The Saito-Kurokawa space of $PGSp_4$ and its transfer to inner forms

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In this paper, we discuss some results on the Saito-Kurokawa space of $PGSp_4$ and its inner forms, interpreting them in the framework of Arthur's conjecture on square-integrable automorphic forms. Most results discussed here are known to the experts: Arthur, Waldspurger, Piatetski-Shapiro, Gelbart, Rallis, Kudla, Moeglin, Soudry.....though they may not have been explicitly stated in the literature or written up with the same point of view. The only new material concerns the transfer of the Saito-Kurokawa space to the inner forms and a characterization of the image by means of the standard L-function.

The Saito-Kurokawa cusp forms for $PGSp_4 \cong SO_5$ are the first examples of the so-called CAP representations or shadows of Eisenstein series. They can be constructed (and exhausted) by using theta lifting from $\widetilde{SL}_2$ to SO_5. In the following, we shall first review relevant results for cusp forms on $\widetilde{SL}_2$, and some general results from theta correspondence. We then give a brief description of Arthur's conjecture for $PGSp_4$: this is the natural framework in which the Saito-Kurokawa space can be understood. After describing Piatetski-Shapiro's construction and characterization of the Saito-Kurokawa space for $PGSp_4$, we describe the lifting of the Saito-Kurokawa space to any inner form G' of $PGSp_4$. This is motivated by the recent results of E. Sayag [S]. We end by giving a proof of the characterization of the Saito-Kurokawa space which is valid for all forms of $PGSp_4$.

Throughout these notes, F will denote a number field and F_v the local field corresponding to a place v of F. The adele ring of F is denoted by $\mathbb{A}$.

§1. Waldspurger's Results for $\widetilde{SL}_2$

In this section, we review the results of Waldspurger in [W1] and [W2]. His local results gives a partition of the irreducible admissible representations of $\widetilde{SL}_2(F_v)$ into packets. His global results give a complete description of the (genuine) discrete spectrum $L^2_{disc}(SL_2(F)\backslash\widetilde{SL}_2(\mathbb{A}))$.

(1.1) The Weil representations of $\widetilde{SL}_2(F_v)$. We first describe some very special representations of the metaplectic group $\widetilde{SL}_2(F_v)$.

Fix a non-trivial unitary character ψ_v of F_v. Then associated to a quadratic character χ_v of $F_v^\times$ (possibly trivial) is a Weil representation ω_{χ_v} of $\widetilde{SL}_2(F_v)$. The representation can be realized on the space $S(F_v)$ of Schwarz functions on F_v. As a representation of $\widetilde{SL}_2(F_v)$, ω_{χ_v} is reducible. In fact, it is the direct sum of two irreducible representations:

$$\omega_{\chi_v} = \omega^+_{\chi_v} \oplus \omega^-_{\chi_v},$$

where $\omega^+_{\chi_v}$ (resp. $\omega^-_{\chi_v}$) consists of the even (resp. odd) functions in $S(F_v)$. If v is a finite place, then ω^-_χ is supercuspidal and ω^+_χ is not.

(1.2) Waldspurger's packets for $\widetilde{SL}_2(F_v)$. In [W1] and [W2], Waldspurger defined a **surjective** map Wd_{ψ_v} from the set of irreducible genuine (unitary) representations of $\widetilde{SL}_2(F_v)$ which are not equal to $\omega^+_{\chi_v}$ for any χ_v to the set of infinite dimensional (unitary) representations of $PGL_2(F_v)$. We will not go into the definition of Wd_{ψ_v} here. Suffices to say that it involves the study of the local theta correspondence between $\widetilde{SL}_2(F_v)$ and $SO_3(F_v) \cong PGL_2(F_v)$ (cf. next section).

In any case, the map Wd_{ψ_v} leads to the following theorem:

(1.3) Theorem *There is a partition of the set of irreducible (unitary) representations of $\widetilde{SL}_2(F_v)$ which are not equal to $\omega^+_{\chi_v}$ for any χ_v, indexed by the infinite dimensional irreducible (unitary) representations of $PGL_2(F_v)$. Namely, if τ_v is such a representation of $PGL_2(F_v)$, we set*

$$\tilde{A}_{\tau_v} = \text{ inverse image of } \tau_v \text{ under } Wd_{\psi_v}.$$

In fact,

$$\#\tilde{A}_{\tau_v} = \begin{cases} 2 \text{ if } \tau_v \text{ is discrete series;} \\ 1 \text{ if } \tau_v \text{ is not.} \end{cases}$$

Moreover, if τ_v is unitary, so are the elements of $\tilde{A}_{\tau_v}$.

In the first case, the set $\tilde{A}_{\tau_v}$ has a distinguished element $\sigma^+_{\tau_v}$, which is characterized by the fact that $\sigma^+_{\tau_v} \otimes \tau_v$ is a quotient of the Weil representation of $\widetilde{SL}_2(F_v) \times SO(2,1)(F_v)$. The other element of $\tilde{A}_{\tau_v}$ will be denoted by $\sigma^-_{\tau_v}$: it is characterized by the fact that $\sigma^-_{\tau_v} \otimes \tau_v$ is a quotient of the Weil representation of $\widetilde{SL}_2(F_v) \times SO(3)(F_v)$ (anisotropic $SO(3)$ here). In the second case, we shall let $\sigma^+_{\tau_v}$ be the unique element in $\tilde{A}_{\tau_v}$ and set $\sigma^-_{\tau_v} = 0$.

Note that this parametrization of the packets on $\widetilde{SL}_2$ in terms of representations of PGL_2 depends on the choice of the character ψ_v. Also, it is quite explicit. For example, in the case $F_v = \mathbb{R}$ and $\psi_v(x) = exp(2\pi ix)$, if τ_v is the discrete series representation of extremal weights $\pm 2k$ (with k an integer), then $\sigma^+_{\tau_v}$ (resp. $\sigma^-_{\tau_v}$) is the holomorphic (resp. antyi-holomorphic) discrete series representation with lowest (resp. highest) weight $k + \frac{1}{2}$.

We also remark that the above discussion in not entirely accurate when $F_v = \mathbb{C}$, for in this case the map Wd_ψ may send some unitary representations of $\widetilde{SL}_2(\mathbb{C}) = SL_2(\mathbb{C}) \times \{\pm 1\}$ to non-unitary representations of $PGL_2(\mathbb{C})$. However, these representations of $\widetilde{SL}_2(\mathbb{C})$ do not intervene in the space of cusp forms and so can be safely ignored for global purposes.

(1.4) Cusp forms of $\widetilde{SL}_2(\mathbb{A})$. Let $\widetilde{SL}_2(\mathbb{A})$ be the two-fold cover of $SL_2(\mathbb{A})$, and fix a non-trivial unitary character $\psi = \prod_v \psi_v$ of $F \backslash \mathbb{A}$. Let $\widetilde{\mathcal{A}}_2$ denote the space of square-integrable genuine automorphic forms on $\widetilde{SL}_2(\mathbb{A})$. Then there is an orthogonal decomposition

$$\widetilde{\mathcal{A}}_2 = \widetilde{\mathcal{A}}_{00} \bigoplus \left(\bigoplus_\chi \widetilde{\mathcal{A}}_\chi \right).$$

Here, χ runs over all quadratic characters (possibly trivial) of $F^\times \backslash \mathbb{A}^\times$.

Now the space $\bigoplus_\chi \widetilde{\mathcal{A}}_\chi$ is what people called the space of "elementary theta functions". It is a space which is very well-understood. Indeed, let us describe the space $\widetilde{\mathcal{A}}_\chi$ more concretely. If $\omega_\chi = \otimes_v \omega_{\chi_v}$ is the global Weil representation attached to χ, then the formation of theta series gives a map

$$\theta_\chi : \omega_\chi \to \widetilde{\mathcal{A}}_2,$$

whose image is the space $\widetilde{\mathcal{A}}_\chi$. To describe the decomposition of $\widetilde{\mathcal{A}}_\chi$, for a finite set S of places of F, let us set

$$\omega_{\chi,S} = (\otimes_{v \in S} \omega_{\chi_v}^-) \otimes (\otimes_{v \notin S} \omega_{\chi_v}^+)$$

so that

$$\omega_\chi = \bigoplus_S \omega_{\chi,S}.$$

Then we have

$$\widetilde{\mathcal{A}}_\chi \cong \bigoplus_{\#S \text{ even}} \omega_{\chi,S}.$$

Moreover, $\omega_{\chi,S}$ is cuspidal if and only if S is non-empty.

(1.5) Multiplicity one result. Thus, the main problem in the study of cusp forms on $\widetilde{SL}_2(\mathbb{A})$ is the description of $\widetilde{\mathcal{A}}_{00}$. In [W1], Waldspurger showed:

(1.6) Theorem *$\widetilde{\mathcal{A}}_{00}$ (and also $\widetilde{\mathcal{A}}_2$) satisfies multiplicity one.*

This theorem is proved by studying the global theta correspondence for $\widetilde{SL}_2$ and $SO_3 \cong PGL_2$, and then appealing to the multiplicity one theorem for PGL_2.

(1.7) Near equivalence classes. Note that $\widetilde{SL}_2$ does not satisfy strong multiplicity one: there are non-isomorphic cuspidal representations π_1 and π_2 whose local components $\pi_{1,v}$ and $\pi_{2,v}$ are isomorphic for almost all places v. We say that such π_1 and π_2 are **nearly equivalent**; this is an equivalence relation on abstract representations. In the paper [W2], Waldspurger described the near equivalence classes of representations in $\widetilde{\mathcal{A}}_{00}$. Let us describe his results.

Given a cuspidal automorphic representation $\tau = \otimes_v \tau_v$ of PGL_2, we define a set of irreducible unitary representations of $\widetilde{SL}_2(\mathbb{A})$ as follows. Recall that for each place v, we have a local "packet"

$$\tilde{A}_{\tau_v} = \{\sigma_{\tau_v}^+, \sigma_{\tau_v}^-\}$$

where $\sigma_{\tau_v}^- = 0$ if τ_v is not discrete series. Now set

$$\tilde{A}_\tau = \{\sigma = \otimes_v \sigma_{\tau_v}^{\epsilon_v} : \epsilon_v = \pm \text{ and } \epsilon_v = + \text{ for almost all } v\}.$$

This is the global "packet" of $\widetilde{SL}_2(\mathbb{A})$ associated to the cuspidal representation τ of PGL_2.

For

$$\sigma = \otimes_v \sigma_{\tau_v}^{\epsilon_v} \in \tilde{A}_\tau,$$

let us set

$$\epsilon_\sigma = \prod_v \epsilon_v.$$

Then we have [W2, Pg. 286, Cor. 1 and 2]:

(1.8) Theorem

$$\widetilde{\mathcal{A}}_{00} = \bigoplus_{cuspidal\ \tau} \widetilde{\mathcal{A}}(\tau)$$

where each $\widetilde{\mathcal{A}}(\tau)$ is a near equivalence class of cuspidal representations and is given by:

$$\widetilde{\mathcal{A}}(\tau) = \bigoplus_{\sigma \in \tilde{A}_\tau : \epsilon_\sigma = \epsilon(\tau, 1/2)} \sigma.$$

(1.9) Remarks: Note that in the above theorem, there are some τ which has $\widetilde{\mathcal{A}}_\tau = 0$. Indeed, this happens precisely when τ_v is principal series for all v (so that the global packet $\tilde{A}_\tau$ is a singleton set) and $\epsilon(\tau, 1/2) = -1$. Such τ can also be characterized by the fact that (cf. [W2, Lemma 41, Pg. 282])

$$\epsilon(\tau \otimes \chi, 1/2) = -1 \qquad \text{for any quadratic character } \chi \text{ of } F^\times \backslash \mathbb{A}^\times.$$

§2. Theta Correspondence

In this section, we give a brief discussion of the basic setup and results in classical theta correspondence. For the general idea, the articles in Corvallis suffice. For a more detailed treatment of the local theory, one can consult the notes of Kudla [K1] or the article of Li [L1].

