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# ON ENDOSCOPY AND THE REFINED GROSS–PRASAD CONJECTURE FOR $(\mathrm{SO}_5, \mathrm{SO}_4)$

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*Abstract* We prove an explicit formula for periods of certain automorphic forms on $\mathrm{SO}_5 \times \mathrm{SO}_4$ along the diagonal subgroup $\mathrm{SO}_4$ in terms of $L$-values. Our formula also involves a quantity from the theory of endoscopy, as predicted by the refined Gross–Prasad conjecture.

*Keywords:* periods of automorphic forms; $L$-values; endoscopy; the Gross–Prasad conjecture

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## 1. Introduction

In [**11**], Gross and Prasad gave a remarkable conjecture on the non-vanishing of the periods of automorphic forms on $\mathrm{SO}_n \times \mathrm{SO}_{n-1}$ along the diagonal subgroup $\mathrm{SO}_{n-1}$ in terms of the non-vanishing of certain automorphic $L$-functions at the centre of the critical strip. Their conjecture was refined in [**21**], where a conjectural formula relating the periods to the central critical $L$-values was given. This refined conjecture follows from a result of Waldspurger [**55**] for $n = 3$ and that of the second author [**20**] for $n = 4$. The purpose of this paper is to establish the refined Gross–Prasad conjecture for certain $L$-packets of automorphic representations in the case $n = 5$. The (candidate) $L$-packets we consider were constructed by Roberts [**49**] and include all endoscopic $L$-packets of $\mathrm{SO}_5$ as well as some stable ones. Because of this, our result gives strong evidence that the Gross–Prasad conjecture is related to the theory of endoscopy (see Remark 1.2). Special cases of our result have been obtained earlier by Böcherer *et al.* [**5**] for the so-called Yoshida lifts [**57**] (which are certain instances of endoscopic representations).

To state our main theorem, we need to introduce quite a lot of notation. Let $F$ be a totally real number field with ring of adeles $\mathbb{A} = \mathbb{A}_F$ and let $E$ be a totally real étale quadratic algebra over $F$. Let $W_0$ be a two-dimensional symplectic space over $E$, which we may regard as a four-dimensional symplectic space $W = \mathrm{R}_{E/F}(W_0)$ over $F$. Set

$$G = \mathrm{GSp}(W) \cong \mathrm{GSp}_4, \qquad \tilde{G}' = \mathrm{R}_{E/F}(\mathrm{GSp}(W_0)) \cong \mathrm{R}_{E/F}(\mathrm{GL}_2),$$

and

$$G' = \{g' \in \tilde{G}' \mid \nu(g') \in \mathbb{G}_m\},$$

where $\nu : \tilde{G}' \to \mathrm{R}_{E/F}(\mathbb{G}_m)$ is the similitude character. Then we have a natural embedding $G' \hookrightarrow G$. Let $V$ be a four-dimensional quadratic space over $F$ and set

$$H = \mathrm{GO}(V).$$

The discriminant algebra of $V$ is the étale quadratic algebra $K$ over $F$ defined by

$$K = \begin{cases} F \times F & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ F(\sqrt{\mathrm{disc}(V)}) & \text{if } \mathrm{disc}(V) \notin F^{\times,2}, \end{cases}$$

and we let $\omega_{K/F}$ be the quadratic character of $\mathbb{A}^\times/F^\times$ associated to $K/F$ by class field theory. Choose a quaternion algebra $D$ over $F$ such that $H$ is described by the short exact sequence:

$$1 \to \mathrm{R}_{K/F}(\mathbb{G}_m) \xrightarrow{i} (\mathrm{R}_{K/F}(D^\times) \times \mathbb{G}_m) \rtimes \langle \boldsymbol{t} \rangle \to H \to 1$$

(see [**49**, § 2]). Here $i(z) = (z, \mathrm{N}_{K/F}(z)^{-1})$ for $z \in \mathrm{R}_{K/F}(\mathbb{G}_m)$ and $\boldsymbol{t}$ is an involution on $\mathrm{R}_{K/F}(D^\times) \times \mathbb{G}_m$ given by $(g, \lambda) \mapsto (g^c, \lambda)$, where $c$ is the non-trivial automorphism of $K$ over $F$.

Now let $\sigma \cong \bigotimes_v \sigma_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space $V_\sigma$ with central character $\omega_\sigma$. We assume the following.

- The Jacquet–Langlands transfer of $\sigma|_{D^\times(\mathbb{A}_K)}$ to $\mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- $\sigma_v \otimes \mathrm{sgn} \cong \sigma_v$ for some place $v$ of $F$.
- If $\sigma_v \otimes \mathrm{sgn} \ncong \sigma_v$, then $\sigma_v \ncong \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of $\mathrm{GSO}(V)(F_v)$ (see Definition 5.4).

Let $\pi$ be the theta lift of $\sigma$ to $G(\mathbb{A})$ on the space $V_\pi$. In § 7, we will show that $\pi$ is a non-zero irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ with central character $\omega_\sigma$. The representations $\pi$ constructed in this way are precisely the ones which occur in the $L$-packets of $\mathrm{GSp}_4$ defined in the paper of Roberts [**49**] (though he assumed that $\sigma$ and hence $\pi$ is tempered). The automorphic representations of $\mathrm{SO}_5(\mathbb{A})$ considered in this paper are precisely those representations $\pi$ with trivial central characters.

