Multiplicity Formula for Cubic Unipotent Arthur Packets

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in memory of my father, Gan Lik

§1. Introduction

Let F be a number field with adele ring $\mathbb{A}$ and let L_F be the conjectural Langlands group of F, which has a natural map to the absolute Galois group $Gal(\overline{F}/F)$. There is a natural bijection between étale cubic F-algebras E and conjugacy classes of homomorphisms

$$\rho_E : Gal(\overline{F}/F) \longrightarrow S_3,$$

where S_3 is the symmetric group on 3 letters and is the automorphism group of the split algebra $F \times F \times F$. The homomorphism ρ_E gives rise to an action of $Gal(\overline{F}/F)$ on S_3 by conjugation, and thus defines a twisted form S_E of the finite constant group scheme S_3. For any commutative F-algebra B, we have $S_E(B) = Aut_B(E \otimes_F B)$.

Let G be the split exceptional group of type G_2 over F with complex dual group $\widehat{G} = G_2(\mathbb{C})$. There is a family of Arthur parameters of G naturally indexed by étale cubic F-algebras E:

$$\psi_E : L_F \times SL_2(\mathbb{C}) \longrightarrow \widehat{G}.$$

The restriction of ψ_E to $SL_2(\mathbb{C})$ is associated (by the Jacobson-Morozov theorem) to the subregular unipotent conjugacy class of $\widehat{G} = G_2(\mathbb{C})$. The centralizer of the image of $SL_2(\mathbb{C})$ is isomorphic to S_3 and $\psi_E|_{L_F}$ is the homomorphism ρ_E. Thus these parameters are almost unipotent in the sense of Arthur: the restriction of ψ_E to L_F is almost trivial.

In [GGJ], we defined for each place v of F a candidate local A-packet $A_{\psi_{E,v}}$ associated to ψ_E, using results of Vogan [V] and Huang-Magaard-Savin [HMS]. This is a finite set of irreducible unitarizable representations of $G(F_v)$ indexed by the irreducible characters of the finite group $S_E(F_v)$:

$$A_{\psi_{E,v}} = \{\pi_{\eta_v} : \eta_v \in \widehat{S_E(F_v)}\}.$$

With the (candidate) local packets defined, the (candidate) global packet A_{ψ_E} is simply defined as the restricted tensor product of the local ones. Hence, the elements of A_{ψ_E} are indexed by irreducible characters of the compact group $S_E(\mathbb{A}) = \prod_v S_E(F_v)$; for a given character $\eta = \otimes\eta_v$, we have

$$\pi_\eta = \otimes_v \pi_{\eta_v}.$$

According to Arthur's conjectures, one should have a $G(\mathbb{A})$-equivariant embedding

$$\iota_E : \bigoplus_\eta \dim \eta^{S_E(F)} \cdot \pi_\eta \hookrightarrow L^2_{disc}(G(F)\backslash G(\mathbb{A})).$$

The main theorem of [GGJ] gives a construction of this embedding, In particular, it implies that if $m_{disc}(\pi_\eta)$ denotes the multiplicity of π_η in the discrete spectrum, then we have

$$m_{disc}(\pi_\eta) \geq \dim \eta^{S_E(F)}.$$

It is remarkable that the numbers $\dim \eta^{S_E(F)}$ are unbounded as η ranges over the irreducible characters of $S_E(\mathbb{A})$ and this provides the first examples of representations with unbounded discrete and cuspidal multiplicities.

This paper is a sequel to [GGJ] and its purpose is to prove the following result.

MAIN THEOREM

(i) Fix the étale cubic F-algebra E and let $\eta = \otimes_v \eta_v$ be an irreducible character of $S_E(\mathbb{A})$. Let π_η be the irreducible representation of $G_2(\mathbb{A})$ associated to η. Then

$$m_{disc}(\pi_\eta) = \dim \eta^{S_E(F)}.$$

(ii) Suppose that $E = F \times K$ where K is an étale quadratic algebra. If τ is an irreducible constituent of the discrete spectrum of G_2 which is nearly equivalent to the representations in A_{ψ_E}, then τ is contained in V_E = image of ι_E. In other words, V_E is a full near equivalence class.

Statement (i) of the theorem already provides *compelling global evidence* for the authenticity of the candidate local packets $A_{\psi_E,v}$. In the case $E = F \times K$, the stronger statement (ii) establishes this *beyond any reasonable doubt.* Indeed, one knows a priori that for almost all v, the representation π_{1_v} of the local packet $A_{\psi_E,v}$ is the irreducible unramified representation with Satake parameter

$$s_{E,v} = \psi_E \left(\mathrm{Frob}_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right).$$

Thus, if a representation τ_{v_0} is a member of the local packet A_{ψ_E,v_0}, then τ_{v_0} should appear as a local component of a global representation τ which occurs in the discrete spectrum and whose local components are π_{1_v} for almost all v. The fact that all such global representations are contained in V_E (when $E = F \times K$) shows that the definition of $A_{\psi_E,v}$ given in [GGJ] already captures all possible candidates for the members of $A_{\psi_E,v}$.

The proof of (i) is given in §2-§4 and is similar in spirit to the proof of the multiplicity one theorem for cusp forms of GL_n. The proof of (ii) is based on an alternative construction of the space V_E (when $E = F \times K$) and a certain Rankin-Selberg integral. It is given in §5 and relies crucially on the results of [GG].

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§2. Multiplicity Formula

The proof of statement (i) of the main theorem is reminescent of that for the multiplicity one theorem of Shalika and Piatetski-Shapiro for cusp forms of GL_n. Thus it is instructive to review the proof of this classic result.

(2.1) Multiplicity one theorem of GL_n. Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $GL_n(\mathbb{A})$. For simplicity, let us assume that π has trivial central character, so that we are working with the group PGL_n. If $\mathcal{A}_{cusp}(PGL_n)$ denotes the space of cusp forms for PGL_n, then the multiplicity one theorem states that

$$\dim \mathrm{Hom}_{PGL_n(\mathbb{A})}(\pi, \mathcal{A}_{cusp}) \leq 1.$$

The proof of this has two distinct steps, as we shall now explain.

Step 1: Global non-vanishing of Whittaker-Fourier coefficient.

Let N be the unipotent radical of a Borel subgroup of PGL_n. If ψ is a generic character of $N(\mathbb{A})$ trivial on $N(F)$, then for any automorphic form φ, one sets

$$l_\psi(\varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(n) \cdot \overline{\psi(n)} dn.$$

Thus one obtains a map

$$\mathrm{Hom}_{PGL_n}(\pi, \mathcal{A}_{cusp}) \to \mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_\psi)$$

by the assignment $f \mapsto l_\psi \circ f$. The first step of the proof shows that this map is *injective*; equivalently, any non-zero cusp form is generic.

Step 2: Local uniqueness of Whittaker functionals.

After Step 1, it remains to show that

$$\dim \mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_\psi) \leq 1.$$

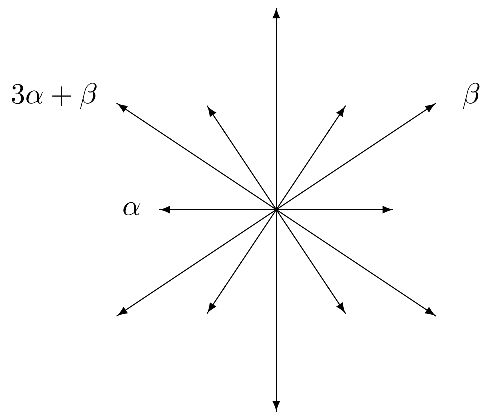
The second step of the proof shows the corresponding local statement for each place v.

Analogously, our proof of statement (i) of the main theorem will have two parts, resembling the two steps above. Before stating the result of each step (Theorems A and B below), we need to introduce some notations for G_2.

(2.2) The group G_2. The Chevalley group G_2 can be canonically defined over $\mathbb{Z}$. Fix a maximal split torus T of G_2 contained in a Borel subgroup B, both defined over $\mathbb{Z}$. This determines a based root datum for G_2 together with an épinglage.