(2.1) Dual pairs. Suppose that $Sp(\mathbb{W})$ is a symplectic group. We are interested in reductive subgroups G_1 and G_2 of $Sp(\mathbb{W})$ such that each is the centralizer of the other. Such a pair of subgroups is called a dual pair. The possible dual pairs in $Sp(\mathbb{W})$ has been classified by Howe. The standard example is obtained as follows. If (V, q) is a quadratic space and W a symplectic space, then $\mathbb{W} := V \otimes W$ becomes a symplectic space naturally, and we have natural inclusions

$$O(V, q) \hookrightarrow Sp(\mathbb{W}) \quad \text{and} \quad Sp(W) \hookrightarrow Sp(\mathbb{W}).$$

The groups $O(V, q)$ and $Sp(W)$ form a dual pair.

(2.2) Double covers. Now assume we are working over a local field F_v. The symplectic group $Sp(\mathbb{W}_v)$ has a unique (non-linear) double cover $\widetilde{Sp}(\mathbb{W}_v)$ (unless $F_v = \mathbb{C}$). This double cover always split over the subgroup $O(V,q)$. The splitting may not be unique, but using the quadratic form q, one can specify a particular splitting. The cover splits over $Sp(W)$ if and only if $dim(V)$ is even. In the case of interest in these notes, $dim(V) = 5$ and $dim(W) = 2$. Thus, we are forced to work with non-linear groups. In particular, we have:

$$G_1 \times G_2 := \widetilde{Sp}(W) \times SO(V,q) \longrightarrow \widetilde{Sp}(\mathbb{W}_v).$$

(2.3) Weil representations. Fix a non-trivial additive character ψ of F_v. Then $\widetilde{Sp}(\mathbb{W}_v)$ has a representation ω_{ψ_v} called the Weil representation associated to ψ_v. Like the case of $\widetilde{SL}_2(F_v)$ discussed earlier, it is the sum of two irreducible representations. We may pullback this representation to the group $G_{1,v} \times G_{2,v}$. The resulting representation is denoted by $\omega_{\psi_v,q}$ (since it depends on the choice of splitting over $SO(V,q)$). The representation $\omega_{\psi,q}$ have various concrete realizations which are essential in applications. For the cases of interest here, one can find formulas for these realizations of $\omega_{\psi,q}$ in [GG,§3] among other places.

(2.4) Local Howe conjecture. Now $\omega_{\psi,q}$ is of course highly reducible and we are interested in how it breaks up into irreducibles. Since we are not working with finite groups here, we need to formulate more precisely what we mean. Suppose that π_v is an irreducible representation of $G_{1,v}$. We let $\omega_{\psi_v,q}[\pi_v]$ denote the maximal π_v-isotypic quotient of $\omega_{\psi_v,q}$. Thus

$$\omega_{\psi_v,q}[\pi_v] = \omega_{\psi_v,q} / \bigcap_{\phi} Ker(\phi)$$

where ϕ runs over all non-zero equivariant maps $\phi : \omega_{\psi_v,q} \to \pi_v$. As a representation of $G_{1,v}$, it is an isotypic sum of π_v.

Now because $G_{1,v}$ commutes with $G_{2,v}$, $\omega_{\psi_v,q}[\pi_v]$ also inherits an action of $G_{2,v}$, and as a representation of $G_{1,v} \times G_{2,v}$, we can write:

$$\omega_{\psi_v,q}[\pi_v] = \pi_v \otimes \theta(\pi_v)$$

for some smooth (possibly zero) representation $\theta(\pi_v)$ of $G_{2,v} = SO(V,q)$. We further let $\Theta(\pi_v)$ be the maximal semisimple quotient of $\theta(\pi_v)$. Clearly, we have analogous definitions with the roles of G_1 and G_2 reversed.

The following is a conjecture of Howe:

Howe's Conjecture for the pair $G_1 \times G_2$: For any representation π_v of G_1, either $\theta(\pi_v)$ is zero or else its maximal semisimple quotient $\Theta(\pi_v)$ is non-zero irreducible. Moreover, the analogous statement with the roles of G_1 and G_2 exchanged also hold.

A trivial reformulation of this conjecture is: for each representation π_v of $G_{1,v}$, there exists at most one irreducible representation τ_v of $G_{2,v}$ such that

$$Hom_{G_{1,v} \times G_{2,v}}(\omega_{\psi_v,q}, \pi_v \otimes \tau_v) \neq 0,$$

and if such a τ_v exists (call it $\Theta(\pi_v)$), the above Hom space is 1-dimensional. Moreover, if $\Theta(\pi_v) \cong \Theta(\pi'_v)$ are non-zero, then $\pi_v \cong \pi'_v$.

In fact, this conjecture is almost totally proved:

(2.5) Theorem *(i) Howe's conjecture is true over all archimedean local fields (proved by Howe) and all p-adic fields with $p \neq 2$ (proved by Waldspurger [W3]).*

(ii) For any p, Howe's conjecture for π_v is true if π_v is supercuspidal (proved by Kudla [K2]); i.e. either $\Theta(\pi_v) = 0$ or is irreducible.

There are of course other isolated cases (of pairs $G_1 \times G_2$) for which the conjecture is proven; some instances may be found in [R1]. In the case of interest here, i.e. $\widetilde{SL}_2 \times SO(5)$, the remaining case of $p = 2$ can be checked directly by hand. Thus in the rest of these notes, we shall assume that the local Howe conjecture is known. A corollary is:

(2.6) Corollary *The map $\pi_v \mapsto \Theta(\pi_v)$ gives an* **injective** *map Θ_v from a subset of the admissible dual of $\widetilde{Sp}(W)$ to the admissible dual of $SO(V, q)$ (namely, Θ_v is defined on those π_v such that $\Theta(\pi_v) \neq 0$). This map is called the* **local theta lift**.

(2.7) Remarks: Note that the local theta correspondence depends on the choices of the character ψ_v and the quadratic form q (and not just on the orthogonal group). However, $\omega_{\psi_\lambda,q} = \omega_{\psi,\lambda q}$. The main unsolved problem in local theta correspondence is the explicit description of $\Theta(\pi_v)$ given π_v.

(2.8) Stable Range. There are certain favorable circumstances which ensure that $\Theta(\pi_v)$ is non-zero for any π_v, so that the map Θ_v in the above corollary is defined on the whole admissible dual. One such example is the case of **stable range**.

DEFINITION: Say that $(Sp(W), SO(V, q))$ is in the stable range (with $Sp(W)$ the smaller group) if (V, q) contains an isotropic subspace whose dimension is $\geq \dim(W)$.

For example, if V is split of dimension 5, then $(SL_2, SO(V))$ is in the stable range.

(2.9) Theorem *Suppose that (G_1, G_2) is in the stable range with G_1 the smaller group. Then for any representation π_v of $G_{1,v}$, $\Theta(\pi_v) \neq 0$ (cf. [K2]). Moreover, if π_v is unitary, then so is $\Theta(\pi_v)$ (this was proved by Li [L3]).*

(2.10) Global Theta lift. Now we come to the global setting. Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$. Then we have the global Weil representation of $\widetilde{Sp}(\mathbb{W})(\mathbb{A})$

$$\omega_\psi = \otimes_v \omega_{\psi_v}.$$

By pulling back to $G_{1,\mathbb{A}} \times G_2(\mathbb{A})$, we have the representation $\omega_{\psi,q}$.

It turns out that there is a natural map

$$\theta : \omega_\psi \longrightarrow \mathcal{A}_2(\widetilde{Sp}(\mathbb{W}))$$

of ω_ψ to the space of square-integrable automorphic forms on $\widetilde{Sp}(\mathbb{W})$. For $\varphi \in \omega_\psi$, we may pullback the function $\theta(\varphi)$ to the group $\widetilde{Sp}(W)(\mathbb{A}) \times SO(V,q)(\mathbb{A})$. This function is of moderate growth on the adelic points of the dual pair. The space of functions thus obtained is a quotient of the representation $\omega_{\psi,q}$.

Now let $\pi \subset \mathcal{A}(G_{1,\mathbb{A}})$ be a cuspidal representation. For $f \in \pi$ and $\varphi \in \omega_{\psi,q}$, we set:

$$\theta(\varphi, f)(g) = \int_{G_{1,F}\backslash G_{1,\mathbb{A}}} \theta(\varphi)(gh) \cdot \overline{f(h)}\, dh.$$

Then $\theta(\varphi, f)$ is an automorphic form on $G_2 = SO(V,q)$. Denote the space of automorphic forms spanned by the $\theta(\varphi, f)$ for all φ and f by $V(\pi)$; it is a $G_2(\mathbb{A})$-submodule in $\mathcal{A}(G_2)$ and is called the **global theta lift** of π.

A main question in theta correspondence is to decide if $V(\pi)$ is non-zero. When (G_1, G_2) is in stable range, then in fact one can show that $V(\pi)$ is always non-zero.

(2.11) Local-global compatibility. How is the representation $V(\pi)$ related to the irreducible representation $\Theta(\pi) := \otimes_v \Theta(\pi_v)$? We have:

(2.12) Proposition *Assume the local Howe conjecture holds for the pair $G_1 \times G_2$ at every place v. Suppose that $V(\pi)$ is non-zero and is contained in the space of square-integrable automorphic forms on G_2. Then $V(\pi) \cong \Theta(\pi)$.*

Proof. We are told that $V(\pi)$ is semisimple. Let τ be an irreducible summand of $V(\pi)$. Then consider the linear map

$$\omega_{\psi,q} \otimes \pi^\vee \otimes \tau^\vee \longrightarrow \mathbb{C}$$

defined by:

$$\varphi \otimes \overline{f_1} \otimes \overline{f_2} \mapsto \int_{G_2(F)\backslash G_2(\mathbb{A})} \theta(\varphi, f_1)(g) \cdot \overline{f_2(g)}\, dg.$$

This map is non-zero and $G_{1,\mathbb{A}} \times G_2(\mathbb{A})$-equivariant. Thus it gives rise to a non-zero equivariant map

$$\omega_{\psi,q} \longrightarrow \pi \otimes \sigma,$$

and thus for all v, a non-zero $G_{1,v} \times G_{2,v}$-equivariant map

$$\omega_{\psi_v,q} \longrightarrow \pi_v \otimes \tau_v.$$

In other words, we must have

$$\tau_v \cong \Theta(\pi_v).$$

Hence, $V(\pi)$ must be an isotypic sum of $\Theta(\pi)$. Moreover, the multiplicity-one statement in Howe's conjecture implies that

$$\dim \mathrm{Hom}_{G_{1,\mathbb{A}} \times G_2(\mathbb{A})}(\omega_{\psi,q}, \pi \otimes \Theta(\pi)) = 1.$$

Thus $V(\pi)$ is in fact irreducible and isomorphic to $\Theta(\pi)$. ∎

(2.13) Multiplicity preservation. Note that because the local theta lifting is an injective map on its domain, we know that if π_1 and π_2 are two cuspidal representations which are non-isomorphic, then $V(\pi_1) \ncong V(\pi_2)$. We come now to the question of multiplicity preservation, first observed by Rallis in [R1]. Suppose that the multiplicity of an abstract representation π in the space of cusp forms of G_1 is $m_{cusp}(\pi)$. Let $\mathcal{A}_{cusp}(\pi)$ be the π-isotypic subspace. We have:

(2.14) Proposition *Suppose that:*

- *the local Howe conjecture holds for the pair $G_1 \times G_2$ at every place v;*
- *for any irreducible summand $\pi_0 \subset \mathcal{A}_{cusp}(\pi)$, $V(\pi_0)$ is non-zero and is contained in the space of square-integrable automorphc forms on G_2.*

Then the multiplicity of the irreducible representation $\Theta(\pi)$ in $V(\mathcal{A}_{cusp}(G_1))$ is equal to $m_{cusp}(\pi)$.

