Eisenstein Series

GOAL: We will discuss a standard construction of automorphic representations: the theory of Eisenstein series.

Let $P = M \cdot N$ be a parabolic subgroup of G and let π be an automorphic representation of $M(\mathbf{A})$. Consider the induced representation

$$I_P(\pi) = Ind_{P(\mathbf{A})}^{G(\mathbf{A})} \delta_P^{1/2} \cdot \pi$$

Then the theory of Eisenstein series gives rise to a $G(\mathbf{A})$-intertwining map

$$E : I_P(\pi) \longrightarrow \mathcal{A}(G),$$

thus giving us concrete examples of automorphic representations.

Unfortunately, this construction does not provide cuspidal representations; indeed the image of E is orthogonal to $\mathcal{A}_0(G)$.

However, the automorphic forms in the image of E turn out to be very useful. For example, the theory of Eisenstein series is one of the most important tool we have for understanding the properties of automorphic L-functions. Moreover, they are necessary for the spectral decomposition of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, as we saw in Labesse's lectures.

Parabolic induction

Let us recall the notion of parabolic induction. Let P be a parabolic $\mathbf{Q}$-subgroup of G, with Levi decomposition $P = M \cdot N$. We shall assume that P is a **maximal** parabolic.

The reason for this assumption is that the Eisenstein series will then be a function on $\mathbf{C}$ (taking values in $\mathcal{A}(G)$). If P is not maximal, then the Eisenstein series will be a function on $\mathbf{C}^n$ ($n \geq 2$), in which case results are harder to state.

(The standard example to keep in mind is the case of $B \subset PGL_2$).

The group $P(\mathbf{A})$ is not unimodular. So P has a modulus character δ_P. This character is trivial on $N(\mathbf{A})$ and its value on $M(\mathbf{A})$ is given by

$$\delta_P(m) = |\,\mathrm{det}(Ad(m)|Lie(N)(\mathbf{A}))|.$$

Given an abstract representation σ of $M(\mathbf{A})$, we can inflate (or pullback) σ to a representation of $P(\mathbf{A})$. Then one has the induced representation:

$$I_P(\sigma) := Ind_{P(\mathbf{A})}^{G(\mathbf{A})} \delta_P^{1/2} \cdot \sigma$$

More generally, we set

$$I_P(\sigma, s) := Ind_{P(\mathbf{A})}^{G(\mathbf{A})} \delta_P^{1/2+s} \cdot \sigma$$

Recall that the vector space for $I_P(\sigma, s)$ is the set of smooth functions

$$f : G(\mathbf{A}) \longrightarrow V_\sigma$$

such that

- $f(pg) = \delta_P(p)^{1/2+s} \cdot \sigma(p)(f(g))$

- f is right K-finite.

The action of $G(\mathbf{A})$ on $I_P(\sigma, s)$ is by right translation.

(In the standard example, σ is a character χ of $T(\mathbf{A}) \cong \mathbf{A}^\times$).

Flat sections

Because of Iwasawa decomposition $G(\mathbf{A}) = P(\mathbf{A}) \cdot K$, an element in $I_P(\sigma, s)$ is determined by its restriction to K. Indeed, restriction to K gives an isomorphism of vector spaces from $I_P(\sigma, s)$ to the space of smooth K-finite functions

$$f : K \to V_\sigma$$

satisfying

$$f(mk) = \sigma(m)(f(k)), \text{ for all } m \in M(\mathbf{A}) \cap K.$$

Given a function f in the latter space, we can extend it to an element $f_s \in I_P(\sigma, s)$. The family $\{f_s : s \in \mathbf{C}\}$ is called a **flat section**: the restriction of f_s to K is independent of s (it is equal to the f we started with). Sometimes, people also call it a **standard section**.

The case of automorphic σ

We shall assume that σ is a irreducible submodule of $\mathcal{A}(M)$ and so $V_\sigma \subset \mathcal{A}(M)$.

(In the standard example, σ is a character of the idele class group $T(\mathbf{Q})\backslash T(\mathbf{A}) = \mathbf{Q}^\times\backslash\mathbf{A}^\times$).

In this case, we can realize $I_P(\sigma, s)$ as $\mathbf{C}$-valued functions, rather than functions valued in V_σ. Indeed, if $f_s \in I_P(\sigma, s)$, then set

$$\tilde{f}_s(g) = [f_s(g)](1).$$

The function $\tilde{f}$ satisfies:

$$\tilde{f}_s(nmg) = f_s(g)(m).$$

In particular,

$$\tilde{f}_s : N(\mathbf{A})M(\mathbf{Q})\backslash G(\mathbf{A}) \longrightarrow \mathbf{C}.$$

(In the standard example, there is no need for this, because f_s is already $\mathbf{C}$-valued).