We let α and β denote the short and long simple roots respectively. For a root δ, we let U_δ denote the associated root subgroup; we have an isomorphism $U_\delta \cong \mathbb{G}_a$ defined over $\mathbb{Z}$ which is well-determined up to ± 1. For a finite place v, we set $K_v = G_2(A_v)$ where A_v is the ring of integers of F_v; K_v is a maximal compact subgroup of $G_2(F_v)$.

For future reference, we include a picture of the roots of G_2.



(2.3) The Heisenberg parabolic P. Let $P = M \cdot N$ be the Heisenberg parabolic subgroup of G_2. Its unipotent radical N is a 5 dimensional Heisenberg group with center Z. The Levi factor M is isomorphic to GL_2 and contains $U_{\pm\alpha}$. Let $M_{ss} \cong SL_2$ be the derived group of M. Then $P_{ss} = M_{ss} \cdot N$ is a Jacobi group in the sense of [I].

The $M(F)$-orbits on the set of unitary characters of $N(\mathbb{A})$ trivial on $N(F)$ are naturally indexed by cubic F-algebras (cf. [GGS] or [G]). The generic orbits correspond to those cubic F-algebras which are étale. There are 3 degenerate orbits:

- the zero orbit, which corresponds to the cubic F-algebra $F[x,y]$ with trivial multiplication: $x^2 = xy = y^2 = 0$. The associated character is the trivial character;
- the orbit corresponding to the cubic algebra $E_1 = F[x]/x^3$. A representative character is one which is non-trivial when restricted to U_β and trivial on other U_δ's in N.
- the orbit corresponding to the cubic algebra $E_2 = F[x]/x^2(x-1)$. A representative character is one which is non-trivial when restricted to $U_{\alpha+\beta}$ and trivial on other U_δ's in N.

For a cubic algebra E, we let ψ_E denote a character of $N(\mathbb{A})$ in the associated orbit. Let $M_{\psi_E} \subset M$ be the stabilizer of ψ_E. If E is étale, then $M_{\psi_E} \cong S_E$ as algebraic group.

(2.4) The maximal parabolic Q. Let $Q = L \cdot U$ be the other standard maximal parabolic subgroup. Its unipotent radical U is a 3-step nilpotent group and $L \cong GL_2$ contains $U_{\pm\beta}$. The center Z_U of U is a 2-dimensional. If we let $C_U = [U,U]$ be the commutator group of U, then $C_U = U_{2\alpha+\beta} \times Z_U$ is abelian. Moreover, U/Z_U is a 3-dimensional Heisenberg group with center C_U/Z_U.

Let $L_{ss} \cong SL_2$ be the derived group of L. Then $J_{ss} = L_{ss} \cdot U/Z_U$ is the classical Jacobi group.

(2.5) Fourier coefficients. Let V be a unipotent subgroup of G_2 and ψ a unitary character of $V(\mathbb{A})$ trivial on $V(F)$. If φ is an automorphic form on G_2, we define the (V,ψ)-Fourier coefficient of φ by:

$$\varphi_{V,\psi}(g) = \int_{V(F)\backslash V(\mathbb{A})} \varphi(vg) \cdot \overline{\psi(v)} dv.$$

We shall be considering Fourier coefficients along N. Hence, if ψ_E is a unitary character of $N(\mathbb{A})$ trivial on $N(F)$, we have the function φ_{N,ψ_E}. For simplicity, we shall call this the E-th Fourier coefficient of φ and write it as φ_{ψ_E}, suppressing the mention of N. We may also consider the continuous linear functional

$$l_{\psi_E} : \mathcal{A}(G_2) \longrightarrow \mathbb{C}, \qquad \varphi \mapsto \varphi_{\psi_E}(1).$$

(2.6) Twisted action of M_{ψ_E}. The stabilizer $M_{\psi_E}(\mathbb{A}) = \prod_v M_{\psi_E}(F_v)$ acts naturally on the vector space $Hom_{N(\mathbb{A})}(\pi, \mathbb{C}_{\psi_E})$. However, we shall consider a twisted action of $M_{\psi_E}(\mathbb{A})$.

An étale cubic algebra E determines a discriminant algebra K_E, which is an étale quadratic algebra. To be more precise,

$$K_E = \begin{cases} F \times F, \text{ when } E = F \times F \times F \text{ or } E = \text{Galois cubic field}, \\ K, \text{ when } E = F \times K; \\ \text{the unique quadratic subfield in the Galois closure of } E \text{ otherwise.} \end{cases}$$

Let χ_{K_E} be the quadratic Grossencharacter associated to K_E. We may regard χ_{K_E} as a quadratic character of $M_{\psi_E}(\mathbb{A})$ via composition with the determinant map of $M(\mathbb{A}) \cong GL_2(\mathbb{A})$.

The twisted $M_{\psi_E}(\mathbb{A})$-action is defined as follows. For $m \in M_{\psi_E}(\mathbb{A})$ and $l \in \mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_{\psi_E})$,

$$(m \cdot l)(v) = \chi_{K_E}(m) \cdot l(m^{-1}v).$$

In this way, $\mathrm{Hom}_{N(\mathbb{A})}(\pi, \mathbb{C}_{\psi_E})$ becomes an $M_{\psi_E}(\mathbb{A})$-module.

(2.7) The Two Steps. Having introduced the basic notations, we can now state the two results which together imply statement (i) of the main theorem.

Theorem A *Fix an étale cubic algebra E and an irreducible character η of $S_E(\mathbb{A})$. Then the assignment $f \mapsto l_{\psi_E} \circ f$ defines an injective map*

$$\mathrm{Hom}_{G_2(\mathbb{A})}(\pi_\eta, \mathcal{A}(G_2)) \longrightarrow \mathrm{Hom}_{N(\mathbb{A})}(\pi_\eta, \mathbb{C}_{\psi_E})^{M_{\psi_E}(F)}.$$

Theorem B *For each place v of F,*

$$\mathrm{Hom}_{N(F_v)}(\pi_{\eta_v}, \mathbb{C}_{\psi_{E_v}}) \cong \eta_v^\vee \qquad \textit{as } M_{\psi_E}(F_v)\textit{-modules.}$$

These two theorems clearly imply statement (i) of the main theorem and will be proved in Section 3 and 4 respectively.

§3. Fourier Coefficients: Proof of Theorem A

The purpose of this section is to prove the following result about Fourier coefficients of a general automorphic form on G_2. Theorem A will be a consequence of this result.

(3.1) Theorem *Let $\pi \subset \mathcal{A}(G)$ be an irreducible non-trivial automorphic subrepresentation. Any automorphic form φ in π has a non-zero E-th Fourier coefficient for some étale cubic algebra E. Equivalently, the linear functional l_{ψ_E} is non-zero on π for some étale E.*

Before going into the rather involved proof of this theorem, let us see how it implies Theorem A.

