, \\ \chi_3 : (g, a, b, c) \mapsto ca, \\ \chi_4 : (g, a, b, c) \mapsto ab. \end{cases} \tag{3.5}$$

On restriction to T_E, we have:

$$\begin{cases} \chi_1 = (\alpha_1 + \alpha_0)/2, \\ \chi_3 = (\alpha_3 + \alpha_0)/2, \\ \chi_4 = (\alpha_4 + \alpha_0)/2, \end{cases} \tag{3.6}$$

where $\alpha_0 = 2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4$ is the highest root.

Suppose now that $E = F \times K$. Then we see by the above that $\mathrm{Hom}_K(L_E, \mathbb{G}_m)$ is generated by χ_1, χ_3 and χ_4. Assume without loss of generality that α_1 is a root defined over F, so that the action of $\mathrm{Gal}(K/F)$ on $\mathrm{Hom}_K(L_E, \mathbb{G}_m)$ fixes χ_1 and exchanges χ_3 and χ_4. Hence $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$ has basis $\{\chi_1, \chi_3 + \chi_4\}$. For later purposes, it is convenient to define the following two elements of $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$:

$$\begin{cases} \mu_1 = 3\chi_1 - (\chi_3 + \chi_4), \\ \mu_2 = 4\chi_1 - 2(\chi_3 + \chi_4). \end{cases} \tag{3.7}$$

These two elements generate a subgroup of index 2 in $\mathrm{Hom}_F(L_E, \mathbb{G}_m)$ and on restriction to T_E, we have

$$\begin{cases} \mu_1 = \alpha_2 + 2\alpha_1 \\ \mu_2 = 2\alpha_1 - \alpha_3 - \alpha_4. \end{cases}$$

Finally, when E is a field,

$$\mathrm{Hom}_F(L_E, \mathbb{G}_m) = \mathbb{Z} \cdot (\chi_1 + \chi_3 + \chi_4)$$

and on restriction to T_E,

$$\chi_1 + \chi_3 + \chi_4 = 2\alpha_0 - \alpha_2.$$

4. **Local Minimal Representation**

Let F be a local field (of characteristic zero) and let E be an étale cubic F-algebra with associated homomorphism $\rho_E : \mathrm{Gal}(\overline{F}/F) \to S_3$. Then $G_E(F)$ has a distinguished irreducible representation Π_E known as the minimal representation. This representation is trivial on the center of $G_E(F)$ and extends (non-canonically) to a representation of the adjoint group $G_E^{ad}(F)$. In this section, we recall various properties of the minimal representation Π_E.

A construction of the minimal representation Π_E was first given by Kazhdan [Ka]. However, at a crucial point in the construction, his argument is incomplete and the analytic difficulty encountered in fixing this gap is non-trivial. Fortunately, an independent treatment which bypasses this analtyic difficulty has now been given in [GaS]. In particular, all essential results of [Ka] are correct. Most of the properties of Π_E discussed below are contained in [Ka], [HMS] and [GaS]. Hence, we shall simply state the results and omit many of the proofs.

Let $r : S_3 \to \mathrm{GL}_2(\mathbb{C})$ denote the 2-dimensional irreducible representation of S_3. Then the composite $r \circ \rho_E$ is a 2-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$. By Jacquet-Langlands [JL], this determines an irreducible admissible representation r_E of $\mathrm{GL}_2(F)$ given by the following table:

E	r_E
$F \times F \times F$	$\sigma(1,1)$
$F \times K$	$\sigma(1, \chi_K)$
Galois field	$\sigma(\chi_E, \chi_E^{-1})$
non-Galois field	monomial supercuspidal

Here, χ_K and χ_E denote the characters of $F^\times$ associated to K and E by local class field theory, and $\sigma(\mu_1, \mu_2)$ denotes the representation of $\mathrm{GL}_2(F)$ unitarily induced from the character $\mu_1 \times \mu_2$ of the diagonal torus. Let χ_r denote the (quadratic) central character of r_E and define a representation of $L_E^{ad}(F) \cong (\mathrm{GL}_2(F) \times E^\times)/\Delta F^\times$ by:

$$\sigma_E = r_E \otimes (\chi_r \circ Nm). \tag{4.1}$$

Here, L_E^{ad} is the image of L_E in G_E^{ad} and is a Levi subgroup of the parabolic subgroup Q_E^{ad} (the image of Q_E in G_E^{ad}). For the description of $L_E^{ad}(F)$ used above, see [HMS, Pg. 69].

Pulling back via the natural map $L_E(F) \to L_E^{ad}(F)$, we regard σ_E as a representation of $L_E(F)$. This representation of $L_E(F)$ is still irreducible (cf. [GaS, §9]). Consider the induced representation

$$Ind_{Q_E(F)}^{G_E(F)} \delta_{Q_E}^{1/6} \sigma_E.$$

Here and elsewhere, our induction is unnormalized. This has a unique irreducible submodule Π_E and it is shown in [GaS] that Π_E is the unique unitarizable minimal representation of $G_E(F)$. Moreover, it is self-contragredient. When F is non-archimedean, Π_E is spherical if and only if E is unramified over F. When F is archimedean, Π_E is spherical if and only if $E = F \times F \times F$.

If E is Galois, let ν_E denote the following character of $M_E(F)$:

$$\nu_E = \begin{cases} \mathbf{1}, \text{ if } E = F \times F \times F; \\ \chi_K \circ det, \text{ if } E = F \times K; \\ \chi_E \circ det, \text{ if } E \text{ is a field.} \end{cases} \tag{4.2}$$

From the above, one has:

Proposition 4.3. *If E is Galois, then Π_E is the unique irreducible subrepresentation of $Ind_{P_E(F)}^{G_E(F)} \nu_E \cdot \delta_{P_E}^{1/5}$.*

Proof. Clearly, we have an embedding of representations of $G_E(F)$:

$$Ind_{P_E(F)}^{G_E(F)} \nu_E \cdot \delta_{P_E}^{1/5} \hookrightarrow Ind_{B_E(F)}^{G_E(F)} (\nu_E \cdot \delta_{P_E}^{1/5})|_{B_E(F)}.$$

On the other hand, by induction in stages through the parabolic subgroup Q_E, one sees that the latter representation is equal to $Ind_{Q_E(F)}^{G_E(F)} \delta_{Q_E}^{1/6} \sigma_E$. The proposition follows. ☐

Using this last proposition, one can extend Π_E to a representation of $H_E(F)$ when E is Galois. More precisely, since the parabolic subgroup P_E and the characters ν_E and δ_{P_E} are stable under the action of $S_E(F)$, we can define an action of $S_E(F)$ on $Ind_{P_E(F)}^{G_E(F)} \nu_E \delta_{P_E}^{1/5}$ by:

$$(s \cdot f)(g) = f(s^{-1}(g)), \quad s \in S_E(F).$$

This action of $S_E(F)$ normalizes that of $G_E(F)$, thus preserving the unique submodule Π_E, and we obtain the desired extension of Π_E to a representation of $H_E(F)$. Of course, when E is non-Galois, there is no need to define an extension since $H_E(F) = G_E(F)$.

We now assume that F is non-archimedean. Fix a non-trivial additive (unitary) character ψ_0 of F. Composition with ψ_0 gives identifications

$$\begin{cases} \mathrm{Hom}(N_E, \mathbb{G}_a) \cong \{\text{unitary characters of } N_E(F)\}, \\ \mathrm{Hom}(U_E, \mathbb{G}_a) \cong \{\text{unitary characters of } U_E(F)\}. \end{cases}$$

The following two propositions describe the Jacquet modules of Π_E with respect to N_E and U_E.

Proposition 4.4. *(i) Let ψ be a non-trivial character of $N_E(F)$. Then*

$$\dim(\Pi_E)_{N_E(F),\psi} = \begin{cases} 1, & \text{if } \psi \in \Omega_E; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) If $E = F \times F \times F$, then as a representation of

$$M_E(F) \cong \{(g_1, g_2, g_3) \in \mathrm{GL}_2(F)^3 : det(g_1) = det(g_2) = det(g_3)\},$$

we have:

$$0 \longrightarrow W \longrightarrow (\Pi_E)_{N_E} \longrightarrow \delta_{P_E}^{\frac{1}{5}} \to 0$$

where after semi-simplification,

$$W \cong \delta_{P_E}^{\frac{1}{5}} \oplus \delta_{P_E}^{\frac{1}{5}}((St \otimes \mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes St \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1} \otimes St)).$$

Here St denotes the Steinberg representation of $\mathrm{GL}_2(F)$ *and* $St \otimes \mathbf{1} \otimes \mathbf{1}$ *is the representation* $St \circ pr_1$, *where* $pr_1 : M_E(F) \to \mathrm{GL}_2(F)$ *is given by* $pr_1(g_1, g_2, g_3) = g_1$.

(iii) If $E = F \times K$, *then as a representation of*

$$M_E(F) = \{(g,h) \in \mathrm{GL}_2(F) \times \mathrm{GL}_2(K) : det(g) = det(h)\},$$

we have:

$$0 \longrightarrow W \longrightarrow (\Pi_E)_{N_E(F)} \longrightarrow \nu_E \delta_{P_E}^{\frac{1}{5}} \longrightarrow 0.$$

Here, W *is obtained by pulling back the representation* $\sigma(|-|^{1/2}, \chi_K|-|^{3/2})$ *of* $\mathrm{GL}_2(F)$ *via the projection* $pr : M_E(F) \to \mathrm{GL}_2(F)$.

(iv) If E *is a Galois cubic field, then as a representation of* $M_E(F)$,

$$(\Pi_E)_{N_E(F)} \cong \delta_{P_E}^{\frac{1}{5}} \cdot (\nu_E \oplus \nu_E^{-1}).$$

(v) If E *is a non-Galois field, then* $(\Pi_E)_{N_E(F)} = 0$.

Proposition 4.5. *(i) Let* ψ *be a non-trivial character of* $U_E(F)$. *Then*

$$\dim(\Pi_E)_{U_E(F),\psi} = \begin{cases} 1, & \text{if } \psi \in \omega_E; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) If $E = F \times F \times F$, *then as a representation of* $L_E(F)$,

$$(\Pi_E)_{U_E} \cong \delta_{Q_E}^{1/6} \sigma_E \oplus |\chi_1|^2 \oplus |\chi_3|^2 \oplus |\chi_4|^2,$$

where the χ_i*'s were defined in (3.5).*

(iii) If $E = F \times K$, *then as a representation of* $L_E(F)$, *we have:*

$$(\Pi_E)_{U_E} \cong \delta_{Q_E}^{1/6} \sigma_E \oplus (\chi_K | \cdot |^2 \circ \mu_1) \otimes |\mu_2|^{-1}$$

where μ_1 *and* μ_2 *were defined in (3.7).*

(iv) If E *is a field, then*

$$(\Pi_E)_{U_E} \cong \delta_{Q_E}^{1/6} \sigma_E.$$

We now consider the case when F is archimedean.