On the other hand, let $\pi'$ be an irreducible unitary cuspidal automorphic representation of $G'(\mathbb{A})$ on the space $V_{\pi'}$ with central character $\omega_\sigma^{-1}$. By [**17**, Theorem 4.13], there exists an irreducible unitary cuspidal automorphic representation $\tau$ of $\tilde{G}'(\mathbb{A})$ on the space $V_\tau$ such that $V_{\pi'} \subset V_\tau^1|_{G'(\mathbb{A})}$. Here $V_\tau^1$ is the subspace of $V_\tau$ on which the group

$$\mathfrak{X}_\tau = \{\omega \in (Z_{\tilde{G}'}(\mathbb{A})G'(\mathbb{A})\tilde{G}'(F)\backslash\tilde{G}'(\mathbb{A}))^D \mid \tau \otimes \omega \cong \tau\}$$

acts trivially, and $V_\tau^1|_{G'(\mathbb{A})}$ is the restriction of $V_\tau^1$ to $G'(\mathbb{A})$ as functions. We remark that the cardinality of $\mathfrak{X}_\tau$ is finite and does not depend on the choice of $\tau$, i.e. it depends only on $\pi'$. We assume the following.

- The base change $\tau_K$ of $\tau$ to $\tilde{G}'(\mathbb{A}_K) \cong \mathrm{GL}_2(\mathbb{A}_{E\otimes K})$ is cuspidal.
- The Jacquet–Langlands transfer $\tau_K^D$ of $\tau_K$ to $D^\times(\mathbb{A}_{E\otimes K})$ exists.

Let $\theta(\tau)$ be the theta lift of $\tau$ to $H(\mathbb{A}_E)$. In § 6, we will show that $\theta(\tau)$ is a non-zero irreducible unitary cuspidal automorphic representation of $H(\mathbb{A}_E)$.

We may now introduce certain automorphic $L$-functions which appear in the refined Gross–Prasad conjecture under consideration:

$$L(s, \pi \times \pi') = L(s, \sigma \times \theta(\tau)),$$
$$L(s, \pi, \mathrm{Ad}) = L(s, \sigma, \mathrm{std})L(s, \sigma, \mathrm{Ad}),$$
$$L(s, \pi', \mathrm{Ad}) = L(s, \tau, \mathrm{Ad}).$$

Here $L(s, \sigma \times \theta(\tau))$ is the triple product $L$-function associated to $\sigma$ and $\theta(\tau)$ of degree eight over $K$ (see § 3), $L(s, \sigma, \mathrm{std})$ is the standard $L$-function of $\sigma$ of degree four over $F$, and $L(s, \sigma, \mathrm{Ad})$ (respectively $L(s, \tau, \mathrm{Ad})$) is the adjoint $L$-function of $\sigma$ (respectively $\tau$) of degree three over $K$ (respectively $E$). Let $S$ be a sufficiently large finite set of places of $F$. By [**45**, Theorem 5.1], the partial $L$-function $L^S(s, \pi \times \pi')$ is holomorphic at $s = \frac{1}{2}$. It is well known that the partial $L$-functions $L^S(s, \pi, \mathrm{Ad})$ and $L^S(s, \pi', \mathrm{Ad})$ are holomorphic and non-zero at $s = 1$. (See also Lemma 7.1.) For each place $v$ of $F$, we similarly define $L$-factors $L_v(s, \pi_v \times \pi'_v)$, $L_v(s, \pi_v, \mathrm{Ad})$, and $L_v(s, \pi'_v, \mathrm{Ad})$ in terms of the Langlands parameters of $\sigma_v$, $\tau_v$, and $\theta(\tau_v)$. By the Kim–Shahidi estimate [**26**, **28**], $L_v(s, \pi_v \times \pi'_v)$ is holomorphic and non-zero at $s = \frac{1}{2}$. It is well known that $L_v(s, \pi_v, \mathrm{Ad})$ and $L_v(s, \pi'_v, \mathrm{Ad})$ are holomorphic and non-zero at $s = 1$.

Now let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ and $\mathcal{B}_{\pi'} : V_{\pi'} \otimes \bar{V}_{\pi'} \to \mathbb{C}$ be the Petersson pairings given by

$$\mathcal{B}_\pi(\phi_1, \phi_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\,\mathrm{d}g,$$
$$\mathcal{B}_{\pi'}(f_1, f_2) = \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} f_1(g')\overline{f_2(g')}\,\mathrm{d}g',$$

for $\phi_1, \phi_2 \in V_\pi$ and $f_1, f_2 \in V_{\pi'}$. Here $\bar{V}_\pi$ and $\bar{V}_{\pi'}$ are the complex conjugate representations of $V_\pi$ and $V_{\pi'}$, $Z_G$ and $Z_{G'}$ are the identity components of the centres of $G$ and $G'$, and $\mathrm{d}g$ and $\mathrm{d}g'$ are the Tamagawa measures on $Z_G(\mathbb{A})\backslash G(\mathbb{A})$ and $Z_{G'}(\mathbb{A})\backslash G'(\mathbb{A})$, respectively. We fix isomorphisms

$$\pi \cong \bigotimes_v \pi_v \quad \text{and} \quad \pi' \cong \bigotimes_v \pi'_v$$

and decompositions

$$\mathcal{B}_\pi = \prod_v \mathcal{B}_{\pi_v} \quad \text{and} \quad \mathcal{B}_{\pi'} = \prod_v \mathcal{B}_{\pi'_v},$$

where $\mathcal{B}_{\pi_v} : \pi_v \otimes \bar{\pi}_v \to \mathbb{C}$ and $\mathcal{B}_{\pi'_v} : \pi'_v \otimes \bar{\pi}'_v \to \mathbb{C}$ are local pairings. Moreover, we fix a decomposition $\mathrm{d}g' = \prod_v \mathrm{d}g'_v$, where $\mathrm{d}g'_v$ is a Haar measure on $Z_{G',v}\backslash G'_v$.