(3.2) Proof of Theorem A. Let π_η be as given in Theorem A. It was shown in [HMS] that as an abstract representation, π_{η_v} is E_v-distinguished for all finite v, in the sense that for any étale E_v' not isomorphic to E_v,

$$Hom_{N(F_v)}(\pi_{\eta_v}, \mathbb{C}_{\psi_{E_v'}}) = 0.$$

Thus for any $f \in \mathrm{Hom}_{G_2(\mathbb{A})}(\pi, \mathcal{A}(G_2))$, the functions in $f(\pi)$ have vanishing E'-th Fourier coefficient if $E' \neq E$. Hence Theorem 3.1 implies that the map $l_{\psi_E} \circ f$ is non-zero as long as f is non-zero, so that the assignment $f \mapsto l_{\psi_E} \circ f$ defines an injective map

$$\mathrm{Hom}_{G_2(\mathbb{A})}(\pi_\eta, \mathcal{A}(G_2)) \longrightarrow \mathrm{Hom}_{N(\mathbb{A})}(\pi_\eta, \mathbb{C}_{\psi_E}).$$

Further, it is easy to see that its image is contained in the subspace of $M_{\psi_E}(F)$-fixed vectors: for $m \in M_{\psi_E}(F)$ and $v \in \pi_\eta$,

$$\begin{aligned} m \cdot (l_{\psi_E} \circ f)(v) &= (l_{\psi_E} \circ f)(m^{-1} \cdot v) \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(v)(nm^{-1}) \cdot \overline{\psi_E(n)} dn \\ &= \int_{N(F)\backslash N(\mathbb{A})} f(v)(n') \cdot \overline{\psi_E(m^{-1}n'm)} dn' \quad (\text{where } n' = mnm^{-1}) \\ &= (l_{\psi_E} \circ f)(v) \end{aligned}$$

since m fixes ψ_E. Theorem A is proved. ∎

(3.3) Proof of Theorem 3.1. The rest of the section is devoted to the proof of Theorem 3.1, which proceeds by contradiction. Under the assumption that the E-Fourier coefficient of π is zero for any étale E, we shall show that for almost all v, the local component π_v is a minimal representation of $G_2(F_v)$. This is a contradiction to the fact (shown in [GS]) that $G_2(F_v)$ does not have an unramified minimal representation.

The proof involves looking at certain Fourier-Jacobi coefficients of φ, a notion that we shall now recall.

(3.4) Weil representation and Jacobi forms. Fix a non-trivial character ψ of $F\backslash\mathbb{A}$. Via the isomorphism $U_{2\alpha+\beta} \cong \mathbb{G}_a$, we regard ψ as a character of $U_{2\alpha+\beta}(\mathbb{A}) \cong C_U(\mathbb{A})/Z_U(\mathbb{A})$. As is well-known, the Heisenberg group $U(\mathbb{A})/Z_U(\mathbb{A})$ has a unique irreducible smooth representation ω_ψ with central character ψ. The representation ω_ψ can be realized on the space $S(U_\alpha(\mathbb{A}))$ of Schwartz-Bruhat functions on $U_\alpha(\mathbb{A})$. The space $S(U_\alpha(\mathbb{A}))$ has a natural topology; for its definition, see [We, §11].

Consider now the Jacobi group $J(\mathbb{A}) = L_{ss}(\mathbb{A}) \cdot U(\mathbb{A})/Z_U(\mathbb{A})$. If $\tilde{L}_{ss}(\mathbb{A})$ denotes the two-fold metaplectic cover of $L_{ss}(\mathbb{A})$, then we obtain a two-fold cover $\tilde{J}(\mathbb{A}) = \tilde{L}_{ss}(\mathbb{A}) \cdot U(\mathbb{A})/Z_U(\mathbb{A})$ of $J(\mathbb{A})$. The representation ω_ψ can be extended uniquely to a representation of $\tilde{J}(\mathbb{A})$; we denote this extended representation by ω_ψ also, and call it the Weil representation (associated to ψ).

Consider functions $\Phi : J(F)\backslash\tilde{J}(\mathbb{A}) \longrightarrow \mathbb{C}$ satisfying:

(i) Φ is smooth;

(ii) Φ is right-invariant under some open compact subgroup of $\tilde{J}(\mathbb{A}_f)$;

(iii) Φ is of uniform moderate growth.

(iv) $\Phi(zg) = \psi(z) \cdot \Phi(g)$ for $z \in (C_U/Z_U)(\mathbb{A})$.

Let $\mathcal{A}_\psi^\infty(J(F)\backslash\tilde{J}(\mathbb{A}))$ denote the space of such functions Φ. The elements of this space are called Jacobi forms.

There is a natural topology on the space of Jacobi forms defined as follows. For each $n \geq 0$ and each open compact subgroup $K \subset J(\mathbb{A}_f)$, let $V_{n,K}$ be the subspace of those Jacobi forms Φ which are right-invariant under K and such that for each $X \in U(Lie(J(F \otimes_\mathbb{Q} \mathbb{R})))$ (the universal enveloping algebra of $Lie(J(F \otimes_\mathbb{Q} \mathbb{R}))$),

$$\beta_{X,n}(\Phi) := \sup_g |(X\Phi)(g)| \cdot ||g||^{-n} \leq \infty.$$

Each $\beta_{X,n}$ defines a semi-norm on $V_{n,K}$, and we give $V_{n,K}$ the (locally convex) topology defined by these semi-norms. By conditions (ii) and (iii) above, the space of Jacobi forms is the inductive limit of the $V_{n,K}$'s. We then give $\mathcal{A}_\psi^\infty(J(F)\backslash\tilde{J}(\mathbb{A}))$ the inductive limit topology.

There is an equivariant topological isomorphism $\phi \mapsto \theta_\phi$ from the representation ω_ψ to a closed $\tilde{J}(\mathbb{A})$-submodule of the space of Jacobi forms. This is defined as follows: for $\phi \in S(U_\alpha(\mathbb{A}))$,

$$\theta_\phi(g) = \sum_{x \in U_\alpha(F)} \omega_\psi(g)\phi(x), \qquad g \in \tilde{J}(\mathbb{A}).$$

(3.5) Remarks: Actually, one can simply work with the space $C_\psi^\infty(J(F)\backslash\tilde{J}(\mathbb{A}))$ of smooth functions on $J(F)\backslash\tilde{J}(\mathbb{A})$ satisfying (iv) above. This is equipped with the C^∞-topology defined by the seminorms

$$\beta_{X,K}(\Phi) = \sup_{g \in K} |(X\Phi)(g)|$$

where K varies over compact subsets of $\tilde{J}(\mathbb{A})$ and $X \in U(Lie(J(F \otimes_{\mathbb{Q}} \mathbb{R})))$. It is clear that the natural inclusion

$$\mathcal{A}^{\infty}_{\psi}(J(F)\backslash\tilde{J}(\mathbb{A})) \hookrightarrow C^{\infty}_{\psi}(J(F)\backslash\tilde{J}(\mathbb{A}))$$

is a continuous map. For our purpose, it is immaterial which of the two spaces we work with.

(3.6) Fourier-Jacobi coefficients. For $\varphi \in \pi$, the Fourier coefficient $\varphi_{C_U,\psi}$, when restricted to $Q_{ss}(\mathbb{A}) = L_{ss}(\mathbb{A}) \cdot U(\mathbb{A})$, descends to a smooth function on the Jacobi group $J(\mathbb{A})$. We can regard it as a function of $\tilde{J}(\mathbb{A})$ and it is not difficult to check that this gives an element $FJ_{\psi}(\varphi)$ of $\mathcal{A}^{\infty}_{\psi}(J(F)\backslash\widetilde{J}(\mathbb{A}))$; this is the Fourier-Jacobi coefficient of φ.

Thus we have a map

$$FJ_{\psi} : \pi \longrightarrow \mathcal{A}^{\infty}_{\psi}(J(F)\backslash\tilde{J}(\mathbb{A}))$$

which is $\tilde{Q}_{ss}(\mathbb{A})$-equivariant. Thus, $FJ_{\psi}(\pi)$ is a $\tilde{J}(\mathbb{A})$-submodule of the space of Jacobi forms. The map FJ_{ψ} probably has closed range, but we do not need to know this in what follows.

