Automorphic Forms and Automorphic Representations

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References:

- A. Borel, *Automorphic forms on* $SL_2(\mathbf{R})$

- D. Bump, *Automorphic forms and representations*

- S. Gelbart, *Automorphic forms on adele groups*

Introduction

About ninety years ago, Ramanujan considered the following power series of q

$$\Delta(q) = q \cdot \prod_{n \geq 1} (1 - q^n)^{24}.$$

Expanding this out formally, we have:

$$\begin{aligned} \Delta(q) &= \sum_{n>0} \tau(n) q^n \\ &= q - 24q^2 + ... \end{aligned}$$

Ramanujan made a number of conjectures about the coefficients $\tau(n)$. These conjectures have turned out to be very influential. They say:

- τ is multiplicative, i.e. if m and n are relatively prime, then

$$\tau(mn) = \tau(m) \cdot \tau(n)$$

Moreover,

$$\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1}).$$

- For all primes p, $|\tau(p)| \leq 2 \cdot p^{11/2}$. This implies that for any n,

$$|\tau(n)| \leq C_\epsilon \cdot n^{11/2+\epsilon}$$

for any ϵ.

The first conjecture was proved by Mordell (around 1920), while the second by Deligne (around 1970).

These conjectures led to the theory of modular forms. We shall begin with a brief description of the basic results in this theory, and then give a reformulation using representation theory. This reformulation leads to a vast generalization of the theory.

<u>Contents:</u>

- Classical modular forms
- Automorphic forms on real groups
- Automorphic representations of adele groups
- Eisenstein series
- Multiplicity One for $GL(n)$

Classical Modular Forms

Upper Half Plane

Let

$$\mathbf{H} = \{x + iy : x \in \mathbf{R}, y > 0\}$$

be the upper half plane. It is a homogeneous space for $SL_2(\mathbf{R})$ under the action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

In fact, this defines an action of $GL_2(\mathbf{R})^+$ on $\mathbf{H}$ with center acting trivially. Moreover, it extends to an action on $\mathbf{H}^* = \mathbf{H} \cup \mathbf{R} \cup \{\infty\}$. The action on $\mathbf{R} \cup \{\infty\}$ is also transitive. The stabilizer of ∞ is the Borel subgroup B of upper triangular matrices.

The stabilizer of $i = \sqrt{-1}$ is

$$K = SO(2) = \{g \in SL_2(\mathbf{R}) : g^t g = 1\}$$

which is a maximal compact subgroup of $SL_2(\mathbf{R})$. So we have:

$$\mathbf{H} \cong SL_2(\mathbf{R})/K.$$

There is an $SL_2(\mathbf{R})$-invariant measure on $\mathbf{H}$, namely

$$\frac{dx\,dy}{y^2}.$$

This is invariant under $SL_2(\mathbf{R})$ because

$$Im(gz) = \frac{Im(z)}{|cz+d|^2}.$$

Standard notations

- $B =$ the Borel subgroup of upper triangular matrices

- $N =$ the unipotent radical of $B = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$

- $T =$ the group of diagonal matrices

- $A = \{\begin{pmatrix} a & 0 & \\ 0 & a^{-1} & : a > 0 \end{pmatrix}\}$

- $K = SO(2)$, the maximal compact.

Arithmetic Subgroups

If Γ is an arithmetic subgroup of $SL_2(\mathbf{R})$, then Γ acts on $\mathbf{H}$ in a properly discontinuous fashion. The quotient $\Gamma\backslash\mathbf{H} = \mathbf{\Gamma\backslash SL_2(R)/K}$ possesses a fundamental domain $\mathcal{F}$ and has finite volume (since $\Gamma\backslash SL_2(\mathbf{R})$ has finite volume).

An example of Γ is $SL_2(\mathbf{Z})$. A fundamental domain for the discrete subgroup $SL_2(\mathbf{Z})$ is:

$$\mathcal{F} = \{z = x + iy : |x| <\leq 1/2, |z| \geq 1\}$$

It looks like:

Cusps

As a Riemann surface, $\mathcal{F}$ is a punctured sphere. It has a natural compactification: by adding the point $i\infty$. This extra point is called the **cusp** at infinity.

