Multiplicity One Theorem for GL_n

Last time, we saw a general construction of automorphic representations using Eisenstein series. This does not produce cuspidal representations. In fact, there are no known general methods which produce embeddings of a representation into $\mathcal{A}_0(G)$.

However, one can prove some results about the structure of $\mathcal{A}_0(G)$ in the case when $G = GL_n$. The goal of this lecture is to prove one such result:

Multiplicity One theorem

The multiplicity $m_0(\pi)$ of an irreducible representation π of $GL_n(\mathbf{A})$ in $\mathcal{A}_0(G)$ is ≤ 1.

Note that this theorem does not tell us which π has $m_0(\pi) = 1$.

Whittaker-Fourier coefficients

The proof of the multiplicity one theorem has two ingredients, one of which is global and the other local. We begin by explaining these 2 ingredients.

Let f be an automorphic form on $G = GL_n$. If $N \subset G$ is a unipotent subgroup, say the unipotent radical of a parabolic subgroup, one can consider the Fourier coefficients of f along N.

Namely, if χ is a unitary character of $N(\mathbf{A})$ which is trivial on $N(\mathbf{Q})$, we have

$$f_{N,\chi}(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \overline{\chi(n)} \cdot f(ng)\, dn$$

Note that if N is abelian, then we have:

$$f(g) = \sum_{\chi} f_{N,\chi}(g).$$

In general, when N is not abelian, the expansion on the RHS is only equal to $f_{[N,N]}$.

We apply the above to the unipotent radical N of the Borel subgroup B of upper triangular matrices

Definition: A character χ of $N(\mathbf{A})$ is **generic** if the stabilizer of χ in $T(\mathbf{A})$ is the center $Z(\mathbf{A})$ of $GL_n(\mathbf{A})$. An equivalent definition is that χ is non-trivial when restricted to every simple root subgroup in N.

Examples:

(i) When $G = GL_2$, a generic character of $N(\mathbf{Q})\backslash N(\mathbf{A})$ just means a non-trivial character of $\mathbf{Q}\backslash\mathbf{A}$. If we fix a character ψ of $F\backslash\mathbf{A}$, then all others are of the form

$$\chi_\lambda(x) = \psi(\lambda x)$$

for some $\lambda \in \mathbf{Q}$.

(ii) When $G = GL_3$, a character of $N(\mathbf{A})$ trivial on $N(\mathbf{Q})$ has the form

$$\chi_{\lambda_1,\lambda_2}\begin{pmatrix} 1 & a_1 & * \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} = \psi(\lambda_1 a_1 + \lambda_2 a_2)$$

for some λ_1 and $\lambda_2 \in \mathbf{Q}$.

Saying that $\chi_{\lambda_1,\lambda_2}$ is generic means that λ_1 and λ_2 are non-zero.

Definition: A **Whittaker-Fourier coefficient** of f is a Fourier coefficient $f_{N,\chi}$ with χ generic. (N is unipotent radical of Borel).

We note:

Lemma: The group $Z(\mathbf{Q})\backslash T(\mathbf{Q})$ acts simply transitively on the generic characters of $N(\mathbf{A})$ trivial on $N(\mathbf{Q})$.

We let χ_0 be the generic character whose restriction to each simple root space is equal to ψ.

Observe that if $t \cdot \chi = \chi'$ with $t \in T(\mathbf{Q})$, then

$$f_{N,\chi'}(g) = f_{N,\chi}(t^{-1}g).$$

Thus, we see that $f_{N,\chi_0} \neq 0$ iff $f_{N,\chi} \neq 0$ for all generic χ.

Definition: A representation $\pi \subset \mathcal{A}(G)$ is said to be **globally generic** if there exists $f \in \pi$ whose Fourier-Whittaker coefficient $f_{N,\chi} \neq 0$ for some (and hence all) generic characters χ.

An equivalent reformulation is as follows. Consider the linear map

$$l_\chi : \mathcal{A}(G) \longrightarrow \mathbf{C}$$

defined by

$$l_\chi(f) = f_{N,\chi}(1)$$

with χ generic. Then π is globally generic iff $l_\chi \neq 0$ when restricted to π.