Proof. The proof is similar to that of the previous proposition. For simplicity, let us denote

$$V_1 = \mathrm{Hom}_{G_1}(\pi, \mathcal{A}_{cusp}(G_1))$$

and

$$V_2 = \mathrm{Hom}_{G_2}(\Theta(\pi), V(\mathcal{A}_{cusp}(\pi))).$$

Because of the injectivity of the map Θ on its domain,

$$V_2 = \mathrm{Hom}_{G_2}(\Theta(\pi), V(\mathcal{A}_{cusp}(G_1))).$$

We need to show V_1 and V_2 have the same dimension. Consider the pairing:

$$\langle -, - \rangle : V_1 \times V_2 \longrightarrow \mathrm{Hom}_{G_1 \times G_2}(\omega_{\psi,q} \otimes \pi^\vee \otimes \Theta(\pi)^\vee, \mathbb{C}) \cong \mathbb{C}$$

given by:

$$\langle f_1, f_2 \rangle (\varphi \otimes v_1 \otimes v_2) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \theta(\varphi, f_1(v_1))(g) \cdot \overline{f_2(v_2)(g)} \, dg.$$

(Note that $\pi_i^\vee \cong \overline{\pi_i}$ because π_i is unitary). The target space of this pairing is isomorphic to $\mathbb{C}$ because of the local Howe conjecture, which we assume to be true. Now it is easy to see that this pairing is perfect and thus exhibit V_1 and V_2 as linear dual of each other. This proves the proposition. ∎

(2.15) Rallis inner product formula. There is a very beautiful formula for the inner product $\langle \theta(\varphi, f), \theta(\varphi, f) \rangle_{G_2}$ (assuming that it converges absolutely) which was first obtained by Rallis ([R2] and [R3]) in certain cases. In [Li2], this was extended to more cases. For the case at hand, namely for the dual pair $\widetilde{SL}_2 \times SO(V, q)$ with $dim(V)$ odd, the results we need can be found in [R3, Thm. 6.2, Pg. 178].

(2.16) Theorem *Suppose that* $\dim(V) = n \geq 3$ *is odd. Let* σ *be a cuspidal representation of* $\widetilde{SL}_2$ *contained in* $\widetilde{A}_{00}$*, so that* σ *is in a global Waldspurger packet associated to a cuspidal representation* τ *of* PGL_2*. There is a finite set of places, including the archimedean ones and the places dividing* 2*, such that*

$$\langle \theta(\varphi_1, f_1), \theta(\varphi_2, f_2) \rangle = \left(\prod_{v \in S} I_v(\varphi_{1,v} \otimes f_{1,v}, \varphi_{2,v} \otimes f_{2,v}) \right) \cdot \frac{L^S(\tau \otimes \chi_{\mathrm{disc}(q)}, \frac{n-2}{2})}{\zeta^S(\frac{n-2}{2})\zeta^S(n-1)}.$$

Here,

$$I_v(\varphi_{1,v} \otimes f_{1,v}, \varphi_{2,v} \otimes f_{2,v}) = \int_{\widetilde{SL}(F_v)} \langle \omega_{\psi,q}(h)\varphi_1, \varphi_2 \rangle \cdot \langle f_2, \overline{\sigma_v}(h) f_1 \rangle \, dh.$$

is a sesquilinear form on $\omega_{\psi_v,q} \otimes \overline{\sigma_v}$ *and the* $\langle -, - \rangle$ *in the integral on the right refers to inner products on* $\omega_{\psi_v,q}$ *and* σ_v *(conjugate linear in second argument).*

Note that if $n \geq 5$, the special L-values are all non-vanishing. Thus the non-vanishing of the global theta lift depends on the non-vanishing of the finitely many local sesquilinear forms I_v. This question was addressed in [R3, Prop. 6.1 and Cor. 6.1]:

(2.17) Theorem *The form* I_v *is non-zero if and only if* $\Theta_v(\sigma_v) \neq 0$*, i.e. the local theta lift of* σ_v *is non-zero.*

Conclusion: The point of the 2 theorems above is that when $\dim(V) \geq 5$, the non-vanishing of the global theta lift depends entirely on the non-vanishing of the local theta lifts at all places (or rather at the finitely many places where σ_v is not unramified). Thus it is purely a local problem. When $\dim(V) = 3$, notice that we will get $L^S(\tau \otimes \chi_{disc(q)}, 1/2)$ in the inner product formula; thus in this case, there will be a global obstruction to non-vanishing of global theta lift. This was actually first observed in Waldspurger's work [W1].

§3. Arthur's conjecture on the discrete spectrum of $PGSp(4)$

In this section, we review what Arthur's conjecture says for the discrete spectrum L^2_{disc} of $G = PGSp_4$. Loosely speaking, Arthur's conjecture is a classification of the near equivalence classes of representations in $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$, in the spirit of the theorem of Waldspurger for $\widetilde{SL}_2$. We will relate this framework of Arthur to the notion of CAP (cuspidal associated to parabolics) representations (introduced by Piatetski-Shapiro). The references for Arthur's conjecture are of course [A1] and [A2].

(3.1) A basic assumption. In the formulation of Arthur's conjecture, one needs to make a (serious) assumption:

(ASSUMPTION): There is a topological group L_F (depending only on the number field F) satisfying the following properties:

- the identity component L_F^0 of L_F is compact and the group of components $L_F^0\backslash L_F$ is isomrophic to the Weil group W_F;
- for each place v, there is a natural conjugacy class of embeddings $L_{F_v} \hookrightarrow L_F$, where L_{F_v} is the Weil group if F_v is archimedean, and the Weil-Deligne group $W_{F_v} \times SL_2(\mathbb{C})$ if F_v is non-archimedean.
- there is a natural bijection between the set of isomorphism classes of irreducible representations of L_F of dimension n and the set of cuspidal representations of $GL_n(\mathbb{A})$.

This assumption is basically the main conjecture in the Langlands program for GL_n.

(3.2) ***A*-parameters.** By a (discrete) A-parameter for G, we mean an equivalence class of maps

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \hat{G} = Sp_4(\mathbb{C})$$

which satisfy some conditions:

- the restriction of ψ to L_F has bounded image (i.e. gives a tempered L-parameter);
- the restriction of ψ to $SL_2(\mathbb{C})$ is an algebraic homomorphism;
- the centralizer of the image of ψ in $Sp_4(\mathbb{C})$ is finite (so the image of ψ is not too small). We let Z_ψ be this centralizer and let S_ψ denote the quotient of Z_ψ by the center $Z_{\hat{G}} = \{\pm 1\}$ of $Sp_4(\mathbb{C})$.

(3.3) Decomposition of discrete spectrum. Now according to Arthur, the discrete spectrum possesses a decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus}_\psi L^2_\psi,$$

where the Hilbert space direct sum runs over the A-parameters ψ. For any ψ, the space L^2_ψ will be a direct sum of nearly equivalent representations, and we want to describe its internal structure next.

(3.4) Local A-packets. The global A-parameter ψ gives rise to a local A-parameter

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

for each place v of F. Denote by Z_{ψ_v} the centralizer of the image of ψ_v, and set $S_{\psi_v} = \pi_0(Z_{\psi_v}/Z_{\hat{G}}) = Z_{\psi_v}/Z^0_{\psi_v} Z_{\hat{G}}$.

Arthur speculated that to each irreducible representation η_v of S_{ψ_v}, one can attach a unitary admissible (possibly reducible, possibly zero) representation π_{η_v} of $G(F_v)$. Thus we have a finite set

$$A_{\psi_v} = \left\{ \pi_{\eta_v} \ : \ \eta_v \in \widehat{S}_{\psi_v} \right\}.$$

This is the local A-packet attached to ψ_v. Of course, there are some conditions to satisfy:

- for almost all v, π_{η_v} is irreducible and unramified if η_v is the trivial character 1_v. For such v, π_{1_v} is the unramified representation whose Satake parameter is:

$$s_{\psi_v} = \psi_v \left(Fr_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right),$$

where Fr_v is a Frobenius element at v and q_v is the number of elements of the residue field at v. In fact, it is required that for any v, π_{1_v} has Langlands parameter ϕ_{ψ_v} given by:

$$\phi_{\psi_v}(w) = \psi_v \left(w \times \begin{pmatrix} |w|_v^{1/2} & \\ & |w|_v^{-1/2} \end{pmatrix} \right)$$

for any $w \in L_{F_v}$.

- a number of other conditions concerning the character distributions which are too technical to state and which will not concern us here.

These requirements may not characterize the set A_{ψ_v} but they come pretty close. The main point to notice here is that for almost all v, we know what the representation π_{1_v} is. However, as it stands, the conjecture does not specify what the other representations in the local packets are. For example, we are not told what are their L-parameters. However, I learn the following conjecture from Moeglin:

Conjecture: If v is a finite place and

$$\psi_v : (W_{F_v} \times SL_2(\mathbb{C})) \times SL_2(\mathbb{C}) \longrightarrow \hat{G}$$

is a local A-parameter with associated local A-packet A_{ψ_v}, then any discrete series representation in A_{ψ_v} has L-parameter equal to

$$W_{F_v} \times SL_2(\mathbb{C}) \xrightarrow{id \times \Delta} W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\psi_v} \hat{G}.$$

(3.5) Global A-packets. With the local packets A_{ψ_v} at hand, we may define the global A-packet by:

$$A_\psi = \{\pi = \otimes_v \pi_{\eta_v} : \pi_{\eta_v} \in A_{\psi_v} \, \eta_v = 1_v \text{ for almost all } v\}.$$

Observe that this is a set of nearly equivalent representations of $G(\mathbb{A})$, indexed by the irreducible representations of the compact group

$$\mathcal{S}_{\psi,\mathbb{A}} := \prod_v S_{\psi,v}.$$

Note that there is a diagonal map

$$\Delta : S_\psi \longrightarrow S_{\psi,\mathbb{A}}.$$

If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi,\mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v, $\eta_v = \mathbf{1_v}$ and $\pi_{\mathbf{1_v}}$ is required to be unramified.

(3.6) Multiplicity formula. The space L^2_ψ will be the sum of the elements of A_ψ with some multiplicities. More precisely, Arthur attached to ψ a quadratic character ϵ_ψ of S_ψ; we will not give the definition here, but will say what it is when we discuss the Saito-Kurokawa representations later. Now if η is an irreducible character of $S_{\psi,\mathbb{A}}$, we set

$$m_\eta = \langle \Delta^*(\eta), \epsilon_\psi \rangle_{S_\psi} = \frac{1}{\# S_\psi} \cdot \left(\sum_{s \in S_\psi} \epsilon_\psi(s) \cdot \eta(s) \right).$$

Then Arthur conjectures that

$$L^2_\psi = \bigoplus_\eta m_\eta \pi_\eta$$

(3.7) Inner forms. We should say that the above description of Arthur's conjecture is only accurate for split groups (though it can be extended to quasi-split groups naturally). For inner forms of a split group, some modifications are necessary; we indicate these briefly.

A global A-parameter ψ for a split group G is also an A-parameter for an inner form G' provided that ψ is relevant, i.e. its image is not contained in the Levi of an irrelevant parabolic subgroup. We saw above that the representations in the local packet for $G(F_v)$ are indexed by irreducible characters η_v of $Z_{\psi_v}/Z^0_{\psi_v} Z_{\hat{G}}$. We should think of η_v as a character of $Z_{\psi_v}/Z^0_{\psi_v}$ which is trivial on $Z_{\hat{G}}$.

Now the main modification for G' is that the representations in the local packet of $G'(F_v)$ should be indexed by (some of) the characters of $Z_{\psi_v}/Z^0_{\psi_v}$ which are **not trivial** on $Z_{\hat{G}}$, at least when G is an adjoint group.