More formally, a point $x \in \mathbf{R} \cup \{\infty\}$ is cuspidal for Γ if the stabilizer of x in Γ contains non-trivial unipotent elements. For the purpose of this lecture, our Γ is always contained in $SL_2(\mathbf{Q})$, in which case the cuspidal points are simply $\mathbf{Q} \cup \{\infty\}$.

A **cusp of** Γ is a Γ-orbit in $\mathbf{Q} \cup \{\infty\}$. Because $SL_2(\mathbf{Z})$ acts transtively on $\mathbf{Q} \cup \infty = SL_2(\mathbf{Q})/B(\mathbf{Q})$, there is one cusp when $\Gamma = SL_2(\mathbf{Z})$.

More generally, the number of cusps of Γ is $\#\Gamma\backslash SL_2(\mathbf{Q})/B(\mathbf{Q})$ (which is finite), and $\Gamma\backslash\mathbf{H}$ can be compactified by adding these cusps:

$$\overline{\Gamma\backslash\mathbf{H}} = \Gamma\backslash(\mathbf{H} \cup \mathbf{Q} \cup \{\infty\}).$$

Siegel sets

It is often useful to have a set which covers the fundamental domain, but is easier to describe. One such example is the **Siegel set** associated to the cusp $i\infty$:

$$\mathfrak{S}_{c,d} = \{x + iy : |x| < c, y > d\}$$

For $\Gamma = SL_2(\mathbf{Z})$, if c is large enough and d is small enough, this set will cover $\mathcal{F}$. The volume of $\mathfrak{S}_{c,d}$ (with respect to the invariant measure) is easily computed and seen to be finite. This shows that $\mathcal{F}$ has finite volume.

Hecke congruence subgroups

Another example of arithmetic group is the Hecke congruence subgroup:

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod N\}.$$

Here the fundamental domain may have more than one cusp. For example, when $N = 2$, the fundamental domain, as shown below, has 3 cusps.

In this case, the fundamental domain cannot be covered by a single Siegel set. One needs 3 Siegel sets, one for each cusp.

In the following, Γ shall mean $\Gamma_0(N)$ for some N.

Modular Forms

Definition: A holomorphic modular form for Γ is a holomorphic function f on $\mathbf{H}$ satisfying some extra properties:

- (automorphy) for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

  $$f(\gamma z) = (cz + d)^k \cdot f(z),$$

  where k is a positive integer.

- (holomorphy) f is holomorphic at every cusp of Γ.

One motivation for studying these type of functions comes from the theory of elliptic curves, but we will not go into this here.

Notations: For $g \in GL_2(\mathbf{R})^+$ and $z \in \mathbf{H}$, set

$$j(g,z) = (cz+d) \cdot det(g)^{-1/2}$$

$$(f|_k g)(z) = j(g,z)^{-k} \cdot f(gz).$$

Then the automorphy condition can be expressed as:

$$f|_k \gamma = f$$

for any $\gamma \in \Gamma$.

The integer k is called the **weight** of f, whereas if $\Gamma = \Gamma_0(N)$, then N is the **level** of f.

Observe that f is necessarily 0 if k is odd (because $-1 \in \Gamma$).

Next we want to explain more precisely the holomorphy condition.

Fourier Expansion

Because the element

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

lies in Γ, we have:

$$f(z+1) = f(z).$$

Thus f is really a function on the strip

$$\{x+iy : -1/2 \le x < 1/2, y > 0\}.$$

The map $z \mapsto q = e^{2\pi i z}$ sends this strip onto the punctured open unit disc, and sends the cusp $i\infty$ to 0.