Now define a $G'(\mathbb{A}) \times G'(\mathbb{A})$-invariant functional

$$\mathcal{P} : (V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'}) \to \mathbb{C}$$

by

$$\mathcal{P}(\phi_1, \phi_2; f_1, f_2) = \left( \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} \phi_1(g') f_1(g') \, \mathrm{d}g' \right) \left( \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} \overline{\phi_2(g') f_2(g')} \, \mathrm{d}g' \right)$$

for $\phi_1, \phi_2 \in V_\pi$ and $f_1, f_2 \in V_{\pi'}$. We call $\mathcal{P}$ the global period integral. On the other hand, for each place $v$ of $F$, we define a $G'_v \times G'_v$-invariant functional

$$\mathcal{P}_v^\natural : (\pi_v \boxtimes \bar{\pi}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \to \mathbb{C}$$

by

$$\mathcal{P}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) = \int_{Z_{G',v}\backslash G'_v} \mathcal{B}_{\pi_v}(\pi_v(g'_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\pi'_v}(\pi'_v(g'_v) f_{1,v}, f_{2,v}) \, \mathrm{d}g'_v$$

for $\phi_{1,v}, \phi_{2,v} \in \pi_v$ and $f_{1,v}, f_{2,v} \in \pi'_v$. In § 9, we will show that this integral is absolutely convergent. It was shown in [**21**, Theorem 1.2] that one has

$$\mathcal{P}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) = \zeta_v(2)\zeta_v(4) \frac{L_v(\frac{1}{2}, \pi_v \times \pi'_v)}{L_v(1, \pi_v, \mathrm{Ad}) L_v(1, \pi'_v, \mathrm{Ad})}$$

for unramified data satisfying the conditions (U1)–(U6) in [**21**, § 1]. This suggests that one normalizes the functional $\mathcal{P}_v^\natural$ by setting

$$\mathcal{P}_v = \frac{1}{\zeta_v(2)\zeta_v(4)} \frac{L_v(1, \pi_v, \mathrm{Ad}) L_v(1, \pi'_v, \mathrm{Ad})}{L_v(\frac{1}{2}, \pi_v \times \pi'_v)} \mathcal{P}_v^\natural.$$

Then the product $\prod_v \mathcal{P}_v$ is another $G'(\mathbb{A}) \times G'(\mathbb{A})$-invariant functional on $(V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$. Note that $\prod_v \mathcal{P}_v$ does not depend on the choices of the decompositions of $\mathcal{B}_\pi$, $\mathcal{B}_{\pi'}$, and $\mathrm{d}g'$.

After this preparation, here is our main theorem.

**Theorem 1.1.** *We have*

$$\mathcal{P} = \frac{\zeta(2)\zeta(4)}{2^\alpha |\mathfrak{X}_\tau|} \frac{L(\frac{1}{2}, \pi \times \pi')}{L(1, \pi, \mathrm{Ad}) L(1, \pi', \mathrm{Ad})} \prod_v \mathcal{P}_v$$

*as functionals on* $(V_\pi \boxtimes \bar{V}_\pi) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'})$. *Here*

$$\alpha = \begin{cases} 3 & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ 2 & \text{if } \mathrm{disc}(V) \notin F^{\times,2}. \end{cases}$$

**Remark 1.2.** Assume that $\omega_\sigma$ is trivial. We regard $\pi$ (respectively $\pi'$) as an automorphic representation of $\mathrm{SO}_5(\mathbb{A})$ (respectively $\mathrm{SO}_4(\mathbb{A})$). Then the refined Gross–Prasad conjecture [**21**, Conjecture 1.5] for $\pi$ and $\pi'$ follows from Theorem 1.1.

Moreover, we let $\mathcal{L}_F$ be the hypothetical Langlands group of $F$ and $W_F$ the Weil group of $F$. Let

$$\phi : \mathcal{L}_F \to {}^L\mathrm{SO}_5 = \mathrm{Sp}_4(\mathbb{C}) \times W_F \quad \text{and} \quad \phi' : \mathcal{L}_F \to {}^L\mathrm{SO}_4 = \mathrm{SO}_4(\mathbb{C}) \rtimes W_F$$

be the conjectural Arthur parameters of $\pi$ and $\pi'$, respectively (see [**49**]). Let $\mathcal{S}_\phi$ (respectively $\mathcal{S}_{\phi'}$) be the centralizer of the image of $\phi$ (respectively $\phi'$) in $\mathrm{Sp}_4(\mathbb{C})$ (respectively $\mathrm{SO}_4(\mathbb{C})$). Then the Arthur conjecture [**2**] asserts that

$$|\mathcal{S}_\phi| = \begin{cases} 4 & \text{if } \mathrm{disc}(V) \in F^{\times,2}, \\ 2 & \text{if } \mathrm{disc}(V) \notin F^{\times,2}, \end{cases}$$

and $|\mathcal{S}_{\phi'}| = 2|\mathfrak{X}_\tau|$.

