Automorphic Representations of Adele Groups

We have defined the space $\mathcal{A}(G,\Gamma)$ of automorphic forms with respect to an arithmetic group Γ of G (a reductive linear algebraic group defined over $\mathbf{Q}$). We saw that $\mathcal{A}(G,\Gamma)$ has a commuting action of $(\mathfrak{g},K)$ and the Hecke algebra $\mathcal{H}(G,\Gamma)$.

From this point of view, we saw that the classical modular forms correspond to different ways of embedding certain irreducible $(\mathfrak{g},K)$-modules into $\mathcal{A}(G,\Gamma)$:

$$M_k(N) \cong Hom_{\mathfrak{g},K}(\pi_k, \mathcal{A}(PGL_2, \Gamma_0'(N)))$$

where π_k is discrete series of $PGL_2(\mathbf{R})$ with lowest weight k. Thus we are interested in how $\mathcal{A}(G,\Gamma)$ decomposes as a $(\mathfrak{g},K)\times\mathcal{H}(G,\Gamma)$ module.

GOAL: to formulate the theory of automorphic forms using adelic language.

The reasons, among others, are:

- we want a theory that deals with $\mathcal{A}(G, \Gamma)$ for all choices of Γ simultaneously.

- we want a framework in which the roles of the $(\mathfrak{g}, K)$-action and the $\mathcal{H}(G, \Gamma)$-action are parallel, i.e. so that they are actions of the same kind.

- To describe the process of attaching an L-function to a classical modular form in terms of representation theory, it is cleanest to use the adelic framework, as demonstrated in Tate's thesis.

Adeles

Let F be a number field. Then one can associate a locally compact topological ring $\mathbf{A}_F$, called the adele ring of F. For concreteness, in these lectures, we shall just work with $\mathbf{Q}$.

Let's recall the definition of $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$:

$$\mathbf{A} \subset \mathbf{R} \times \prod_p \mathbf{Q}_p$$

consisting of those $x = (x_v)$ such that for almost all primes p,

$$x_p \in \mathbf{Z}_p.$$

It is clearly a subring of $\prod_v \mathbf{Q}_v$.

This construction is called the **restricted direct product**.

The ring $\mathbf{A}$ has a natural topology: a basis of open neighbourhoods at a point x consists of:

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbf{Z}_v$$

where S is a finite set of places of $\mathbf{Q}$, including the archimedean prime, and U_v is an open neighbourhood of x. In particular, $\mathbf{A}$ is a locally compact ring.

Alternatively, one can think of $\mathbf{A}$ as the inductive (or direct) limit of

$$\prod_{v \in S} \mathbf{Q}_v \times \prod_{v \notin S} \mathbf{Z}_v.$$

Observe that $\mathbf{Q} \hookrightarrow \mathbf{A}$ diagonally, so that $\mathbf{A}$ is a $\mathbf{Q}$-algebra. An important property is:

Theorem: $\mathbf{Q}$ is discrete in $\mathbf{A}$ with $\mathbf{Q} \backslash \mathbf{A}$ compact.

Thus, the situation of $\mathbf{Q} \subset \mathbf{A}$ is analogous to the situation of $\mathbf{Z} \subset \mathbf{R}$.

There are some variants of the above construction. If S is a finite set of places of $\mathbf{Q}$, we let:

$$\begin{cases} \mathbf{Q}_S = \prod_{v \in S} \mathbf{Q}_v \\ \mathbf{A}^S = \{(x_v) \in \prod_{v \notin S} \mathbf{Q}_v : x_v \in \mathbf{Z}_v \text{ for almost all } v\} \end{cases}$$

We call $\mathbf{A}^S$ the S-adeles. If S consists only of the place ∞, then we call $\mathbf{A}^S$ the finite adeles and denote it by $\mathbf{A}_f$.

The following is called the **strong approximation theorem for adeles:**

Theorem: Let S be a non-empty finite set of places of $\mathbf{Q}$. Then $\mathbf{Q}$ is dense in $\mathbf{A}^S$.

When S consists only of the archimedean place, this is the so-called Chinese remainder theorem.

Adele Groups

Let G be a linear algebraic group defined over $\mathbf{Q}$; the examples to keep in mind are the reductive groups GL_n and SL_n, as well as unipotent groups.

Note that G is an affine algebraic variety over $\mathbf{Q}$. One can thus consider the group $G(\mathbf{A})$ of adelic points of G. We simply call this the **adele group of** G. It is a locally compact group and we can give it a more concrete description as follows.

We consider the set of sequences (K_p) (indexed by primes) of open compact subgroups K_p of $G(\mathbf{Q}_p)$, and consider two such sequences (K_p) and (K'_p) to be equivalent if $K_p = K'_p$ for almost all p.

Now given a linear algebraic group G, one can associate an equivalent class of such sequences.

Namely, choose any embedding $i : G \hookrightarrow GL(V)$ where V is a $\mathbf{Q}$-vector space, and pick a lattice Λ in V. Then let K_p be the stabilizer of $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p$ in $G(\mathbf{Q}_p)$. The resulting sequence (K_p) will depend on (i, Λ) but different choices give equivalent sequences.