This $\tilde{f}_s$ has the property that for any $k \in K$, the function

$$m \mapsto \tilde{f}_s(mk)$$

is an element of $V_\sigma \subset \mathcal{A}(M)$. Moreover, if f_s is a flat section, then the element $\tilde{f}_s(-k) \in \mathcal{A}(M)$ is independent of s.

Formation of Eisenstein series

We want to make an automorphic form on $G(\mathbf{A})$ out of $\tilde{f}$. Since $\tilde{f}$ is only left-invariant under $P(\mathbf{Q})$ but not $G(\mathbf{Q})$, the easiest way to do this is to average over $P(\mathbf{Q})\backslash G(\mathbf{Q})$.

More precisely, let $f_s \in I_P(\sigma, s)$ be a flat section, whose restriction to K is a function f. We define the following function on $G(\mathbf{A})$:

$$E(f, s, g) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \tilde{f}_s(\gamma g)$$

Formally, this function on $G(\mathbf{A})$ is left-invariant under $G(\mathbf{Q})$. But we need to address convergence.

Convergence and properties

Here is the result on convergence:

Proposition:

(i) There exists $c > 0$ such that the above sum converges absolutely for any f and g when $Re(s) > c$. The convergence is locally uniform in g.

(ii) The function $E(f, s, g)$ is an automorphic form on G.

(iii) For fixed s, the map

$$f \mapsto E(f, s, -)$$

is a $G(\mathbf{A})$-equivariant map

$$I_P(\sigma, s) \longrightarrow \mathcal{A}(G).$$

(iv) In the half plane $Re(s) > c$, the function $s \mapsto E(f, s, g)$ (with f and g fixed) is holomorphic.

Constant term of Eisenstein series

The main result in the theory of Eisenstein series is the meromorphic continuation of $E(f,s,g)$ to $s \in \mathbf{C}$. A important ingredient in the proof of meromorphic continuation is the computation of the constant term of $E(f,s,g)$ along N.

Simplifying assumption:

Assume that σ is **cuspidal** and P is conjugate to its opposite parabolic. (This is automatic in the standard example).

Let $W_M = N_G(M)/M$. Because P is maximal parabolic, $W_M \cong S_2$, and we let w be the non-trivial element in W_M. We have:

Proposition: Assume $Re(s) >> 0$. Then

$$E_N(f,s,g) = \tilde{f}_s(g) + M_w(\sigma,s)\tilde{f}_s(g)$$

where $M_w(\sigma, s)\tilde{f}_s(g)$ is defined for $Re(s) >> 0$ by the absolutely convergent integral

$$M_w(\sigma, s)\tilde{f}(g) = \int_{N(\mathbf{A})} \tilde{f}_s(wng)\, dn.$$

The map $\tilde{f}_s \mapsto M_w(\sigma, s)\tilde{f}_s$ is a $G(\mathbf{A})$-equivariant map

$$I_P(\sigma, s) \longrightarrow I_P(w \cdot \sigma, -s).$$

Remarks: The operator $M_w(\sigma, s)$ is called a **standard intertwining operator**. It appears naturally in the constant term of the Eisenstein series, and is intricately connected with the properties of the Eisenstein series. In the course of proving that $E(f, s, g)$ has meromorphic continuation, one proves simultaneously that $M_w(\sigma, s)$ has meromorphic continuation. Observe that if $f = \prod_v f_v$, then $M_w(\sigma, s)$ factors into the product of **local intertwining operators** $M_{w,v}(\sigma_v, s)$. Thus this operator can often be analyzed locally.

The standard example

Let us compute the constant term in our standard example.

We have the Bruhat decomposition

$$G(\mathbf{Q}) = B(\mathbf{Q}) \bigcup B(\mathbf{Q}) w N(\mathbf{Q}).$$

So

$$B(\mathbf{Q})\backslash G(\mathbf{Q}) \leftrightarrow \{1, wN(\mathbf{Q})\}$$

and we have:

$$\sum_{\gamma \in B(\mathbf{Q})\backslash G(\mathbf{Q})} f_s(\gamma g) = f_s(g) + \sum_{\gamma \in N(\mathbf{Q})} f_s(w\gamma g).$$

This is what the previous proposition asserts.