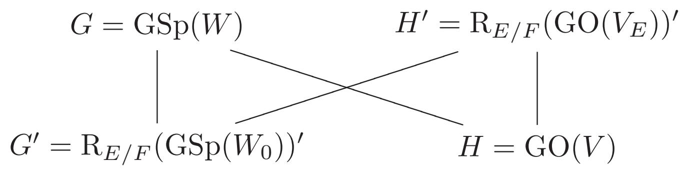
For the representations $\pi$ and $\pi'$ considered in this paper, the above expectations of the Arthur conjecture are essentially verified in [**49**] for $\mathrm{SO}_5$ and in [**17**] for $\mathrm{SO}_4$. Hence Theorem 1.1 is compatible with [**21**, Conjecture 2.1], in the sense that we have

$$2^\alpha |\mathfrak{X}_\tau| = |\mathcal{S}_\phi|\,|\mathcal{S}_{\phi'}|.$$

This power of 2 is the most subtle part of Theorem 1.1. It gives strong evidence that the Gross–Prasad conjecture is related to the theory of endoscopy.

**Remark 1.3.** In the theorem, we have assumed that $F$ and $E$ are totally real, so as to use the Siegel–Weil formula by Kudla *et al.* [**35**]. This is the only place where this assumption is necessary.

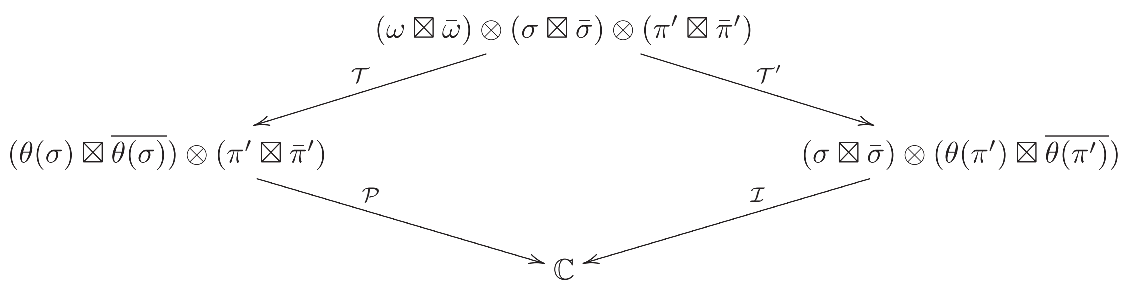
Let us describe the main ideas and inputs in the proof of Theorem 1.1. We have a seesaw diagram of reductive dual pairs:




Here

$$\begin{aligned} \mathrm{R}_{E/F}(\mathrm{GSp}(W_0))' &= \{g' \in \mathrm{R}_{E/F}(\mathrm{GSp}(W_0)) \mid \nu(g') \in \mathbb{G}_m\}, \\ \mathrm{R}_{E/F}(\mathrm{GO}(V_E))' &= \{h' \in \mathrm{R}_{E/F}(\mathrm{GO}(V_E)) \mid \nu(h') \in \mathbb{G}_m\}, \end{aligned}$$

and $V_E = V \otimes_F E$. This gives rise to a global seesaw identity, which can be described as a commutative diagram of equivariant maps:



Here

- $\omega$ is the Weil representation,
- $\mathcal{T}$ and $\mathcal{T}'$ are equivariant surjective maps induced by the global theta lifts,
- $\mathcal{I}$ is an invariant functional induced by the triple product period integral.

On the other hand, by integrating matrix coefficients, one has a local analogue (the explicit local seesaw identity) of the above commutative diagram for each place $v$ of $F$. Because of some local multiplicity one theorems, we may compare the product of the local diagrams with the global diagram. Indeed, one has

$$\mathcal{T} \approx \bigotimes_v \mathcal{T}_v, \qquad \mathcal{T}' \approx \bigotimes_v \mathcal{T}'_v, \qquad \mathcal{I} \approx \prod_v \mathcal{I}_v, \tag{1.1}$$

so that

$$\mathcal{P} \approx \prod_v \mathcal{P}_v. \tag{1.2}$$

Here $\approx$ denotes equality up to a scalar. The main theorem amounts to an explicit determination of the constant of proportionality in (1.2). But by the commutativity of the local and global diagrams above, it suffices to determine the three constants of proportionality in (1.1). To determine the constants of proportionality for $\mathcal{T}$ and $\mathcal{T}'$, we use the Rallis inner product formula, whereas for $\mathcal{I}$, we use a formula for triple product period integrals by the second author [**20**] (or rather its extension from $\mathrm{GSO}(V)$ to $\mathrm{GO}(V)$).