The definition of the quadratic character ϵ_ψ does not change; thus ϵ_ψ is a quadratic character of Z_ψ which is trivial on $Z_{\hat{G}}$. For a representation π_η in the global A-packet, where η is an irreducible character of $Z_{\psi,\mathbb{A}}$, the multiplicity $m(\pi_\eta)$ is given by $\langle \epsilon_\psi, \Delta^*(\eta) \rangle_{Z_\psi}$. Note that this is non-zero only if $\Delta^*(\eta)$ is trivial on $Z_{\hat{G}}$. We do not know if this condition is automatic (though for the case of interest in these notes, it is).

(3.8) The A-parameters of $PGSp_4$. Now we want to see more concretely what Arthur's conjecture says for $PGSp_4$. We first describe the A-parameters of $PGSp_4$.

We can first partition the set of A-parameters ψ of $PGSp_4$ according to the restriction of ψ to $SL_2(\mathbb{C})$. Recall that the Jacobson-Morozov theorem states that there is a bijection between the set of conjugacy classes of homomorphisms $SL_2(\mathbb{C}) \longrightarrow \hat{G}$ and the set of unipotent conjugacy classes in $\hat{G}$. The bijection is given by attaching to a morphism $\iota : SL_2 \to \hat{G}$ the conjugacy class of the unipotent element

$$\iota \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now $\hat{G} = Sp_4(\mathbb{C})$ has 4 unipotent conjugacy classes, and thus there are 4 families of A-parameters. There is a partial order on the set of unipotent classes:

$$\mathcal{O}_1 < \mathcal{O}_2 \quad \text{if and only if} \quad \mathcal{O}_1 \subset \overline{\mathcal{O}_2}.$$

We arrange the 4 unipotent classes in increasing sizes:

$$\mathcal{O}_0 = \{1\} < \mathcal{O}_{long} < \mathcal{O}_{short} < \mathcal{O}_{reg}.$$

Here $\mathcal{O}_0$ is the trivial class, $\mathcal{O}_{long}$ is the class of a non-trivial element in a long-root-subgroup, $\mathcal{O}_{short}$ is the class of a non-trivial element in a short-root-subgroup and $\mathcal{O}_{reg}$ is the principal (or regular) unipotent conjugacy class.

One can describe the morphism ι_* attached to the conjugacy class $\mathcal{O}_*$ more concretely. Obviously, ι_0 is the trivial map. Now there is a natural embedding

$$j : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \hookrightarrow Sp_4(\mathbb{C})$$

and we have:

$$\begin{cases} \iota_{long}(g) = j(g,1); \\ \iota_{short}(g) = j(g,g). \end{cases}$$

Finally, ι_{reg} gives the action of SL_2 on its irreducible 4-dimensional representation.

This partition of the set of A-parameters leads to a decomposition of the discrete spectrum into 4 pieces:

$$L^2_{disc} = \mathcal{A}_0 \oplus \mathcal{A}_{long} \oplus \mathcal{A}_{short} \oplus \mathcal{A}_{reg}.$$

We consider these 4 pieces separately:

- if ψ corresponds to the orbit $\mathcal{O}_{reg}$, then the local and global A-packets are singletons, consisting of 1-dimensional representations (quadratic characters). Thus $\mathcal{A}_{reg}$ is the direct sum of quadartic Grossencharacters.
- $\mathcal{A}_{long}$ is the so-called Saito-Kurokawa space (as explained below). This space was constructed by Piatetski-Shapiro [PS] and we will review his results below. We shall call the A-parameters in this class the *Saito-Kurokawa parameters.*
- $\mathcal{A}_{short}$ was constructed by Howe-Piatetski-Shapiro [HPS] and Soudry [So].
- $\mathcal{A}_0$ is conjecturally the tempered part of L^2_{disc}. Thus it is the most nondegenerate part of L^2_{disc}. Note that $\mathcal{A}_0$ decomposes naturaly into two parts, depending on whether the tempered parameter ψ factors through the subgroup $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. If it does, we say that ψ is tempered endoscopic. If it doesn't, we say that ψ is stable. Thus we have $\mathcal{A}_0 = \mathcal{A}_{0,end} \oplus \mathcal{A}_{0,st}$. The space $\mathcal{A}_{0,end}$ was first studied by Yoshida [Y] (for classical Siegel modular forms) and the general construction has been carried out by Roberts [R].

(3.9) CAP representations. Most of the constructions mentioned in the previous subsection were couched in the language of **CAP** representations, rather than in the framework of Arthur's conjectures.

DEFINITION: A cuspidal representation π of a *quasi-split* group G is said to be **CAP** *with respect to a parabolic subgroup* P of G if it is nearly equivalent to the irreducible constituents of an induced representation $Ind_P^G\tau$, with τ a cuspidal representation of the Levi factor of P.

In particular, a **CAP** representation is nearly equivalent to an Eisenstein series. This explains why **CAP** representations are sometimes called **shadows of Eisenstein series**.

(3.10) Inner forms. The above definition is not the right notion for non-quasi-split groups. Here is the right definition. If G' is an inner form of a quasi-split G, then G'_v and G_v are isomorphic for almost all v. Thus it makes sense to say that a representation π' of $G'(\mathbb{A})$ is nearly equivalent to a representation π of $G(\mathbb{A})$. Thus we say that a cuspidal π' of G' is **CAP** wrt a parabolic P of G if π' is nearly equivalent to the constituents of $Ind_P^G\tau$ with τ cuspidal. That this modification of the notion of **CAP** is necessary (and reasonable) is suggested by the paper of Sayag [S].

We make another definition:

DEFINITION: The Saito-Kurokawa space of $PGSp_4$ (or its inner forms) is the subspace $\mathcal{A}_{SK} \subset L^2_{disc}$ consisting of all representations which are CAP with respect to the Siegel parabolic P.

We note that Arthur's conjecture **predicts** that the cuspidal part of $\mathcal{A}_{long}$ is precisely the subspace of representations which are CAP with respect to the Siegel parabolic subgroup P, whereas the cuspidal part of $\mathcal{A}_{short}$ consists precisely of representations which are CAP with respect to the Klingen parabolic Q and the Borel subgroup B. This prediction is easy to see, because given an A-parameter ψ, Arthur's conjecture specifies the near equivalence class of the representations in the global A-packet A_ψ. Thus, we expect that $\mathcal{A}_{SK} = \mathcal{A}_{long,cusp}$ (and [PS] showed that it is indeed the case).

§4. Saito-Kurokawa A-packets

In this section, we examine the fine structure of the Saito-Kurokawa A-packets in greater detail. This will suggest a way of constructing these packets.

(4.1) Saito-Kurokawa A-parameters. Recall that we have the subgroup

$$SL_2 \times SL_2 \subset Sp_4.$$

Further, the centralizer of one of these SL_2's is the other SL_2. For a Saito-Kurokawa A-parameter ψ, since $\psi|_{SL_2(\mathbb{C})}$ is an isomorphism onto one of these SL_2, say the second one, $\psi|_{L_F}$ must send L_F into $SL_2 \times \mu_2$, where μ_2 is the center of the second SL_2. Thus to give ψ means to give a (irreducible) map $L_F \to SL_2(\mathbb{C})$ and a quadratic character $L_F \to W_F \to \mu_2$. According to our basic assumption, this means that:

Saito-Kurokawa A-parameters are (conjecturally) parametrized by pairs (τ, χ) where τ is a cuspidal representations of PGL_2 and χ is a quadratic Grossencharacter.

In other words, a typical Saito-Kurokawa parameter looks like:

$$\psi_{\tau,\chi} : L_F \times SL_2(\mathbb{C}) \xrightarrow{\rho_\tau \oplus (\chi \otimes id)} SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow Sp_4(\mathbb{C}).$$

Observe that for χ fixed, such parameters are indexed by cuspidal representations of PGL_2, just as the near equivalence classes in the space $\widetilde{\mathcal{A}}_{00}$ for $\widetilde{SL}_2$.

Given a parameter as above, it is easy to check that the centralizer (modulo center) $S_{\psi_{\tau,\chi}}$ of $\psi_{\tau,\chi}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the local component groups $S_{\psi_{\tau,\chi},v}$ are given by

$$S_{\psi_{\tau,\chi},v} = \begin{cases} 1, & \text{if } \rho_{\tau,\mathrm{v}} \text{ is reducible} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \rho_{\tau,\mathrm{v}} \text{ is irreducible.} \end{cases}$$

The condition $\rho_{\tau,v}$ is irreducible is equivalent to τ_v being a discrete series representation of $PGL_2(F_v)$.

(4.2) Local Arthur packets. Now Arthur's conjecture predicts that for each place v, the local A-packets $A_{\tau,\chi,,v}$ has the form:

$$A_{\tau,\chi,v} = \begin{cases} \{\pi^+_{\tau_v,\chi_v}\}, & \text{if } \tau_\mathrm{v} \text{ is not discrete series} \\ \{\pi^+_{\tau_v,\chi_v}, \pi^-_{\tau_v,\chi_v}\}, & \text{if } \tau_\mathrm{v} \text{ is discrete series.} \end{cases}$$

Here, $\pi^+_{\tau_v,\chi_v}$ is indexed by the trivial character of $S_{\psi_{\tau,\chi},v}$.

Of course, we know what π^+_v has to be for almost all v: it is irreducible unramified with Satake parameter $s_{\psi_{\tau_v,\chi_v}}$. This unramifed representation π^+_v can be alternatively expressed as $J_P(\tau_v, \chi_v, 1/2)$, i.e. the unique irreducible quotient of the generalized principal series unitarily induced from the representation $\tau_v \otimes \chi_v| - |_v^{\frac{1}{2}}$ of the Levi $PGL_2 \times GL_1$ of P. From this, we see that the representations in the global A-packet are nearly equivalent to the constitutents of $Ind_P^G \tau \otimes \chi| - |^{1/2}$. Thus representations in $\mathcal{A}_{long}$ are indeed CAP wrt P.

In fact, one knows what $\pi^+_{\tau_v,\chi_v}$ is in general, for Arthur's conjecture specifies the L-parameter of $\pi^+_{\tau_v,\chi_v}$. In the case at hand, the L-packet for this L-parameter is a singleton. Thus we must have:

$$\pi^+_{\tau_v,\chi_v} \cong J_P(\tau_v, \chi_v, 1/2) \quad \text{for all } v.$$

When we construct the local packets later using theta correspondence, we should check that this is indeed the case for our construction.

The main observation to make here is that (for fixed χ_v) the structure of the local SK A-packet attached to τ_v is identical to the Waldspurger packet of $\widetilde{SL}_2$ associated to τ_v.

(4.3) Global A-packets. Let S_τ be the set of places v where τ_v is discrete series, so that the global A-packet has $2^{\# S_\tau}$ elements. This global packet will contribute to a subspace of L^2_{disc}; we denote the corresponding subspace y $\mathcal{A}(\tau, \chi)$.

To describe the multiplicity of $\pi_\eta \in A_{\tau,\chi}$ in $\mathcal{A}(\tau, \chi)$, we need to know the quadratic character $\epsilon_{\psi_{\tau,\chi}}$ of $S_{\psi_{\tau,\chi}}$. It turns out that $\epsilon_{\psi_{\tau,\chi}}$ is the non-trivial character of $S_{\psi_{\tau,\chi}} \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\epsilon(\tau \otimes \chi, 1/2) = -1$.

Now if $\pi = \otimes_v \pi^{\epsilon_v}_{\tau_v,\chi_v} \in A_{\tau,\chi}$, then the multiplicity associated to π by Arthur's conjecture is:

$$m(\pi) = \begin{cases} 1, & \text{if } \epsilon_\pi := \prod_v \epsilon_v = \epsilon(\tau \otimes \chi, 1/2); \\ 0, & \text{if } \epsilon_\pi = -\epsilon(\tau \otimes \chi, 1/2). \end{cases}$$

Thus, we should have:

$$\mathcal{A}(\tau,\chi) \cong \bigoplus_{\pi \in A_{\tau,\chi} : \epsilon_\pi = \epsilon(\tau\otimes\chi, 1/2)} \pi.$$

Again, observe that the automorphy of the representations in the global SK A-packet are controlled by the same condition on ϵ-factors as the automorphy of the representations in the global Waldspurger packet associated to τ.