The function f gives rise to a holomorphic function $\tilde{f}(q)$ on the punctured disc, and $\tilde{f}(q)$ has a Laurent expansion about 0:

$$\tilde{f}(q) = \sum_n a_n q^n.$$

By "f is holomorphic at the cusp $i\infty$", we mean that the singularity at 0 is removable, so that $a_n = 0$ if $n < 0$.

Thus we can expand f as a Fourier series:

$$f(z) = \sum_{n \geq 0} a_n(f) e^{2\pi i n z}.$$

The numbers $\{a_n(f)\}$ are the Fourier coefficients of f (at the cusp $i\infty$).

One has analogous Fourier expansion at every cusp of Γ. More precisely, any cusp $a \in \mathbf{Q}$ can be moved to ∞ by an element γ of $SL_2(\mathbf{Z})$, and one can consider the Fourier expansion of $f|_k\gamma$ at $i\infty$ as above.

CUSP FORMS: f is a cusp form (or is cuspidal) if f vanishes at every cusp.

Thus f is cuspidal iff the zeroth Fourier coefficient $a_0(f)$ in the Fourier expansion of f at every cusp is zero.

Moderate growth and rapid decrease

In the presence of holomorphy of f on $\mathbf{H}$, the holomorphy condition at a cusp is implied by a weaker assumption, namely that of moderate growth.

We say that f is of **moderate growth** at the cusp $i\infty$ if there exists n such that

$$|f(x+iy)| \leq C \cdot y^n$$

as $y \to \infty$ with z in a Siegel set for $i\infty$ (i.e. as $y \to \infty$ with x bounded).

Say that f is **rapidly decreasing** at $i\infty$ if, for any k, there exists C_k such that

$$|f(x+iy)| \leq C_k y^{-k}$$

as $y \to \infty$ in a Siegel set for $i\infty$.

Because $|e^{2\pi ikz}| = e^{-2\pi ky}$, we see that if f is holomorphic on $\mathbf{H}$ and satisfies automorphy condition, then

- f is of moderate growth at $i\infty$ iff f is holomorphic at $i\infty$.

- f is rapidly decreasing at $i\infty$ iff f vanishes at $i\infty$.

Natural Question: Suppose we are given a q-series $f(z) = \sum_{n \geq 0} a_n q^n$ which converges for $z \in \mathbf{H}$. Under what conditions can we conclude that $f(z)$ is a modular form, with respect to Γ? Such a result is called a **converse theorem**. We will come to this type of result later.

Remarks: We know that Γ is finitely generated. For example, $SL_2(\mathbf{Z})$ is generated by the following two elements:

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad\qquad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So to check that a given Fourier series is modular with respect to Γ, it suffices to check the transformation property for w. But this is not apparent from the Fourier series at all!

Finite Dimensionality

Let $M_k(N)$ denote the space of modular forms of weight k and level N, and let $S_k(N)$ be the subspace of cusp forms. These spaces are finite-dimensional. For example:

$$dim M_k(1) = \begin{cases} \frac{k}{12} + 1, \text{ if } k \neq 2 \mod 12, \\ \frac{k}{12}, \text{ if } k = 2 \mod 12. \end{cases}$$

Ring structure

Observe that if $f_i \in M_{k_i}(N)$, then $f_1 \cdot f_2 \in M_{k_1+k_2}(N)$. Thus $\bigoplus_k M_k(N)$ has a ring structure. Moreover, if one of f_i's is cuspidal, so is $f_1 \cdot f_2$.

Examples

(i) **Eisenstein series E_{2k}.**

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{m,n\in\mathbf{Z},(m,n)\neq(0,0)} (mz+n)^{-2k}.$$

This converges absolutely for all $z \in \mathbf{H}$ provided that $k \geq 2$. We claim that this is a modular form of weight $2k$ and level 1. Given $g \in SL_2(\mathbf{Z})$, we have:

$$(E_{2k}|_{2k}g)(z) = (cz+d)^{-2k} \cdot \sum \left(m(\frac{az+b}{cz+d}) + n \right)^{-2k}$$

$$= \sum \left((ma+nc)z + (mb+nd)\right)^{-2k}.$$

Since $g \in SL_2(\mathbf{Z})$, $(m,n) \mapsto (ma+nc, mb+nd)$ is a bijection of $\mathbf{Z}^2$ onto itself. Thus the last expression is equal to E_{2k}. This shows the automorphy condition.