Let us pick one such sequence (K_p) from the equivalence class determined by G. Then it can be shown that:

$$G(\mathbf{A}) =$$

$$\{(g_v) \in \prod_v G(\mathbf{Q}_v) : g_v \in K_v \text{ for almost all } v\}.$$

This is independent of the choice of the sequence.

From this, one sees that if S is a finite set of primes, then

$$G(\mathbf{A}) = G(\mathbf{Q}_S) \times G(\mathbf{A}^S).$$

Moreover, a basis of neighbourhoods at 1 in $G(\mathbf{A}_f)$ consists of open compact subgroups $U = \prod_v U_v$ with $U_v = K_v$ for almost all v.

For almost all p, the open compact subgroup K_p is a so-called **hyperspecial maximal compact subgroup**. For example, when $G = GL_n$, $K_p = GL_n(\mathbf{Z}_p)$.

We can modify K_v at the remaining places (including ∞) and assume that they are special maximal compact subgroups. Then

$$K = \prod_v K_v$$

is a maximal compact subgroup of $G(\mathbf{A})$. We fix this K henceforth.

For example, when $G = GL_1$, then

$$GL_1(\mathbf{A}) =$$

$$\{x = (x_v) \in \prod_v \mathbf{Q}_v^\times,\, x_p \in \mathbf{Z}_p^\times \text{ for almost all } p\}.$$

This is the so-called idele group of $\mathbf{Q}$.

The quotient $G(\mathbf{Q})\backslash G(\mathbf{A})$

Because $\mathbf{Q} \subset \mathbf{A}$, we have:

$$G(\mathbf{Q}) \subset G(\mathbf{A}).$$

The situation of $G(\mathbf{Q}) \subset G(\mathbf{A})$ is entirely analogous to the situation of $SL_2(\mathbf{Z}) \subset SL_2(\mathbf{R})$. Indeed, we have:

- $G(\mathbf{Q})$ is a discrete subgroup of $G(\mathbf{A})$.
- $G(\mathbf{Q})\backslash G(\mathbf{A})$ has a fundamental domain which can be covered by a sufficiently large Siegel set (associated to any parabolic $\mathbf{Q}$-subgroup).
- $G(\mathbf{Q})\backslash G(\mathbf{A})$ has finite volujme if G is semisimple; it is compact if G is anisotropic.

Strong Approximation for G

The strong approximation theorem allows one to relate the adelic picture to the case of $\Gamma\backslash G(\mathbf{R})$.

Theorem: Assume that G is simply-connected and S is a finite set of places of $\mathbf{Q}$ such that $G(\mathbf{Q}_S)$ is not compact, then $G(\mathbf{Q})$ is dense in $G(\mathbf{A}^S)$.

Here is a reformulation. Given any open compact subgroup $U^S \subset G(\mathbf{A}^S)$, we have:

$$G(\mathbf{A}) = G(\mathbf{Q}) \cdot G(\mathbf{Q}_S) \cdot U^S.$$

A consequence of this is:

Corollary: Under the asumptions of the theorem, if we let $\Gamma = G(\mathbf{Q}) \cap U^S$, then

$$G(\mathbf{Q})\backslash G(\mathbf{A})/U^S \cong \Gamma\backslash G(\mathbf{Q}_S).$$

An example

As an example, consider the case when $G = SL_2$ and $S = \{\infty\}$. Then

$$SL_2(\mathbf{Q})\backslash SL_2(\mathbf{A})/U_f \cong \Gamma\backslash SL_2(\mathbf{R})$$

where U_f is any open compact subgroup of $G(\mathbf{A}_f)$ and $\Gamma = G(\mathbf{Q}) \cap U_f$.

Let's take $\mathbb{U}_f$ to be the group

$$K_0(N) = \prod_{p|N} I_p \cdot \prod_{(p,N)=1} SL_2(\mathbf{Z}_p)$$

where I_p is an Iwahori subgroup of $SL_2(\mathbf{Q}_p)$:

$$I_p = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}_p) : c \equiv 0 \mod p\}.$$

Then it is clear that

$$\Gamma_0(N) = K_0(N) \cap SL_2(\mathbf{Q}).$$

So we have:

$$SL_2(\mathbf{Q})\backslash SL_2(\mathbf{A})/K_0(N) \cong \Gamma_0(N)\backslash SL_2(\mathbf{R}).$$

Passage from real to adele groups

The above consideration allows us to regard an automorphic form f on $\Gamma\backslash G(\mathbf{R})$ as a function on $G(\mathbf{Q})\backslash G(\mathbf{A})$, at least for certain Γ.

We say that Γ is a **congruence subgroup** of G if $\Gamma = G(\mathbf{Q}) \cap U_\Gamma$ for some open compact subgroup U_Γ of $G(\mathbf{A}_f)$.

Thus if Γ is congruence, and G satisfies strong approximation, we have:

$$\Gamma\backslash G(\mathbf{R}) \cong G(\mathbf{Q})\backslash G(\mathbf{A})/U_\Gamma$$

and we can regard an automorphic form on $\Gamma\backslash G(\mathbf{R})$ as a function on $G(\mathbf{Q})\backslash G(\mathbf{A})$ which is right-invariant under U_Γ.