The example of GL_2

Suppose $G = GL_2$ and $\pi \subset \mathcal{A}_0(G)$ is an irreducible cuspidal representation.

Claim: π is globally generic.

Proof: Take any non-zero $f \in \pi$. Then we have the expansion

$$f(g) = \sum_{\chi} f_{N,\chi}(g)$$

Since f cuspidal, $f_N = 0$. So some $f_{N,\chi} \neq 0$.

In this case, the following are equivalent:

(i) π is globally generic;

(ii) $\pi = \otimes_v \pi_v$ is infinite-dimensional;

(iii) π_v is infinite-dimensional for all v.

Whittaker functionals

One can define the notion of a "generic representation" locally.

Let π_v be a representation of $G(\mathbf{Q}_v)$ and let

$$\chi_v : N(\mathbf{Q}_v) \longrightarrow \mathbf{C}$$

be a generic unitary character.

Definition: Let p be a finite prime. Then π_p is an **abstractly generic** representation if, given any generic χ_p, there is a non-zero linear functional $l_p : \pi_p \to \mathbf{C}$ such that

$$l_p(n \cdot v) = \chi_p(n) \cdot l_p(v)$$

for all $n \in N(\mathbf{Q}_p)$ and $v \in \pi_p$. Such a functional is called a local **Whittaker functional**.

Archimedean case

One can make the same definition at the infinite prime, except for one subtlety: π_∞ is a $(\mathfrak{g}, K)$-module and $N(\mathbf{R})$ does not act on π_∞.

Recall that given an admissible $(\mathfrak{g}, K)$-module V_K, there exists a continuous representation of $G(\mathbf{R})$ on a Hilbert space H whose underlying $(\mathfrak{g}, K)$-module is V_K. However, this Hilbert representation is not unique.

Regardless, for each such H, we can consider the space of smooth vectors H^∞, and equip this space with the smooth topology, making it into a Frechet space with continuous $G(\mathbf{R})$-action.

The following is an amazing result of Casselman-Wallach:

Theorem: The $G(\mathbf{R})$-representation H^∞ is independent of the choice of H.

We call this the canonical globalization of V_K.

Let π be an admissible $(\mathfrak{g}, K)$-module and let π^∞ be its canonical Frechet globalization.

Definition: Say that π is **abstractly generic** if π^∞ has a non-zero continuous Whittaker functional.

Note that such a functional is non-zero when restricted from π^∞ to π (by density of K-finite vectors).

Now let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $G(\mathbf{A})$, one says that π is an **abstractly generic** representation if each of its local components π_v is abstractly generic.

The two ingredients

We can now state the two ingredients needed for the proof of the Multiplicity One Theorem.

Theorem A (Global genericity):

Let $\pi \subset \mathcal{A}(G)$ be an irreducible cuspidal representation. Then π is globally generic.

Theorem B (Local uniqueness of Whittaker functionals):

Let π_v be an irreducible smooth representation of $G(\mathbf{Q}_v)$. Then the space of (continuous) Whittaker fuctional on π_v is at most 1-dimensional.

We remark that while Theorem B is still true for an arbitrary (quasi-split) group, Theorem A is only valid for GL_n.

Proof of Multiplicity One

Assume Theorems A and B. We need to show that for any ireducible admissible representation π of $G(\mathbf{A})$,

$$\dim Hom_{G(\mathbf{A})}(\pi, \mathcal{A}_0(G)) \leq 1.$$

Let χ be a generic character of $N(\mathbf{A})$ trivial on $N(\mathbf{Q})$. Recall we have the map

$$l_\chi : \mathcal{A}(G) \longrightarrow \mathbf{C}_\chi$$

given by

$$l_\chi(\phi) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \overline{\chi(n)} \cdot \phi(n)\, dn.$$

Now we have a map

$$Hom_{G(\mathbf{A})}(\pi, \mathcal{A}_0(G)) \longrightarrow Hom_{N(\mathbf{A})}(\pi, \mathbf{C}_\chi)$$

given by $f \mapsto l_\chi \circ f$.