For the holomorphy at $i\infty$, we need to find the Fourier expansion of E_{2k}. It turns out that

$$E_{2k}(z) = 1 + \frac{(-1)^k 4k}{B_{2k}} \cdot \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

where B_{2k} is the $2k$-th Bernoulli number and $\sigma_{2k-1}(n)$ is the sum of the $(2k-1)$-powers of the divisors of n.

As a corollary, we see that for $k > 2$ even,

$$M_k(1) = \mathbf{C} \cdot E_k \oplus S_k(1)$$

$$dim M_k(1) = dim S_k(1) + 1.$$

So $\dim S_k(1) = 0$ if $k < 12$ and $\dim S_{12}(1) = 1$.

(ii) **Ramanujan Δ function.** The easiest way to construct cusp forms is to use linear combinations of Eisenstein series. For example, let us set

$$f(z) = E_4(z)^3 - E_6(z)^2.$$

One computes to see that its Fourier expansion looks like:

$$f(z) = \frac{1}{1728}(q + 24q^2 + ...);$$

so it is indeed a cusp form of weight 12 and level 1.

It turns out that this cusp form is precisely equal to a multiple of the Ramanujan Δ-function:

$$q \prod_{n \geq 1} (1 - q^n)^{24} = q + 24q^2 + ...$$

Ramanujan showed that this is an element of $S_{12}(1)$. Because this latter space has dimension 1, and so this is equal to our definition given above.

(iii) **Theta functions.** Another source of modular forms is the so-called theta functions. Let $A = (a_{ij})$ be a positive definite symmetric matrix with integer entries. Assume for simplicity that A has determinant 1 and A is even, i.e. $x^t A x$ is even for all $x \in \mathbf{Z}^k$. Define a quadratic form by

$$Q(x) = \frac{1}{2} x^t A x$$

and consider the series

$$\theta_Q(z) = \sum_{x \in \mathbf{Z}^k} q^{Q(x)} =$$

$$= \sum_{n \geq 0} a_n(Q) q^n$$

where

$$a_n(Q) = \#\{x \in \mathbf{Z}^k : Q(x) = n\}.$$

Then $\theta_Q(z)$ is a modular form of weight $k/2$ and level 1. This is a consequence of the Poisson summation formula.

Bounds on Fourier Coefficients

We have the following trivial bound of Hecke.

Proposition: If $f(z) = \sum_{n>0} a_n(f) e^{2\pi i n z}$ is a cusp form of weight k, then

$$|a_n(f)| \le C_f \cdot n^{k/2}.$$

Proof: Consider the function $F(z) = y^{k/2}|f(z)|$, if $z = x + iy$. Then $F(\gamma z) = F(z)$ for any $\gamma \in \Gamma$. Thus, F is a continuous function on $\Gamma \backslash \mathbf{H}$. Since f is cuspidal, $F(z)$ tends to 0 as z approaches the cusps. Thus, F is bounded and we have:

$$|f(x+iy)| \le C_f \cdot y^{-k/2}.$$

Now, for fixed y,

$$|a_n(f) \cdot e^{2\pi n y}| = |\int_0^1 e^{-2\pi i n x} \cdot f(z)\, dx| \le$$

$$\leq \int_0^1 |f(x+iy)|\, dx.$$

Thus

$$|a_n(f)| \leq C_f e^{-2\pi n y} y^{-k/2}$$

and putting $y = 1/n$ gives the result.

Ramanujan-Petersson Conjecture: Let f be a cuspidal Hecke eigenform form of weight k. Then for all primes p,

$$|a_p(f)| \leq 2 \cdot p^{\frac{k-1}{2}}.$$

This implies that

$$|a_n(f)| \leq C_\epsilon \cdot n^{\frac{k-1}{2}+\epsilon}$$

for any $\epsilon > 0$.