Proposition 4.6. *(i) If* $E = F \times F \times F$, *then*

$$\dim \mathrm{Hom}_{Q_E(F)}(\Pi_E, |\chi_i|^2) \leq 1$$

where $i \in \{1, 3, 4\}$.

(ii) If $E = F \times K$, *then*

$$\dim \mathrm{Hom}_{Q_E(F)}(\Pi_E, (\chi_K | \cdot |^2 \circ \mu_1) \otimes |\mu_2|^{-1}) \leq 1.$$

Proof. (i) This follows by Frobenius reciprocity and the fact that the induced representation $Ind_{Q_E(F)}^{G_E(F)}|\chi_i|^2$ contains exactly one spherical subquotient.

(ii) This is similar to (i); one checks that the relevant induced representation contains the minimal K-type of Π_E with multiplicity one. □

5. **Global Minimal Representation**

Let F be a number field and E an étale cubic F-algebra. In this section, we construct an automorphic realization

$$\theta_E : \Pi_E \hookrightarrow \mathcal{A}(G_E) \cap L^2(G_E(F)\backslash G_E(\mathbb{A}))$$

of the global minimal representation $\Pi_E = \otimes_v \Pi_{E_v}$, and study its Fourier coefficients along the two unipotent subgroups N_E and U_E. When E is non-Galois, the automorphic realization has been constructed by Kazhdan in [Ka]. Hence, we shall concentrate on the case when E is Galois. When $E = F \times F \times F$, the automorphic realization of Π_E was constructed by Ginzburg-Rallis-Soudry in [GRS] (starting from a different degenerate principal series representation). The proofs of many results in this section are standard exercises in the theory of Eisenstein series and we give the complete proofs for the case $E = F \times F \times F$ in the appendix at the end of the paper; the other cases are in fact easier. The only exception is Prop. 5.5 which is the main result of this section and whose proof is given at the end of the section.

The 2-dimensional representation $r \circ \rho_E$ of $\mathrm{Gal}(\overline{F}/F)$ gives rise to an automorphic representation $r_E = \otimes_v r_{E_v}$ of GL_2. When E is Galois, r_E is a global principal series representation and thus can be embedded into $\mathcal{A}(\mathrm{GL}_2)$ by the formation of Eisenstein series. When E is non-Galois, r_E is cuspidal automorphic. As in (4.1), we can define the automorphic representation $\sigma_E = \otimes_v \sigma_{E_v}$ of $L_E^{ad}(\mathbb{A})$ and hence of $L_E(\mathbb{A})$. Henceforth, we shall regard σ_E as a submodule of $\mathcal{A}(L_E)$.

Let ν_E be the character of $F^\times\backslash\mathbb{A}^\times$ defined analogously as in (4.2), so that $\nu_E = \otimes_v \nu_{E_v}$. Since the minimal representation Π_E is self-contragredient, we deduce by Prop. 4.3 that when E is Galois,

$$\dim \mathrm{Hom}_{G_E(\mathbb{A})}(Ind_{P_E(\mathbb{A})}^{G_E(\mathbb{A})} \nu_E \delta_{P_E}^{4/5}, \Pi_E) = 1.$$

For a standard K_∞-finite section $f_s \in Ind_{P_E(\mathbb{A})}^{G_E(\mathbb{A})} \chi_E \delta_{P_E}^s$, let $E(g, f_s)$ be the associated Eisenstein series. Then we have:

Proposition 5.1. *(i) Suppose that $E = F \times F \times F$. For any standard K_∞-finite section f_s, $E(g, f_s)$ has at most a double pole at $s = \frac{4}{5}$. This double pole is attained by the spherical section. Moreover, the space of automorphic forms spanned by the functions $(s - \frac{4}{5})^2 E(-, f_s)|_{s=4/5}$ is an irreducible square-integrable automorphic representation isomorphic to Π_E.*

(ii) Suppose that $E = F \times K$ or E is a Galois cubic field. For any standard K_∞-finite section f_s, $E(g, f_s)$ has at most a simple pole at $s = \frac{4}{5}$. This pole is attained by some K_∞-finite section, and the space of automorphic forms spanned by the residues of $E(g, f_s)$ at $s = \frac{4}{5}$ is an irreducible square-integrable automorphic representation isomorphic to Π_E.

Proof. We give the proof of (i) in the appendix at the end of the paper. The other cases are similar. □

Together with [Ka, Pg. 144-145], Prop. 5.1 gives a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-equivariant embedding

$$\theta_E : (\Pi_E)_K \hookrightarrow \mathcal{A}(G_E)_K \cap L^2(G_E(F)\backslash G_E(\mathbb{A}))$$

in all cases. By the automatic continuity theorem of Casselman-Wallach (cf. [C] and [W]), this extends to a $G(\mathbb{A})$-equivariant embedding

$$\theta_E : \Pi_E \hookrightarrow \mathcal{A}(G_E) \cap L^2(G_E(F)\backslash G_E(\mathbb{A})).$$

Moreover, we have defined in the last section an extension of Π_E to $H_E(\mathbb{A})$ and the finite group $S_E(F)$ acts on $\mathcal{A}(G_E) \cap L^2(G_E(F)\backslash G_E(\mathbb{A}))$ by:

$$(s \cdot f)(g) = f(s^{-1}(g)).$$

It is then easy to see that the map θ_E is in fact $S_E(F) \ltimes G_E(\mathbb{A})$-equivariant.

Let

$$\theta_E^{U_E} : \Pi_E \to \mathcal{A}(L_E)$$

be the $L_E(\mathbb{A})$-equivariant map defined by

$$\theta_E^{U_E}(v)(l) = \int_{U_E(F)\backslash U_E(\mathbb{A})} \theta_E(v)(ul)du \quad \text{for } l \in L_E(\mathbb{A}).$$

Thus $\theta_E^{U_E}$ is the constant term map along U_E. The following Proposition describes the image $Im(\theta_E^{U_E})$ of this map and is the global analog of Prop. 4.5. It is proved in the course of proving Prop. 5.1 by examining the constant term along U_E of the relevant Eisenstein series at the point $s = \frac{4}{5}$; we give the proof of (i) in the appendix.

Proposition 5.2. *(i) If $E = F \times F \times F$, then*

$$Im(\theta_E^{U_E}) \cong \delta_{Q_E}^{1/6}\sigma_E \oplus |\chi_1|^2 \oplus |\chi_3|^2 \oplus |\chi_4|^2.$$

Here the χ_i's were defined in (3.5).

(ii) If $E = F \times K$, then

$$Im(\theta_E^{U_E}) \cong \delta_{Q_E}^{1/6}\sigma_E \oplus (\chi_K|\cdot|^2 \circ \mu_1) \otimes |\mu_2|^{-1}.$$

Here the μ_i's were defined in (3.7).

(iii) If E is a cubic field, then

$$Im(\theta_E^{U_E}) \cong \delta_{Q_E}^{1/6}\sigma_E.$$

Similarly, we have the constant term map

$$\theta_E^{N_E} : \Pi_E \to \mathcal{A}(M_E),$$

More generally, if $\psi_0 : F\backslash\mathbb{A} \to S^1$ is a non-trivial unitary character, then composition with ψ_0 gives an identification

$$\mathrm{Hom}(N_E, \mathbb{G}_a) \cong \{\text{unitary characters of } N_E(\mathbb{A}) \text{ trivial on } N_E(F)\}$$

For any unitary character ψ of $N_E(\mathbb{A})$ trivial on $N_E(F)$, we have the linear functional

$$\theta^{N_E}_{E,\psi} : \Pi_E \to \mathbb{C}$$

defined by:

$$\theta^{N_E}_{E,\psi}(v) = \int_{N_E(F)\backslash N_E(\mathbb{A})} \theta_E(v)(n) \cdot \overline{\psi(n)} dn.$$

The following proposition describes the Fourier expansion of Π_E along N_E and is also proved in the course of proving Prop. 5.1; we give the proof of (ii) in the appendix.

Proposition 5.3. *(i) Suppose that ψ is non-trivial. The linear functional $\theta^{N_E}_{E,\psi}$ is non-zero if and only if $\psi \in \Omega_E(F)$.*

(ii) If $E = F \times F \times F$, then as a representation of $M_E(\mathbb{A})$, $\theta^{N_E}_E(\Pi_E) \subset \mathcal{A}(M_E)$ is given by the exact sequence:

$$0 \longrightarrow W \longrightarrow \theta^{N_E}_E(\Pi_E) \longrightarrow \delta^{1/5}_{P_E} \longrightarrow 0.$$

Here, W is defined as follows. Let V be the representation of $\mathrm{GL}_2(\mathbb{A})$ defined by:

$$0 \longrightarrow V \longrightarrow \sigma(|-|^{1/2}, |-|^{-1/2}) \longrightarrow \mathbf{1} \longrightarrow 0.$$

Then there is an exact sequence

$$0 \longrightarrow \delta^{1/5}_{P_E} \longrightarrow W \longrightarrow \delta^{1/5}_{P_E} \cdot ((V \otimes \mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes V \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1} \otimes V)) \longrightarrow 0,$$

where $V \otimes \mathbf{1} \otimes \mathbf{1}$ is regarded as a representation of $M_E(\mathbb{A})$ via the first projection $pr_1 : M_E \to \mathrm{GL}_2$ (as in Prop. 4.4(ii)) and so on.

Moreover, W is realized as a subspace of $\mathcal{A}(M_E)$ as follows. Let $E(g, f_s)$ be the Eisenstein series associated to a flat section $f_s \in \sigma(|-|^s, |-|^{-s})$. Then $E(g, f_s)$ has a pole at $s = \frac{1}{2}$ if f_s is the spherical section. On the other hand, if f_s is a flat section such that $f_{\frac{1}{2}} \in V$, then $E(g, f_s)$ is holomorphic and non-zero at $s = \frac{1}{2}$. Thus the map $f_{\frac{1}{2}} \mapsto E(g, f_{\frac{1}{2}})$ defines an embedding of vector spaces:

$$Eis : V \hookrightarrow \mathcal{A}(\mathrm{GL}_2),$$

which is not $\mathrm{GL}_2(\mathbb{A})$-equivariant. Then W is spanned as a vector subspace of $\mathcal{A}(M_E)$ by the constant functions on $M_E(\mathbb{A})$ and the subspaces $pr_1^*(Eis(V)) \oplus pr_2^*(Eis(V)) \oplus pr_3^*(Eis(V))$.