Remarks: (i) In general, for any reductive G and any open compact $U \subset G(\mathbf{A}_f)$, we have

$$\#G(\mathbf{Q})\backslash G(\mathbf{A})/G(\mathbf{R})U < \infty.$$

In this case, if $\{g_i \in G(\mathbf{A}_f)\}$ is a set of double coset representatives, then

$$G(\mathbf{Q})\backslash G(\mathbf{A})/U = \bigcup_i \Gamma_i \backslash G(\mathbf{R})$$

with $\Gamma_i = G(\mathbf{Q}) \cap g_i U g_i^{-1}$.

Thus, the passage between real and adele groups is not seriously affected by the lack of strong approximation.

(ii) When $G = GL_n$, we still have the decomposition:

$$GL_n(\mathbf{A}) = GL_n(\mathbf{Q}) \cdot GL_n(\mathbf{R}) \cdot U$$

for any open compact $U \subset GL_n(\mathbf{A}_f)$ on which the image of determinant is equal to $\prod_p \mathbb{Z}_p^\times$.

This is a consequence of strong approximation for SL_n and the fact that $\mathbf{Q}$ has class number 1. So it would not be true for a general number field.

In particular, when $U = K_0'(N)$ (the analog of $K_0(N)$ for GL_2), we have

$$\Gamma_0(N)\backslash SL_2(\mathbf{R}) \cong \Gamma_0'(N)Z(\mathbf{R})\backslash GL_2(\mathbf{R})$$

$$\cong Z(\mathbf{A})GL_2(\mathbf{Q})\backslash GL_2(\mathbf{A})/K_0'(N).$$

Properties of functions

When we regard an automorphic form on $\Gamma\backslash G(\mathbf{R})$ as a function f on $G(\mathbf{Q})\backslash G(\mathbf{A})$, the function f will inherit the properties of an automorphic form. Let us spell out some of these.

Definition:

- A function f on $G(\mathbf{A})$ is said to be **smooth** if it is C^∞ in its archimedean variable, and locally constant in the finite-adeles variable.

- f is K-**finite** if the right K-translates of K span a finite dimension vector space. Equivalently, f is K_∞-finite and is right-invariant under an open compact subgroup of $G(\mathbf{A}_f)$.

- If we fix a $\mathbf{Q}$-embedding $i : G \hookrightarrow GL_n$, we may define a **norm function** by

$$||g|| = \prod_v max_{j,k}\{|i(g)_{jk}|_v, |i(g^{-1})_{jk}|_v\}.$$

- f is said to be of **moderate growth** if there exists $n \geq 0$ and $C > 0$ such that

$$|f(g)| \leq C||g||^n$$

for all g.

- f is **rapidly decreasing on a Siegel set** $\mathfrak{S}$ if, for any $k > 0$, there exists $C_k > 0$ such that

$$|f(g)| \leq C_k \cdot ||g||^{-k}$$

for any $g \in \mathfrak{S}$.

Automorphic forms on adele groups

Let G be a reductive linear algebraic group over $\mathbf{Q}$.

Definition: A function f on $G(\mathbf{Q})\backslash G(\mathbf{A})$ is called an **automorphic form** if

- f is smooth
- f is right K-finite
- f is of moderate growth
- f is $Z(\mathfrak{g})$-finite.

We let $\mathcal{A}(G)$ denote the space of automorphic forms on G. This space contains $\mathcal{A}(G(\mathbf{R}), \Gamma)$ (for any congruence Γ) as the space of functions right-invariant under an open compact subgroup of $G(\mathbf{A}_f)$.

Cusp forms

Definition: An automrophic form f on G is called a **cusp form** if, for any parabolic $\mathbf{Q}$-subgroup $P = MN$ of G, the constant term

$$f_N(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} f(ng)\, dn$$

is zero as a function on $G(\mathbf{A})$.

It suffices to check this vanishing on a set of representatives of G-conjugacy classes of maximal parabolic subgroups.

We let $\mathcal{A}_0(G)$ denote the space of cusp forms on G.

<u>Analytic properties</u>

The analytic properties of an automorphic form f follow immediately from those on real groups, using the passage from real to adele groups. For example, we have:

- f is real analytic when restricted to $G(\mathbf{R})$
- f is of uniform moderate growth
- if f is cuspidal, f is rapidly decreasing on a Siegel set defined using any parabolic $\mathbf{Q}$-subgroup. In particular, $f \in L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ if G is semisimple.

Automorphic representations

The space $\mathcal{A}(G)$ possesses the structure of a $(\mathfrak{g}, K)$-module as before. In addition, for each prime p, the group $G(\mathbf{Q}_p)$ acts on $\mathcal{A}(G)$ by right translation. Thus, $\mathcal{A}(G)$ has the structure of a representation of

$$(\mathfrak{g}, K) \times G(\mathbf{A}_f).$$

Moreover, as a representation of $G(\mathbf{A}_f)$, it is a smooth representation.

We shall abuse terminology, and say that $\mathcal{A}(G)$ is a smooth representation of $G(\mathbf{A})$, even though $G(\mathbf{R})$ does not preserve $\mathcal{A}(G)$.

Definition: An irreducible smooth representation π of $G(\mathbf{A})$ is called an **automorphic representation** if π is a subquotient of $\mathcal{A}(G)$.