This conjecture has been proved by Deligne in 1971 as a consequence of his proof of the Weil conjectures. This bound on Fourier coefficients have many number theoretic applications.

Petersson Inner Product

The space $S_k(N)$ comes equipped with a natural inner product:

$$\langle f_1, f_2 \rangle_k = \int_{\Gamma \backslash \mathbf{H}} f_1(z) \overline{f_2(z)} y^k \cdot \frac{dx\, dy}{y^2}.$$

This is convergent because $y^{k/2} f_i(z)$ tends to zero at the cusps. It remains convergent as long as one of the functions is cuspidal. Thus it makes sense to take the inner product of a cusp form with any modular form.

L-functions and Hecke theory

Given a cusp form of weight k,

$$f(z) = \sum_{n \geq 0} a_n(f) e^{2\pi i n z},$$

one may consider the Dirichlet series:

$$L(f,s) = \sum_{n \geq 1} \frac{a_n(f)}{n^s}.$$

This converges for $Re(s) > \frac{k}{2} + 1$, because $|a_n(f)| = O(n^{k/2})$. It is related to f by a Mellin transform. Indeed,

$$\int_0^\infty f(iy) \cdot y^s \cdot \frac{dy}{y}$$

$$= \int_0^\infty \sum_{n \geq 1} a_n(f) e^{-2\pi n y} y^s \cdot \frac{dy}{y}$$

$$= \sum_n a_n(f) \cdot \int_0^\infty e^{-t} (2\pi n)^{-s} t^s \cdot \frac{dt}{t} \quad (t = 2\pi n y)$$

$$= (2\pi)^{-s}\Gamma(s) \cdot \sum_n \frac{a_n(f)}{n^s}$$

where

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \cdot \frac{dt}{t}$$

is the so-called Gamma function, which satisfies $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n+1) = n!$.

Here is an important theorem about the L-functions attached to cusp forms (for simplicity, we state it for level 1 forms only):

Theorem: Let f be a cusp form of weight k of level 1. Then we have:

(i) (analytic continuation) $L(s, f)$ extends to an entire function on $\mathbf{C}$.

(ii) (functional equation) The function $\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$ satisfies the functional equation

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f).$$

(iii) (boundedness in vertical strips) $\Lambda(s, f)$ is bounded in vertical strips.

proof: When $Re(s)$ is large, we have:

$$\Lambda(s, f) = \int_0^\infty f(iy) \cdot y^s \cdot \frac{dy}{y}.$$

But the RHS is convergent for all s (and thus gives (i)). This is because:

- $f(iy)$ is exponentially decreasing in y as $y \to \infty$, since f is cuspidal.

- since f is modular with respect to w,

  $$f(iy) = (-1)^{k/2} y^{-k} f(i/y).$$

  So as $y \to 0$, $f(iy) \to 0$ faster than any power of y.

To see (ii), note that

$$
\begin{aligned}
&\int_0^\infty f(iy) \cdot y^s \cdot \frac{dy}{y} \\
=&\int_0^\infty (-1)^{k/2} y^{-k} f(i/y)\, y^s \,\frac{dy}{y} \\
=&(-1)^{k/2} \cdot \int_0^\infty f(it)\, t^{k-s} \cdot \frac{dy}{y} \qquad\qquad (t = 1/y).
\end{aligned}
$$

This proves (ii), and (iii) is clear.

The L-function of f turns out to be a very important invariant of f.

Hecke Operators

The theory of Hecke operators explains why the Fourier coefficients of certain modular forms are multiplicative functions. Let us assume that $\Gamma = SL_2(\mathbf{Z})$ for simplicity. For each positive integer n, we are going to define a Hecke operator T_n which is a linear operator on M_k preserving S_k.