By Theorem A, this map is injective!

So it suffices to show that the RHS has dimension ≤ 1.

The generic character χ is of the form $\prod_v \chi_v$ for generic characters χ_v of $N(\mathbf{Q}_v)$.

Now if $L \in Hom_{N(\mathbf{A})}(\pi, \mathbf{C}_\chi)$ is non-zero, then for each v,

$$\dim Hom_{N(F_v)}(\pi_v, \mathbf{C}_{\chi_v}) \neq 0,$$

i.e. π is abstractly generic. By Theorem B, the above dimenson is 1, and for almost all v, a non-zero local functional l_v is non-zero on $\pi_v^{K_v}$.

Let us choose $l_v \neq 0$ so that for almost all v, $l_v(u_v^0) = 1$, where u_v^0 is the distinguished K_v-fixed vector in π_v. Then one has, for some constant c,

$$L(u) = c \cdot \prod_v l_v(u_v) \qquad \text{for any } u = \otimes_v u_v.$$

This shows that

$$\dim Hom_{N(\mathbf{A})}(\pi, \mathbf{C}_\chi) = 1$$

as desired.

Proof of Theorem A

Recall that we have shown Theorem A for GL_2, for we can express $f \in \pi$ as:

$$f(g) = \sum_{\chi} f_{N,\chi}(g) = \sum_{\gamma \in \mathbf{Q}^\times} f_{N,\chi_0}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Note that we have this expansion for any smooth function on $B(\mathbf{Q})\backslash GL_2(\mathbf{A})$.

Foir $n > 2$, because N is non-abelian, it appears that the Whittaker-Fourier coefficients $f_{N,\chi}$ only determines $f_{[N,N]}$. This is why Theorem A is not trivial.

However, we shall show:

Proposition: We have the expansion

$$f(g) = \sum_{\gamma \in N_{n-1}(\mathbf{Q}) \backslash GL_{n-1}(\mathbf{Q})} f_{N,\chi_0}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

Here N_{n-1} is the unipotent radical of the Borel subgroup of GL_{n-1}.

Clearly, this proposition implies Theorem A. The proof of the proposition makes use of the so-called **mirabolic subgroup** P_n of GL_n (which is something specific to GL_n):

$$P_n = \{ \begin{pmatrix} g_{n-1} & * \\ 0 & 1 \end{pmatrix} : g_{n-1} \in GL_{n-1} \}.$$

It has a decomposition $P_n = GL_{n-1} \cdot U_n$.

One proves inductively the following statement.:

(*) Suppose that f is a smooth function on $P_n(\mathbf{Q})\backslash GL_n(\mathbf{A})$ whose constant terms along any standard unipotent subgroup $U \subset P_n$ vanishes. Then f has the expansion of the proposition.

For the purpose of this proof, we say that such an f is cuspidal. The proof proceeds as follows:

- Expand f along U_n:

  $$f(g) = \sum_{\lambda} = f_{U_n,\lambda}(g).$$

  Then $f_{U_n} = 0$ since f cuspidal. Also $GL_{n-1}(\mathbf{Q})$ acts transitively on the non-trivial λ's. One such λ is $\lambda_0 = \chi_0|_{U_n}$. Its stabilizer in $GL_{n-1}(\mathbf{Q})$ is precisely $P_{n-1}(\mathbf{Q})$.

  So we have the preliminary expansion:

  $$f(g) = \sum_{\gamma \in P_{n-1}(\mathbf{Q})\backslash GL_{n-1}(\mathbf{Q})} f_{U_n,\lambda_0}(\gamma g).$$

- Now let

$$\Phi_g = f_{U_n,\lambda_0}(-g)|_{GL_{n-1}}.$$

It is easy to see that Φ is smooth cuspidal as a function on $P_{n-1}(\mathbf{Q})\backslash GL_{n-1}(\mathbf{A})$. So we can apply induction hypothesis to Φ:

$$\Phi_g(h) = \sum_{\delta\in N_{n-2}\backslash GL_{n-2}} (\Phi_g)_{N_{n-1},\chi_0'}(\delta h)$$

where $\chi_0' = \chi_0|_{N_{n-1}}$.