Admissibility

Theorem: An automorphic representation π is **admissible**, i.e. given any irreducible representation ρ of K, the multiplicity with which ρ occurs in π is finite.

An equivalent definition of "admissibility" is: for any open compact subgroup $U = \prod_p U_p \subset G(\mathbf{A}_f)$ and any irreducible representation ρ_∞ of K_∞, the subspace of vectors in π which are fixed by U and which is contained in the ρ_∞-isotypic subspace of π^U is finite-dimensional.

Proof: Suppose that $V_1 \subset V_2 \subset \mathcal{A}(G)$ are submodules with $V_2/V_1 \cong \pi$. We may assume that V_2 is generated over $G(\mathbf{A})$ by $f \in V_2 \setminus V_1$. Otherwise, we simply replace V_2 by the $G(\mathbf{A})$-submodule V_2' generated by f and V_1 by $V_1 \cap V_2'$.

If f is killed by an ideal J of finite codimension in $Z(\mathfrak{g})$, then V_2 is killed by J. Thus

$$V_2^U \subset \mathcal{A}(G, J)^U$$

and

$$\mathcal{A}(G, J)^U \cong \oplus_{i=1}^r \mathcal{A}(G(\mathbf{R}), \Gamma_i, J).$$

The RHS is an admissible $(\mathfrak{g}, K)$-module by a fundamental theorem of Harish-Chandra. This proves the theorem.

Restricted tensor product

We usually expect an irreducible representation of a direct product of groups G_i to be the tensor product of irreducible representations V_i of G_i. In the case of interest here, the adele group $G(\mathbf{A})$ is almost a direct product; it is a restricted direct product with respect to a family (K_p). It turns out that an irreducible admissible representation of $G(\mathbf{A})$ is almost a tensor product.

Definition: Suppose we have a family (W_v) of vector spaces, and for almost all v, we are given a non-zero vector $u_v^0 \in W_v$. The **restricted tensor product** $\otimes'_v W_v$ of the W_v's with respect to (u_v^0) is the inductive limit of $\{W_S = \otimes_{v\in S} W_v\}$, where for $S \subset S'$, one has $W_S \longrightarrow W_{S'}$ defined by

$$\otimes_{v\in S} u_v \mapsto (\otimes_{v\in S} u_v) \otimes (\otimes_{v\in S'\setminus S} u_v^0).$$

We think of $\otimes_v' W_v$ as the vector space generated by the elements

$$u = \otimes_v u_v \qquad \text{with } u_v = u_v^0 \text{ for almost all } v,$$

subject to the usual linearity conditions in the definition of the usual tensor product.

Now if each W_v is a representation of $G(\mathbf{Q}_v)$, and for almost all v, the distinguished vector u_v^0 is fixed by the maximal compact K_v, then the restricted tensor product inherits an action of $G(\mathbf{A})$: if $g = (g_v)$, then

$$g(\otimes_v u_v) = \otimes_v g_v u_v.$$

Because $g_v \in K_v$ and $u_v = u_v^0$ for almost all v, the resulting vector still has the property that almost all its local components are equal to the distinguished vector u_v^0.

Representations of adele groups

The following is a theorem of Flath:

Theorem: An irreducible admissible representation of $G(\mathbf{A})$ is a restricted tensor product of irreducible admissible representations π_v of $G(\mathbf{Q}_v)$ with respect to a family of vectors (u_v^0) such that $u_v^0 \in \pi_v^{K_v}$ for almost all v.

For the proof of this, see [Bump, §3.4].

Corollary: An automorphic representation π has a restricted tensor product decomposition: $\pi \cong \otimes_v' \pi_v$, where for almost all v, $\pi_v^{K_v} \neq 0$.

Unramified representations

Remarks: Note that if π_p is an irreducible admissible representation of $G(\mathbf{Q}_p)$, and K_p is a hyperspecial maximal compact subgroup of $G(\mathbf{Q}_p)$, then

$$\dim \pi_p^{K_p} \leq 1.$$

So the choice of u_p^0 is unique up to scaling.

We call an irreducible representation of $G(\mathbf{Q}_p)$ **unramified** or **spherical** with respect to K_p if $\dim \pi_p^{K_p} = 1$. These has been classified in Yu's lectures, using the **Satake isomorphism**.

We shall come back to this later, when we give a representation theoretic interpretation of the Ramanujan-Petersson conjecture.

Cuspidal automorphic representations

The space $\mathcal{A}_0(G)$ of cusp forms is clearly a submodule under $G(\mathbf{A})$. When G is reductive, with center Z, we usually specify a central character χ for $Z(\mathbf{A})$. Namely, if χ is a character of $Z(\mathbf{Q})\backslash Z(\mathbf{A})$, then we let $\mathcal{A}(G)_\chi$ be the subspace of automorphic forms f which satisfy:

$$f(zg) = \chi(z) \cdot f(g).$$

We let $\mathcal{A}_0(G)_\chi$ be the subspace of cuspidal functions in $\mathcal{A}(G)_\chi$. Then as in the case of $\mathcal{A}_0(G(\mathbf{R}), \Gamma)$ (with G semisimple), $\mathcal{A}_0(G)_\chi$ decomposes as the direct sum of irreducible representations of $G(\mathbf{A})$, each occurring with finite multiplicities.