Now we compute:

$$E_N(f,s,g)$$

$$= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} E(f,s,ng)\,dn$$

$$= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\gamma\in B(\mathbf{Q})\backslash G(\mathbf{Q})} f_s(\gamma ng)\,dn$$

$$= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} f_s(ng)\,dn +$$

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\gamma\in N(\mathbf{Q})} f_s(w\gamma ng)\,dn$$

$$= f_s(g) + \int_{N(\mathbf{A})} f_s(wng)\,dn$$

Meromorphic continuation

The following is the first main result of the theory.

Theorem:

(i) The function $s \mapsto E(f,s,g)$ can be continued to a meromorphic function on $\mathbf{C}$.

(ii) At a point s_0 where $E(f,s,g)$ is holomorphic for all g, the function $E(f,s_0,g)$ of g is an automorphic form.

(iii) At a point s_0 where $E(f,s,g)$ is holomorphic for all f and g, the map $f \mapsto E(f,s_0,-)$ is a $G(\mathbf{A})$-equivariant map of $I_P(\sigma,s_0)$ to $\mathcal{A}(G)$.

(iv) For any s_0, there is a constant N such that

$$\inf_{f,g}\{ord_{s=s_0}E(f,s,g)\} = -N.$$

Laurent expansion

The last part of the theorem says that one has a Laurent expansion about the point s_0:

$$E(f,s,g) =$$

$$\frac{a_{-N}(f,s_0,g)}{(s-s_0)^N} + \frac{a_{-(N-1)}(f,s_0,g)}{(s-s_0)^{N-1}} + ...$$

with $a_{-N}(f,s_0,g) \neq 0$ for some f and g.

Parts (ii) and (iii) can be extended to all $s_0 \in \mathbf{C}$, provided we use the **leading term** of the Laurent expansion at the point s_0.

More precisely, we have:

Theorem:

(i) For any i, the function $a_i(f, s_0, g)$ is an automorphic form on G.

(ii) The map $f \mapsto a_{-N}(f, s_0, -)$ is a $G(\mathbf{A})$-equivariant map $I_P(\sigma, s_0) \to \mathcal{A}(G)$.

Remarks: In part (ii), if we had used a_i, with $i \neq -N$, the map will not be $G(\mathbf{A})$-equivariant. One can only say that the composite map

$$a_i : I_P(\sigma, s_0) \to \mathcal{A}(G) \to \mathcal{A}(G)/\langle Im(a_k) : k < i\rangle$$

is equivariant.

Global intertwining operators

In the course of proving the meromorphic continuation of Eisenstein series, one also proves:

Theorem:

(i) The intertwining operator $M_w(\sigma, s)\tilde{f}_s$ has a meromorphic continuation to all of $\mathbf{C}$.

(ii) At each $s_0 \in \mathbf{C}$, the order of poles is bounded (as f varies).

(iii) the leading term of the Laurent expansion o $M_w(\sigma, s)$ at s_0 is an intertwining operator $I_P(\sigma, s) \to I_P(w \cdot \sigma, -s)$.

(iv) $M_w(w \cdot \sigma, -s) \circ M_w(\sigma, s) = id$.

As a consequence, the identity

$$E_N(f, s, g) = \tilde{f}_s(g) + M_w(\sigma, s)\tilde{f}_s(g)$$

holds for all $s \in \mathbf{C}$.

Functional equation

A consequence of the formula for the constant term E_N is that we have a functional equation for the Eisenstein series:

Theorem:

$$E(f, s, g) = E(M_w(s)f, -s, g).$$

Proof: Both sides have the same constant terms along (using the formula for the constant term E_N as well as (iv) of the last theorem). Moreover, under our simplifying assumption, the constant term along other parabolic subgroups are zero.

So the difference of the two sides is a cusp form. But as we shall see next, each Eisenstein series is orthogonal to the cusp forms. This shows that the difference is zero.

Orthogonality to cusp forms

The Eisenstein series $E(f,s,g)$ is orthogonal to the space $\mathcal{A}_0(G)$ of cusp forms. Suppose that $\phi \in \mathcal{A}_0(G)$. Then for $Re(s)$ large,

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} E(f,s,g)\cdot\phi(g)\,dg$$

$$=\int_{G(\mathbf{Q})\backslash G(\mathbf{A})} \left(\sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})} \tilde{f}_s(\gamma g)\right)\cdot\phi(g)\,dg$$

$$=\int_{P(\mathbf{Q})\backslash G(\mathbf{A})} \tilde{f}_s(g)\cdot\phi(g)\,dg$$

$$=\int_{N(\mathbf{A})M(\mathbf{Q})\backslash G(\mathbf{A})} \tilde{f}_s(g)\cdot\left(\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \phi(ng)\,dn\right)\,dg$$

$$=0$$

The result for general s follows by meromorphic continuation.