A general construction: Let $g \in GL_2^+(\mathbf{Q})$ and write the double coset $\Gamma g \Gamma$ as a union of single cosets:

$$\Gamma g \Gamma = \bigcup_i \Gamma a_i.$$

Here the union is over a finite indexing set and the a_i's are in $GL_2^+(\mathbf{Q})$. We then set

$$f|_k[g] = \sum_i f|_k a_i.$$

This defines an operator $M_k(\Gamma) \to M_k(\Gamma)$. It is independent of the choice of the a_i's.

Hecke operator T_n: Now let $M(n)$ be the integral 2×2 matrices with determinant n By the theory of elementary divisors,

$$M(n) = \bigcup_{d|a, ad=n} \Gamma \, t(a,d) \, \Gamma$$

where

$$t(a,d) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

We set

$$f|T_n = n^{k/2-1} \sum_{d|a, ad=n} f|_k[t(a,d)].$$

For example, T_p is simply the operator defined by the double coset $t(p, 1)$. More explicitly, because

$$\begin{aligned} M(p) &= \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma \\ &= \bigcup_{k=0}^{p-1} \Gamma \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \cup \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

we have:

$$(f|T_p)(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{k=0}^{p-1} f(\frac{z+k}{p}).$$

Proposition:

- (Effects on Fourier coefficients) we have:

  $$a_n(T_p f) = a_{pn}(f) + p^{k-1} a_{n/p}(f)$$

  where the second summand is interpreted to be 0 if p does not divide n. More complicated formulas exist for T_n.

- T_n preserves S_k.

- T_n is self-adjoint with respect to the Petersson inner product:

  $$\langle f_1, T_n f_2 \rangle = \langle T_n f_1, f_2 \rangle.$$

- if $(n,m) = 1$, then $T_n T_m = T_{nm} = T_{mn}$. Moreover,

  $$T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}.$$

Thus we see that the linear span of the T_n's form an algebra and this algebra is generated by T_p's with p prime. Moreover, this algebra is commutative.

As consequences of the above properties, we have:

Corollary:

- The action of $\{T_n\}$ on S_k can be simultaneously diagonalized.

- If f is a cuspidal Hecke eigenform with eigenvalues λ_n for T_n, then

  $$a_n(f) = \lambda_n \cdot a_1(f).$$

  Thus if f is non-zero, then $a_1(f) \neq 0$, and we can normalize it by scaling to make $a_1(f) = 1$.

- **Multiplicity One Theorem**: If f is a normalized cuspidal eigenform, then f is completely determined by its Hecke eigenvalues.

- If f is a normalized cuspidal eigenform, then the Fourier coefficients of f are multiplicative, and satisfy:

$$a_p a_{p^r} = a_{p^{r+1}} + p^{k-1} a_{p^{r-1}}.$$

Euler products

The fact that the Fourier coefficients of f (a normalized cuspidal eigenform) are multiplicative implies that $L(f,s)$ has an Euler product (when $Re(s)$ is large):

$$L(f,s) = \prod_p (\sum_k a_{p^k} p^{-ks})$$

$$= \prod_p \frac{1}{1 - a_p(f)p^{-s} + p^{k-1-2s}}.$$

For example, Δ is a normalized cuspidal Hecke eigenform, and this explains the first of Ramanujan's conjectures on the coeffcients $\tau(n)$.

We say that an L-function defined by a Dirichlet series in some right half plane is **nice** if it has **analytic continuation, satisfies appropriate functional equation, is bounded in vertical strips and possesses an Euler product**. Thus $L(s,f)$ is nice if f is a cuspidal Hecke eigenform.

Hecke operators for $\Gamma_0(N)$

The above theory of Hecke works beautifully for modular forms with respect to $SL_2(\mathbf{Z})$, but there are complications when $\Gamma = \Gamma_0(N)$ ($N > 1$). We can still define the operators T_n as before, the algebra is still commutative and generated by all the T_p 's. But T_n is self-adjoint only if $(n, N) = 1$. So we can only simultaneously diagonalize the actions of T_n with $(n, N) = 1$.