(iii) If $E = F \times K$, then

$$0 \longrightarrow W \longrightarrow \theta^{N_E}_E(\Pi_E) \longrightarrow \nu_E \delta^{1/5}_{P_E} \longrightarrow 0$$

where W is defined by pulling back the irreducible representation $V = \sigma(|-|^{1/2}, \chi_K|-|^{3/2})$ of $\mathrm{GL}_2(\mathbb{A})$ via the projection $pr : M_E \to \mathrm{GL}_2$. Moreover, if $Eis : V \hookrightarrow \mathcal{A}(\mathrm{GL}_2)$ is the embedding defined by the theory of Eisenstein series, then W is the subspace of $\mathcal{A}(M_E)$ consisting of the functions $g \mapsto Eis(v)(pr(g))$ for $v \in V$.

(iv) If E is a Galois field extension of F, then

$$\theta^{N_E}_E(\Pi_E) \cong \nu_E \delta^{1/5}_{P_E} \oplus \nu_E^{-1} \delta^{1/5}_{P_E}.$$

(v) If E is non-Galois, then $\theta_E^{N_E}(\Pi_E) = 0$.

The following corollary is immediate from Prop. 5.2 and Prop. 5.3.

Corollary 5.4. *Let $v \in \Pi_E$. When E is Galois,*

$$\begin{cases} \theta_E^{U_E}(v)|_L \in \mathcal{A}(L)_{Eis}, \\ \theta_E^{N_E}(v)|_M \in \mathcal{A}(M)_{Eis}. \end{cases}$$

When E is non-Galois,

$$\begin{cases} \theta_E^{U_E}(v)|_L \in \mathcal{A}(L)_{cusp}, \\ \theta_E^{N_E}(v)|_M = 0 \in \mathcal{A}(M). \end{cases}$$

We come now to the main result of this section.

Proposition 5.5. *For any $v \in \Pi_E$, the restriction $\theta_E(v)|_G$ is a square-integrable automorphic form on G.*

Proof. Firstly, $\theta_E(v)|_G$ is certainly a smooth function of moderate growth, and is K_∞-finite whenever v is a K_{E_∞}-finite vector. Here K_∞ and K_{E_∞} are the maximal compact subgroups of $G(F_\infty)$ and $G_E(F_\infty)$ respectively. Moreover, it was shown in [V] that, for v archimedean, Π_{E_v} possesses an infinitesimal character when regarded as a representation of $G(F_v)$. Hence $\theta_E(v)|_G$ is certainly $\mathcal{Z}(\mathfrak{g}_v)$-finite for any archimedean v. In other words, $\theta_E(v)|_G$ is an automorphic form on G.

It remains to show that $\theta_E(v)|_G$ is square-integrable. As an automorphic form, $\theta_E(v)|_G$ has a decomposition [MW]:

$$\theta_E(v)|_G = f_{cusp} + f_P + f_Q + f_B$$

where f_{cusp} is a cusp from, f_P has cuspidal support along P and so on. Consider the case when E is Galois. By Lemmas 3.3(iii) and 3.4, as well as Prop. 5.3(i), we see that

$$\begin{cases} (\theta_E(v)|_G)^U = \theta_E^{U_E}(v)|_L, \\ (\theta_E(v)|_G)^N = \theta_E^{N_E}(v)|_M, \end{cases}$$

where $(\theta_E(v)|_G)^U$ denotes the constant term of the automorphic form $\theta_E(v)|_G$ along U. Indeed, for $m \in M(\mathbb{A})$, we have

$$\begin{aligned} (\theta_E(v)|_G)^N(m) &= \int_{N(F)\backslash N(\mathbb{A})} \theta_E(v)(nm)dn \\ &= \int_{N(F)\backslash N(\mathbb{A})} \left(\theta_E^{N_E}(v)(nm) + \sum_{\psi \in \Omega_E(F)} \theta_{E,\psi}^{N_E}(nm \cdot v) \right) dn \\ &= \theta_E^{N_E}(v)(m) + \sum_{\psi \in \Omega_E(0)} \theta_{E,\psi}^{N_E}(m) \\ &= \theta_E^{N_E}(v)(m). \end{aligned}$$

Thus Corollary 5.4 implies immediately that $f_P = f_Q = 0$. Hence it remains to show that f_B is square-integrable. Since f_B is concentrated along the Borel subgroup B, we need to examine the exponents of $\theta_E(v)|_G$ along B. By Prop. 5.2, one deduces that the exponents are given by:

$$\begin{cases} -\alpha-\beta, -2\alpha-\beta, \text{ if } E = F\times F\times F; \\ -\alpha-\beta, -2\alpha-\beta, \text{ if } E = F\times K; \\ -2\alpha-\beta, \text{ if } E \text{ is a Galois field.} \end{cases}$$

Since all the exponents have negative coefficients, we deduce by Jacquet's square-integrability criterion [MW, Pg. 74] that $\theta_E(v)|_G$ is square-integrable. The case when E is non-Galois can be treated similarly; we omit the details. The proposition is proved completely. □

6. **Local Arthur Packets**

In this section, we return to the local situation and recall the results of Vogan [V, Thm. 18.10, Pg. 788] and Huang-Magaard-Savin [HMS, Prop. 6.1, Pg. 76] which give a natural construction of what one might hope is the local Arthur packet $A_{E,v}$. Since the setting is entirely local, we shall suppress v from the notation.

When σ is a tempered representation of $\mathrm{GL}_2(F)$ and $Re(s) > 0$, we let $J_P(\sigma, s)$ and $J_Q(\sigma, s)$ denote the Langlands quotients of the induced representations $Ind_{P(F)}^{G(F)} \delta_P^{\frac{1}{2}+s}\sigma$ and $Ind_{Q(F)}^{G(F)} \delta_Q^{\frac{1}{2}+s}\sigma$. Let Π_E be the minimal representation of $H_E(F)$, as defined in Section 4. The following proposition describes the restriction of Π_E to the subgroup $S_E(F) \times G(F)$.

Proposition 6.1. *As a representation of $S_E(F) \times G(F)$,*

$$\Pi_E = \bigoplus_{\eta \in \widehat{S_E(F)}} \eta^\vee \otimes \pi_\eta$$

where the sum runs over irreducible characters η of $S_E(F)$ and each π_η is a non-zero irreducible unitarizable non-generic representation of $G(F)$ and the π_η's are mutually non-isomorphic.

Remarks: How does one check that the π_η's are mutually non-isomorphic? Recall that the unitary characters of $N(F)$ are parametrized by $\mathrm{Hom}(N(F), F)$ and the $M(F)$-orbits are parametrized by cubic F-algebras. Moreover, if ψ is a character of $N(F)$ in the $M(F)$-orbit $\mathcal{O}_E$ corresponding to E, then the stabilizer of ψ in $M(F)$ is a finite subgroup isomorphic to $S_E(F)$. When F is non-archimedean, it was shown in [HMS, 1.10] that for any $\psi \in \mathcal{O}_E$,

$$(\pi_\eta)_{N,\psi} \cong \eta$$

as $S_E(F)$-modules. This proves that the π_η's are mutually non-isomorphic without determining what each π_η is.

In [V, Thm. 18.10] and [HMS, §7], the irreducible representation π_η is completely determined in many cases (and completely in the archimedean case). We summarize the result in the following proposition.

Proposition 6.2. *The following table determines π_η in many instances.*

E	η	π_η
$F \times F \times F$	$\mathbf{1}$	$J_Q(r_E, \frac{1}{5})$
	r	$J_P(St, \frac{1}{6})$
	ϵ	*supercuspidal (if v finite)*
$F \times K$	$\mathbf{1}$	$J_Q(r_E, \frac{1}{5})$
	κ	*supercuspidal (if v finite)*
Galois field	$\mathbf{1}$	$J_Q(r_E, \frac{1}{5})$
	ν	*supercuspidal (if v finite)*
	ν^2	*supercuspidal (if v finite)*
non-Galois field	$\mathbf{1}$	$J_Q(r_E, \frac{1}{5})$

In the above table, ϵ is the sign character of $S_E(F) \cong S_3$, κ is the non-trivial character of $S_E(F) \cong \mathrm{Gal}(K/F)$ and ν is a non-trivial (cubic) character of $S_E(F) \cong \mathrm{Gal}(E/F)$.

When F is archimedean, the special unipotent representations of $G(F)$ associated to the subregular unipotent orbit has been classified in [BV] and [V, Thm. 18.5] and are precisely the representations of $G(F)$ obtained in the restriction of Π_E. When $F = \mathbb{R}$, let us describe the representations π_η using the notations of [V, Thm. 18.5] and give their minimal K-types (with $K = (\mathrm{SU}_2 \times \mathrm{SU}_2)/\Delta\mu_2$). Suppose first that $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then we have:

$$\Pi_E = (\mathbf{1} \otimes J(\xi_1, (1, 0, -1))) \oplus (r \otimes J(H_2; (1, -1))) \oplus (\epsilon \otimes J_-(H_2; (2, 0))).$$

Here, $\pi_{\mathbf{1}} = J(\xi_1, (1, 0, -1))$ is spherical, $\pi_r = J(H_2; (1, -1))$ has minimal K-type $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\pi_\epsilon = J_-(H_2; (2, 0))$ is a limit of discrete series with minimal K-type $\mathbb{C} \otimes \mathbb{C}^3$. On the other hand, when $E = \mathbb{R} \times \mathbb{C}$,

$$\Pi_E = (\mathbf{1} \otimes J(\xi_x, (1, 0, -1))) \oplus (\kappa \otimes J(H_1; (1, 1)))$$

where $\pi_{\mathbf{1}} = J(\xi_x, (1, 0, -1))$ has minimal K-type $\mathbb{C}^3 \otimes \mathbb{C}$ and $\pi_\kappa = J(H_1; (1, 1))$ has minimal K-type $\mathbb{C}^5 \otimes \mathbb{C}$.

In any case, observe that π_η is completely determined when η is the trivial representation, and the description is uniform in E. Also, if E is unramified and η is the trivial representation, then π_η is unramified. It is of course desirable to specify the representation π_η completely in all cases, but for the global applications we have in mind, the results of the above Proposition are often sufficient.