Analytic behaviour of constant term

Clearly, if $E(f,s,-)$ has a pole of order k at s_0, then the order of pole at s_0 of the constant term $E_N(f,s,-)$ is $\leq k$. In fact, we have

Theorem: The order of pole of $E(f,s,-)$ at any s_0 is the same as that of the constant term $E_N(f,s,-)$.

Proof: If not, the leading term $a_{-k}(f,s,-)$ of the Laurent expansion of $E(f,s,-)$ at s_0 will be a non-zero cusp form, contradicting the fact that $a_{-k}(f,s,-)$ is orthogonal to all cusp forms.

Remarks: This result is of great importance in practice. It says that to decide the behaviour of E at s, it suffices to examine the behaviour of E_N at s. But we have a formula for E_N in terms of the intertwining operator, which is an Euler product of local intertwining operators.

It turns out that one can calculate these local intertwining operators explicitly for almost all p. This is the so-called **Gindikin-Karpelevich formula**. The answer can be expressed (as shown by Langlands) in terms of certain local L-functions of π_p.

Thus, we see that the analytic properties of E are controlled by those of appropriate L-functions.

This relation can be exploited both ways. Sometimes, one uses properties of the L-functions to deduce, for example, that E has a pole somewhere. On the other hand, one may also use the analytic properties of E to deduce analytic properties of the L-functions. This is the technique used in two of the standard approaches of studying L-functions: the Rankin-Selberg method and the Langlands-Shahidi method.

Spectral decomposition

For the application to the spectral decomposition of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$, it is necesary to know the following result:

Theorem: The Eisenstein series $E(f,s,g)$ is holomorphic on $Re(s) = 0$ (i.e. the so-called unitary axis). Moreover, it has only finitely many poles in $Re(s) > 0$.

The first statement is needed because in the spectral decomposition, one needs to consider integral of the type:

$$\int_{Re(s)=0} \phi(s) \cdot E(f,s,g)\, ds.$$

Moreover, this integral is obtained by a contour shift from the same integral over $Re(s) = c$ for some large c. From the second statement, we see that we may pick up finitely many residues of $E(f,s,g)$ during the contour shift, and these contribute to the so-called residual spectrum.

The standard example

We illustrate the above in the case when $G = PGL_2$ and σ is the trivial character.

In this case, the Gindikin-Karpelevich formula gives:

$$M_w(s) f_v^0 = \frac{\zeta_v(2s)}{\zeta_v(2s+1)} \cdot f_v^0.$$

So we see that the constant term is:

$$E_N(f,s,g) = f_s(g) +$$

$$\prod_{v \in S} M_{w,v}(s) f_v(g_v) \cdot \frac{\zeta^S(2s)}{\zeta^S(2s+1)} f^{0,S}(g^S).$$

The local intertwining operators are defined by absolutely convergent integral when $Re(s) > 0$, and so has no poles there. Thus, one sees that in $Re(s) > 0$, there is precisely only one pole of order 1, namely at $s = 1/2$.

The residue there turns out to be a constant function.

A theorem of Langlands

The above discussion shows that the process of parabolic induction sends cuspidal automorphic representations on M to automorphic representations of G. In other words, if σ is cuspidal, then every irreducible constituent of $I_P(\sigma)$ is automorphic. Langlands showed the converse to this:

Proposition:

If π is an automorphic representation of G, then there exists a parabolic subgroup $P = MN$ and a cuspidal representation σ of M such that π is a constituent of $I_P(\sigma)$.

This theorem shows that cuspidal representations are the fundamental objects in the theory of automorphic forms, in the sense that every other automorphic representation is built out of them by parabolic induction.

[Compare this with the representation theory of p-adic groups. There, the basic objects are the supercuspidal representations and every irreducible smooth representation is a constituent of some $I_P(\sigma)$ with σ supercuspidal. Moreover, the pair (M, σ) is unique up to conjugacy.]

The theorem of Langlands above does not claim that the pair (M, σ) is unique up to conjugacy. For GL_n, this is in fact true: it is a non-trivial theorem of Jacquet-Shalika. In general, however, it is false!

For example, for the group $PGSp_4$, it was shown by Waldspurger that there are cuspidal representations π which are abstractly isomorphic to a constituent of some $I_P(\sigma)$ with σ cuspidal on $M = GL_2$. These are the so-called **CAP** representations.