If f happens to be an eigenfunction for all T_n's, then $L(f, s)$ will still have an Euler product as before. But now, we have no guarantee that $S_k(N)$ has a basis of this type.

This shows that the theory of modular forms can be quite sensitive to the group Γ.

Old and new forms

Another complication is that the eigenvalues of the T_p's, with $(p, N) = 1$, do not separate the forms.

Example: $\Delta(z)$ and $\Delta(2z)$ are both elements of $S_{12}(2)$, but have same eigenvalues for T_n for all odd n.

This is not so surprising since both these functions are built out of a single function Δ. They are so-called old forms:

Definition: Suppose that $m \cdot n$ divides N, then for $f \in M_k(m)$, the function $f(nz)$ is an element of $M_k(mn)$ and thus of $M_k(N)$. The subspace of $M_k(N)$ spanned by elements of this type is the space of **old forms**. Its

orthogonal complement in $S_k(N)$ is the space of **new forms**.

This definition is due to Atkin-Lehner.

Results of Atkin-Lehner

What Atkin and Lehner showed is that when one restricts to the space of new forms, then the neat results for $SL_2(\mathbf{Z})$ are restored. Namely,

- the action of ALL T_n's can be simultaneously diagonalized, so that the L-function of a cuspidal Hecke eigen-newform has an Euler product.

- we have the **multiplicity one theorem**: the newforms can be distinguished from one another by their eigenvalues with respect to the T_p's with $(p, N) = 1$.

Converse theorems.

We have seen that the L-function of a cuspidal Hecke eigen-newform is nice. Hecke showed the following converse:

Theorem: Suppose that $|a_n| = O(n^r)$ for some r so that $L(s) = \sum_{n>0} a_n n^{-s}$ converges absolutely when $Re(s)$ is large. If $L(s)$ has analytic continuation, is bounded in vertical strips and satisfies the functional equation

$$\Lambda(s) = (-1)^{k/2}\Lambda(k-s)$$

then $f(z) = \sum_{n>0} a_n q^n$ is a cusp form of weight k and level 1. If $L(f,s)$ has Euler product, then f is a Hecke eigenform.

The role of the functional equation is that it allows us to deduce automorphy with respect to w.

How does one characterize the cusp forms of level N? Weil proved an analog of the above theorem, but with a crucial "twist". More precisely, instead of just requiring $L(s)$ to satisfy a single functional equation, his theorem requires functional equat

ions for twists of $L(s)$ by various Dirichlet characters χ, i.e. for

$$L_\chi(s) = \sum_n \frac{a_n \cdot \chi(n)}{n^s}.$$

The reason why this extra functional equations are needed is because $\Gamma_0(N)$ usually have more than 2 generators! For the precise statement of the theorem and a proof, see [Bump, Thm. 1.5.1].

Maass Forms

Maass introduced certain analogs of modular forms which are not holomorphic. More precisely, one says that a function f on $\mathbf{H}$ is a Maass form with respect to Γ with paramater s if

- f is smooth;
- $f(\gamma g) = f(g)$ for any $\gamma \in \Gamma$;
- f is of moderate growth at the cusps of Γ;
- f is an eigenfunction for the hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

with eigenvalue $\frac{1}{4} - s^2$.

From this, one can show that f is in fact real-analytic.

Say that f is cuspidal if it vanishes at the cusps of Γ. Then f is rapidly decreasing at the cusps and thus is bounded on $\Gamma\backslash\mathbf{H}$. In particular, a Maass cusp form belongs to $L^2(\mathbf{H})$.