(3.7) A result of Ikeda. We now recall a result of Ikeda [I] about Jacobi forms. Take any $\Phi \in \mathcal{A}^{\infty}_{\psi}(J(F)\backslash\tilde{J}(\mathbb{A}))$. For two elements ϕ_1 and ϕ_2 of $S(U_{\alpha}(\mathbb{A}))$, consider the function on $\tilde{J}(\mathbb{A})$ defined by:

$$\Phi_{\phi_1,\phi_2}(lu) = \theta_{\phi_1}(lu) \cdot \Phi_{\phi_2}(l)$$

where

$$\Phi_{\phi_2}(l) = \int_{U(F)Z_U(\mathbb{A})\backslash U(\mathbb{A})} \Phi(lv) \cdot \overline{\theta_{\phi_2}(lv)} dv.$$

Let W be a (not-necessarily-closed) $U(\mathbb{A})$-submodule of the space of Jacobi forms and let $cl(W)$ be its closure; note that W need not be a $\tilde{J}(\mathbb{A})$-submodule here. Ikeda showed that if $\Phi \in W$, then Φ_{ϕ_1,ϕ_2} is still an element of $cl(W)$, and in fact $cl(W)$ is the closed linear span of the functions Φ_{ϕ_1,ϕ_2}, as Φ ranges over elements of W and ϕ_1 and ϕ_2 range over all elements of $S(U_{\alpha}(\mathbb{A}))$ (cf. [I, Prop. 1.2]).

(3.8) Remarks: To be honest, Ikeda worked with the space C^{∞} rather than $\mathcal{A}_{\infty}$, but this is immaterial, as we mentioned above. He also assumed that $W = cl(W)$ is closed, but his proof of [I, Prop. 1.2] gives the slightly extended version above.

(3.9) Vanishing of $FJ_{\psi}(\varphi)$. We now suppose that φ does not have any non-zero E-th Fourier coefficient with E étale; then any non-zero automorphic form in π has the same property. We have:

(3.10) Lemma $FJ_{\psi}(\pi) = 0$.

Proof. Suppose that $FJ_{\psi}(\pi) \neq 0$; we shall derive a contradiction. Applying Ikeda's results above to $W = FJ(\pi)$, we see that $cl(FJ_{\psi}(\pi))$ is the closed linear span of certain functions of the form

$$F_{\phi,f}(l \cdot u) = \theta_{\phi}(lu) \cdot f(l)$$

where f is a smooth function on $L_{ss}(F)\backslash\tilde{L}_{ss}(\mathbb{A})$.

There is thus a non-zero function $F_{\phi,f}$ in $cl(FJ_\psi(\pi))$. Using the action of $U(\mathbb{A})$, one sees that for any non-zero ϕ', $F_{\phi',f}$ is a non-zero element of $cl(FJ_\psi(\pi))$. Further, because $FJ_\psi(\pi)$ consists of functions on $J(\mathbb{A})$ (rather than $\tilde{J}(\mathbb{A})$), we deduce that f must be a non-zero genuine function on $L_{ss}(F)\backslash\tilde{L}_{ss}(\mathbb{A})$.

Consider the Fourier expansion of f along $U_\beta(F)\backslash U_\beta(\mathbb{A})$. There is a non-trivial character χ of $U_\beta(F)\backslash U_\beta(\mathbb{A})$ for which $f_{U_\beta,\chi}\neq 0$. Replacing $F_{\phi,f}$ by an $\tilde{L}_{ss}(\mathbb{A})$-translate (which still lies in $cl(FJ_\psi(\pi))$ since the latter is a $\tilde{J}(\mathbb{A})$-module), we may assume that $f_{U_\beta,\chi}(1)\neq 0$. Replacing ϕ if necessary, we may assume in addition that $\phi(0)\neq 0$.

Then the linear functional

$$l_\chi : cl(FJ_\psi(\pi)) \longrightarrow \mathbb{C}$$

defined by

$$l_\chi(\Phi)=\int_{U_\beta(F)\backslash U_\beta(\mathbb{A})}\int_{U_{\alpha+\beta}(F)\backslash U_{\alpha+\beta}(\mathbb{A})}\overline{\chi(u_\beta)}\cdot\Phi(u_{\alpha+\beta}u_\beta)du_{\alpha+\beta}du_\beta$$

is continuous and non-zero on $cl(FJ_\psi(\pi))$ since it is non-zero on the vector $F_{\phi,f}$. Indeed,

$$\begin{aligned} l_\chi(F_{\phi,f}) &= \int_{U_\beta(F)\backslash U_\beta(\mathbb{A})}\int_{U_{\alpha+\beta}(F)\backslash U_{\alpha+\beta}(\mathbb{A})}\overline{\chi(u_\beta)}\cdot\theta_\phi(u_{\alpha+\beta}u_\beta)\cdot f(u_\beta)du_{\alpha+\beta}du_\beta \\ &= \int_{U_\beta(F)\backslash U_\beta(\mathbb{A})}\overline{\chi(u_\beta)}\cdot f(u_\beta)\cdot(\omega_\psi(u_\beta)\phi)(0)du_\beta \\ &= f_{U_\beta,\chi}(1)\cdot\phi(0)\neq 0. \end{aligned}$$

Thus, we have shown that $l_\chi\circ FJ_\psi$ is non-zero on π. Now one observes that the composite $l_\chi\circ FJ_\psi:\pi\rightarrow\mathbb{C}$ is simply the map l_{ψ_E} for an étale cubic algebra $E=F\times K$ where K is some étale quadratic algebra. Since we are assuming that all such l_{ψ_E} are zero, we obtain the desired contradiction. Hence, $FJ_\psi(\pi)=0$. The lemma is proved. ∎

(3.11) Corollary *If $l_{\psi_E}=0$ on π for all étale cubic algebras of the form $E=F\times K$, then $l_{\psi_{E_2}}=0$ on π.*

(3.12) Another Fourier-Jacobi coefficient. Next, we shall examine another Fourier-Jacobi coefficient of φ. As we noted before, the group $P_{ss}=M_{ss}N$ is a Jacobi group in the sense of Ikeda. The Heisenberg group $N(\mathbb{A})$ has a unique irreducible representation ω_ψ with central character ψ, and this extends uniquely to the metaplectic cover of $Sp(N/Z)(\mathbb{A})$. The action of M_{ss} on N/Z gives an injection $M_{ss}\hookrightarrow Sp(N/Z)$. It turns out that the metaplectic cover of $Sp(N/Z)(\mathbb{A})$ splits over the subgroup $M_{ss}(\mathbb{A})$ and this splitting is unique since M_{ss} is simply-connected. Thus the representation ω_ψ extends uniquely to the group $P_{ss}(\mathbb{A})$. This representation of $P_{ss}(\mathbb{A})$ can be realized on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions on

$$V(\mathbb{A})=U_{2\alpha+\beta}(\mathbb{A})\times U_{3\alpha+\beta}(\mathbb{A}).$$

As before, we have an equivariant map

$$\theta : \omega_\psi \longrightarrow \mathcal{A}_\psi^\infty(P_{ss}(F)\backslash P_{ss}(\mathbb{A}))$$

defined by

$$\theta_\phi(g) = \sum_{x\in V(F)} (\omega_\psi(g)\phi)(x)$$

for $\phi \in S(V(\mathbb{A}))$.

Consider the restriction of the function $\varphi_{Z,\psi}$ to $P_{ss}(\mathbb{A})$. This defines a $P_{ss}(\mathbb{A})$-equivariant map

$$FJ'_\psi : \pi \longrightarrow \mathcal{A}_\psi^\infty(P_{ss}(F)\backslash P_{ss}(\mathbb{A})).$$

We claim:

(3.13) Lemma $FJ'_\psi(\pi) \subset \theta(\omega_\psi)$.

Proof. As before, the result of Ikeda implies that $cl(FJ'_\psi(\pi))$ is the closed linear span of certain functions of the form

$$F_{\phi,f}(nm) = \theta_\phi(nm) \cdot f(m)$$

where f is a smooth function on $M_{ss}(F)\backslash M_{ss}(\mathbb{A}) \cong SL_2(F)\backslash SL_2(\mathbb{A})$. So to prove our claim, we have to show that for any $F_{\phi,f} \in cl(FJ'_\psi(\pi))$, the function f is constant. For this, it suffices to show that for any non-trivial character χ of $U_\alpha(F)\backslash U_\alpha(\mathbb{A})$, $f_{U_\alpha,\chi} = 0$.