We shall let

$$A_E = \{\pi_\eta : \eta \in \widehat{S_E(F)}\}$$

and one can hope that A_E is the Arthur packet associated to the parameter ψ_E described in §2. How does one check that A_E is the right set? To do so, one would first need to check that the distribution

$$\Delta_E = \sum_\eta \dim(\eta)\,\mathrm{Tr}(\pi_\eta)$$

is a stable distribution. Further, one has to verify certain identities involving transfer of distributions to endoscopic groups of G. Both of these are important problems in local harmonic analysis and are beyond our means. However, there are reasons to believe that the set A_E is the right one. Besides the naturality of its construction, this belief should be justifiable when $F = \mathbb{C}$ using the results of [BV]. Moreover, there is a Langlands parameter

$$\phi_{\psi_E} : L_F \to \mathrm{G}_2(\mathbb{C})$$

naturally associated to ψ_E [A2, Pg. 25-26] and defined by

$$\phi_{\psi_E}(w) = \psi_E\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right).$$

If Σ_E is the L-packet associated to ϕ_{ψ_E}, then one expects an injection $\Sigma_E \hookrightarrow A_E$ [A2, Pg. 38], whose image is the singleton set containing the representation $\pi_{\mathbf{1}}$ associated to the trivial character of the component group $S_E(F)$. This last expectation can actually be verified. Using Prop. 6.2, one can easily check that Σ_E is precisely the set $\{\pi_{\mathbf{1}}\}$ as predicted. This provides evidence for the claim that A_E is indeed the Arthur packet associated to ψ_E.

7. Arthur Parameters and Residual Spectrum

In this section, we summarize the results of Kim and Zampera on the residual spectrum. It is most illuminating to formulate their results in the framework of Arthur's conjectures and hence we shall begin by describing (as much as possible) all the Arthur parameters of $G = \mathrm{G}_2$.

There are 5 conjugacy class of homomorphisms $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$, corresponding to the 5 unipotent conjugacy classes in $\mathrm{G}_2(\mathbb{C})$. These are:

- the trivial morphism, which corresponds to the identity element of $\mathrm{G}_2(\mathbb{C})$;
- the long root SL_2, which corresponds to the unipotent conjugacy class of a non-trivial element in the root subgroup associated to a long root;

- the short root SL_2, which corresponds to the unipotent conjugacy class of a non-trivial element in the root subgroup associated to a short root;
- the subregular SL_2, which corresponds to the subregular unipotent conjugacy class;
- the regular SL_2, which corresponds to the regular unipotent conjugacy class.

The following table gives the centralizer in $\mathrm{G}_2(\mathbb{C})$ of the image of these SL_2's:

SL_2	trivial	long root	short root	subregular	regular
Centralizer	$\mathrm{G}_2(\mathbb{C})$	short root SL_2	long root SL_2	S_3	trivial

Henceforth, let F be a number field and L_F the conjectural Langlands group of F. We can now describe the possible Arthur parameters $\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C})$ and fantasize about the associated Arthur packet. For the purpose of describing the discrete spectrum, it suffices to look at those parameters ψ for which the centralizer of the image of ψ is finite. If ψ is such that $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is the trivial morphism, then ψ is a tempered parameter and for such, we have nothing intelligent to add. On the other hand, if $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is the regular morphism, then the resulting local and global Arthur packets are singletons consisting of the trivial representation of $G(F_v)$ and $G(\mathbb{A})$ respectively.

We now consider the long and short root SL_2's. From the above table, we see that there is a map

$$\iota : \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C}),$$

where one copy of SL_2 is short root and the other is long root. This realizes $(\mathrm{SL}_2, \mathrm{SL}_2)$ as a dual pair in G_2 and the kernel of ι is the diagonal μ_2. Hence the situations for the long and short root SL_2's are entirely symmetrical.

Given $\phi : L_F \to \mathrm{SL}_2(\mathbb{C})$, set

$$\psi = \iota \circ (\phi \times \mathrm{id}) : L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C}).$$

The centralizer S_ψ of the image of ψ is finite if and only if ϕ is an irreducible representation of L_F, in which case $S_\psi \cong \mathbb{Z}/2\mathbb{Z}$. Conjecturally, such a ϕ corresponds to a cuspidal representation σ_ϕ of $PGL_2(\mathbb{A})$.

We now consider the local parameter

$$\psi_v : L_{F_v} \hookrightarrow L_F \to \mathrm{G}_2(\mathbb{C}),$$

where L_{F_v} is the Weil-Deligne group (resp. Weil group) of F_v if v is finite (resp. archimedean), and determine the component group S_{ψ_v} of the centralizer of image(ψ_v). There are two cases, depending on whether the local representation $\phi_v : L_{F_v} \to \mathrm{SL}_2(\mathbb{C})$ is irreducible or not; equivalently, whether the local component $\sigma_{\phi,v}$ is a discrete series representation or not. Indeed,

$$S_{\psi_v} = \begin{cases} \{1\}, & \text{if } \phi_v \text{ is reducible;} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \phi_v \text{ is irreducible.} \end{cases}$$

Letting ϵ denote the non-trivial character of S_{ψ_v} in the second case, we see that the local Arthur packet A_{ψ_v} has the form

$$A_{\psi_v} = \{\pi_{\mathbf{1}}\} \quad \text{or} \quad A_{\psi_v} = \{\pi_{\mathbf{1}}, \pi_\epsilon\}$$

in the respective cases.

As explained at the end of the previous section, we can describe the representation $\pi_{\mathbf{1}}$, at least in terms of $\sigma_{\phi,v}$. Indeed, $\pi_{\mathbf{1}}$ has Langlands parameter ϕ_{ψ_v} and is given by:

$$\pi_{\mathbf{1}} = \text{the Langlands quotient} \begin{cases} J_P(\sigma_{\phi_v}, \frac{1}{6}), & \text{if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the short root } \mathrm{SL}_2; \\ J_Q(\sigma_{\phi_v}, \frac{1}{10}), & \text{if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the long root } \mathrm{SL}_2. \end{cases}$$

On the other hand, π_ϵ is expected to be a supercuspidal representation (when v is finite), which remains to be determined.

Remark: Since we do not know the Ramanujan conjecture for GL_2, the representation σ_{ϕ_v} above is not known to be tempered. Hence, it may not be entirely accurate to call π_1 a Langlands quotient of the relevant induced representation. However, it is true that this induced representation has a unique irreducible quotient (this was checked in [Z]). In fact, the recent progress made by Kim and Shahidi towards Ramanujan conjecture shows that if σ_{ϕ_v} is in the complementary series, it has the form $\sigma(\chi|\cdot|^s, \chi|\cdot|^{-s})$ for some quadratic character χ and more importantly, with $|s| < \frac{1}{6}$ (this used to be an assumption in [K]). This implies that π_1 is a Langlands quotient, but its Langlands data is supported on the Borel subgroup, instead of P or Q.

With the above description of the local packets, it is not difficult to write down the global Arthur packet which has $2^{r(\phi)}$ elements, where $r(\phi)$ is the number of places v such that $\sigma_{\phi,v}$ is a discrete series representation. Arthur's multiplicity formula says that only half of the elements in the global packet will appear (with multiplicity one) in the discrete spectrum, at least when $r(\phi) > 0$. Of course, the only element of the global packet that we can specify is the representation

$$\pi_{\mathbf{1}} = \begin{cases} \pi_{P,\sigma_\phi} := J_P(\sigma_\phi, \frac{1}{6}), & \text{if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the short root } \mathrm{SL}_2; \\ \pi_{Q,\sigma_\phi} := J_Q(\sigma_\phi, \frac{1}{10}), & \text{if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the long root } \mathrm{SL}_2, \end{cases} \tag{7.1}$$

corresponding to the trivial character of $\prod_v S_{\psi_v}$. This is the only representation in the packet that can occur in L^2_{res}, since for example π_ϵ is supercuspidal at a finite prime v. The other representations in this Arthur packet, if they occur in L^2_d at all, are cuspidal and nearly equivalent to $\pi_{\mathbf{1}}$. Hence they are CAP with respect to

$$\begin{cases} P \text{ if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the short root } \mathrm{SL}_2; \\ Q \text{ if } \psi_v|_{\mathrm{SL}_2(\mathbb{C})} \text{ is the long root } \mathrm{SL}_2. \end{cases}$$

This concludes our discussion of the long and short root SL_2's.

We come now to the subregular parameters ψ_E which are parametrized by étale cubic F-algebras E and have been treated in Section 2 in some detail. Fixing E, we shall take the set A_{E_v} constructed in the previous section as the Arthur packet. Then for each irreducible representation $\eta = \otimes_v \eta_v$ of $S_E(\mathbb{A})$, we have the global representation $\pi_\eta = \otimes_v \pi_{\eta_v}$. Let $A_{E,res}$ be the set of η given by the following table and such that $m_\eta \neq 0$. Here, the character ϵ and κ are as defined in Prop. 6.2. As an example, when $E = F \times F \times F$, any $\eta \in A_{E,res}$ is equal to

$$\eta_S = (\bigotimes_{v\in S} r)\bigotimes(\bigotimes_{v\notin S}\mathbf{1})$$

for some finite set S of places of F and

$$m_{\eta_S} = \begin{cases} \frac{1}{6}(2^{\#S} + (-1)^{\#S}2), \text{ if } S \text{ is non-empty;} \\ 1, \text{ if } S \text{ is empty,} \end{cases}$$

is non-zero if and only if $\#S \neq 1$.

E	η
$F \times F \times F$	no local component η_v is the sign character ϵ
$F \times K$	no local component η_v is ϵ or κ
Galois field	η is trivial
non-Galois field	η is trivial

We can now state the results of Kim and Zampera on the residual spectrum L^2_{res}. Let us write

$$L^2_{res} = L^2_{res}(B) \oplus L^2_{res}(P) \oplus L^2_{res}(Q)$$

where $L^2_{res}(B)$ is the closed span of those automorphic forms whose cuspidal supports are along B and so on. Here is a reformulation of the results of [K] and [Z] in terms of the above picture:

Proposition 7.2.

$$L^2_{res}(B) = \mathbf{1} \oplus \left(\bigoplus_{E \text{ Galois}} \left(\bigoplus_{\eta\in A_{E,res}} \pi_\eta \right)\right).$$

$$L^2_{res}(Q) = \left(\bigoplus_{E \text{ non-Galois}} \pi_{\mathbf{1}} \right) \oplus \left(\bigoplus_{\sigma} \pi_{Q,\sigma} \right).$$

Here the second sum is over all non-monomial cuspidal representations σ of PGL_2 *for which*

$$L(\frac{1}{2}, \sigma, Sym^3) \neq 0$$

and $\pi_{Q,\sigma}$ *is defined in (7.1).*

$$L^2_{res}(P) = \bigoplus_{\sigma} \pi_{P,\sigma}.$$

Here the sum is over all cuspidal representation σ *of* PGL_2 *such that*

$$L(\frac{1}{2}, \sigma) \neq 0$$

and $\pi_{P,\sigma}$ *is defined in (7.1).*

The following consequence of Prop. 7.2 is what we need:

Corollary 7.3. *Let* E *be an étale cubic* F*-algebra and* η *an irreducible representation of* $S_E(\mathbb{A})$*. Then*

$$\dim \mathrm{Hom}_{G(\mathbb{A})}(\pi_\eta, L^2_{res}) = \begin{cases} 1, & \text{if } \eta \in A_{E,res}; \\ 0, & \text{if } \eta \notin A_{E,res}. \end{cases}$$

Proof. Fixing the pair (E, η), we need to show:

(i) for any pair $(E', \eta') \neq (E, \eta)$, $\pi_{\eta'} \ncong \pi_\eta$;

(ii) for any cuspidal representation σ of $PGL_2(\mathbb{A})$, $\pi_\eta \ncong \pi_{P,\sigma}$ and $\pi_\eta \ncong \pi_{Q,\sigma}$.