This paper is organized as follows. In § 2, we study the restriction of automorphic forms on $\mathrm{GO}(V)(\mathbb{A})$ to $\mathrm{GSO}(V)(\mathbb{A})$. This is needed in § 3, where we extend the result of the second author [**20**] and prove a formula for triple product period integrals for $\mathrm{GO}(V)$. In §§ 4 and 5, we study local theta lifts from $\mathrm{GL}_2$ to $\mathrm{GO}(V)$ and those from $\mathrm{GO}(V)$ to $\mathrm{GSp}_4$, respectively. In §§ 6 and 7, we study global theta lifts from $\mathrm{GL}_2$ to $\mathrm{GO}(V)$ and those from $\mathrm{GO}(V)$ to $\mathrm{GSp}_4$, respectively. In particular, we construct explicit pairings on the local theta lifts, and using the Rallis inner product formula, we compare the product of the local pairings with the Petersson pairing on the global theta lift. To prove the Rallis inner product formula, we use the Siegel–Weil formula (the second term identity) for $(\mathrm{O}(V), \mathrm{Sp}_4)$ by Kudla and Rallis [**31**] and Kudla *et al.* [**35**], and prove a certain spherical second term identity for $(\mathrm{GSp}_4, \mathrm{GO}_8)$. After choosing Haar measures on various groups in § 8, we prove in § 9 the explicit local seesaw identity with respect to these measures. Finally, using the local and global seesaw identities and the formula for triple product period integrals, we prove Theorem 1.1 in § 10. We also include two appendices: the first one determines completely the local theta correspondence for $\mathrm{GO}(V) \times \mathrm{GSp}_4$ and establishes certain properties of the correspondence we need, while the second one proves the spherical second term identity for $(\mathrm{GSp}_4, \mathrm{GO}_8)$.

## 2. Automorphic forms on GO(*V*)

Let $F$ be a number field and $V$ a four-dimensional quadratic space over $F$. Set

$$H = \mathrm{GO}(V), \qquad H^0 = \mathrm{GSO}(V), \qquad \boldsymbol{\mu}_2 = \langle \boldsymbol{t} \rangle.$$

Then we have a short exact sequence:

$$1 \to H^0 \to H \to \boldsymbol{\mu}_2 \to 1.$$

Let $Z_H$ be the identity component of the centre of $H$. We identify $\boldsymbol{\mu}_2$ with $\{+,-\}$. For each place $v$ of $F$, let $\boldsymbol{t}_v$ be the image of $\boldsymbol{t}$ in $H_v$.

Let $\mathrm{d}h$ and $\mathrm{d}h_0$ be the Tamagawa measures on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$ and $Z_H(\mathbb{A})\backslash H^0(\mathbb{A})$, respectively. Let $\mathrm{d}\epsilon_v$ be the Haar measure on $\boldsymbol{\mu}_2(F_v)$ such that $\mathrm{vol}(\boldsymbol{\mu}_2(F_v)) = 1$. Then the product measure $\mathrm{d}\epsilon = \prod_v \mathrm{d}\epsilon_v$ is the Tamagawa measure on $\boldsymbol{\mu}_2(\mathbb{A})$. Moreover, we have

$$\int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} f(h)\,\mathrm{d}h = \int_{\boldsymbol{\mu}_2(F)\backslash \boldsymbol{\mu}_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} f(h_0\epsilon)\,\mathrm{d}h_0\,\mathrm{d}\epsilon$$

for $f \in L^1(Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A}))$.

Let $\Pi \cong \bigotimes_v \Pi_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space $V_\Pi$. Let $\mathfrak{S}$ be the set of places $v$ of $F$ such that $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$. Let $\mathcal{B}_\Pi : V_\Pi \otimes \bar{V}_\Pi \to \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\Pi(\phi_1, \phi_2) = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h)\overline{\phi_2(h)}\,\mathrm{d}h$$

for $\phi_1, \phi_2 \in V_\Pi$. We have an $H^0(\mathbb{A})$-equivariant surjective map

$$V_\Pi \to V_\Pi|_{H^0(\mathbb{A})},$$

where $V_\Pi|_{H^0(\mathbb{A})}$ is the restriction of $V_\Pi$ to $H^0(\mathbb{A})$ as functions.

*The case* $\mathfrak{S} = \varnothing$

Let $\pi$ be the automorphic representation of $H^0(\mathbb{A})$ on the space $V_\pi = V_\Pi|_{H^0(\mathbb{A})}$. Then $\pi$ is irreducible. The restriction to $H^0(\mathbb{A})$ as functions induces an isomorphism

$$V_\Pi \cong V_\pi$$

as representations of $H^0(\mathbb{A})$. Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ be the Petersson pairing.

**Lemma 2.1.** *We have*

$$\mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) = 2\mathcal{B}_\Pi(\phi_1, \phi_2)$$

*for* $\phi_1, \phi_2 \in V_\Pi$.

**Proof.** For each $\epsilon \in \boldsymbol{\mu}_2(\mathbb{A})$, we define an $H^0(\mathbb{A})$-invariant pairing $\mathcal{B}_\pi^\epsilon : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ by

$$\mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) = \mathcal{B}_\pi(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})})$$

for $\phi_1, \phi_2 \in V_\Pi$. Then we have $\mathcal{B}_\pi^\epsilon = C_\epsilon \mathcal{B}_\pi$ with some constant $C_\epsilon$. Hence we have

$$\begin{aligned}\mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) &= C_\epsilon \mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon \mathcal{B}_\pi^\epsilon(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon^2 \mathcal{B}_\pi(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= C_\epsilon^2 \mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}),\end{aligned}$$

so that $C_\epsilon^2 = 1$. Since $\mathcal{B}_\pi^\epsilon$ is positive definite, we have $C_\epsilon = 1$. Hence we have

$$\begin{aligned}\mathcal{B}_\Pi(\phi_1, \phi_2) &= \int_{\boldsymbol{\mu}_2(F)\backslash\boldsymbol{\mu}_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\, \mathrm{d}h_0\, \mathrm{d}\epsilon \\ &= \int_{\boldsymbol{\mu}_2(F)\backslash\boldsymbol{\mu}_2(\mathbb{A})} \mathcal{B}_\pi^\epsilon(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})})\, \mathrm{d}\epsilon \\ &= \mathrm{vol}(\boldsymbol{\mu}_2(F)\backslash\boldsymbol{\mu}_2(\mathbb{A}))\mathcal{B}_\pi(\phi_1|_{H^0(\mathbb{A})}, \phi_2|_{H^0(\mathbb{A})}).\end{aligned}$$