As in the proof of Lemma 3.10, the fact that $F_{\phi,f} \in cl(FJ'_\psi(\pi))$ implies that $F_{\phi',f} \in cl(FJ'_\psi(\pi))$ for any $\phi' \in S(V(\mathbb{A}))$. We may thus assume that $\phi(0) \neq 0$. Now consider the continuous linear functional on $cl(FJ'_\psi(\pi))$ defined by:

$$l_\chi(\Phi) = \int_{U_\alpha(F)\backslash U_\alpha(\mathbb{A})} \overline{\chi(u_\alpha)} \cdot \int_{V(F)\backslash V(\mathbb{A})} \Phi(vu_\alpha)dvdu_\alpha.$$

If $F_{\phi,f} \in cl(FJ'_\psi(\pi))$, then it is easy to see that for $m \in M_{ss}(\mathbb{A})$, $F_{\phi,m\cdot f} \in cl(FJ'_\psi(\pi))$. Now we have:

$$l_\chi(F_{\phi,m\cdot f}) = f_{U_\alpha,\chi}(m) \cdot \phi(0).$$

Since we have assumed that $\phi(0) \neq 0$, it remains to show that $L_\chi = l_\chi \circ FJ'_\psi = 0$ on π.

The linear form L_χ can be interpreted as follows. Let $P' = M' \cdot N'$ be the maximal parabolic subgroup whose Levi factor M' contains the root subgroups $U_{\pm(\alpha+\beta)}$ and whose unipotent radical N' is a Heisenberg group with center $U_{3\alpha+\beta}$ (it may be useful to refer to the picture of the root system of G_2 given in (2.2)). It is a conjugate of the Heisenberg parabolic P. Let $R \subset N'$ be the 4-dimensional subgroup generated by the unipotent subgroups Z, V and U_α. Let χ_R be the character on $R(\mathbb{A})$ determined by:

$$\chi_R|_{U_\alpha} = \chi, \quad \chi_R|_Z = \psi \quad \text{and } \chi_R|_{V(\mathbb{A})} \text{ is trivial.}$$

Then we have:

$$L_\chi(\varphi) = \int_{R(F)\backslash R(\mathbb{A})} \overline{\chi_R(r)} \cdot \varphi(r) dr.$$

Considering Fourier expansion of φ along N', we see that

$$L_\chi(\varphi) = \sum_{\psi'} \varphi_{N',\psi'}(1)$$

where the sum ranges over those characters ψ' of $N'(\mathbb{A})$, trivial on $N'(F)$, such that $\psi'|_R = \chi_R$.

Now by Corollary 3.11 (applied to N' in place of N), we know that $\varphi_{N',\psi'} = 0$ unless ψ' is trivial or ψ' lies in the $M'(F)$-orbit indexed by E_1. However, if $\psi'|_R = \chi_R$, then ψ' lies in an orbit corresponding to either E_2 or an étale E. Thus we conclude that $L_\chi = 0$ on π, as desired. The lemma is proved. ∎

(3.14) End of proof. We have shown that FJ'_ψ defines a $P_{ss}(\mathbb{A})$-equivariant map

$$FJ'_\psi : \pi \longrightarrow \omega_\psi.$$

It is not difficult to see that FJ'_ψ cannot be the zero map (unless π is the space of constant functions); this was proved in [GS, Lemma 5.6], for example.

Now we arrive at a contradiction by noting the following two propositions. The first proposition says roughly that the unramified local components of our π are minimal representations. The second shows that $G_2(F_v)$ does not have unramified minimal representations.

(3.15) Proposition *Suppose that π is not the space of constant functions. If FJ'_ψ defines a $P_{ss}(\mathbb{A})$-equivariant map*

$$FJ'_\psi : \pi \longrightarrow \omega_\psi,$$

then for almost all places v, the wave-front set of π_v (in the sense of [MW]) is equal to the closure of the minimal orbit of $G_2(F_v)$.

PROOF. This is a consequence of the proof of [GS, Thm. 5.4] and [GS, Prop. 3.7]. ∎

(3.16) Proposition *Let v be a finite place of F. The group $G_2(F_v)$ does not have an irreducible unramified representation whose wave-front set is equal to the closure of the minimal orbit.*

PROOF. Suppose that τ is an unramified representation whose wave-front set is equal to the closure of the minimal orbit. We shall show that τ must be a submodule of a degenerate principal series representation $Ind_P^{G_2}\chi$. Assuming this, we note that the wave-front sets of the constituents of $Ind_P^{G_2}\chi$ have been determined in [HMS]. By inspection of the results of [HMS], one sees that none of these constituents have wave-front set equal to the closure of the minmal orbit and the proposition would be proven.

It remains to prove that τ is a constituent of $Ind_P^{G_2}\chi$ for some χ. For this, it suffices to show that the constituents of the Jacquet module τ_N are all 1-dimensional. This is equivalent to showing that for any non-trivial character χ of $N_\alpha(F_v)$, the twisted Jacquet module $(\tau_N)_{N_\alpha,\chi}$ is zero. The proof of this is quite similar to the proof of Lemma 3.13.

We regard χ as a character of $N \cdot N_\alpha$ which is trivial on N. Let $P' = M' \cdot N'$ be the maximal parabolic considered in the proof of Lemma 3.13; it is a conjugate of the Heisenberg parabolic P. If $R = N' \cap (N_\alpha \cdot N)$ is the 4-dimensional subgroup of N' considered in the proof of Lemma 3.13, we may consider the restriction χ_R of χ to R and there is a natural surjection $\tau_{R,\chi_R} \longrightarrow (\tau_N)_{N_\alpha,\chi}$. Thus it suffices to show that $\tau_{R,\chi_R} = 0$ (note that this χ_R is different from the one in the proof of Lemma 3.13).

For this, we regard τ_{R,χ_R} as a representation of the abelian group N'/Z'. If $\tau_{R,\chi_R} \neq 0$, then there is a character ψ of $N'(F_v)$ such that ψ restricted to R is equal to χ_R and $\tau_{N',\psi} \neq 0$. We claim that this is impossible, so that $\tau_{R,\chi_R} = 0$. On one hand, any ψ which restricts to χ_R lies in the $M'(F_v)$-orbit of characters corresponding to the cubic algebra E_2. On the other hand, because of our assumption on the wave-front set of τ, a result of Moeglin-Waldspurger [MW] implies that if $\tau_{N',\psi} \neq 0$, then ψ is either trivial or lies in the orbit indexed by the algebra E_1. The proposition is proved. ∎

In view of the contradiction implied by the two propositions, Theorem 3.1 is proved. ∎

§4. Local Functionals: Proof of Theorem B

In this section, we shall prove Theorem B. Since the situation is local, we shall suppress v from the notations. Hence, in this section, F will be a local field. For any pair (E, η), we know from the global results of [GGJ] that $Hom_{N(F)}(\pi_\eta, \mathbb{C}_{\psi_E}) \neq 0$. To prove Theorem B, we need to identify this non-zero representation of $S_E(F)$.

(4.1) Non-archimedean case. When F is non-archimedean, Theorem B was claimed in [HMS, (1.10)] except that they mis-stated the result, using the untwisted action of $M_{\psi_E}(F)$ (this only makes a difference when $E = F \times K$). For the sake of completeness, we shall provide the proof here.

(4.2) Minimal representation. Let's first recall the construction of the representations in A_{ψ_E}. The semi-direct product $H_E = S_E \ltimes Spin_8^E$ contains $S_E \times G_2$ as a subgroup, and $H_E(F)$ has a distinguished representation Π_E, which is a particular extension (as defined in [GGJ, 236-237]) of the unique unitarizable minimal representation of $Spin_8^E(F)$ [GS]. When restricted to $S_E(F) \times G_2(F)$, we have:

$$\Pi_E = \bigoplus_\eta \eta^\vee \otimes \pi_\eta,$$

and this defines the representations π_η in A_{ψ_E}. The π_η's were shown to be irreducible in [HMS].