For (i), assume first that the étale cubic algebras E and E' are non-isomorphic. Then there are infinitely many places v of F such that $\eta_v = \eta'_v = \mathbf{1}$ but $E_v \ncong E'_v$. For such a place v, Prop. 6.2 gives:

$$\pi_{\eta_v} \cong J_Q(r_{E_v}, 1/5) \quad \text{and} \quad \pi_{\eta'_v} \cong J_Q(r_{E'_v}, 1/5).$$

Since r_{E_v} and $r_{E'_v}$ are tempered and non-isomorphic, the representations π_{η_v} and $\pi_{\eta'_v}$ have different Langlands data and are thus non-isomorphic. On the other hand, if $E = E'$ but $\eta_v \neq \eta'_v$ for some place v, then $\pi_{\eta_v} \ncong \pi_{\eta'_v}$ by Prop. 6.1.

Now we come to (ii). To distinguish between π_η and $\pi_{P,\sigma}$, we consider the Jacquet modules of π_{η_v} and π_{P,σ_v} along P for a suitable finite place v. More precisely, there are infinitely many finite places v such that

(a) $E_v = F_v \times F_v \times F_v$ and $\eta_v = \mathbf{1}$;

(b) σ_v is infinite-dimensional and unramified.

For such a place v, we see that as a representation of $M(F_v) \cong \mathrm{GL}_2(F_v)$,

$$(\pi_{\eta_v})_N = \delta_P^{1/3} \cdot (\mathbf{1} \oplus \mathbf{1} \oplus St)$$

after semi-simplification, whereas $(\pi_{P,\sigma_v})_N$ contains the infinite-dimensional spherical representation $\delta_P^{1/3} \cdot \sigma_v$ as a quotient. This shows that $(\pi_{\eta_v})_N \ncong (\pi_{P,\sigma_v})_N$ and thus $\pi_{\eta_v} \ncong \pi_{P,\sigma_v}$.

On the other hand, to distinguish between π_η and $\pi_{Q,\sigma}$, we take a finite place v satisfying (a) and (b) above and consider the Jacquet modules of the local representations along Q. In this case, we find that as a representation of $L(F_v) \cong \mathrm{GL}_2(F_v)$,

$$(\pi_{\eta_v})_U \cong \delta_Q^{3/10} \cdot r_{E_v} \oplus \delta_Q^{2/5}$$

whereas $(\pi_{Q,\sigma_v})_U$ contains the infinite-dimensional representation $\delta_Q^{2/5}\cdot\sigma_v$ as a quotient. Since σ_v and r_{E_v} are both representations of $PGL_2(F_v)$, one sees that $\delta_Q^{2/5}\cdot\sigma_v$ and $\delta_Q^{3/10}\cdot r_{E_v}$ have different central characters and thus $\pi_{\eta_v}\ncong\pi_{Q,\sigma_v}$. The corollary is proved completely. □

Remarks: (i) The proposition provides further evidence that the set A_{E_v} is the predicted local Arthur packet.

(ii) In [K, Pg. 1264-1267], an attempt was made to interpret the constituents of $L^2_{res}(B)$ in terms of Arthur parameters. However, the interpretation given there is not entirely accurate. To be precise, the interpretation given in [K, Pg. 1266, Case 1] and [K, Pg. 1267, Case 2], which concerns the cases $E = F\times K$ and E a Galois field respectively, appears to be incorrect.

(iii) When $E = F\times F\times F$ and $\eta\in A_{E,res}$, the embedding $\pi_\eta\hookrightarrow L^2_{res}$ was constructed by Moeglin-Waldspurger [MW, Appendix III], building on the work of Langlands [La, Appendix 3] who constructed the embedding of $\pi_{\mathbf{1}}$. In this respect, let us take the opportunity to correct a minor inaccuracy in [A2, Pg. 56], where the example of Langlands was discussed. The discussion on [A2, Pg. 56-57] was meant to correct an error in [A1] concerning Langlands representation $\pi_{\mathbf{1}}$. However, it was claimed incorrectly that a principal unipotent element of the subgroup $\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})/\Delta\mu_2\hookrightarrow\mathrm{G}_2(\mathbb{C})$ is contained in a proper Levi subgroup of $\mathrm{G}_2(\mathbb{C})$, when in fact such an element lies in the subregular orbit. In other words, the original description in [A1] was correct after all.

The cuspidal representations in the global Arthur packet associated to ψ_E are nearly equivalent to π_η for $\eta\in A_{E,res}$ and are thus CAP with respect to B (resp. Q) if E is Galois (resp. non-Galois). In the next section, we give a construction of these cusp forms.

8. Global Arthur Packets

We continue to assume that F is a number field and E an étale cubic F-algebra. Given an automorphic representation $\eta=\otimes_v\eta_v$ of $S_E(\mathbb{A})$, the conjecture of Arthur predicts that $\pi_\eta=\otimes_v\pi_{\eta_v}$ occurs in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$ with multiplicity $\geq m_\eta$. In this section, we verify this and give a construction of what one might hope is the global Arthur packet.

Let $\Pi_E=\otimes_v\Pi_{E_v}$ be the global minimal representation of $H_E(\mathbb{A})$. As an abstract representation of $S_E(\mathbb{A})\times G(\mathbb{A})$, we have:

$$\Pi_E=\bigoplus_{\eta\in\widehat{S_E(\mathbb{A})}}\eta^\vee\otimes\pi_\eta.$$

Hence, for each irreducible representation η of $S_E(\mathbb{A})$, there is a unique (up to scaling) $S_E(\mathbb{A})\times G(\mathbb{A})$-equivariant embedding

$$\iota_\eta:\eta^\vee\otimes\pi_\eta\hookrightarrow\Pi_E.$$

Now let

$$\theta_E:\Pi_E\hookrightarrow L^2(G_E(F)\backslash G_E(\mathbb{A}))$$

be the automorphic realization of Π_E constructed in Section 5. Recall that this embedding is $S_E(F) \times G(\mathbb{A})$-equivariant. As a consequence, the function on $S_E(\mathbb{A})$ defined by

$$s \mapsto \theta_E(s(v))(g)$$

is left-invariant under $S_E(F)$ for any $v \in \Pi_E$ and $g \in G(\mathbb{A})$.

Set

$$\theta_\eta = \theta_E \circ \iota_\eta : \eta^\vee \otimes \pi_\eta \hookrightarrow L^2(G_E(F)\backslash G_E(\mathbb{A})).$$

Fix $w_\eta \in \eta$ and $w_\eta^\vee \in \eta^\vee$ so that the natural pairing $\langle w_\eta^\vee, w_\eta \rangle$ is non-zero. Then for any

$$f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A}))),$$

we define an element

$$f_* \in \mathrm{Hom}_{G(\mathbb{A})}(\pi_\eta, \mathcal{A}(G))$$

by setting:

$$f_*(v)(g) = \int_{S_E(F)\backslash S_E(\mathbb{A})} \theta_\eta(sw_\eta^\vee \otimes v)(g) \cdot f(w_\eta)(s)ds, \quad \text{for } v \in \pi_\eta \text{ and } g \in G(\mathbb{A}).$$

Thus $f \mapsto f_*$ gives a map of vector spaces

$$\mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A}))) \to \mathrm{Hom}_{G(\mathbb{A})}(\pi_\eta, \mathcal{A}(G)).$$

The choices of w_η and $w_\eta^\vee$ are not really important. Indeed, we have:

Lemma 8.1. *For fixed $f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$, the image of f_* in $\mathcal{A}(G)$ is equal to the space spanned by all the functions of the form*

$$\Theta_E(v, \varphi)(g) = \int_{S_E(F)\backslash S_E(\mathbb{A})} \theta_E(s(v))(g)\varphi(s)ds$$

for all $v \in \Pi_E$ and $\varphi \in f(\eta)$.

Here is the main result of this paper:

Theorem 8.2. *(i) For any non-zero $f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$, f_* is non-zero and $f_*(\pi_\eta)$ consists of square integrable automorphic forms. In other words, we have a $G(\mathbb{A})$-equivariant embedding*

$$\alpha_\eta : \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A}))) \otimes \pi_\eta \hookrightarrow L^2_d(G(F)\backslash G(\mathbb{A}))$$

given by $f \otimes v \mapsto f_*(v)$.

(ii) Let $A_E = \{\eta : m_\eta \neq 0\}$, and let $\Theta_E(\eta)$ denote the image of α_η. The following table determines those $\eta \in A_E$ for which $\Theta_E(\eta)$ is contained in L^2_{cusp}.

E	η *such that* $\Theta_E(\eta)$ *is cuspidal*
$F \times F \times F$	*some local component* η_v *is the sign character* ϵ
$F \times K$	*some local component* η_v *is* ϵ *or* κ
Galois field	η *is non-trivial*
non-Galois field	η *is non-trivial*

For the other η*'s, the subspace of those* f *such that* $f_*(\pi_\eta) \subset L^2_{cusp}$ *has codimension 1 in* $Hom_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$*.*

(iii) Each non-zero automorphic form in $\Theta_E(\eta)$ *is* E*-distinguished. More precisely, if* ψ *is a non-trivial unitary character of* $N(\mathbb{A})$ *trivial on* $N(F)$ *which is in a generic* $M(F)$*-orbit, then for any non-zero* $\varphi \in \Theta_E(\eta)$*, the Fourier coefficient*

$$\varphi_\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi(n)}dn$$

is non-zero (as a function on $G(\mathbb{A})$*) if and only if* $\psi \in \mathcal{O}_E$*.*

(iv) Assume that $f_*(\pi_\eta) \subset L^2_{cusp}$*. If* E *is Galois, then* $f_*(\pi_\eta)$ *is CAP with respect to the Borel subgroup. If* E *is non-Galois, then* $f_*(\pi_\eta)$ *is CAP with respect to* Q*.*

Proof. (i) Given a non-zero $f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$, it follows from Prop. 5.5 that $f_*(\pi_\eta)$ is contained in the space of square-integrable automorphic forms, so that

$$f_*(\pi_\eta) \subset L^2_d(G(F)\backslash G(\mathbb{A})).$$

It remains to show that f_* is non-zero.