□

*The case* $\mathfrak{S} \neq \varnothing$

We fix an isomorphism

$$V_\Pi \cong \bigotimes_v \mathcal{V}_v = \varinjlim_S \left(\bigotimes_{v\in S} \mathcal{V}_v\right) \otimes \left(\bigotimes_{v\notin S} \boldsymbol{\phi}_v\right) \tag{2.1}$$

as representations of $H(\mathbb{A})$, where $\mathcal{V}_v$ is the space of $\Pi_v$, $S$ is a sufficiently large finite set of places of $F$, and $\boldsymbol{\phi}_v$ is an $H(\mathfrak{o}_v)$-invariant element of $\mathcal{V}_v$ for $v \notin S$.

If $v \in \mathfrak{S}$, then we can write $\Pi_v|_{H_v^0} = \pi_v^+ \oplus \pi_v^-$, where $\pi_v^\pm$ is an irreducible admissible representation of $H_v^0$. Note that $\pi_v^+ \ncong \pi_v^-$ and $\pi_v^+ \circ \mathrm{Ad}(\boldsymbol{t}_v) \cong \pi_v^-$. We have $\mathcal{V}_v = \mathcal{V}_v^+ \oplus \mathcal{V}_v^-$, where $\mathcal{V}_v^\pm$ is the space of $\pi_v^\pm$ and $\mathcal{V}_v^- = \Pi_v(\boldsymbol{t}_v)(\mathcal{V}_v^+)$. We have $\boldsymbol{\phi}_v = \boldsymbol{\phi}_v^+ + \boldsymbol{\phi}_v^-$ for almost all $v \in \mathfrak{S}$, where $\boldsymbol{\phi}_v^\pm$ is an $H^0(\mathfrak{o}_v)$-invariant element of $\mathcal{V}_v^\pm$ and $\boldsymbol{\phi}_v^- = \Pi_v(\boldsymbol{t}_v)(\boldsymbol{\phi}_v^+)$. If $v \notin \mathfrak{S}$, then $\pi_v = \Pi_v|_{H_v^0}$ is an irreducible admissible representation of $H_v^0$ on the space $\mathcal{V}_v$.

Let $S$ be a sufficiently large finite set of places of $F$. For $\epsilon = (\epsilon_v) \in \boldsymbol{\mu}_2(F_{S\cap\mathfrak{S}})$, let $V_{\Pi,S}^\epsilon$ be the inverse image of

$$\left(\bigotimes_{v\in S\cap\mathfrak{S}} \mathcal{V}_v^{\epsilon_v}\right) \otimes \left(\bigotimes_{v\in S,\, v\notin\mathfrak{S}} \mathcal{V}_v\right) \otimes \left(\bigotimes_{v\notin S} \boldsymbol{\phi}_v\right)$$

in $V_\Pi$ by (2.1). Then $H^0(F_S)\boldsymbol{\mu}_2(\mathbb{A}^{S\cap\mathfrak{S}})$ acts on $V_{\Pi,S}^\epsilon$ and the representation of $H^0(F_S)$ on $V_{\Pi,S}^\epsilon$ is given by

$$\pi_S^\epsilon = \left(\bigotimes_{v\in S\cap\mathfrak{S}} \pi_v^{\epsilon_v}\right) \otimes \left(\bigotimes_{v\in S,\, v\notin\mathfrak{S}} \pi_v\right).$$

Hence we have

$$\Pi|_{H^0(\mathbb{A})} \cong \varinjlim_S \bigoplus_{\epsilon\in\boldsymbol{\mu}_2(F_{S\cap\mathfrak{S}})} \pi_S^\epsilon, \qquad V_\Pi = \varinjlim_S \bigoplus_{\epsilon\in\boldsymbol{\mu}_2(F_{S\cap\mathfrak{S}})} V_{\Pi,S}^\epsilon,$$

as representations of $H^0(\mathbb{A})$.

By [**15**, § 1], there exists an irreducible unitary cuspidal automorphic representation $\pi$ of $H^0(\mathbb{A})$ on the space $V_\pi$ such that

$$V_\Pi|_{H^0(\mathbb{A})} = V_\pi \oplus V_{\pi\circ\mathrm{Ad}(\boldsymbol{t})}. \tag{2.2}$$

We may assume that

$$\pi \cong \Big(\bigotimes_{v\in\mathfrak{S}} \pi_v^+\Big) \otimes \Big(\bigotimes_{v\notin\mathfrak{S}} \pi_v\Big),$$

$$V_\pi \cong \varinjlim_S \Big(\bigotimes_{v\in S\cap\mathfrak{S}} \mathcal{V}_v^+\Big) \otimes \Big(\bigotimes_{v\in S,\, v\notin\mathfrak{S}} \mathcal{V}_v\Big) \otimes \Big(\bigotimes_{v\notin S,\, v\in\mathfrak{S}} \boldsymbol{\phi}_v^+\Big) \otimes \Big(\bigotimes_{v\notin S,\, v\notin\mathfrak{S}} \boldsymbol{\phi}_v\Big).$$