Let $P_E = M_E \cdot N_E$ be a Heisenberg parabolic subgroup of $Spin_8^E$ such that $P_E \cap G_2 = P$. In particular, the center Z of N is the center of the unipotent radical N_E of P_E. For the purpose of proving Theorem B in the non-archimedean case, the main property of Π_E we need is contained in the following proposition [GS, Prop. 11.11]:

(4.3) Proposition *Let V be the smooth $P_E(F) \rtimes S_E(F)$-module which is the kernel of the natural projection map $(\Pi_E)_Z \to (\Pi_E)_{N_E}$. Let Ω be the minimal $M_E(F)$-orbit of non-trivial unitary characters of $N_E(F)$. Then V can be realized on $C_c^\infty[\Omega]$ with action given by:*

$$\begin{cases} n \cdot f(\chi) = \chi(n) \cdot f(\chi) \quad n \in N_E(F); \\ m \cdot f(\chi) = \chi_{K_E}(m) \cdot \delta_{P_E}(m)^{1/5} \cdot f(m^{-1}\chi), \quad m \in M_E(F); \\ s \cdot f(\chi) = f(s^{-1}\chi), \quad s \in S_E(F). \end{cases}$$

Here, the quadratic character χ_{K_E} is regarded as a character of $M_E(F)$ by composition with an isomorphism $M_E(F)/[M_E(F), M_E(F)] \cong F^\times$.

To be honest, [GS, Prop. 11.11] proves this proposition without the $S_E(F)$-action. But it is easy to see that for the particular extension of Π_E to $H_E(F)$, the $S_E(F)$-action is as given. We should also note that the error in [HMS] occurs because of the omission of the character χ_{K_E} in the formula above.

Now let ψ_E be as given in Theorem B. Set

$$\Omega[\psi_E] = \{\chi \in \Omega : \chi|_{N(F)} = \psi_E\}.$$

Clearly, $S_E(F) \times M_{\psi_E}(F)$ acts naturally on $\Omega[\psi_E]$. The following lemma can be checked directly; we omit the proof.

(4.4) Lemma *Each of $S_E(F)$ and $M_{\psi_E}(F)$ acts simply transitively on $\Omega[\psi_E]$. In particular, the action of $S_E(F) \times M_{\psi_E}(F)$ on $\Omega[\psi_E]$ is isomorphic to the natural action of $S_E(F) \times S_E(F)$ on $S_E(F)$.*

(4.5) Jacquet module. Now we are ready to prove Thm. B in the non-archimedean case. Consider the Jacquet module $(\Pi_E)_{N,\psi_E}$. On one hand, we have:

$$(\Pi_E)_{N,\psi_E} = \bigoplus_\eta \eta^\vee \otimes (\pi_\eta)_{N,\psi_E}.$$

On the other hand, by the proposition and lemma above, we see that

$$(\Pi_E)_{N,\psi} = \mathbb{C}[S_E(F)],$$

where the action of $S_E(F) \times M_{\psi_E}(F)$ is via the regular representation twisted by the quadratic character χ_{K_E} of $M_{\psi_E}(F)$. In other words,

$$(\Pi_E)_{N,\psi_E} = \bigoplus_\eta \eta^\vee \otimes (\eta \cdot \chi_{K_E}).$$

Thus we deduce that:

$$(\pi_\eta)_{N,\psi_E} \cong \eta \cdot \chi_{K_E} \quad \text{as } M_{\psi_E}(F)\text{-modules.}$$

Since $Hom_{N(F)}(\pi_\eta, \mathbb{C}_{\psi_E})$ is the contragredient of $(\pi_\eta)_{N,\psi_E}$, this proves Thm. B for the p-adic case. ∎

(4.6) Real case. We now consider Theorem B when $F = \mathbb{R}$. We first introduce some notations. If R denotes the parabolic subgroup P or Q, then its Levi factor is isomorphic to GL_2; we choose this isomorphism so that

$$\delta_P = |det|^3 \quad \text{and} \quad \delta_Q = |det|^5.$$

If τ is a representation of $GL_2(F)$, we let $I_R(\tau, s)$ be the representation of $G(F)$ unitarily induced from $\tau \otimes |det|^s$. Similarly, if χ is a character of $F^\times$, then we write $I_R(\chi)$ for the representation of $G(F)$ unitarily induced from $\chi \circ det$. If τ is tempered and $Re(s) > 0$, then $I_R(\tau, s)$ has a unique irreducible quotient which we denote by $J_R(\tau, s)$.

If χ_1 and χ_2 are characters of $F^\times$, then $\pi(\chi_1, \chi_2)$ denotes the representation of $GL_2(F)$ unitarily induced from the character $\chi_1 \times \chi_2$ of the diagonal torus. Finally, we let sgn denote the sign character of $\mathbb{R}^\times$.

(4.7) Unipotent representations. When $F \cong \mathbb{R}$, $E \cong \mathbb{R}^3$ or $\mathbb{R} \times \mathbb{C}$. The representations π_η in A_{ψ_E} are listed below:

- when $E \cong \mathbb{R}^3$,

  $$\pi_{\mathbf{1}} = J_Q(\pi(1,1), 1), \quad \pi_r = J_P(St, 1/2), \quad \pi_\epsilon = \text{unique non-generic summand of } I_P(D_3, 0),$$

  where St denotes the Steinberg representation and D_3 denotes the discrete series representation with extremal weight ± 3. Here, following the notations of [GGJ], we have written ϵ for the sign character of $S_E(F) = S_3$, and r for the 2-dimensional irreducible representation.

- when $E = \mathbb{R} \times \mathbb{C}$,

  $$\pi'_{\mathbf{1}} = J_Q(\pi(1, sgn), 1), \quad \pi'_\kappa = J_Q(St, 1/2)$$

  where κ is the non-trivial character of $S_E(F) = \mathbb{Z}/2\mathbb{Z}$.

By Vogan [V, Thm. 18.5], these 5 representations have the same infinitesimal character λ and can be characterized as the non-generic representations with this infinitesimal character. Further, they can be distinguished from each other easily because the minimal K-type of each one does not occur in the others. The following lemma shows that these representations can all be embedded as submodules of certain degenerate principal series associated to P.

(4.8) Lemma *(i) We have the short exact sequence*

$$0 \longrightarrow \pi'_\kappa \longrightarrow I_P(|-|^{1/2}) \longrightarrow \pi_1 \longrightarrow 0.$$

(ii) We have the short exact sequence

$$0 \longrightarrow \pi_\epsilon \longrightarrow I_P(sgn \cdot |-|^{1/2}) \longrightarrow \pi'_1 \longrightarrow 0.$$

(iii) We have the inclusion:

$$\pi_r \hookrightarrow I_P(\tau, 0),$$

where τ denotes the representation $std \otimes |det|^{-1/2}$ of $M(\mathbb{R}) \cong GL_2(\mathbb{R})$.

Proof. Each of the degenerate principal series in question has infinitesimal character λ and does not contain any generic constituents. Thus, after semisimplification, they are linear combinations of the 5 unipotent representations π_η. In [V, 13.4 and 14.3], Vogan worked out the occurence of the minimal K-types of the π_η's in degenerate principal series. From his results, one sees that after semisimplification,

$$\begin{cases} I_P(|-|^{1/2}) = \pi_1 \oplus \pi'_\kappa, \\ I_P(sgn \cdot |-|^{1/2}) = \pi'_1 \oplus \pi_\epsilon \\ I_P(\tau, 0) = \pi_r \oplus \pi'_1 \oplus \pi'_\kappa. \end{cases}$$

To prove (i) and (ii), it remains to show that π_1 and π'_1 are submodules of $I_P(|-|^{-1/2})$ and $I_P(sgn \cdot |-|^{-1/2})$. Write π for π_1 or π'_1, and χ for 1 or sgn respectively. Then π is the unique irreducible submodule of

$$I_Q(\pi(\chi, 1), -1) = I_P(\chi|-|^{-1/2} \cdot \pi(|-|^{-1/2}, |-|^{1/2}))$$

and the latter representation has $I_P(\chi|-|^{-1/2})$ as submodule.