By Lemma 8.1, it suffices to find $v \in \Pi_E$ and $\varphi \in f(\eta)$ such that $\Theta_E(v,\varphi)$ is non-zero. To see this, we compute a particular Fourier coefficient of $\Theta_E(v,\varphi)$ along N. Let ψ be a unitary character of $N(\mathbb{A})$ trivial on $N(F)$ which lies in the M-orbit $\mathcal{O}_E$. Set

$$\Theta^N_{E,\psi}(v,\varphi) = \int_{N(F)\backslash N(\mathbb{A})} \Theta_E(v,\varphi)(n)\cdot\overline{\psi(n)}dn.$$

Then a simple calculation, coupled with Prop. 4.4(i) and Lemma 3.3(ii), gives:

$$\Theta^N_{E,\psi}(v,\varphi) = \int_{S_E(\mathbb{A})} \theta^{N_E}_{E,\psi_0}(sv)\cdot\varphi(s)ds$$

where ψ_0 is any element in $\Omega_E(\psi)$. To simplify notations, set

$$F_v(s) = \theta^{N_E}_{E,\psi_0}(sv),$$

which is a function on $S_E(\mathbb{A})$. Let us fix $\varphi \in f(\eta)$ such that $\varphi(1) \neq 0$, and $v_0 \in \Pi_E$ such that $F_{v_0}(1) \neq 0$. This is possible by Prop. 5.3(i), and thus the product $F_{v_0} \cdot \varphi$ is a non-zero function. Let $v = \sum_i c_i \cdot n_i(v_0) \in \Pi_E$, with $c_i \in \mathbb{C}$ and $n_i \in N_E(\mathbb{A})$. Then a short computation gives:

$$F_v(s) = \left(\sum_i c_i \psi_0(s n_i s^{-1})\right) \cdot F_{v_0}(s).$$

Hence,

$$\Theta^N_{E,\psi}(v, \varphi) = \int_{S_E(\mathbb{A})} \left(\sum_i c_i \psi_0(s n_i s^{-1})\right) \cdot F_{v_0}(s) \cdot \varphi(s) ds.$$

Since the function $F_{v_0} \cdot \varphi$ is a non-zero function, it suffices to show that the vector space of functions of the form

$$s \mapsto \sum_i c_i \psi_0(s n_i s^{-1})$$

is dense in the space of continuous functions on the compact group $S_E(\mathbb{A})$ (with respect to uniform norm). This follows by the Stone-Weierstrass theorem as in [GS. Pg. 201-202].

(ii) Let $A_{E,cusp}$ denote the set of those η described in the table, and observe that the complement of $A_{E,cusp}$ in A_E is precisely the set $A_{E,res}$ defined at the end of the previous section. For $\eta \in A_{E,cusp}$, let us take a non-zero

$$f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A}))).$$

Then we know that $f_*(\pi_\eta) \subset L^2_d$. Let P_{res} denote the orthogonal projection of L^2_d onto L^2_{res}. By Cor. 7.3, we see that π_η does not occur in L^2_{res} and hence $P_{res}(f_*(\pi_\eta)) = 0$, i.e. $f_*(\pi_\eta) \subset L^2_{cusp}$. Of course, in many instances, one can conclude that $\Theta_E(\eta) \subset L^2_{cusp}$ without using Cor. 7.3. For example, if one finite local component of η is equal to the sign character ϵ, then π_η can only occur in the space of cusp forms since one of its local component is supercuspidal.

Now suppose that $\eta \in A_{E,res}$, and let

$$W_\eta \subset \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$$

be the subspace consisting of those f such that $f_*(\pi_\eta) \subset L^2_{cusp}$. By Cor. 7.3 again, we see that W_η has codimension at most 1. Hence it remains to show that $\mathrm{codim}(W_\eta) \neq 0$. Since the proof is similar in the different cases, we shall simply give the proof in the case when $E = F \times F \times F$ and η is non-trivial; the other cases are either similar or simpler. Since an irreducible representation of S_3 is self-dual, we shall write η in place of $\eta^\vee$.

Assume hence that $\eta = \otimes_v \eta_v$ is non-trivial and none of its local components is equal to the sign character ϵ of S_3. Let $S_r = \{v : \eta_v = r\}$. It is easy to check that $m_\eta \neq 0$ if and only if $\# S_r \neq 1$. We need to produce $f \in \mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$ such that the automorphic form $\Theta_E(w\otimes v, f(w'))$ is non-cuspidal for some $w\otimes v\in\eta\otimes\pi_\eta$ and $w'\in\eta$. Its constant term along U is given by:

$$\Theta_E^U(w\otimes v, f(w')) = \int_{S_E(F)\backslash S_E(\mathbb{A})} \theta_E^{U_E}(sw\otimes v)(1)\cdot f(w')(s)ds \tag{8.3}$$

and we need to show that this is non-zero for some f, w, w' and v. For this, it suffices to show that the linear functional

$$l:\eta\otimes\pi_\eta\to\mathbb{C}$$

given by

$$u\mapsto\theta_E^{U_E}(u)(1)$$

is non-zero. Indeed, if this were the case, the function on $S_E(F)\backslash S_E(\mathbb{A})$ defined by

$$F_{v,w}: s\mapsto l(sw\otimes v)$$

would be non-zero for some v and w, and this means that the map $f_v: w\mapsto F_{v,w}$ would be a non-zero element in $\mathrm{Hom}_{S_E(\mathbb{A})}(\eta, L^2(S_E(F)\backslash S_E(\mathbb{A})))$. It is then easy to choose suitable f and w' to ensure that (8.3) is non-zero.

We now describe the linear functional l in more concrete terms. Recall from Prop. 5.2(i) that the image of the $Q_E(\mathbb{A})$-equivariant map

$$\theta_E^{U_E}:\Pi_E\to\mathcal{A}(L_E)$$

is equal to

$$\delta_{Q_E}^{1/6}\sigma_E\oplus|\chi_1|^2\oplus|\chi_3|^2\oplus|\chi_4|^2.$$

It is not difficult to see that when η is non-trivial and $u\in\eta\otimes\pi_\eta$, the projection of $\theta_E^{U_E}(u)$ onto $\delta_{Q_E}^{1/6}\sigma_E$ is zero; if it were not the case, it would follow by Frobenius reciprocity that π_η has a non-zero $G(\mathbb{A})$-equivariant embedding into $Ind_{Q(\mathbb{A})}^{G(\mathbb{A})}\delta_Q^{3/10}r_E$, but this latter representation has a unique submodule isomorphic to $\pi_{\mathbf{1}}$. Hence, the restriction of $\theta_E^{U_E}$ to the subspace $\eta\otimes\pi_\eta$ takes value in the 3-dimensional space $W\subset\mathcal{A}(L_E)$ spanned by the 3 characters $|\chi_1|^2$, $|\chi_3|^2$ and $|\chi_4|^2$ of $L_E(\mathbb{A})$. Denoting these 3 characters by μ_1,μ_2 and μ_3 for simplicity, we see that if $\theta_E^{U_E}(u)=a\mu_1+b\mu_2+c\mu_3$, then

$$l(u)=a+b+c.$$

Hence, we need to show that $a+b+c\neq 0$ for some $u\in\eta\otimes\pi_\eta$.

To do this, let us consider the local situation. For any place v of F, it follows from Props. 4.5(ii) and 4.6(i) that

$$\dim\mathrm{Hom}_{Q_E(F_v)}(\Pi_{E_v},\mu_{i,v})=1\quad\text{for } i=1,2,3.$$

Let $l_{i,v}$ be a non-zero element of this 1-dimensional vector space. Since Π_{E_v} is generated by a spherical vector ω_v as a $Q_E(F_v)$-module, we see that $l_{i,v}(\omega_v)\neq 0$. Hence, let us normalize the choice of the basis element $l_{i,v}$ by requiring that $l_{i,v}(\omega_v)=1$ for all i. Now the map

$$l_v=\oplus_i l_{i,v}:\Pi_{E_v}\to\oplus_i\mu_{i,v}$$

is such that $l_v(\omega_v) = (1,1,1)$. Since l_v is surjective and $Q_E(F_v) \rtimes S_E(F_v)$-equivariant (where the action of $S_E(F_v)$ on the 3-dimensional space $\oplus_i \mu_{i,v}$ is via the permutation action), the restriction of l_v to $r \otimes \pi_r$ is non-zero; the image being equal to the subspace of vectors (a,b,c) with $a+b+c=0$. In particular, we can choose $u_v \in r \otimes \pi_r$ such that $l_v(u_v) = (2,-1,-1)$.

Let $l_i = \otimes_v l_{i,v} \in \mathrm{Hom}_{Q_E(\mathbb{A})}(\Pi_E, \mu_i)$. Then the global map

$$\theta_E^{U_E} : \Pi_E \to W$$

is equal to $l_1 \oplus l_2 \oplus l_3$. Let us evaluate this map on a certain vector $u = \otimes_v u_v$ in $\eta \otimes \pi_\eta = \otimes_v(\eta_v \otimes \pi_{\eta_v})$ constructed as follows. For $v \in S_r$, we take the vector $u_v \in r \otimes \pi_r$ constructed in the previous paragraph. On the other hand, when $v \notin S_r$, we take the spherical vector ω_v which lies in $\mathbf{1} \otimes \pi_{\mathbf{1}}$. The tensor product of these local vectors give an element $u \in \eta \otimes \pi_\eta$. Moreover,

$$\theta_E^{U_E}(u) = 2^{\#S_r}\mu_1 \oplus (-1)^{\#S_r}\mu_2 + (-1)^{\#S_r}\mu_3.$$

In particular,

$$l(u) = 2^{\#S_r} + 2(-1)^{\#S_r}.$$

We have noted before that $\#S_r \neq 1$ for $\eta \in A_{E,res}$. Thus $l(u) \neq 0$, as desired.

(iii) The vanishing of φ_ψ for a generic $\psi \notin \mathcal{O}_E$ is an immediate consequence of Prop. 5.3(i) and Lemma 3.3. The non-vanishing of φ_ψ for $\psi \in \mathcal{O}_E$ follows from the proof of (i).

(iv) This is an immediate consequence of Prop. 6.2. □

Remark: It is possible to prove (ii) without appealing to the results of [K] and [Z] on residual spectrum. The proof is more involved and so we will not reproduce it here.

In conclusion, the global theta lifting gives a natural construction of what one might hope is the global Arthur packet for the parameter ψ_E, and also shows that the automorphic representations in this Arthur packet have some very special properties. It is natural to ask if these properties characterize the automorphic forms in these Arthur packets. We intend to pursue these and related questions in a sequel to the present paper.

9. **Appendix**

In this appendix, we assume that $E = F \times F \times F$ and give the proofs of Propositions 5.1(i), 5.2(i) and 5.3(ii); the other cases can be treated similarly. As we mentioned before, the proofs of these propositions are standard exercises in the theory of Eisenstein series, but they are non-trivial and can sometimes be very involved.