Conclusion: The above discussion shows that the structure of the space $\widetilde{\mathcal{A}}_{00}$ on $\widetilde{SL}_2$ and the space $\mathcal{A}_{long}$ on $PGSp_4$ are identical. This should suggest that one can construct the space $\mathcal{A}_{long}$ by "lifting" from $\widetilde{SL}_2$.

How does one carry out this "lifting"? This can be done using theta correspondence.

§5. Construction of Saito-Kurokawa space

Now we are ready to state the results of Piatetski-Shapiro on the space $\mathcal{A}_{SK}$. Consider the dual pair $\widetilde{SL}_2 \times SO(5)$, with $SO(5)$ split. Now to define theta correspondence, it is necessary to specify the quadratic space (V, q) giving rise to $SO(5)$. How many choices are there?

(5.1) Odd quadratic spaces and orthogonal groups. Fix a quadratic space V over F. Then the quadratic spaces over F of same dimension as V are classified by $H^1(F, O(V))$. When $\dim(V)$ is odd, the different forms of $SO(V)$ are classified by $H^1(F, SO(V))$ and we have:

$$H^1(F, O(V)) = H^1(F, SO(V)) \times H^1(F, \mu_2) = H^1(F, SO(V)) \times F^\times/F^{\times 2}.$$

The projection onto $F^\times/F^{\times 2}$ simply gives the discriminant of a quadratic space, whereas the projection onto $H^1(F, SO(V))$ gives the isomorphism class of the associated special orthogonal group. From this, we see that the set of isomorphism classes of quadratic spaces which give rise to a particular special orthogonal group is a principal homogeneous space for $F^\times/F^{\times 2}$ and these quadratic spaces are distinguished by their discriminants.

Given a form G of SO_{2n+1}, there is thus a unique quadratic space of discriminant one whose special orthogonal group is G.

(5.2) Twisted theta lifts. In the following, whenever we talk about theta correspondence for $\widetilde{SL}_2 \times G$, it will be defined using this distinguished quadratic space of discriminant one. For convenience, we define the twisted theta lift:

DEFINITION: For a quadratic character χ, the χ-twisted theta lift of a representation σ of $\widetilde{SL}_2$ (both locally and globally) is:

$$\Theta_\chi(\sigma) = \Theta(\sigma) \otimes \chi.$$

(5.3) Construction of $\mathcal{A}_{SK}$. By our discussion of the previous section, it is reasonable to expect that $\mathcal{A}_{long}$ can be constructed by theta lifting from $\widetilde{SL}_2$. More precisely, if we fix a cuspidal representation τ of PGL_2 and a quadratic Grossencharacter χ (which corresponds to an element λ_χ of $F^\times/F^{\times 2}$), then we shall construct the subspace $\mathcal{A}(\tau,\chi)$ of the discrete spectrum by:

$$\mathcal{A}(\tau,\chi) = \Theta_\chi(\widetilde{A}(\tau \otimes \chi)).$$

The question is: is this a reasonable definition?

(5.4) Local results. We begin first with local considerations. The local representation $\tau_v \otimes \chi_v$ determines a local Waldspurger packet $\{\sigma_v^+, \sigma_v^-\}$ ($\sigma_v^- = 0$ if τ_v is principal series). Consider the local theta lifts of the elements of this packet:

$$\pi^\pm_{\tau_v,\chi_v} = \Theta_\chi(\sigma_v^\pm).$$

These representations are irreducible and unitary. Moreover, all such representations are distinct as τ_v ranges over all unitary representations of $PGL_2(F_v)$ (with χ_v fixed).

Now let A_{τ_v,χ_v} be the local A-packet of $PGSp_4(F_v)$ attached to the Saito-Kurokawa parameter determined by (τ_v,χ_v). Then we **define**:

$$A_{\tau_v,\chi_v} = \{\pi^+_{\tau_v,\chi_v}, \pi^-_{\tau_v,\chi_v}\}.$$

This is a reasonable definition, because we have the following proposition (which verifies directly that Howe's conjecture holds)

(5.5) Proposition *(i) $\Theta_{\chi_v}(\sigma_v^+) \cong J_P(\tau_v,\chi_v,1/2)$; it has L-parameter $\rho_{\tau_v} \oplus \rho_{\chi_v}$.*

(ii) When v is archimedean, $\Theta_{\chi_v}(\sigma_v^-)$ can be completely determined (it is a discrete series or limit of discrete series; we omit the description here but see the example at the end of the section). When v is finite,

$$\Theta_{\chi_v}(\sigma_v^-) = \begin{cases} \text{supercuspidal, if } \tau_v \otimes \chi_v \text{ is not Steinberg;} \\ \text{tempered, if } \tau_v \otimes \chi_v \text{ is Steinberg.} \end{cases}.$$

In fact, when $\tau_v \otimes \chi_v$ is Steinberg, $\Theta_{\chi_v}(\sigma_v^-)$ is the unique non-generic summand in $I_Q(St_{\chi_v})$.

(iii) Moreover, $\Theta_{\chi_v}(\sigma_1) = \Theta_{\chi_v}(\sigma_2)$ implies that $\sigma_1 = \sigma_2$.

The proof of the proposition is an easy exercise, but does require one to know the formulas for the action of the Weil representation $\omega_{\psi,q}$. From (i), we see that the representation $\pi^+_{\tau_v,\chi_v}$ has the expected Langlands parameters required by Arthur's conjecture. The only thing which is not explicit in the proposition is that it does not give some alternative description of $\pi^-_{\tau_v,\chi_v}$; for example, it does not give its Langlands parameter. The main result of Schmidt's paper [S] is the resolution of this local issue:

(5.6) Proposition *The Langlands parameter of $\Theta_{\chi_v}(\sigma_v^-)$ factors through the subgroup $SL_2 \times SL_2$ and as a map into this subgroup, is given by $\rho_{\tau_v} \oplus \rho_{St_{\chi_v}}$. Here, we are using the definition of local L-packets given by Roberts [R]. In particular, this is in agreement with the conjecture of Moeglin mentioned in (3.4).*

(5.7) Global results. Now suppose that

$$\sigma = \otimes_v \sigma_v^{\epsilon_v} \subset \widetilde{A}(\tau \otimes \chi) \quad \text{with} \quad \prod_v \epsilon_v = \epsilon(\tau \otimes \chi, 1/2).$$

Its χ-twisted global theta lift $V_\chi(\sigma)$ is non-zero (stable range) and one can check that $V_\chi(\sigma)$ is contained in the space of suqare-integrable automorphic forms (i.e. $V_\chi(\sigma) \subset L^2_{disc}$). Thus, we have:

$$V_\chi(\sigma) \cong \Theta_\chi(\sigma) := \otimes_v \Theta_{\chi_v}(\sigma_v).$$

By the injectivity of local theta correspondence, we know that if $\sigma_1 \ncong \sigma_2$, then $V_\chi(\sigma_1) \ncong V_\chi(\sigma_2)$. Thus, we have:

(5.8) Proposition *Let $A_{\tau,\chi}$ be the global A-packet obtained from the local A-packets A_{τ_v,χ_v} defined above. The χ-twisted global theta lift of $\widetilde{A}(\tau \otimes \chi)$ is a subspace $\mathcal{A}(\tau,\chi)$ of L^2_{disc} with*

$$\mathcal{A}(\tau,\chi) \cong \bigoplus_{\pi \in A_{\tau,\chi} : \epsilon_\pi = \epsilon(\tau\otimes\chi,1/2)} \pi.$$

Here $\pi = \otimes_v \pi^{\epsilon_v}_{\tau_v,\chi_v}$ and $\epsilon_\pi = \prod_v \epsilon_v$. Moreover, $\pi = V_\chi(\sigma)$ is contained in the space of cusp forms, unless $L(\tau \otimes \chi, 1/2) \neq 0$ and $\sigma = \sigma^+ := \otimes_v \sigma_v^+$. Moreover, $\mathcal{A}(\tau \otimes \chi, 1) \otimes \chi = \mathcal{A}(\tau, \chi)$.

The proposition says that our definition of the local and global packets leads to a subspace of the discrete spectrum in perfect agreement with Arthur's conjecture. Moreover, the cuspidal π are contained in $\mathcal{A}_{SK}$, since they are CAP with respect to P. In particular, if $\pi \subset \mathcal{A}(\tau,\chi)$, then its partial standard L-functions has the form:

$$L^S(\pi, s) = L^S(\tau, s) \cdot L^S(\chi, s + 1/2) \cdot L^S(\chi, s - 1/2).$$

Thus $L(\pi \otimes \chi, s)$ has a pole at $s = 3/2$. This also shows that the spaces $\mathcal{A}(\tau,\chi)$ have non-isomorphic irreducible constituents (and thus are mutually disjoint), since the standard L-functions are different for different χ. Thus if $Proj_{cusp}$ denotes orthogonal projection onto the cuspidal spectrum, then we have:

$$Proj_{cusp}\left(\bigoplus_{(\tau,\chi)} \mathcal{A}(\tau,\chi)\right) \subset \mathcal{A}_{SK}.$$

The following theorem is the main result of [PS]:

(5.9) Theorem

$$Proj_{cusp}\left(\bigoplus_{(\tau,\chi)} \mathcal{A}(\tau,\chi)\right) = \mathcal{A}_{SK}.$$

In fact, the representations in $\mathcal{A}_{SK}$ can be characterized as those cuspidal representations π such that the standard L-function of some quadratic twists of π has a pole somewhere (in which case, a pole must occur at $s = 3/2$).

The proof of this uses the theory of standard L-functions for $GSp(4)$. It is quite intricate and is beyond the scope of these notes. In the last section, we present a proof of a slightly weaker result based on the Rallis inner product formula, the regularized Siegel-Weil formula and the doubling method of Piatetski-Shapiro and Rallis.

A corollary of the theorem is:

(5.10) Corollary *The space $\mathcal{A}_{SK}$ has multiplicity one.*

Proof. This follows from the theorem, the multiplicity-one result for $\widetilde{A}_{00}$, the multiplicity preservation of theta correspondence and the fact that the χ-twists of the space constructed from the theta lifts have non-isomorphic irreducible summands for different χ. ∎

(5.11) Examples. We conclude this section with an example. Suppose that f is a classical holomorphic newform of level N and weight $2k$. For no other reason but for simplicity, we assume that N is square-free with S distinct prime factors. Then f corresponds to a cuspidal representation τ_f of PGL_2 such that

$$\tau_{f,v} = \begin{cases} \text{unramified representation, if } v \text{ does not divide } N; \\ \text{the Steinberg representation, if } v \text{ divides } N; \\ \text{holomorphic discrete series of lowest weight } 2k, \text{ if } v \text{ is infinite.} \end{cases}$$

Moreover, $\epsilon(\tau_f, 1/2) = (-1)^S \cdot (-1)^k = (-1)^{S+k}$.

The local Saito-Kurokawa packet associated to τ_f and the trivial (quadratic) character thus has two elements π_v^+ and π_v^- if v is infinite or divides N. For a finite v, we have described these representations above. At the real place, these 2 representations can be described as the cohomologically induced modules $A_{\mathfrak{q}^\pm}(\lambda_k)$ where $\mathfrak{q}^+$ (resp. $\mathfrak{q}^-$) is the θ-stable Siegel parabolic subalgebra whose Levi subalgebra is the complexification of $U(1,1)$ (resp. $U(2)$) and $\lambda_k = det^{k-2}$. This shows that π_v^- is a holomorphic discrete series when $k \geq 2$ and is a limit of holomorphic discrete series when $k = 1$.