Finally to prove (iii), we note that

$$\pi_r \hookrightarrow I_P(St, -1/2) \hookrightarrow I_P(sgn \cdot \pi(1, |-|^{-1})).$$

However, $I_P(sgn \cdot \pi(1, |-|^{-1}))$ is equal to

$$I_Q(\pi(|-|^{-1}, sgn)) = I_Q(\pi(sgn, |-|^{-1})) = I_P(\pi(|-|^{-1}, sgn \cdot |-|)).$$

This last induced representation has $I_P(\tau, 0)$ as submodule, and contains the minimal K-type of π_r with multiplicity one. Thus the image of π_r under the above inclusion is contained in the submodule $I_P(\tau, 0)$. The lemma is proved. ∎

Now we note the following result of Wallach ([W, Thm. 13] and [W2]):

(4.9) Proposition *Let $F = \mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional irreducible representation of $M(F)$ and consider the induced representation $I_P(V)$. Let ψ be a generic character of $N(F)$. Then under the untwisted action of $M_\psi(F)$,*

$$\mathrm{Hom}_{N(F)}(I_P(V), \mathbb{C}_\psi) \cong V^\vee$$

as $M_\psi(F)$-modules.

Lemma 4.8 and Prop. 4.9 immediately imply Theorem B for $F = \mathbb{R}$. ∎

(4.10) Complex case. Finally, we consider the case where $F = \mathbb{C}$, so that $E = F^3$ necessarily. In this case, the three unipotent representations π_η were studied by Barbasch-Vogan [BV], who gave particularly convenient realizations of these representations. More precisely, define a finite-dimensional representation of $GL_2(\mathbb{C})$ by:

$$\tau_\eta = \begin{cases} |det|^{1/2} \text{ if } \eta = 1; \\ \text{the standard representation if } \eta = r; \\ |det|^{1/2} \cdot \overline{det}^{-1} \text{ if } \eta = \epsilon. \end{cases}$$

Then Barbasch-Vogan [BV, Pg. 88-89] showed that

$$\pi_\eta = I_P(\tau_\eta).$$

The result of Wallach alluded to above thus completes the proof of Theorem B for the complex case. ∎

§5. Near Equivalence Classes

The rest of the paper is devoted to the proof of statement (ii) of the main theorem, i.e. that V_E is a full near equivalence class when $E = F \times K$ is not a field. In this case, the subspace V_E of the discrete spectrum can be constructed in another way. This was shown in [GG], and we begin by recalling this alternative construction.

(5.1) Alternative construction of V_E. Fix a non-trivial unitary character ψ of $F\backslash\mathbb{A}$, and let $\widetilde{SL}_2(\mathbb{A})$ denote the 2-fold metaplectic cover of $SL_2(\mathbb{A})$. Then one may consider the Weil representation Ω_ψ of the dual pair $\widetilde{SL}_2(\mathbb{A}) \times SO_7(\mathbb{A})$ associated to ψ. Since G_2 is a subgroup of SO_7, one may consider Ω_ψ as a representation of $\widetilde{SL}_2(\mathbb{A}) \times G_2(\mathbb{A})$, and use Ω_ψ to define a theta lifting from $\widetilde{SL}_2(\mathbb{A})$ to $G_2(\mathbb{A})$. The study of this lifting was initiated by Rallis-Schiffmann in [RS] and completed in [GG].

The subspace V_E was constructed in [GGJ] by restricting the minimal representation of $Spin_8^E$. In [GG, §13], it was shown that V_E can also be obtained by lifting from $\widetilde{SL}_2$.

More precisely, if the étale quadratic algebra K corresponds to $a \in F^\times/F^{\times 2}$, then one may consider the Weil representation ω_K of $\widetilde{SL}_2(\mathbb{A})$ associated to the character ψ_a (with $\psi_a(x) = \psi(ax)$). The formation of theta series defines a map from ω_K to the discrete spectrum of $\widetilde{SL}_2$. Let $\mathcal{A}_K$ be the image of this map, and let $\Theta(\mathcal{A}_K)$ be the (regularised) theta lift of $\mathcal{A}_K$ to G_2 (via Ω_ψ). Here it is necessary to regularize the theta lifting because $\mathcal{A}_K$ is not totally contained in the space of cusp forms (cf. [GG, §12] for details). In any case, by [GG, Thm. 13.1], we have:

$$\Theta(\mathcal{A}_K) = V_{F\times K}.$$

(5.2) SU_3-periods. The cuspidal representations of G_2 which are lifted from $\widetilde{SL}_2$ can be characterized by the non-vanishing of certain periods. More precisely, if SU_3^L denotes the quasi-split unitary group in 3 variables associated to an étale quadratic L, then there is a natural conjugacy class of embeddings $SU_3^L \hookrightarrow G_2$. If π is an irreducible cuspidal representation, we say that π has non-vanishing period over SU_3^L if

$$\int_{SU_3^L(F)\backslash SU_3^L(\mathbb{A})} f(g)dg \neq 0 \qquad \text{for some } f \in \pi.$$

The following proposition was shown in [RS].

(5.3) Proposition *Let π be an irreducible constitutent of the cuspidal spectrum of G_2. Suppose that π has non-vanishing period over some SU_3^L, then there exists an irreducible cuspidal representation σ of $\widetilde{SL}_2$ such that π is not orthogonal to the theta lift $\Theta(\sigma)$ of σ (via Ω_ψ).*

We come now to the key result of this section.

(5.4) Theorem *Let τ be an irreducible constituent of the cuspidal spectrum of G_2. Assume that τ is nearly equivalent to the representations in V_E (with $E = F \times K$). Then τ has non-vanishing period over SU_3^K.*

Let us see how Theorem 5.4 gives the proof of statement (ii) of the main theorem.

(5.5) Proof of (ii) of Main Theorem. Let τ be as given in statement (ii) of the main theorem. In particular, τ may not be cuspidal, but is nearly equivalent to the representations in A_{ψ_E}. By (i) of the main theorem, it suffices to show that τ is actually isomorphic to some π_η in A_{ψ_E}.

We first assume that the projection of τ to the residual spectrum is non-zero. Then τ is isomorphic to an irreducible constituent τ' of the residual spectrum. On the other hand, the residual spectrum has been completely determined by Kim [K] and Zampera [Z]. Their results are summarized in [GGJ, Prop. 7.2], using the framework of Arthur's conjecture. The proof of [GGJ, Prop. 7.3] now shows that if τ' is nearly equivalent to the representations in A_{ψ_E}, then τ' must be isomorphic to some $\pi_\eta \in A_{\psi_E}$. Hence, it remains to consider the case when τ is contained in the cuspidal spectrum.

When τ is cuspidal, Theorem 5.4 implies that τ is not orthogonal to some $\Theta(\sigma)$. Now for almost all v, $\tau_v \cong \pi_{1_v}$, and thus $\pi_{1_v} \otimes \sigma_v$ is a quotient of Ω_{ψ_v}. In [GG, Thm. 9.1], the local theta correspondence for $\widetilde{SL}_2 \times G_2$ was completely determined, and one sees that the only possible σ_v is the even Weil representation $\omega_{K_v}^+$ attached to K_v. Hence, σ is contained in $\mathcal{A}_K$ (since $\mathcal{A}_K$ is a full near equivalence class), so that $\Theta(\sigma)$ is contained in $\Theta(\mathcal{A}_K) = V_E$.

In conclusion, we deduce that τ is not orthogonal to V_E and thus must be isomorphic to some π_η. This proves statement (ii). ∎

(5.6) Proof of Theorem 5.4. The rest of the section is devoted to the proof of Theorem 5.4. Note that since τ is nearly equivalent to the representations in V_E, the only non-zero Fourier coefficient which τ possesses is the one corresponding to E. As we shall see later, this implies that the period of τ over SU_3^L is zero if $L \neq K$. In trying to decide if τ has a non-zero period over SU_3^K, we are naturally led to consider a Rankin-Selberg integral. To describe this, we first introduce a family of Eisenstein series on SU_3^K.