We begin by establishing some more notations. Let W denote the Weyl group of G_E relative to the maximal torus T_E and denote by w_i the simple reflection in W corresponding to the simple root α_i. In short, we shall write $w[i_1...i_m]$ for $w_{i_1}w_{i_2}...w_{i_m}$.

For any K_∞-finite $f \in I_{P_E}(4/5) := Ind_{P_E(\mathbb{A})}^{G_E(\mathbb{A})} \delta_{P_E}^{4/5}$, let $f_s \in I_{P_E}(s) = Ind_{P_E(\mathbb{A})}^{G_E(\mathbb{A})} \delta_{P_E}^{s}$ be the unique standard section extending f. When the real part of s is sufficiently large, the sum

$$E(g,f,s) = \sum_{\gamma \in P_E(F)\backslash G_E(F)} f_s(\gamma g)$$

converges absolutely and defines a $G_E(\mathbb{A})$-equivariant map $I_{P_E}(s)_K \to \mathcal{A}(G_E)_K$. Moreover, it possesses a meromorphic continuation to all $s \in \mathbb{C}$. We are interested in the analytic behaviour of $E(g,f,s)$ when $s = 4/5$. For the convenience of the reader, let us repeat the statement of Prop. 5.1(i) to be proved here:

Proposition 9.1. *For any standard section f_s, $E(g,f,s)$ has at most a double pole at $s = 4/5$. The double pole is attained by the spherical section f_s^0. The space of automorphic forms on G_E spanned by the functions $(s-4/5)^2 E(g,f_s)|_{s=4/5}$ is an irreducible square-integrable automorphic representation isomorphic to Π_E.*

To determine the poles of the Eisenstein series $E(g,f,s)$, it suffices to determine the poles of its constant term along U_E. To describe the constant term of $E(g,f,s)$ along U_E, we need to introduce yet more notations. The representation $I_{P_E}(s)$ is a submodule of

$$I_{B_E}(\chi_s) := Ind_{B_E(\mathbb{A})}^{G_E(\mathbb{A})} \delta_{B_E}^{1/2} \cdot \chi_s,$$

where $\chi_s = \delta_{B_E}^{-1/2} \cdot \delta_{P_E}^{s}$. For $w \in W$, set

$$M_w(s)(f_s)(g) = \int_{(U_{B_E} \cap w^{-1} U_{B_E} w)(\mathbb{A}) \backslash U_{B_E}(\mathbb{A})} f_s(wug)du$$

where U_{B_E} is the unipotent radical of the Borel subgroup B_E. This converges for $Re(s)$ sufficiently large and has a meromorphic continuation to all $s \in \mathbb{C}$. Moreover, it defines an intertwining operator

$$M_w(s) : I_{B_E}(\chi_s) \longrightarrow I_{B_E}(w^{-1} \cdot \chi_s).$$

By a standard computation [GRS], we obtain:

$$(9.2) \qquad \int_{U_E(F)\backslash U_E(\mathbb{A})} E(ug,f,s)\,du = \sum_{w \in \Psi} E_{L_E^w}(g, M_w(s) f_s|_{L_E}, s') \quad \text{for } g \in L_E(\mathbb{A})$$

where

- $\Psi = \{w \in W : w(\alpha_2) > 0 \text{ and } w^{-1}(\alpha_i) > 0 \text{ for } i = 1,3,4\}$ is the set of distinguished coset representatives for $P_E \backslash G_E / Q_E$;
- $L_E^w = w^{-1} P_E w \cap L_E$ is either equal to L_E or the Borel subgroup $L_E \cap B_E$ of L_E;
- $M_w(s)(f_s)|_{L_E}$ is either a character of $L_E(\mathbb{A})$ or an element of an induced representation $Ind_{L_E^w(\mathbb{A})}^{L_E(\mathbb{A})} \chi_{w,s}$ with $\chi_{w,s}|_{T_E} = \delta_{B_E}^{1/2} \cdot w^{-1}\chi_s$;
- $2s' = \langle \alpha_2^\vee, \delta_{B_E}^{1/2} \cdot w^{-1}\chi_s \rangle$;
- $E_{L_E^w}$ refers to Eisenstein series on L_E corresponding to the induced representation $Ind_{L_E^w(\mathbb{A})}^{L_E(\mathbb{A})} \chi_{w,s}$.

The following two lemmas give precise description of the summands in (9.2).

Lemma 9.3. *The set Ψ consists of the following elements of W:*

$$\begin{cases} e, w[21], w[23], w[24], w[213], w[214], w[234], w[2134], \\ w[21324], w[21423], w[23421], w[2134213], w[2134214], w[2134234], w[21342134]. \end{cases}$$

Lemma 9.4. *We have*

$$w^{-1}P_E w \cap L_E = \begin{cases} L_E \text{ if } w = w[21], w[23],\ w[24],\ w[2134213],\ w[2134214] \text{ or } w[2134234]; \\ B_E \cap L_E \text{ otherwise.} \end{cases}$$

Let $f^0 \in I_{P_E}(4/5)$ be the spherical vector normalized by $f^0(1) = 1$. Then the Gindikin-Karpelevich formula allows us to compute $M_w(s)f_s^0(1)$. Indeed, we have:

$$M_w(s)(f_s^0)(1) = \prod_{\alpha>0, w^{-1}\alpha<0} \frac{\zeta(\langle\alpha^\vee, \chi_s\rangle)}{\zeta(\langle\alpha^\vee, \chi_s\rangle + 1)}$$

where $\zeta(s)$ refers to the complete Dedekind zeta function of the number field F. The following table gives the value of $M_w(s)(f_s^0)(1)$ for each $w \in \Psi$, describes its analytic behaviour at $s = 4/5$ and also determines the value of s'.

$w \in \Psi$	$M_w(s)f_s^0(1)$	order of pole at $s = 4/5$	s'
e	1	0	2
$w[21]$ $w[23]$ $w[24]$	$\frac{\zeta(5s-2)}{\zeta(5s)}$	0	0
$w[213]$ $w[214]$ $w[234]$	$\frac{\zeta(5s-2)^2}{\zeta(5s)\zeta(5s-1)}$	0	1
$w[2134]$	$\frac{\zeta(5s-2)^3}{\zeta(5s)\zeta(5s-1)^2}$	0	2
$w[21324]$ $w[23421]$ $w[21423]$	$\frac{\zeta(5s-2)\zeta(5s-3)\zeta(10s-5)}{\zeta(5s)\zeta(5s-1)\zeta(10s-4)}$	1	3/2
$w[2134214]$ $w[2134213]$ $w[2134234]$	$\frac{\zeta(5s-2)\zeta(5s-3)^2\zeta(10s-5)}{\zeta(5s)\zeta(5s-1)^2\zeta(10s-4)}$	2	0
$w[21342134]$	$\frac{\zeta(5s-3)^3\zeta(10s-5)}{\zeta(5s)\zeta(5s-1)^2\zeta(10s-4)}$	3	1/2

From the above table, it is not difficult to determine the analytic behaviour of the term $E_{L_E^w}(g, M_w(s)f_s^0|_{L_E}, s')$ at $s = 4/5$: they have pole of order ≤ 1 at $s = 4/5$ except for the terms corresponding to the last four w's in the above table. Let us examine the remaining terms in greater detail:

- If $w = w[21342134]$, the term $M_w(s)f_s^0$ has a pole of order 3 at $s = 4/5$. However, the Eisenstein series $E_{L_E^w}$ is essentially the Eisenstein series of SL_2 at the point $s' = 1/2$ (on the unitary axis) and this has a zero of order 1. Moreover, the leading term of its Laurent expansion gives an embedding of $\delta_{Q_E}^{1/6}\sigma_E$ into $\mathcal{A}(L_E)$. Hence, the term $E_{L_E^w}(g, M_w(s)f_s^0|_{L_E}, s')$ has a pole of order 2 and $(s-4/5)^2 E_{L_E^w}(g, M_w(s)f_s^0|_{L_E}, s')|_{s=4/5}$ span a subspace of $\mathcal{A}(L_E)$ isomorphic to $\delta_{Q_E}^{1/6}\sigma_E$.
- If $w = w[2134214]$, $w[2134213]$ or $w[2134234]$, then $M_w(s)(f_s^0)$ has a pole of order 2 at $s = 4/5$ and $(s-4/5)^2 M_w(f_s^0)|_{L_E}$, evaluated at $s = 4/5$, is a character of $L_E(\mathbb{A})$ equal to $|\chi_3|^2$, $|\chi_4|^2$ and $|\chi_1|^2$ respectively. Here, recall that the χ_i's were defined in (3.5) and (3.6).

By computing central characters, one sees that the poles of order 2 from the above terms do not cancel. Hence $E(g, f^0, s)$ has a pole of order 2 at $s = 4/5$. By Prop. 4.3 and the fact that Π_E is unramified, we deduce that $E(g, f, s)$ has a pole of order ≤ 2 for any $f \in I_{P_E}(4/5)$, and as we have seen, this double pole is attained by the spherical section f_s^0.

Consider now the map

$$f \mapsto (s-4/5)^2 E(g, f, s)|_{s=4/5}.$$

This defines a non-zero $G_E(\mathbb{A})$-equivariant map

$$\theta_E : I_{P_E}(4/5) \to \mathcal{A}(G_E).$$

Claim. The image of θ_E is contained in $L^2(G_E(F)\backslash G_E(\mathbb{A})) \cap \mathcal{A}(G_E)$.

Proof of the claim: We use the square integrability criterion of Jacquet [MW, Pg. 74]. Since the image of θ_E consists of automorphic forms with cuspidal support along the Borel subgroup B_E, we need to compute the automorphic exponents of $\theta_E(v)$, for any $v \in I_{P_E}(4/5)$, along B_E and see that they are linear combinations of the simple roots with negative coefficients. For this, we need to compute the constant term of $\theta_E(v)$ along the unipotent radical of B_E. However, we have seen that the constant term of $\theta_E(v)$ along U_E is contained in the subspace of $\mathcal{A}(L_E)$ given by

$$|\chi_1|^2 \oplus |\chi_3|^2 \oplus |\chi_4|^2 \oplus \delta_{Q_E}^{1/6}\sigma_E.$$

From this, it is easy to see that the automorphic exponents of $\theta_E(v)$ are contained in the set

$$\begin{cases} \alpha_2 - 2\alpha_0 + \alpha_4, \\ \alpha_2 - 2\alpha_0 + \alpha_3, \\ \alpha_2 - 2\alpha_0 + \alpha_1, \\ \alpha_2 - 2\alpha_0, \end{cases}$$

where we recall that $\alpha_0 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ is the highest root. Since all the coefficients of simple roots are negative, the image of θ_E is contained in $L^2(G_E(F)\backslash G_E(\mathbb{A}))$ as claimed.