Then we have

$$\pi\circ\mathrm{Ad}(\boldsymbol{t}) \cong \Big(\bigotimes_{v\in\mathfrak{S}} \pi_v^-\Big) \otimes \Big(\bigotimes_{v\notin\mathfrak{S}} \pi_v\Big),$$

$$V_{\pi\circ\mathrm{Ad}(\boldsymbol{t})} \cong \varinjlim_S \Big(\bigotimes_{v\in S\cap\mathfrak{S}} \mathcal{V}_v^-\Big) \otimes \Big(\bigotimes_{v\in S,\, v\notin\mathfrak{S}} \mathcal{V}_v\Big) \otimes \Big(\bigotimes_{v\notin S,\, v\in\mathfrak{S}} \boldsymbol{\phi}_v^-\Big) \otimes \Big(\bigotimes_{v\notin S,\, v\notin\mathfrak{S}} \boldsymbol{\phi}_v\Big).$$

**Lemma 2.2.** *For $\phi \in V_{\Pi,S}^1$, the support of $\phi$ is contained in*

$$H^0(\mathbb{A})\boldsymbol{\mu}_2(\mathbb{A}^{S\cap\mathfrak{S}}) \cup H^0(\mathbb{A})\boldsymbol{\mu}_2(\mathbb{A}^{S\cap\mathfrak{S}})\boldsymbol{t}.$$

**Proof.** Let $\epsilon \in \boldsymbol{\mu}_2(F_{S\cap\mathfrak{S}})$. By (2.2), we have $V_{\Pi,S}^\epsilon|_{H^0(\mathbb{A})} = 0$ unless $\epsilon \in \boldsymbol{\mu}_2(F)$. Since $V_{\Pi,S}^\epsilon = \{\Pi(\epsilon)\phi \mid \phi \in V_{\Pi,S}^1\}$ and

$$H(\mathbb{A}) = \bigcup_{\epsilon\in\boldsymbol{\mu}_2(F_{S\cap\mathfrak{S}})} H^0(\mathbb{A})\boldsymbol{\mu}_2(\mathbb{A}^{S\cap\mathfrak{S}})\epsilon,$$

the assertion follows. □

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ be the Petersson pairing. We fix a decomposition

$$\mathcal{B}_\pi = \prod_{v\in\mathfrak{S}} \mathcal{B}_v^+ \prod_{v\notin\mathfrak{S}} \mathcal{B}_v,$$

where

- $\mathcal{B}_v^+ : \mathcal{V}_v^+ \otimes \bar{\mathcal{V}}_v^+ \to \mathbb{C}$ is an $H_v^0$-invariant pairing if $v \in \mathfrak{S}$,
- $\mathcal{B}_v : \mathcal{V}_v \otimes \bar{\mathcal{V}}_v \to \mathbb{C}$ is an $H_v$-invariant pairing if $v \notin \mathfrak{S}$,
- $\mathcal{B}_v^+(\boldsymbol{\phi}_v^+, \boldsymbol{\phi}_v^+) = \mathcal{B}_v(\boldsymbol{\phi}_v, \boldsymbol{\phi}_v) = 1$ for almost all $v$.

For each $v \in \mathfrak{S}$, we define an $H_v^0$-invariant pairing $\mathcal{B}_v^- : \mathcal{V}_v^- \otimes \bar{\mathcal{V}}_v^- \to \mathbb{C}$ by

$$\mathcal{B}_v^-(\phi_1, \phi_2) = \mathcal{B}_v^+(\Pi_v(\boldsymbol{t}_v)\phi_1, \Pi_v(\boldsymbol{t}_v)\phi_2)$$

for $\phi_1, \phi_2 \in \mathcal{V}_v^-$. Then $\mathcal{B}_v^-(\boldsymbol{\phi}_v^-, \boldsymbol{\phi}_v^-) = 1$ for almost all $v \in \mathfrak{S}$. For each place $v$ of $F$, we define an $H_v$-invariant pairing $\mathcal{B}_{\Pi_v}^\natural : \mathcal{V}_v \otimes \bar{\mathcal{V}}_v \to \mathbb{C}$ as follows.

- If $v \in \mathfrak{S}$, let $\mathcal{B}^{\natural}_{\Pi_v}(\phi_1^+ + \phi_1^-, \phi_2^+ + \phi_2^-) = \frac{1}{2}(\mathcal{B}_v^+(\phi_1^+, \phi_2^+) + \mathcal{B}_v^-(\phi_1^-, \phi_2^-))$ for $\phi_1^+, \phi_2^+ \in \mathcal{V}_v^+$ and $\phi_1^-, \phi_2^- \in \mathcal{V}_v^-$.
- If $v \notin \mathfrak{S}$, let $\mathcal{B}^{\natural}_{\Pi_v} = \mathcal{B}_v$.

Then $\mathcal{B}^{\natural}_{\Pi_v}(\boldsymbol{\phi}_v, \boldsymbol{\phi}_v) = 1$ for almost all $v$.