Definition: A representation π of $G(\mathbf{A})$ is **cuspidal** if it occurs as a submodule of $\mathcal{A}_0(G)_\chi$.

The representation π_f

If f is a classical cuspidal Hecke eigenform on $\Gamma_0(N)$, we have seen that f gives rise to an automorphic form ϕ_f on $\Gamma_0'(N)\backslash PGL_2(\mathbf{R})$ which generates an irreducible $(\mathfrak{g}, K)$-module isomorphic to the discrete series representation of lowest weight k.

Now if we then transfer ϕ_f to a cusp form Φ_f on $PGL_2(\mathbf{Q})\backslash PGL_2(\mathbf{A})$, we can consider the subrepresentation π_f of $\mathcal{A}_0(PGL_2)$ generated by Φ_f. It turns out that this is an irreducible representation of $G(\mathbf{A})$ if f is a newform.

Thus a Hecke eigen-newform in $S_k(N)$ corresponds to a cuspidal representation of $PGL_2(\mathbf{A})$. Moreover, if $\pi_f \cong \otimes_v' \pi_v$, then π_p is unramified for all p not dividing N.

Basic questions

Having defined the notion of automorphic representations, some basic questions one can ask is:

- Given an irreducible admissible representation $\pi = \otimes'_v \pi_v$ of $G(\mathbf{A})$, can we decide if π is automorphic? When is it cuspidal?

- More generally, classify the automorphic representations of $G(\mathbf{A})$. One purpose of the Langlands program is to formulate an answer to this question.

- Construct some examples of automorphic representations. It turns out that there is a general method of constructing submodules of $\mathcal{A}(G)$. This is the theory of Eisenstein series. However, there is no known general method for constructing submodules of $\mathcal{A}_0(G)$.

Hecke algebra

On $\mathcal{A}(G(\mathbf{R}), \Gamma)$, we have the action of the Hecke algebra $\mathcal{H}(G, \Gamma)$. Under the isomorphism

$$\mathcal{A}(\Gamma\backslash G(\mathbf{R})) \cong \mathcal{A}(G)^{U_\Gamma},$$

what does the operator $T_\alpha = \Gamma\alpha\Gamma$ (with $\alpha \in G(\mathbf{Q})$) get translated to?

Let's call this new operator on the adelic side T'_α.

If $f \in \mathcal{A}(G)^{U_\Gamma}$, then the identification of f with a function on $\Gamma\backslash G(\mathbf{R})$ is simply given by restriction to $G(\mathbf{R})$. So we want to find T'_α such that

$$(T'_\alpha f)|_{G(\mathbf{R})} = T_\alpha(f|_{G(\mathbf{R})}).$$

Let us evaluate $T'_\alpha f$ at an element $g = (g_\infty, g_f) \in G(\mathbf{R}) \times G(\mathbf{A}_f)$. Because we are assuming that

$$G(\mathbf{A}_f) = G(\mathbf{Q}) \cdot U_\Gamma,$$

we can accordingly write

$$g_f = \gamma \cdot u.$$

Writing:

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma a_i,$$

we compute

$$\begin{aligned} T'_\alpha f(g) &= T'_\alpha f(g_\infty, \gamma u) & &= T'_\alpha f(\gamma^{-1} g_\infty, 1) \\ &= \sum_i f(a_i \gamma^{-1} g_\infty, 1) & &= \sum_i f(g_\infty, \gamma a_i^{-1}) \\ &= \sum_i f(g_\infty, \gamma u u^{-1} a_i^{-1}) & &= \sum_i f(g_\infty, \gamma u a_i^{-1}) \\ &= \sum_i f(g_\infty, g_f a_i^{-1}) & &= \sum_i f(g a_i^{-1}) \end{aligned}$$

Note that because

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma a_i \quad \text{in } G(\mathbf{Q})$$

we have

$$U_\Gamma \alpha U_\Gamma = \bigcup_i U_\Gamma a_i \quad \text{in } G(\mathbf{A}_f)$$

and so

$$U_\Gamma \alpha^{-1} U_\Gamma = \bigcup_i a_i^{-1} U_\Gamma.$$

We have now translated the action of $\mathcal{H}(G, \Gamma)$ to the adelic picture. Is the resulting operator something we have seen before?

Recollection from Yu's lectures

Recall from Yu's lectures that if V is a smooth representation of a locally profinite group G and $U \subset G$ is an open compact subgroup, then the map $V \mapsto V^U$ defines a functor from the category of smooth representatioons of G to the category of modules for the Hecke agebra $\mathcal{H}(G//U)$.

Recall that $\mathcal{H}(G//U)$ is the ring of functions in $C_c^\infty(G)$ which are bi-U-invariant, and the product is given by convolution of functions.

A basis for $\mathcal{H}(G//U)$ is given by the characteristic functions $f_\alpha = 1_{U\alpha U}$. The action of this on a vector in V^U is:

$$f_\alpha \cdot v = \int_G f_\alpha(g) \cdot gv\, dg$$

$$= \int_{U\alpha U} vdg = \sum_i a_i v$$

if $U\alpha U = \bigcup a_i U$ (and dg gives U volume 1).