The above claim implies that the image of θ_E is a semisimple representation. By Prop. 4.3, $I_{P_E}(4/5)$ has Π_E as the unique irreducible quotient. It thus follows that the image of θ_E

is isomorphic to Π_E. This proves Prop. 5.1(i) completely and in the process we have also demonstrated Prop. 5.2(i), namely that

$$Im(\theta_E^{U_E}) = \delta_{Q_E}^{1/6}\sigma_E \oplus |\chi_1|^2 \oplus |\chi_3|^2 \oplus |\chi_4|^2.$$

Finally, we shall prove Prop. 5.3(ii), namely:

Proposition 9.5. *As a representation of $M_E(\mathbb{A})$, $\theta_E^{N_E}(\Pi_E) \subset \mathcal{A}(M_E)$ is given by the exact sequence:*

$$0 \longrightarrow W \longrightarrow \theta_E^{N_E}(\Pi_E) \longrightarrow \delta_{P_E}^{1/5} \longrightarrow 0.$$

Here, W is defined as follows. Let V be the representation of $\mathrm{GL}_2(\mathbb{A})$ *defined by:*

$$0 \longrightarrow V \longrightarrow \sigma(|-|^{1/2}, |-|^{-1/2}) \longrightarrow \mathbf{1} \longrightarrow 0,$$

Then there is an exact sequence

$$0 \longrightarrow \delta_{P_E}^{1/5} \longrightarrow W \longrightarrow \delta_{P_E}^{1/5} \cdot ((V \otimes \mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes V \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1} \otimes V)) \longrightarrow 0,$$

where $V \otimes \mathbf{1} \otimes \mathbf{1}$ is regarded as a representation of $M_E(\mathbb{A})$ via the first projection $pr_1 : M_E \to \mathrm{GL}_2$ (as in Prop. 4.4(ii)) and so on. Moreover, W is realized as a subspace of $\mathcal{A}(M_E)$ as follows. Let $Eis : V \hookrightarrow \mathcal{A}(\mathrm{GL}_2)$ denotes the embedding of vector spaces (not $\mathrm{GL}_2(\mathbb{A})$-equivariant) furnished by the theory of Eisenstein series (cf. Prop. 5.3(ii)). Then W is spanned as a vector subspace of $\mathcal{A}(M_E)$ by the constant functions on $M_E(\mathbb{A})$ and the subspaces $pr_1^*(Eis(V)) \oplus pr_2^*(Eis(V)) \oplus pr_3^*(Eis(V))$.

Here, recall that

$$M_E \cong \{(g_1, g_2, g_3) \in \mathrm{GL}_2^3 : \det(g_1) = \det(g_2) = \det(g_3)\}$$

and pr_1, pr_2 and pr_3 are the three projections $M_E \to \mathrm{GL}_2$. For any subgroup $H \subset \mathrm{GL}_2^3$, we shall write H_0 for $H \cap M_E$. Moreover, we shall let B be a Borel subgroup of GL_2.

To determine $\theta^{N_E}(\Pi_E)$, we investigate the constant term

$$\int_{N_E(F)\backslash N_E(\mathbb{A})} E(g, f, s) = \sum_{w \in \Psi} E_{M_E^w}(g, M_w(s) f_s, s'), \tag{9.6}$$

of the Eisenstein series $E(g, f, s)$ along N_E. Here

$$\Psi = \{w \in W : \; w \text{ and } w^{-1} \text{ both send } \alpha_1, \alpha_3 \text{ and } \alpha_4 \text{ to positive roots}\},$$

$s' = (s_1', s_3', s_4')$ is the triple of numbers defined by $2s_i' = \langle \alpha_i^\vee, \delta_{B_E}^{1/2} \cdot w^{-1}\chi_s \rangle$ and the other notations are similar to those in (9.2).

The following two lemmas describe explicitly all summands in the last formula.

Lemma 9.7. *We have*

$$\Psi = \{e, w[2], w[2132], w[2142], w[2342], w[21342], w[213242132]\}.$$

Lemma 9.8. *We have*

$$w^{-1}P_E w\cap M_E=\begin{cases} M_E, \text{ if } w=e \text{ or } w[213242132];\\ (B\times B\times B)_0, \text{ if } w=w[2] \text{ or } w[21342];\\ (GL_2\times GL_2\times B)_0, \text{ if } w=w[2132];\\ (GL_2\times B\times GL_2)_0, \text{ if } w=w[2142];\\ (B\times GL_2\times GL_2)_0, \text{ if } w=w[2342].\end{cases}$$

Taking the normalized spherical section $f_s^0\in I_{P_E}(s)$ and using the Gindikin-Karpelevich formula, we list in the following table the analytic behaviour of $M_w(s)f_s^0(1)$ at $s=4/5$ and give the value of s'.

w	$M_w(s)f_s^0(1)$	order of pole at $s=4/5$	s'
e	1	0	$(0,0,0)$
$w[2]$	$\frac{\zeta(5s-1)}{\zeta(5s)}$	0	$(3/2,3/2,3/2)$
$w[2132]$ $w[2142]$ $w[2342]$	$\frac{\zeta(5s-2)\zeta(5s-3)}{\zeta(5s)\zeta(5s-1)}$	1	$(0,0,2)$ $(0,2,0)$ $(2,0,0)$
$w[21342]$	$\frac{\zeta(5s-2)^3\zeta(10s-5)}{\zeta(5s)\zeta(5s-1)^2\zeta(10s-4)}$	0	$(1,1,1)$
$w[213421342]$	$\frac{\zeta(5s-3)^2\zeta(5s-4)\zeta(10s-5)}{\zeta(5s)\zeta(5s-1)^2\zeta(10s-4)}$	3	$(0,0,0)$

From the table, we conclude that for each $w\in\Psi$, the term $E_{M_E^w}(g,M_w(s)f_s,s')$ has a pole of order ≤ 3 at $s=4/5$. For each $w\in\Psi$, let us write:

$$E_{M_E^w}(g,M_w(s)f_s,s')=\frac{b_{-3}^w(f,g)}{(s-4/5)^3}+\frac{b_{-2}^w(f,g)}{(s-4/5)^2}+\dots.$$

From the table again, we see that b_{-3}^w and b_{-2}^w are both zero unless

$$w=w':=w[21342] \quad\text{or}\quad w=w_0:=w[213421342],$$

in which case the pole of order 3 is attained by the spherical section f_s^0. Indeed, when $w=w_0$, $M_{w_0}(s)f_s^0|_{M_E}$ has a pole of order 3 as shown in the table and when $w=w'$, $E_{M_E^w}(g,M_w(s)f_s,s')$ is essentially the Eisenstein series of $\mathrm{SL}_2\times\mathrm{SL}_2\times\mathrm{SL}_2$ attached to the induced representation

$$Ind_{B(\mathbb{A})}^{\mathrm{SL}_2(\mathbb{A})}\delta_B\otimes Ind_{B(\mathbb{A})}^{\mathrm{SL}_2(\mathbb{A})}\delta_B\otimes Ind_{B(\mathbb{A})}^{\mathrm{SL}_2(\mathbb{A})}\delta_B.$$

Of course, we already know that $E(g,f,s)$ has a pole of order at most 2, so that necessarily

$$b_{-3}^{w'}(f,g)+b_{-3}^{w_0}(f,g)=0.$$

To describe b_{-2}^w, we need to introduce more notations.

As in the statement of the propostion, let V be the $\mathrm{GL}_2(\mathbb{A})$ representation defined by the exact sequence

$$0\longrightarrow V\longrightarrow Ind_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\delta_B\longrightarrow \mathbf{1}\longrightarrow 0.$$

For $\varphi \in Ind_{B(\mathbb{A})}^{GL_2(\mathbb{A})}\delta_B$, the corresponding Eisenstein series $E(g,\varphi,s)$ has a pole at $s=1$ if and only if $\varphi \notin V$. Consider the Laurent expansion

$$E(g,\varphi,s) = \frac{a_{-1}(\varphi,g)}{(s-1)} + a_0(\varphi,g) + ...$$

about $s=1$. Then the map

$$a_{-1} : Ind_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\delta_B \longrightarrow \mathcal{A}(\mathrm{GL}_2)$$

is $\mathrm{GL}_2(\mathbb{A})$-equivariant and

$$\begin{cases} Im(a_{-1}) = \{\text{the constant functions}\}, \\ Ker(a_{-1}) = V. \end{cases}$$

On the other hand,

$$a_0 : Ind_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\delta_B \longrightarrow \mathcal{A}(\mathrm{GL}_2)$$

is an embedding of vector spaces which is not $\mathrm{GL}_2(\mathbb{A})$-equivariant. However, the composite map

$$a_0 : Ind_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\delta_B \longrightarrow \mathcal{A}(\mathrm{GL}_2) \longrightarrow \mathcal{A}(\mathrm{GL}_2)/Im(a_{-1})$$

is $\mathrm{GL}_2(\mathbb{A})$-equivariant. Hence, the subspace

$$Im(a_{-1}) \oplus Im(a_0) \subset \mathcal{A}(\mathrm{GL}_2)$$

supports a representation of $\mathrm{GL}_2(\mathbb{A})$ with unique irreducible quotient isomorphic to $\mathbf{1}$, unique irreducible submodule isomorphic to $\mathbf{1}$ and with V as a subquotient.

Let $\varphi_0 \in Ind_{B(\mathbb{A})}^{GL_2(\mathbb{A})}\delta_B$ be the normalized spherical vector. Using the above information and consulting the last table, it is not difficult to see that

$$\{b_{-2}^{w_0}(f,g) : f \in I_{P_E}(4/5)\} = \mathbb{C} \cdot \delta_{P_E}^{1/5} \cdot pr_i^*(a_{-1}(\varphi_0)) = \mathbb{C} \cdot \delta_{P_E}^{1/5}.$$

On the other hand,

$$\begin{aligned}\{b_{-2}^{w'}(f,g) : f \in I_{P_E}(4/5)\} =& \delta_{P_E}^{1/5} \cdot (pr_1^*(a_0(V)) \oplus pr_2^*(a_0(V)) \oplus pr_3^*(a_0(V))) \\ & \oplus \mathbb{C} \cdot \delta_{P_E}^{1/5} \cdot (pr_1^*(a_0(\varphi_0)) + pr_2^*(a_0(\varphi_0)) + pr_3^*(a_0(\varphi_0))).\end{aligned}$$

From this, Proposition 5.2(i) follows easily.

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Mathematics Department, Princeton University, Princeton, NJ 08544, U.S.A.

Mathematics Department, Princeton University, Princeton, NJ 08544, U.S.A.

Mathematics Department, University of Minnesota, Minneapolis MN 55455, U.S.A.
E-mail address: wtgan@math.princeton.edu
E-mail address: gurevich@math.princeton.edu
E-mail address: dhjiang@math.umn.edu