Nonholomorphic Eisenstein series

An example of a Maass form is a nonholomorphic Eisenstein series, constructed as follows. Consider the function

$$\phi(z) = y^{\frac{1}{2}+s}.$$

This satisfies:

$$\Delta\phi = \left(\frac{1}{4} - s^2\right)\phi.$$

Moreover, because

$$\phi(\gamma z) = \frac{\phi(z)}{|cz+d|^{2s+1}}$$

we see that $\phi(\gamma z) = \phi(z)$ if $\gamma \in N(\mathbf{Z})$. If we set

$$E(z,s) = \sum_{\gamma \in N(\mathbf{Z})\backslash SL_2(\mathbf{Z})} \phi(\gamma z),$$

then this converges absolutely if $Re(s) > 1/2$. In that case, $E(z,s)$ is $SL_2(\mathbf{Z})$-invariant and $\Delta E(z,s) = (1/2 - s^2)E(z,s)$. Note that $E(z,s)$ is not cuspidal.

Fourier expansion

Assume that $\Gamma = SL_2(\mathbf{Z})$ for simplicity. Because $f(z+1) = f(z)$, we also have a Fourier expansion of f at the cusp $i\infty$. But because f is not holomorphic, the Fourier expansion is not as clean:

$$f(x+iy) = \sum_{n=-\infty}^{\infty} \alpha_n(y,s) e^{2\pi i n x},$$

for some functions $\alpha_n(y,s)$. These functions are not random, because $\alpha_n(y,s)e^{2\pi i n x}$ is also an eigenfunction of Δ with eigenvalue $1/4 - s^2$.

For example,

$$\alpha_0(y) = a y^{1/2+s} + b y^{1/2-s}.$$

Moreover, f is cuspidal iff $\alpha_0(y) = 0$.

In general, it turns out that

$$\alpha_n(y) = a_n \cdot \sqrt{y} \cdot K_s(2\pi |n| y)$$

where a_n is a constant, and K_s is the normalized Bessel function.

L-functions

As in the holomorphic case, one can develop a Hecke theory by attaching to f an L-function. The map

$$\iota : x + iy \mapsto -x + iy$$

gives an involution on the space of Maass forms. We say that f is even (respectively odd) if $\iota f = f$ (respectively $-f$), in which case $a_{-n}(f) = a_n(f)$ (resp. $-a_n(f)$).

For an even or odd f, we set

$$L(s, f) = \sum_{n \geq 1} a_n n^{-s}$$

which converges when $Re(s) > 3/2$, because $a_n = O(n^{1/2})$. Then one can show that $L(s, f)$ has analytic continuation and functional equations. Moreover, there are also actions of Hecke operators and if f is a Hecke eigenform, then $L(s, f)$ has Euler product.

Existence of cusp forms

It is not easy to show the existence of cuspidal Maass forms. Selberg used the **trace formula** to show that many such functions exist. In fact, the trace formula give a count of these cuspidal functions (the so-called **Weyl's law**). Perhaps this will be covered in Labesse's lectures.

There are, however, not many explicit constructions. One such construction is due to Maass, but let's not go into this here. He showed that there is a Maass form whose L-function is the same as that of a Hecke character of a real quadratic extension of $\mathbf{Q}$. This was the initial question which led to the discovery/definition of Maass forms. For more details, see [Bump, Thm. 1.9.1].

Selberg's conjecture

Because Δ is a positive definite operator (on $L^2(\Gamma\backslash\mathbf{H})$), the eigenvalue $\lambda = 1/4 - s^2$ is > 0 if f is cuspidal. It implies that one of the following 2 situations occurs:

- s is real and $|s| < 1/2$, in which case, $\lambda < 1/4$;
- s is purely imaginary, in which case $\lambda \geq 1/4$.

Selberg proved that when $\Gamma = SL_2(\mathbf{Z})$, the first possibility never occurs, so that $\lambda \geq 1/4$ always. He conjectured that the same holds for any congruence subgroups.

Selberg's Conjecture:

For any congruence subgroup Γ, if f is cuspidal with respect to Γ with parameter s, then s is purely imaginary and $\lambda \geq 1/4$.

It is known that this is not true for some non-congruence Γ.

Later, when we reformulate everything in terms of representation theory, we shall see that the Selberg conjecture and the Ramanujan conjecture (on size of Fourier coefficients) are basically the same phenomenon.