Adelic Hecke algebras

We can apply the material from Yu's lectures to the smooth representation $\mathcal{A}(G)$ of $G(\mathbf{A}_f)$. Then the adelic Hecke algebra $\mathcal{H}(G(\mathbf{A}_f)//U_\Gamma)$ acts on $\mathcal{A}(G)^{U_\Gamma}$.

More explicitly, if $U_\Gamma \alpha^{-1} U_\Gamma = \bigcup_i a_i^{-1} U_\Gamma$, then the characteristic function of $U_\Gamma \alpha^{-1} U_\Gamma$ acts by

$$(T_\alpha f)(g) = \sum_i (a_i^{-1} f)(g) = \sum_i f(g a_i^{-1})$$

In conclusion, we see that the action of $\mathcal{H}(G, \Gamma)$ on $\mathcal{A}(G, \Gamma)$ gets translated to an action of the adelic Hecke algebra $\mathcal{H}(G(\mathbf{A}_f)//K_\Gamma)$ on $\mathcal{A}(G)^{K_\Gamma}$. This action of the adelic Hecke algebra arises from the smooth $G(\mathbf{A}_f)$-module structure on $\mathcal{A}(G)$. Note also that

$$\Gamma \backslash G(\mathbf{Q}) / \Gamma \leftrightarrow U_\Gamma \backslash G(\mathbf{A}_f) / U_\Gamma.$$

Local Hecke algebras

Because $G(\mathbf{A}_f)$ is a restricted direct product, we have in fact

$$\mathcal{H}(G(\mathbf{A}_f)//U) \cong \otimes'_v \mathcal{H}(G(\mathbf{Q}_p)//U_p)$$

if $U = \prod_p U_p$. So the structure of $\mathcal{H}(G(\mathbf{A}_f)//U)$ is known once we understand the local Hecke algebras $\mathcal{H}(G(\mathbf{Q}_p)//U_p)$.

For almost all p, however, we know that $U_p = K_p$ is a hyperspecial maximal compact subgroup. In that case, the structure of the local Hecke algebra is known, by the **Satake isomorphism**. In particular, $\mathcal{H}(G(\mathbf{Q}_p)//K_p)$ is commutative and its irreducible modules are classified.

Because $V \mapsto V^{K_p}$ induces a bijection of irreduible unramified representations with irreducible modules of $\mathcal{H}(G(\mathbf{Q}_p)//K_p)$, we get in this way the classification of irreducible unramified representations of $G(\mathbf{Q}_p)$. We recall this classification next.

Classification of unramified representations

Let us assume for simplicity that G is a split group (e.g. $G = GL_n$). Let $B = T \cdot N$ be a Borel subgroup of G, with maximal torus T. So $T \cong (GL_1)^r$ and $T(\mathbf{Q}_p) \cong (\mathbf{Q}_p^\times)^r$. We let $W := N_G(T)/T$ be the Weyl group of G.

Let $\chi : T(\mathbf{Q}_p) \longrightarrow \mathbf{C}^\times$ be a (smooth) character of $T(\mathbf{Q}_p)$. We say that χ is an **unramifed** character if χ is trivial when restricted to $T(\mathbf{Z}_p) \cong (\mathbf{Z}_p^\times)^r$. If χ is unramified, then it is of the form

$$\chi(a_i, ..., a_r) = t_1^{ord_p(a_1)} \cdot ... \cdot t_r^{ord_p(a_r)}, \quad a_i \in \mathbf{Q}_p^\times$$

for some $s_i \in \mathbf{C}^\times$.

We may regard χ as a character of $B(\mathbf{Q}_p)$ using the projection $B(\mathbf{Q}_p) \rightarrow N(\mathbf{Q}_p)\backslash B(\mathbf{Q}_p) \cong T(\mathbf{Q}_p)$.

Given an unramified character χ of $T(\mathbf{Q}_p)$, we may form the induced representation

$$I_B(\chi) := Ind_{B(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} \delta_B^{1/2} \cdot \chi.$$

Here, δ_B is the modulus character of B, defined by:

$$\delta_B(b) = |det(Ad(b)|Lie(N))|_p.$$

We recall that the space of $I_B(\chi)$ is the subspace of $C^\infty(G(\mathbf{Q}_p))$ satisfying:

- $f(bg) = \delta(b)^{1/2} \cdot \chi(b) \cdot f(g)$ for any $b \in B(\mathbf{Q}_p)$ and $g \in G(\mathbf{Q}_p)$.

- f is right-invariant under some open compact subgroup U_f of $G(\mathbf{Q}_p)$.

Then $I_B(\chi)$ is an admissible representation of $G(\mathbf{Q}_p)$, possibly reducible. The representations $I_B(\chi)$ are called the **principal series representations**.

Because of the Iwasawa decomposition $G(\mathbf{Q}_p) = B(\mathbf{Q}_p) \cdot K_p$, an element f of $I_B(\chi)$ is completely determined by its restriction to K_p. Thus we see that

$$\dim I_B(\chi)^{K_p} = 1$$

and a vector in this 1-dimensional space is given by

$$f_0(bk) = \delta_B(b)^{1/2} \cdot \chi(b);$$

equivalently, $f_0|_{K_p}$ is the constant function 1.