The global Saito-Kurokawa packet associated to τ_f has 2^{S+1} elements but by the multiplicity formula, only half of them occur in the discrete spectrum. More precisely, $\pi = \otimes_v \pi_v^{\epsilon_v}$ occurs in the discrete specturm iff $\prod_v \epsilon_v = (-1)^{S+k}$.

§6. Transfer of Saito-Kurokawa representations between inner forms

In this section, we consider the transfer of the Saito-Kurokawa space to the other forms of $PGSp_4$; these are of the form $SO(V,q)$ with $\dim(V) = 5$. Given such a form G', it is natural to construct the transfer of $\mathcal{A}_{SK}$ from $PGSp_4$ to G' via the theta correspondence from $\widetilde{SL}_2$. Thus, given (τ, χ), we want to examine the structure of the χ-twisted theta lift of $\widetilde{\mathcal{A}}(\tau)$ to G'.

If σ is an irreducible constituent of $\widetilde{A}(\tau)$, the global theta lift $\Theta'_\chi(\sigma)$ may now be zero (since we are no longer in the stable range). However, the Rallis inner product formula tells us that it is non-zero if and only if for each v, the local theta lift of σ_v to G'_v is non-zero. Thus, we shall first consider the question of local theta lifts. After that, we can ask: if $\Theta'_\chi(\sigma)$ is non-zero, which representation of $PGSp_4$ is it lifted from (say, in the sense of Langlands parameter)?

(6.1) Local theta lifts. Fix a place v. If v is not complex, there is a unique special orthogonal group G'_v whose F_v-rank is 1. Consider the local theta correspondence for $\widetilde{SL}_2(F_v) \times G'_v$. The following can be found in [R3]:

(6.2) Proposition *Let σ be an irreducible representation of $\widetilde{SL}_2(F_v)$. Then $\Theta'_{\chi_v}(\sigma) \neq 0$ if and only if $\mathrm{Hom}_{U(F_v)}(\sigma, \mathbb{C}_{\psi_a}) \neq 0$ for some $a \notin F^{\times 2}$. In fact, the only σ's whose local theta lift is 0 are given by:*

- *(i) if v is finite, then $\sigma_v = \omega^{\pm}_{\psi_v}$;*
- *(ii) if v is real, then σ is holomorphic discrete series or $\sigma = \omega^{\pm}_{\psi_v}$.*

For a finite place, the group G' is the only non-trivial form of $PGSp_4$. On the other hand, when v is real, there is an additional form G_c which is compact which is defined by a definite quadratic space. For this we have:

(6.3) Proposition *If G' is anisotropic, then $\Theta'(\sigma) \neq 0$ iff σ is holomorphic discrete series of lowest weight $\geq 5/2$.*

(6.4) Local packets. We can now define the local packets for the group G'. Suppose we are given a local A-parameter of Saito-Kurokawa type associated to the pair (τ_v, χ_v). We have the local Waldspurger packet $\widetilde{A}_{\tau_v \otimes \chi_v} = \{\sigma_v^+, \sigma_v^-\}$ associated to $\tau_v \otimes \chi_v$ and we consider the set of χ_v-twisted local theta lifts

$$\{\Theta'_{\chi_v}(\sigma_v^+), \Theta'_{\chi_v}(\sigma_v^-)\}.$$

We want to define this as the local packet on G'_v attached to the A-parameter $\psi = \psi_{\tau_v, \chi_v}$. Recall that the representations in the local packet should be indexed by the irreducible characters of Z_ψ / Z_ψ^0 which are non-trivial when restricted to $Z_{\hat{G}'}$. How are these two representations indexed?

When τ_v is discrete series, $Z_{\psi_v}/Z^0_{\psi_v} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in which $Z_{\hat{G}'}$ sits diagonally. There are thus 2 characters which are non-trivial when restricted to $Z_{\hat{G}'}$, namely η_{+-} and η_{-+} (where η_{+-} is trivial on the first copy of $\mathbb{Z}/2\mathbb{Z}$ and non-trivial on the second copy). We set:

$$\pi^{+-}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^+) \quad \text{and} \quad \pi^{-+}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^-).$$

When τ_v is principal series, $Z_{\psi_v}/Z^0_{\psi_v} \cong \mathbb{Z}/2\mathbb{Z}$, and $Z_{\hat{G}'}$ maps isomorphically onto this. Thus the local packet just consists of the single representation $\Theta'_{\chi_v}(\sigma_v^+)$ which is indexed by the non-trivial character character of $Z_{\psi_v}/Z^0_{\psi_v}$. Now the center of $SL_2 \times SL_2$ lies in Z_{ψ_v} and when restricted to this, the map $Z_{\psi_v} \to Z_{\psi_v}/Z^0_{\psi_v}$ is simply the second projection $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Thus the non-trivial character of $Z_{\psi_v}/Z^0_{\psi_v}$ is the character η_{+-} of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus we set

$$\pi^{+-}_{\tau_v,\chi_v} = \Theta'_{\chi_v}(\sigma_v^+).$$

Observe that this labelling is consistent with the discrete series case.

This completes our definition of the local packets for the group G'. For a representation π_v in our packets, we shall set:

$$\epsilon(\pi_v) = \begin{cases} +, & \text{if } \pi_v = \pi_v^{+-}; \\ -, & \text{if } \pi_v = \pi_v^{-+}. \end{cases}$$

The following proposition describes the representations of the packets more explicitly:

(6.5) Proposition *Assume that v is p-adic so that G' has F_v-rank* 1.

(i) If τ_v is the principal series $\pi(\mu_v, \mu_v^{-1})$, then $\pi^{+-}_{\tau_v,\chi_v} = I_{P'}(\chi_v, \mu_v)$; its L-parameter is

$$\rho_{\tau_v} \oplus \rho_{St_{\chi_v}} = \mu_v \oplus \mu_v^{-1} \oplus \rho_{St_{\chi_v}}.$$

(ii) Suppose that τ_v is discrete series. Then

$$\pi^{-+}_{\tau_v,\chi_v} = J_{P'}(JL(\tau_v), \chi_v, 1/2).$$

Here $JL(\tau_v)$ is the Jacquet-Langlands lift of τ_v to the inner form of PGL_2. The Langlands parameter is $\rho_{\tau_v} \oplus \rho_{\chi_v}$, i.e. the same as the L-parameter of the representation $\pi^{+}_{\tau_v,\chi_v}$ of $PGSp_4(F_v)$. Moreover,

$$\pi^{+-}_{\tau_v,\chi_v} = \begin{cases} \textit{supercuspidal, if } \tau_v \textit{ is supercuspidal;} \\ \textit{discrete series, if } \tau_v \otimes \chi_v \textit{ is twisted Steinberg;} \\ 0, \ \textit{if } \tau_v \otimes \chi_v \textit{ is Steinberg.} \end{cases}$$

More precisely, when $\tau_v \otimes \chi_v$ is a twisted Steinberg, $\pi^{+-}_{\tau_v,\chi_v}$ is the irreducible submodule of the induced representation $I_{P'}(JL(\tau_v), \chi_v, 1/2)$.

(6.6) Remarks: In view of Moeglin's conjecture in (3.4), we expect that the L-parameter of $\pi^{+-}_{\tau_v,\chi_v}$ should be equal to $\rho_{\tau_v} \oplus \rho_{St_{\chi_v}}$, i.e. the same as that of $\pi^{-}_{\tau_v,\chi_v}$ for $PGSp_4$. As the above proposition shows, this is the case when τ_v is principal series and when $\tau_v \otimes \chi_v$ is twisted Steinberg.

When $F_v = \mathbb{R}$, the correspondence can also be (and has been) explicitly determined; we omit the description here, but see [Sc].

(6.7) Remarks: There are 2 further observations to make here:

(a) As a consequence of (ii), we see that in one particular case, namely when $\tau_v \otimes \chi_v$ is Steinberg, a representation in the local packet is zero even though it is allowed to be non-zero by Arthur's conjecture. Thus the packet has only one element. This is not so surprising. Indeed, we expect the L-parameter of $\pi^{+-}_{St_v,\chi_v}$ to be equal to $\rho_{St_v \otimes \chi_v} \oplus \rho_{St_v \otimes \chi_v}$, but this L-parameter is **irrelevant** since it factors through the Levi of the Siegel parabolic of $Sp_4(\mathbb{C})$.

When $\tau_v \otimes \chi_v$ is not Steinberg, this obstruction is not present; thus the local packet has 2 or 1 element depending on whether τ_v is discrete series or not (just as the split case).

Similarly, suppose that $F = \mathbb{R}$ and τ_v is discrete series. If G'_v has rank 1, we have $\pi^{+-} = 0$. On the other hand, when G'_v is compact, then $\pi^{-+} = 0$, and if further τ_v has lowest weight 2, then $\pi^{+-} = 0$ also.

(b) The Langlands parameter for G' in (i) is not equal to any L-parameter associated to local components of Saito-Kurokawa representations of the split $PGSp_4$. As we shall see later, this implies that there are cuspidal representations in the Saito-Kurokawa space of G' whose Langlands lift to $PGSp_4$ is not contained in the Saito-Kurokawa space of $PGSp_4$ (indeed not contained in the discrete spectrum).

(6.8) Global lifting. As in the split case, the subspace of the discrete spectrum corresponding to the A-parameter $\psi_{\tau,\chi}$ can be constructed as:

$$\mathcal{A}_{G'}(\tau,\chi) = V_{G',\chi}(\widetilde{A}(\tau \otimes \chi)).$$

We leave it to the reader to verify that with our definition of local packets, the structure of this representation is in agreement with the multiplicity formula in Arthur's conjecture. In particular, a representation π' with parameter $\psi_{\tau,\chi}$ occurs in the $\mathcal{A}_{G'}(\tau,\chi)$ iff $\prod_v \epsilon(\pi'_v) = \epsilon(\tau \otimes \chi, 1/2)$.

As before, it is clear that

$$Proj_{cusp}\left(\bigoplus_{\tau,\chi} \mathcal{A}_{G'}(\tau,\chi)\right) \subset \mathcal{A}_{G',SK}.$$

It is natural to ask if equality holds. The following is the main result of [S]:

(6.9) Theorem *Assume that G' has F-rank 1. Then*

$$Proj_{cusp}\left(\bigoplus_{\tau,\chi} \mathcal{A}_{G'}(\tau,\chi)\right) = \mathcal{A}_{G',SK}.$$

The proof of this is similar to that of Piatetski-Shapiro for the split case. It relies on a particular Rankin-Selberg representation of the standard L-function for G'. This Rankin-Selberg integral requires the existence (and uniqueness) of a Bessel functional (i.e. a Fourier coefficient along the unipotent radical of Siegel parabolic). One does not have this if G' is anisotropic, and thus the proof does not work in this case.

However, there is another Rankin-Selberg integral for the standard L-function of G', the so-called doubling method of Piatetski-Shapiro and Rallis [PSR], which works for all forms G'. Might one be able to to show the above theorem for all G' using this? In the next section, we shall attempt to do so, proving a slightly weaker result than the above theorem.

(6.10) Some Peculiar Representations of Inner Forms. We describe some peculiar cuspidal representations alluded to in the remark above. The existence of these representations were first noticed by Sayag [S] and this motivated the definition of CAP representations for non-quasi-split groups.

Let us take $G' = SO(V', q')$ to be an inner form of $PGSp_4$. Then there are an even number of places v where G'_v is not split. Call this set of places S. Let τ be a cuspidal representation of PGL_2 and suppose that for some place $v_0 \in S$, τ_{v_0} is a principal series representation $\pi(\mu_{v_0}, \mu_{v_0}^{-1})$.

Let π' be a representation in the global A-packet of G' associated to τ (and the trivial quadratic character) and suppose that $\prod_v \epsilon(\pi'_v) = \epsilon(\tau, 1/2)$, so that π' occurs in $\mathcal{A}_{G',SK}$.