(5.7) Eisenstein series on SU_3^K. Let $B_K = T_K \cdot N_K$ be a Borel subgroup of SU_3^K with modulus character δ_K. The maximal torus T_K is isomorphic to $Res_{K/F}\mathbb{G}_m$, so that $T_K(F) \cong K^\times$ and $\delta_K(t) = |Nm_{K/F}(t)|^2$.

One may consider the family of induced representations

$$I_K(s) = Ind_{B_K(\mathbb{A})}^{SU_3^K(\mathbb{A})} \delta_K^s.$$

Here, the induction is unnormalized. For a standard section f_s, one has the Eisenstein series $E(f,s,g)$ which is meromorphic in s, and is given by the sum

$$E(f,s,g) = \sum_{\gamma \in B_K(F)\backslash SU_3^K(F)} f_s(\gamma g)$$

when $Re(s)$ is sufficiently large.

The behaviour of $E(f,s,g)$ at $s=1$ is given by the following proposition.

(5.8) Proposition *(i) Let $K = F \times F$. For any standard section f_s, $E(f,s,g)$ has at most a double pole at $s=1$. This double pole is attained by the spherical section.*

(ii) Let K be a field. For any standard section f_s, $E(f,s,g)$ has at most a simple pole at $s=1$. This simple pole is attained by some section.

iii) The residue representation in each case is the trivial representation.

This can be easily checked by examining the constant term of the Eisenstein series, as usual.

(5.9) A Rankin-Selberg integral. For $\varphi \in \tau$, we now set

$$J(f,\varphi,s) = \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} E(f,s,g)\cdot\varphi(g)dg$$

for a standard section $f_s \in I_K(s)$. This defines a meromorphic function on $\mathbb{C}$.

On unfolding the Rankin-Selberg integral, assuming that $Re(s)$ is sufficiently large, we obtain:

$$J(f,\varphi,s) = \int_{N_K(\mathbb{A})\backslash SU_3^K(\mathbb{A})} f_s(g)\cdot\varphi_{\psi_K}(g)\,dg$$

where ψ_K is a unitary character on $N(\mathbb{A})$ corresponding to $E = F \times K$. This identity and Prop. 5.8 also justify our remark in (5.6) that the period of τ over SU_3^L is zero if the Fourier coefficient τ associated to $F \times L$ is zero.

In general, this is as far as we can go, since the Fourier coefficient φ_{ψ_K} may not be Eulerian. However, in the case at hand, φ_{ψ_K} is almost Eulerian. Indeed, for almost all v, τ_v is isomorphic to π_{1_v} and as we have seen in Theorem B,

$$\dim \mathrm{Hom}_{N(F_v)}(\tau_v, \mathbb{C}_{\psi_{K_v}}) = 1.$$

Moreover, for almost all v, a non-zero linear form in this space takes non-zero value on the spherical vector φ_v^0 of τ_v. Let l_v^0 be the non-zero linear form such that $l_v^0(\varphi_v^0) = 1$. Then

$$\varphi_{\psi_K}(g) = l_S(g_S\varphi_S)\cdot\prod_{v\notin S} l_v^0(g_v\varphi_v^0)$$

for some finite set S of places of F, including the archimedean ones. In particular, we have a factorization

$$J_K(f,\varphi,s) = J_{K,S}(\varphi_S, f_S, s)\cdot\prod_{v\notin S} J_{K,v}(\varphi_v^0, f_v^0, s)$$

where

$$J_{K,v}(\varphi_v^0, f_v^0, s) = \int_{N_K(F_v)\backslash SU_3^K(F_v)} f_{v,s}^0(g)\cdot l_v^0(g_v\varphi_v^0)\,dg$$

and $J_{K,S}(\varphi_S, f_S, s)$ is the analogously defined factor over $\mathbb{A}_S$.

To proceed further, we need to evaluate the unramified local factor for $v \notin S$ and to study the analytic behaviour of the bad factor at $s = 1$. These crucial steps are carried out in [GG, §15]. We simply state the result here:

(5.10) Proposition *(i) With all the data unramifed,*

$$J_{K,v}(\varphi_v^0, f_v^0, s) = \zeta_{K_v}(2s-1) \cdot L(\chi_{K_v}, 4s-2)$$

Here χ_{K_v} is the quadratic character associated to K_v by local class field theory.

(ii) Fix $s_0 \in \mathbb{C}$. There exists φ_S and f_S such that $J_{K,S}(\varphi_S, f_S, s)$ is holomorphic at s_0 and $J_{K,S}(\varphi_S, f_S, s_0)$ is non-zero.

(5.11) Conclusion of proof. We can now conclude the proof of Theorem 5.4. Indeed, Prop. 5.10(i) gives:

$$J_K(\varphi, f, s) = J_{K,S}(\varphi_S, f_S, s) \cdot \zeta_K^S(2s-1) \cdot L^S(\chi_K, 4s-2).$$

Now Prop. 5.8 implies that, at $s = 1$, the left-hand-side has a pole of order at most 2 (resp. 1) if $K = F \times F$ (resp. K is a field). On the other hand, Prop. 5.10(ii) implies that one may choose φ and f such that the right-hand-side has a pole of order 2 or 1 at $s = 1$ in the respective cases. Thus for these choices of φ and f, we have:

$$-\mathrm{ord}_{s=1}(J_K(\varphi, f, s)) = \begin{cases} 2, & \text{if } K = F \times F; \\ 1, & \text{if } K \text{ is a field,} \end{cases}$$

and thus

$$\int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g)dg \neq 0.$$

Theorem 5.4 is proved. ∎

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References

[BV] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Annals of Math. 121 (1985), 41-110.

[G] W. T. Gan, *A Siegel-Weil formula for exceptional groups*, J. Reine Angew. Math. 528 (2000), 149-181.

[GG] W. T. Gan and N. Gurevich, *Non-tempered A-packets of G_2: liftings from $\tilde{SL}_2$*, available at **http://www.math.ucsd.edu/˜wgan**.

[GGJ] W. T. Gan, N. Gurevich and D.-H. Jiang, *Cubic unipotent Arthur parameters and multiplicities of square-integrable automorphic forms*, Invent Math. 149 (2002), 225-265.

[GS] W. T. Gan and G. Savin, *On minimal representations*, Representation Theory Vol. 9 (2005), 46-93.

[GGS] W. T. Gan, B. H. Gross and G. Savin, *Fourier coefficients of modular forms on G_2*, Duke Math. J. 115 (2002), 105-169.

[HMS] J.-S. Huang, K. Magaard and G. Savin, Unipotent representations of G_2 arising from the minimal representation of D_4^E, J. Reine Angew Math. 500 (1998), 65-81.

[I] T. Ikeda, *On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series*, J. Math. Kyoto. Univ. 34 (1994), 615-636.

[K] H. H. Kim, *The residual spectrum of G_2*, Canadian J. of Math. 48 (6) (1996), 1245-1272.

[MW] C. Moeglin and J.-L. Waldspurger, *Modeles de Whittaker degeneres pour des groupes p-adiques*, Math. Z. 196 (1987), 427-452.

[RS] S. Rallis and G. Schiffmann, *Theta correspondence associated to G_2*, American J. of Math. 111 (1989), 801-849.

[V] D. Vogan, *The unitary dual of G_2*, Invent. Math. 116 (1994), 677-791.

[W] N. Wallach, *Generalized Whittaker vectors for holomorphic and quaternionic representations*, Commentarii Math. Helv. 78 (2003), 266-307.

[W2] N. Wallach, *Holomorphic continuations of generalized Jacquet integrals for degenerate principal series*, preprint (2002), available at **http://www.math.ucsd.edu/~nwallach**.

[We] A. Weil, *Sur certain groupes d'operateurs unitaires*, Acta Math. 111 (1964), 143-211.

[Z] S. Zampera, *The residual spectrum of the group of type G_2*, J. Math. Pure et Appl. 76 (1997), 805-835.

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