Thus $I_B(\chi)$ has a unique irreducible subquotient π_χ with the property that $\pi_\chi^{K_p} \neq 0$.

Theorem: Any irreducible unramified representation of $G(\mathbf{Q}_p)$ is of the form π_χ for some unramfied character χ of $T(\mathbf{Q}_p)$.

The Weyl group W acts naturally on $T(\mathbf{Q}_p)$ and thus on the set of characters of $T(\mathbf{Q}_p)$: Namely, for $w \in W$,

$$(w\chi)(t) = \chi(w^{-1}tw).$$

Proposition: $\pi_\chi \cong \pi_{\chi'}$ iff $\chi = w\chi'$ for some $w \in W$.

Thus, the irreducible unramified representations are classified by W-orbits of unramified characters of $T(\mathbf{Q}_p)$.

Unitarizability: One may ask whether π_χ is unitarizable. When χ is a unitary character, then it is clear that $I_B(\chi)$ is unitarizable, and thus so is π_χ. Indeed, a $G(\mathbf{Q}_p)$-invariant inner product on $I_B(\chi)$ is:

$$\langle f_1, f_2 \rangle = \int_{K_p} f_1(k) \cdot \overline{f_2(k)} \, dk$$

Though this inner product is not $G(\mathbf{Q}_p)$-invariant if χ is not unitary, it may be possible to define an invariant inner product in some other ways.

At this point, it appears that the problem of determining the unitarizable unramified representations is not completely solved for all groups. Of course, it has been solved for GL_n some time ago.

The example of GL_2

Let us look at the example of GL_2. Then B is the group of upper triangular matrices, and

$$\delta_B \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |a/d|_p.$$

The Weyl group W is isomorphic to the group S_2; the non-trivial element of W is represented by the matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An unramified character is of the form

$$\chi_{t_1,t_2} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = t_1^{ord_p(a)} \cdot t_2^{ord_p(d)}.$$

Moreover, under the action of w, we see that

$$w\chi_{t_1,t_2} = \chi_{t_2,t_1}.$$

So the irreducible unramified representations of $GL_2(\mathbf{Q}_p)$ are parametrized by diagonal matrices

$$t_\chi = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

modulo the action of w. In other words, they are parametrized by *semisimple conjugacy clases* in $GL_2(\mathbf{C})$.

Observe that $\pi_{\chi_{t_1,t_2}}$ is a representation of the group $PGL_2(\mathbf{Q}_p)$ iff $t_1 t_2 = 1$.

Analogously, the irreducible unramified representations of $GL_n(\mathbf{Q}_p)$ are naturally parametrized by semisimple conjugacy classes in $GL_n(\mathbf{C})$. The semisimple class associated to an unramified representation π is called the **Satake parameter** of π.

Tempered representations

Let us recall the notion of a tempered representation of $G(\mathbf{Q}_p)$. Let π be a unitarizable representation of $G(\mathbf{Q}_p)$. Suppose that $\langle -, - \rangle$ is a $G(\mathbf{Q}_p)$-invariant inner product on π. Then by a matrix coefficient of π is a function on $G(\mathbf{Q}_p)$ of the form

$$f_{v_1,v_2}(g) = \langle g v_1, v_2 \rangle$$

with v_1 and v_2 in π.

Assume that π has a unitary central character, so that $|f_{v_1,v_2}(g)|$ is a functon on $Z(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)$.

Definition: Say that π is a **tempered representation** if the matrix coefficients of π lie in $L^{2+\epsilon}(Z(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p))$ for any $\epsilon > 0$.

Proposition: The unramified representation π_χ is tempered iff χ is a unitary character.

We mentioned earlier that there may be other π_χ's which are unitarizable, but for which χ is not unitary. We call these other π_χ's the (unramified) **complementary series representations**. For example, the trivial representation of $G(\mathbf{Q}_p)$ is certainly not tempered.

Remarks: The notion of being tempered is a natural one. Indeed, if one considers the regular representation $L^2(G(\mathbf{Q}_p))$ (G semisimple, say), which decomposes into the sum of a discrete spectrum and a continuous one, then an irreducible unitarizable representation is tempered if and only if it occurs in the decomposition of $L^2(G(\mathbf{Q}_p))$.

Example: Let us describe the (unramified) complementary series of $PGL_2(\mathbf{Q}_p)$.

Proposition: $\pi_{\chi_{t_1,t_2}}$ is unitarizable and non-tempered iff $p^{-1/2} \leq t_1 \leq p^{1/2}$.

When $t_1 = p^{1/2}$ or $p^{-1/2}$, the corresponding unramified representation is the trivial representation.

Note that t_1 and t_1^{-1} parametrize the same representation.

Reformulating Ramanujan's conjecture

We are now in a position to reformulate the Ramanujan-Petersson conjecture in terms of representation theory. This reformulation is due to Satake.

We start with a cuspidal Hecke eigenform f of weight k for $SL_2(\mathbf{Z})$ with Fourier coefficients $\{a_n(f)\}$. The Ramanujan-Petersson conjecture says:

$$|a_p(f)| \leq 2 \cdot p^{(k-1)/2}.$$

Obviously, since $S_k(1)$ is finite dimensional, it suffices to prove this bound for a basis of $S_k(1)$. Recall that the action of the Hecke operators $\{T_n\}$ can be simultaneously diagonalized. So we have a natural basis of $S_k(1)$ consisting of Hecke eigenforms. We can further assume that these Hecke eigenforms are normalized, i.e. $a_1(f) = 1$.