(6.11) Proposition *The Langlands lift of the representation $\pi' \subset \mathcal{A}_{G',SK}$ described above is not contained in the Saito-Kurokawa space $\mathcal{A}_{SK}$ of $PGSp_4$. Moreover, if G' has F-rank 1 (so that the Siegel parabolic P' exists), there are no cuspidal representations τ' for which π' is nearly equivalent to the constituents of $Ind_{P'}^{G'}\tau'$.*

Proof. The local component π'_{v_0} is the degenerate principal series $I_{P'}(1, \mu_{v_0})$ and its local L-parameter $\rho_{\tau_{v_0}} \oplus \rho_{St_{v_0}}$ is not equal to the L-parameters of local components of representations in the Saito-Kurokawa space $\mathcal{A}_{SK}$ of $PGSp_4$. This shows the first assertion. For the second, if such a τ' exists, then the Jacquet-Langlands lift of τ' to PGL_2 is nearly equivalent to τ and thus must be equal to τ. But since τ_{v_0} is a principal series, τ_{v_0} is not equal to a Jacquet-Langlands lift. With this contradiction, the second assertion is proved. ∎

(6.12) Remarks: When G' has F-rank 1 and τ is everywhere unramified, the unique π' in the global A-packet is cuspidal and everywhere unramified. In his Ph.D. thesis [P], A. Pitale gave a construction of π' (or rather the unique spherical vector in π') using a converse theorem of Maass. The fact that the representation constructed by Pitale is the same as that constructed here by theta lifting is a consequence of Theorem 6.9 or the weaker Theorem 7.1. The method of construction of π' given in [P] is interesting, as it also gives explicit formulas for the Fourier coefficients of the spherical vector in π'. However, it is not clear how this construction can be extended to the case of general τ and it seems difficult to use this approach to obtain a classification result as refined as the one obtained here.

There is nothing wrong with the fact that the Langlands lift of π' is not contained in $\mathcal{A}_{SK}$. It is not a contradiction to the functoriality conjecture, because the conjecture only requires the Langlands lift to be automorphic; there is no reason why it should be in the discrete spectrum!

A similar phenomenon can already be found in the Jacquet-Langlands correspondence: the Langlands lift of the trivial representation of $PD^\times$ (D a quaternion division algebra) is the anomalous representation of PGL_2 which is Steinberg at the places of ramification of D and trivial everywhere else, and this representation is automorphic but is not contained in the discrete spectrum. However, in the comparison of trace formulas, the constant functions of $PD^\times$ and PGL_2 are matched up, so that there is some reason for considering the trivial representation of PGL_2 as the lift of the trivial representation of $PD^\times$, even though they have different L-parameters. In fact, the right way to view the Jacquet-Langlands transfer of discrete spectrums is that it is functorial with respect to A-parameters.

Thus, the Saito-Kurokawa space and its transfer to various inner forms are best understood in terms of A-parameters rather than L-parameters.

§7. Characterization of Saito-Kurokawa Space

In this section, we shall try to show that

$$Proj_{cusp}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)=\mathcal{A}_{G',SK}.$$

for any inner form G' of $PGSp_4$. However, we shall only be able to prove a weaker result. More precisely, we let

$$\mathcal{B}_{G',SK}\subset\mathcal{A}_{G',SK}$$

be the submodule consisitng of representations which are nearly equivalent to the representations in the Saito-Kurukawa A-packets $\bigcup_{\tau,\chi}A_{\tau,\chi}$. Clearly, we have:

$$Proj_{cusp}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)\subset\mathcal{B}_{G',SK}.$$

The main result of this section is:

(7.1) Theorem *Let G' be any inner form of $PGSp_4$ (possibly split). Then*

$$Proj_{cusp}\left(\bigoplus_{\tau,\chi}\mathcal{A}_{G'}(\tau,\chi)\right)=\mathcal{B}_{G',SK}.$$

Of course when G' is isotropic, this theorem is a consequence of the stronger result of Piatetski-Shapiro [PS] and Sayag [S]. In any case, this result shows that our definition of local A-packets is correct.

PROOF. The proof that we present below is based on the *regularized* Rallis inner product formula (which in turn relies on the regularized Siegel-Weil formula).

Let π be an irreducible summand of $\mathcal{B}_{G',SK}$. Then for some quadratic character χ, $\pi\otimes\chi$ is nearly equivalent to an induced representation $I_P(\tau,1/2)$ of $PGSp_4$, with τ a cuspidal representation of PGL_2. Consider the global theta lift $\widetilde{V}(\pi\otimes\chi)$ of $\pi\otimes\chi$ to $\widetilde{SL}_2$; thus $\widetilde{V}(\pi\otimes\chi)$ is a subrepresentation of the space of automorphic forms on $\widetilde{SL}_2$. It is not difficult to check that $\widetilde{V}(\pi\otimes\chi)$ is contained in the cuspidal spectrum (we omit the standard computation here). To prove the theorem, we need to show that $\widetilde{V}(\pi\otimes\chi)$ is non-zero.

In the following, we may assume, without loss of generality, that χ is trivial.

(7.2) Inner product. For $\varphi_i\in\omega_\psi$ and $f_i\in\pi$, we consider the inner product

$$\langle\widetilde{\theta}(\varphi_1,f_1),\widetilde{\theta}(\varphi_2,f_2)\rangle_{SL_2}$$

where

$$\widetilde{\theta}(\varphi_i,f_i)(h)=\int_{G'_F\backslash G'_{\mathbb{A}}}\theta(\varphi_i)(gh)\cdot\overline{f_i(g)}\,dg.$$

We want to show that this inner product is non-zero for some choice of φ_1 and φ_2 (we fix f_1 and f_2 without loss of generality). Formally exchanging orders of integration, this inner product is equal to

$$\int_{(G'_F\times G'_F)\backslash(G'_{\mathbb{A}}\times G'_{\mathbb{A}})} \overline{f_1(g_1)}f_2(g_2)\cdot \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \theta(\varphi_1)(g_1h)\overline{\theta(\varphi_2)(g_2h)}\,dh\,dg_1\,dg_2.$$

However, the inner integral $I(\varphi_1,\varphi_2)$ is not convergent (for general φ_1 and φ_2), so that this exchange is not justified. Fortunately, there is an easy way to regularize this inner integral, as shown by Kudla and Rallis [KR]; we explain this regularization next.

(7.3) Doubling. Let $\mathbb{V}$ be the ten-dimensional quadratic space $(V',q')\oplus(V',-q')$. Then $\mathbb{V}$ is totally split and $\mathbb{G}=SO(\mathbb{V})$ contains $G'\times G'$ as a natural subgroup. Let W_{ψ_v} be the Weil representation of $\widetilde{SL}_2(F_v)\times\mathbb{G}_v$; it may be realized on the space $S(\mathbb{V}_v)$ of Schwarz functions on $\mathbb{V}_v$ (the Schrodinger model). Then as a representation of $\tilde{SL}_2(F_v)\times(G'_v\times G'_v)$, we have a natural isomorphism:

$$W_{\psi_v}\cong\omega_{\psi_v,q}\otimes\omega_{\psi_v,-q},$$

with the tensor on the right being an inner tensor for the $\widetilde{SL}_2$ action and an outer tensor for the action of $G'_v\times G'_v$. This isomorphism is induced by the natural isomorphism of vector spaces:

$$S(V'_v)\hat{\otimes}S(V'_v)\cong S(\mathbb{V}_v).$$

Note in particular that the representation W_ψ factors to the linear group SL_2.

Similarly, on the global level, the natural isomorphism

$$\iota: S(V'_{\mathbb{A}})\hat{\otimes}S(V'_{\mathbb{A}})\cong S(\mathbb{V}_{\mathbb{A}})$$

gives an isomorphism $\omega_{\psi,q}\otimes\omega_{\psi,-q}\cong W_\psi$ as representations of $\widetilde{SL}_2(\mathbb{A})\times(G'_{\mathbb{A}}\times G'_{\mathbb{A}})$. Moreover, we have:

$$\Theta(\iota(\varphi_1\otimes\varphi_2)((g_1,g_2)h)=\theta(\varphi_1)(g_1h)\cdot\overline{\theta(\varphi_2)(g_2h)}.$$

Thus, the integral that we need to regularize is equal to

$$I(\Phi)(g_1,g_2)=\int_{SL_2(F)\backslash SL_2(\mathbb{A})}\Theta(\Phi)((g_1,g_2)h)dh$$

for $\Phi\in S(\mathbb{V}_{\mathbb{A}})$. This is simply the theta lift of the constant function of SL_2 to $\mathbb{G}_{\mathbb{A}}$ (restricted to $G'_{\mathbb{A}}\times G'_{\mathbb{A}}$) if it were convergent.

(7.4) Regularization. It turns out that $I(\Phi)$ converges absolutely iff $W_\psi(h)\Phi(0)$ is zero for all $h\in SL_2(\mathbb{A})$. The map which associates to Φ the function

$$h\mapsto W_\psi(h)\Phi(0)$$

is an $SL_2(\mathbb{A})$-intertwining map

$$T : W_\psi \longrightarrow \text{a principal series of } SL_2(\mathbb{A})$$

(this map is also $\mathbb{G}_{\mathbb{A}}$-invariant). Thus, $I(\Phi)$ is absolutely convergent iff Φ lies in the kernel of T.

Now, at any archimedean place v_0, this principal series is easily seen to have different infinitesimal character from the trivial representation of SL_2. Thus one can find an element Z in the center of the universal enveloping algebra of $\mathfrak{sl}_2(F_{v_0})$ with the property that

$$Z = \begin{cases} 1 \text{ on the trivial representation;} \\ 0 \text{ on the above principal series.} \end{cases}$$

Then for any $\Phi \in S(\mathbb{V}_{\mathbb{A}})$, $W_\psi(Z)\Phi$ lies in $ker(T)$ and the regularization of $I(\Phi)$ is defined to be:

$$I^{reg}(\Phi) = I(W_\psi(Z)(\Phi)).$$

The integral $I^{reg}(\Phi)$ is the regularized theta lift of the constant function of SL_2 to $\mathbb{G}$. Indeed, because the action of Z commutes with both SL_2 and $\mathbb{G}$, the map $\Phi \mapsto I^{reg}(\Phi)$ is both $SL_2(\mathbb{A})$-invariant and $\mathbb{G}_{\mathbb{A}}$-intertwining, just as the map $\Phi \mapsto I(\Phi)$ would be if it were convergent. Moreover, if $\Phi \in ker(T)$ to begin with, then

$$I^{reg}(\Phi) = I(\Phi).$$

Though this regularization may seem somewhat ad-hoc, as it involves the choice of Z, it is in fact canonical, for one can show that there is at most one extension of I from $ker(T)$ to W_ψ which is both SL_2-invariant and $\mathbb{G}$-intertwining.

(7.5) Regularized inner product. How is this regularized integral relevant to our problem? Let us consider the absolutely convergent integral

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \Theta\left(W_\psi(Z)(\iota(\varphi_1 \otimes \varphi_2))\right)((g_1, g_2)h)\, dh\, dg_1\, dg_2.$$

Now we can write

$$W_\psi(Z)(\iota(\varphi_1 \otimes \varphi_2)) = \sum_{k=1}^{r} \iota(\varphi_{1,k} \otimes \varphi_{2,k}).$$

Thus, on exchanging orders of integration, we see that the above integral is equal to

$$\sum_{k=1}^{r} \langle \widetilde{\theta}(\varphi_{1,k}, f_1, \widetilde{\theta}(\varphi_{2,k}, f_2) \rangle_{SL_2}.$$

In other words, to prove the theorem, it remains to show that

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} \cdot f_2(g_2) \cdot I^{reg}(\Phi)(g_1, g_2)\, dg_1\, dg_2$$

is non-zero for some choice of Φ.