- Finally, we substitute this expansion for Φ_g into the preliminary expansion for f. The double sum can be collapsed into a single sum over $N_{n-1}(\mathbf{Q})\backslash GL_{n-1}(\mathbf{Q})$. Also, one observes that the summand is given, for $\gamma \in P_{n-1}(\mathbf{Q})\backslash GL_{n-1}(\mathbf{Q})$, by:

$$(\Phi_g)_{N_{n-1},\chi_0'}(\gamma)$$

$$= \int_{N_{n-1}(\mathbf{Q})\backslash N_{n-1}(\mathbf{A})} \overline{\chi_0(n')} \cdot \Phi_g(n'\gamma)\, dn'$$

$$= \int_{N_{n-1}(\mathbf{Q})\backslash N_{n-1}(\mathbf{A})} \overline{\chi_0(n')} \cdot f_{U_n,\lambda_0}(n'\gamma g)\, dn'$$

$$= \int_{N_{n-1}(\mathbf{Q})\backslash N_{n-1}(\mathbf{A})} \int_{U_n(\mathbf{Q})\backslash U_n(\mathbf{A})} \overline{\chi_0(u)} \cdot \overline{\chi_0(n')} \cdot$$

$$f(un'\gamma g)\, du\, dn'$$

Because $N_n = U_n \cdot N_{n-1}$, this last double integral can be combined into the single integral

$$\int_{N_n(\mathbf{Q})\backslash N_n(\mathbf{A})} \overline{\chi_0(n)} \cdot f(n\gamma g)\, dn = f_{N_n,\chi_0}(\gamma g).$$

This is the desired result.

Strong Multiplicity One

In fact, for GL_n, a stronger result is true. Namely,

Theorem: (Rigidity)

Let π_1 and π_2 be irreducible cuspidal representations of GL_n. Assume that for almost all v, $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic as abstract representations. Then π_1 and π_2 are isomorphic as abstract representations.

The proof of this theorem, due to Jacquet-Shalika, proceeds by using L-functions.

Corollary: (Strong multiplicity one)

If $\pi_1, \pi_2 \subset \mathcal{A}_0(G)$ are such that $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic for almost all v, then $\pi_1 = \pi_2$ as subspaces of $\mathcal{A}_0(G)$.

The corollary follows from the theorem above and multiplicity one theorem.

Other groups

Finally, let us comment on $\mathcal{A}_0(G)$ for other groups G. It turns out that in general, the multiplicity one theorem is false, though we have:

Theorem (Ramakrishnan, 2000) The multiplicity one theorem is true for $G = SL_2$

The proof of this theorem is highly non-trivial; this shows once again that GL_2 is much easier to handle than SL_2. For example, strong multiplicity one is not true for SL_2. What about SL_n for $n \geq 3$?

Theorem (Blasius, 1994) For $G = SL_n$, $n \geq 3$, the multiplicity one theorem is false.

Does one have an explanation for this?

Generalized Ramanujan conjecture

Recall that we have formulated the Ramanujan conjecture for GL_n as: if π is cuspidal, then π is tempered.

Now one might make the same conjecture for general G. However, it turns out to be false! The first examples of such non-tempered cuspidal representations were discovered in the 70's by Saito-Kurokawa and Howe-Piatetski-Shapiro on the group Sp_4. However, these counterexamples to the naive generalization of Ramanujan's conjecture are non-generic. Thus, one is led to the following conjecture:

Generalized Ramanujan Conjecture: Let π be cuspidal and generic. Then π is tempered.

In fact, there is a deep conjecture of Arthur which extends the generalized Ramanujan conjecture. It explains the extent of the failure of

Ramanujan conjecture for general representations in the discrete spectrum of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$ as well as the presence of multiplicities in the discrete spectrum.