Now assume that f is a normalized Hecke eigenform and suppose that

$$T_n f = \lambda_n(f) f.$$

Then we have seen that

$$a_n(f) = \lambda_n(f).$$

Thus, the Ramanujan-Petersson conjecture is equivalent to saying that the Hecke eigenvalues λ_p of T_p occurring in $S_k(1)$ satisfy

$$|\lambda_p| \leq 2p^{(k-1)/2}$$

Now we want to reformulate this on $\mathcal{A}(PGL_2(\mathbf{R}), \Gamma)$, wit $\Gamma = PGL_2(\mathbf{Z})$.

We saw that f gives rise to a function

$$\phi_f(g) := (f|_k g)(i)$$

on $SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R}) \cong PGL_2(\mathbf{Z})\backslash PGL_2(\mathbf{R})$. On $\mathcal{A}(PGL_2, \Gamma)$, we have the Hecke algebra $\mathcal{H}(PGL_2, \Gamma)$ acting and we showed that:

$$\phi_{T_p f} = p^{k/2-1} \cdot T_{\alpha_p} \phi_f$$

where

$$\alpha_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the Ramanujan-Petersson conjecture says the eigenvalue ν_p of ϕ_f with respect to T_{α_p} satisfies:

$$\nu_p = 2p^{1/2},$$

since $\lambda_p = p^{k/2-1}\nu_p$.

Now we pass to $\mathcal{A}(PGL_2)^{K_f}$ where $K_f = \prod_p PGL_2(\mathbf{Z}_p)$.

If we regard ϕ_f as an element Φ_f of $\mathcal{A}(PGL_2)^{K_f}$, then we saw that

$$T_{\alpha_p}\phi_f = T_{\alpha_p^{-1}}\Phi_f.$$

Here $T_{\alpha_p^{-1}}$ is the element in the adelic Hecke algebra corresponding to the double coset

$$K_f \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} K_f = K_f \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_f,$$

and is equal to T_{α_p}.

Further, under the decomposition

$$\mathcal{H}(PGL_2(\mathbf{A}_f)//K_f) = \otimes_p' \mathcal{H}(PGL_2(\mathbf{Q}_p)//K_p),$$

we see that T_{α_p} is supported only at the prime p.

Now suppose $\pi_f \subset \mathcal{A}_0(PGL_2)$ is the irreducible representation generated by Φ_f, and $\pi_f \cong \otimes'_v \pi_v$.

The action of T_{α_p} on $\pi_f^{K_f} = \mathbf{C} \cdot \Phi_f$ is simply the action of the characteristic functon t_p of $PGL_2(\mathbf{Z}_p)\alpha_p PGL_2(\mathbf{Z}_p)$ on $\pi_p^{K_p}$. So the eigenvalue of t_p on π_p is ν_p.

Now the local Hecke algebra $\mathcal{H}(PGL_2(\mathbf{Q}_p)//K_p)$ is generated as an algebra by t_p. So the unramified representation π_p is completely determined by the eigenvalue of t_p on $\pi_p^{K_p}$. So the Satake parameter s_p of π_p is completely determined by the eigenvalue ν_p associated to Φ_f.

So we ask:

What is the relation between the Hecke eigenvalue ν_p and the Satake parameter s_p of π_p?

Proposition: $p^{1/2} \cdot Trace(s_p) = \nu_p$.

Proof: If $s_p = diag(t_1, t_2)$, then as we have seen,

$$\pi_p = I_B(\chi_{t_1,t_2})$$

If f_0 is the K_p-fixed vector in $I_B(\chi)$ with $f_0(1) = 1$, then

$$\begin{aligned}
\nu_p &= (t_p f_0)(1) \\
&= \sum_{r=0}^{p-1} f_0 \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix} + f_0 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \\
&= p^{1/2}(t_2 + t_1) \\
&= p^{1/2} \cdot Trace(s_p).
\end{aligned}$$

Corollary: The Ramanujan-Petersson conjecture for $f \in S_k(1)$ is equivalent to saying that $\pi_{f,p}$ is tempered for all p.

Proof: Since π_p is necessarily unitarizable, π_p is either tempered or in the complementary series. If the Satake parameter is $s_p = diag(t, t^{-1})$, then π_p is complementary series iff

$$p^{-1/2} \leq t \leq p^{1/2}, \quad \text{but } t \neq 1,$$

which is equivalent to

$$|Trace(s_p)| > 2.$$

Remarks: Recall that the Selberg conjecture for the eigenvalues of the hyperbolic Laplacian on a Maass form is equivalent to saying that $\pi_{f,\infty}$ is tempered. Thus we have shown that the Selberg conjecture and the Ramanujan conjecture can be unified in a single statement in terms of the representation π_f.

Ramanujan Conjecture for GL_n

Let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of GL_n with unitary central character. Then for each v, π_v is tempered.

In the conference next week, you will probably hear some progress towards this conjecture.