(7.6) Regularized Siegel-Weil formula. The regularized integral $I^{reg}(\Phi)$ turns out to be equal to the residue of an Eisenstein series on $\mathbb{G}_{\mathbb{A}}$; this is a special case of the regularized Siegel-Weil formula. More precisely, there is a natural $SL_2(\mathbb{A})$-invariant and $\mathbb{G}_{\mathbb{A}}$-equivariant map

$$R : W_\psi \longrightarrow Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4} \quad \text{(unnormalized induction)}$$

where $\mathbb{P}'$ is a maximal parabolic subgroup of $\mathbb{G}$ stabilizing a maximal isotropic subspace X of $\mathbb{V}$. We note here that in $\mathbb{G}$, there are two conjugacy classes of maximal isotropic subspaces and our discussion below does not depend on the choice of the class of the maximal isotropic subspace X.

The definition of this map R is analogous to that of the map T above, but involves a mixed model for W_ψ rather than the Schrodinger model used in the definition of T. Indeed, the Weil representation W_ψ can also be realized on $S(W \otimes X^*)$ (W is standard 2-dimensional symplectic space), and the isomorphism

$$S(\mathbb{V}) \longrightarrow S(W \otimes X^*)$$

is given by a partial Fourier transform: $\Phi \mapsto \hat{\Phi}$. For $\hat{\Phi} \in S(W \otimes X^*)$,

$$R(\hat{\Phi})(g) = (W_\psi(g)\hat{\Phi})(0).$$

Moreover, we have:

(7.7) Proposition *The degenerate principal series* $Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}$ *has a unique irreducible submodule. This irreducible submodule* Π *is spherical and is the so-called minimal representation of* $\mathbb{G}_{\mathbb{A}}$. *The map* R *gives an* $\mathbb{G}_{\mathbb{A}}$*-equivariant isomorphism*

$$(W_\psi)_{SL_2(\mathbb{A})} \cong \Pi \subset Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}.$$

Now each $K_{\mathbb{G}}$-finite $f' \in Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}$ determines a standard section f'_s of the family of degenerate principal series $Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{s}$, and we may form the Eisenstein series $E(f', s, g)$. It turns out that at $s = 1/4$, $E(f', s, g)$ is holomorphic, and the map

$$f' \mapsto E(f', s, g),$$

initially defined for $K_{\mathbb{G}}$-finite f', extends to give a $\mathbb{G}_{\mathbb{A}}$-equivariant map from Π to the space of square-integrable automorphic forms on $\mathbb{G}_{\mathbb{A}}$. The *regularized Siegel-Weil formula* states:

(7.8) Proposition *There is a non-zero constant* c *such that*

$$I^{reg}(\Phi)(g) = c \cdot E(R(\hat{\Phi}), 1/4, g).$$

There is no doubt that the constant c can be precisely determined, but we will not need its precise value here.

(7.9) Intertwining operator. It turns out to be more convenient to work with the degenerate principal series at $s = 3/4$ rather than at $s = 1/4$. Let $\mathbb{P}$ be the standard parabolic subgroup which is associated with the other conjugacy class of maximal isotropic subspaces of $\mathbb{V}$. Thus, $\mathbb{P}$ and $\mathbb{P}'$ are associates but not conjuagte. There is a standard intertwining map

$$M(s) : Ind_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{1/2+s} \longrightarrow Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/2-s}$$

and $M(s)$ is meromorphic in s. At $s = 1/4$, $M(s)$ has a pole of order 1 and so we have

$$M = Res_{s=1/4} M(s) : Ind_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4} \longrightarrow Ind_{\mathbb{P}'}^{\mathbb{G}} \delta_{\mathbb{P}'}^{1/4}.$$

The image of this map is precisely equal to Π. Moreover, by the functional equation for Eisenstein series,

$$Res_{s=3/4} E(f, s, g) = E(M(f), 1/4, g).$$

Together with the regularized Siegel-Weil formula, this gives:

(7.10) Corollary *As f varies over all elements of $Ind_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4}$, the space of functions $Res_{s=3/4} E(f, s, g)$ is the same as the space of functions $I^{reg}(\Phi)(g)$ as Φ ranges over $S(\mathbb{V}_{\mathbb{A}})$. In particular, to prove the theorem, it suffices to show that*

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot E(f, s, (g_1, g_2))\, dg_1\, dg_2$$

has a pole at $s = 3/4$ for some $K_{\mathbb{G}}$-finite $f \in Ind_{\mathbb{P}}^{\mathbb{G}} \delta_{\mathbb{P}}^{3/4}$.

(7.11) Standard L-functions. Piatetski-Shapiro and Rallis showed in [PSR] that the expression in the above corollary is a Rankin-Selberg representation for the standard L-function of π. To state their result, let us fix the maximal isotropic subspace

$$\Delta V' = \{(v, v) \in \mathbb{V} : v \in V'\}$$

and suppose without loss of generality that $\mathbb{P}$ is the stabilizer of $\Delta V'$. Then the action of $G' \times G'$ on $\mathbb{P}\backslash\mathbb{G}$ has a unique Zariski open dense orbit, namely the orbit of $\Delta V'$. The stabilizer of $\Delta V'$ is equal to $\Delta G'$ and thus $G' \times \{1\}$ acts simply transitively on an open dense subset of $\mathbb{P}\backslash\mathbb{G}$. Now we have:

(7.12) Proposition *For $Re(s)$ sufficiently large and for a sufficiently large finite set S of places of F,*

$$\int_{(G'_F \times G'_F)\backslash(G'_{\mathbb{A}} \times G'_{\mathbb{A}})} \overline{f_1(g_1)} f_2(g_2) \cdot E(f, s, (g_1, g_2))\, dg_1\, dg_2$$

$$= \prod_{v \in S} Z_v(f_{1,v}, f_{2,v}, f_v, s) \cdot L^S(\pi, 4s - 3/2) \cdot \frac{1}{\zeta^S(8s) \cdot \zeta^S(8s-2)}.$$

Here, for $v \in S$,

$$Z_v(f_{1,v}, f_{2,v}, f_v, s) = \int_{G'_v} \langle \pi_v(g)(f_{1,v}), f_{2,v} \rangle \cdot f_{v,s}(g, 1)\, dg$$

where $\langle -, - \rangle$ is the G'_v-invariant inner product on π_v which is linear in the second variable and conjugate-linear in the first.

Now the left hand side of the identity in the above proposition extends to a meromorphic function of s if f is $K_{\mathbb{G}}$-finite. On the other hand, for the case at hand, we know a priori that

$$L^S(\pi, s) = \zeta^S(s + 1/2) \cdot \zeta^S(s - 1/2) \cdot L^S(\tau, s)$$

for some cuspidal representation of PGL_2 and thus $L^S(\pi, 1/2)$ has a meromorphic continuation to $\mathbb{C}$. Finally, the local zeta factors Z_v (for $v \in S$) has been studied in detail in [KR2]. It was shown in [KR2, Thm. 3.2.2] that for any smooth f, $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ converges for $Re(s)$ large and has meromorphic continuation to all of $\mathbb{C}$. Thus, the identity in the proposition holds for all $s \in \mathbb{C}$ and is an equality of meromorphic functions, as long as f is $K_{\mathbb{G}}$-finite.

(7.13) End of proof. Now we are given that $L^S(\pi, 4s - 3/2))$ has a pole at $s = 3/4$. Thus, by the above proposition, the proof of the theorem is reduced to:

(7.14) Lemma *For each $v \in S$, there exists a $K_{\mathbb{G}_v}$-finite f_v so that $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ is non-zero at $s = 3/4$.*

Proof. Let ϕ be an arbitrary function in $C_c^\infty(G'_v)$. Define a function

$$f_\phi \in Ind_{\mathbb{P}_v}^{\mathbb{G}_v} \delta_{\mathbb{P}}^{3/4}$$

by requiring that f_ϕ vanishes outside the set $\mathbb{P}_v \cdot (G'_v \times \{1\})$ and

$$F_\phi(p \cdot (g, 1)) = \delta_{\mathbb{P}}(p)^{3/4} \cdot \phi(g).$$

This is well-defined because $\mathbb{P}_v \cdot (G'_v \times \{1\})$ is open dense in $\mathbb{G}_v$ and is analytically isomorphic to $\mathbb{P}_v \times G'_v$.

Let $f_{\phi,s}$ be the standard section extending f_ϕ. Then it is easy to see that the integral which defines $Z(f_{1,v}, f_{2,v}, f_\phi, s)$ for $Re(s) >> 0$ is in fact convergent for all s. At $s = 3/4$, it is equal to:

$$\int_{G'_v} \langle \pi_v(g)(f_{1,v}), f_{2,v} \rangle \cdot \phi(g)\, dg.$$

Now since $\langle \pi_v(g) f_{1,v}, f_{2,v} \rangle$ is a non-zero function of g, it is clear that one can find ϕ so that this integral is non-zero.

Now if v is finite, then f_ϕ is already $K_{\mathbb{G}_v}$-finite and so we are done. When v is archimedean, f_ϕ may not be $K_{\mathbb{G}_v}$-finite, but [KR2] shows that $Z_v(f_{1,v}, f_{2,v}, f_v, s)$ is continuous in f_v for fixed s. Thus the lemma follows by the density of $K_{\mathbb{G}_v}$-finite vectors. ∎

The theorem is proved. ∎

(7.15) Remarks: (i) What the proof shows is that if π is a cuspidal representation of G' such that for some quadratic character χ, $L^S(\pi \otimes \chi, s)$ has a pole at $s = 3/2$, then $\pi \otimes \chi$ has non-zero theta lift to $\widetilde{SL}_2$ and thus

$$\pi \subset Proj_{cusp}\left(\bigoplus_{\tau,\chi} \mathcal{A}_{G'}(\tau, \chi)\right).$$

(ii) It seems likely that the above argument can be pushed to yield the theorem with $\mathcal{B}_{G',SK}$ replaced by $\mathcal{A}_{G',SK}$.

Finally, we note the following corollary of the theorem:

(7.16) Corollary *Let π be a representation in a Saito-Kurokawa packet for G'. Then $m_{cusp}(\pi) \leq 1$.*

This corollary puts one in a position to apply the result of C. Sorensen [Sor] about level-raising congruences for Saito-Kurokawa representations of G'. We finish up with an example which will be relevant for the application of the main theorem of [Sor].

(7.17) Example: Suppose f is a holomorphic cuspidal newform of weight 4 with respect to $\Gamma_0(N)$ with $N = p_1, ...p_r$ squarefree. The corresponding cuspidal representation τ_f of PGL_2 is unramified outside $S_f = \{p_1, ...p_r, \infty\}$, Steinberg at p_i and a discrete series of lowest weight 4 at the archimedean place. Suppose that r is odd, so that $\epsilon(\tau_f, 1/2) = (-1)^r = -1$. The Saito-Kurokawa packet determined by τ_f (and the trivial character) has 2^{r+1} representations of which 2^r occur in the discrete spectrum. One of these is the representation $\pi_f = \pi_\infty^- \otimes (\otimes_p \pi_p^+)$, with π_∞^- a holomorphic discrete series and π_p non-tempered for all p. This π occurs in the discrete spectrum because of the odd number of minus signs.

Let G' be the inner form of $PGSp_4$ which is ramified precisely at $S_f = \{p_1, ..., p_r, \infty\}$ with $G'(\mathbb{R})$ compact. Now there is only one representation π' in the Saito-Kurokawa packet of $G'(\mathbb{A})$ corresponding to τ_f. This is the theta lift of the representation

$$\sigma = \sigma_\infty^+ \otimes (\otimes_{p \in S_f} \sigma_p^-) \otimes (\otimes_{p \notin S_f} \sigma_p^+)$$

which is contained in $\widetilde{A}_{00}$ because of the odd number of minus signs. Thus π' occurs in the discrete spectrum of G' and has the following properties:

- its archimedean component π'_∞ is the trivial representation of $G'(\mathbb{R})$;
- it is the Langlands lift of π, i.e. for all v, π'_v and π_v have the same L-parameters;
- its multiplicity in $L^2(G'_{\mathbb{Q}} \backslash G'_{\mathbb{A}})$ is equal to 1.

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