**Lemma 2.3.** *We have*

$$\mathcal{B}_{\Pi} = \prod_v \mathcal{B}^{\natural}_{\Pi_v}.$$

**Proof.** Fix a sufficiently large finite set $S$ of places of $F$. Put

$$S' = S \smallsetminus (S \cap \mathfrak{S}), \qquad s = |S \cap \mathfrak{S}|, \qquad s' = |S'|.$$

Let

$$\phi_1 = \Big(\bigotimes_{v \in S} \phi_{1,v}\Big) \otimes \Big(\bigotimes_{v \notin S} \boldsymbol{\phi}_v\Big), \qquad \phi_2 = \Big(\bigotimes_{v \in S} \phi_{2,v}\Big) \otimes \Big(\bigotimes_{v \notin S} \boldsymbol{\phi}_v\Big) \in V^1_{\Pi,S},$$

where $\phi_{1,v}, \phi_{2,v} \in \mathcal{V}_v^+$ (respectively $\phi_{1,v}, \phi_{2,v} \in \mathcal{V}_v$) if $v \in S \cap \mathfrak{S}$ (respectively if $v \in S'$). Then $\mathcal{B}_{\Pi}(\phi_1, \phi_2)$ is equal to

$$\begin{aligned}\int_{\boldsymbol{\mu}_2(F)\backslash\boldsymbol{\mu}_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} & \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\,\mathrm{d}h_0\,\mathrm{d}\epsilon \\ &= \frac{1}{2}\int_{\boldsymbol{\mu}_2(\mathbb{A})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\,\mathrm{d}h_0\,\mathrm{d}\epsilon \\ &= \frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_S)} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\,\mathrm{d}h_0.\end{aligned}$$

By Lemma 2.2, this integral is equal to

$$\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} (\phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)} + \phi_1(h_0\epsilon\boldsymbol{t})\overline{\phi_2(h_0\epsilon\boldsymbol{t})})\,\mathrm{d}h_0.$$

We have

$$\begin{aligned}\int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} & \phi_1(h_0\epsilon\boldsymbol{t})\overline{\phi_2(h_0\epsilon\boldsymbol{t})}\,\mathrm{d}h_0 \\ &= \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(\mathrm{Ad}(\boldsymbol{t})(h_0)\epsilon)\overline{\phi_2(\mathrm{Ad}(\boldsymbol{t})(h_0)\epsilon)}\,\mathrm{d}h_0 \\ &= \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\,\mathrm{d}h_0.\end{aligned}$$

Hence we have

$$\mathcal{B}_{\Pi}(\phi_1, \phi_2) = \frac{1}{2^{s+s'}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi_1(h_0\epsilon)\overline{\phi_2(h_0\epsilon)}\,\mathrm{d}h_0.$$

Since $\Pi(\epsilon)\phi_1, \Pi(\epsilon)\phi_2 \in V^1_{\Pi,S}$ for $\epsilon \in \boldsymbol{\mu}_2(F_{S'})$, $\mathcal{B}_\Pi(\phi_1, \phi_2)$ is equal to

$$\begin{aligned}\frac{1}{2^{s+s'}} &\sum_{\epsilon\in\boldsymbol{\mu}_2(F_{S'})} \mathcal{B}_\pi(\Pi(\epsilon)\phi_1|_{H^0(\mathbb{A})}, \Pi(\epsilon)\phi_2|_{H^0(\mathbb{A})}) \\ &= \frac{1}{2^{s+s'}} \sum_{\epsilon\in\boldsymbol{\mu}_2(F_{S'})} \prod_{v\in S\cap\mathfrak{S}} \mathcal{B}^+_v(\phi_{1,v}, \phi_{2,v}) \prod_{v\in S'} \mathcal{B}_v(\Pi_v(\epsilon_v)\phi_{1,v}, \Pi_v(\epsilon_v)\phi_{2,v}) \\ &= \frac{1}{2^s} \prod_{v\in S\cap\mathfrak{S}} \mathcal{B}^+_v(\phi_{1,v}, \phi_{2,v}) \prod_{v\in S'} \mathcal{B}_v(\phi_{1,v}, \phi_{2,v}) \\ &= \prod_{v\in S\cap\mathfrak{S}} \mathcal{B}^\natural_{\Pi_v}(\phi_{1,v}, \phi_{2,v}) \prod_{v\in S'} \mathcal{B}^\natural_{\Pi_v}(\phi_{1,v}, \phi_{2,v}).\end{aligned}$$

This completes the proof. □

## 3. Triple product period integrals for GO($V$)

Let $F$ be a number field and let $E$ be an étale quadratic algebra over $F$. Let $V$ be a four-dimensional quadratic space over $F$. Set

$$H = \mathrm{GO}(V), \qquad H^0 = \mathrm{GSO}(V).$$

Let $K$ be the discriminant algebra of $V$ and choose a quaternion algebra $D$ over $F$ associated to $V$ as in § 1.

Let $\Pi \cong \bigotimes_v \Pi_v$ (respectively $\Pi' \cong \bigotimes_v \Pi'_v$) be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ (respectively $H(\mathbb{A}_E)$) on the space $V_\Pi$ (respectively $V_{\Pi'}$) with central character $\omega_\Pi$ (respectively $\omega_{\Pi'}$). We assume the following:

- $\omega_\Pi\omega_{\Pi'}$ is trivial on $Z_H(\mathbb{A})$;
- $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$ for some place $v$ of $F$;
- $\Pi'_v \otimes \mathrm{sgn} \ncong \Pi'_v$ for all places $v$ of $F$.

Let $\pi$ (respectively $\pi'$) be an irreducible unitary cuspidal automorphic representation of $H^0(\mathbb{A})$ (respectively $H^0(\mathbb{A}_E)$) on the space $V_\pi$ (respectively $V_{\pi'}$) such that $V_\Pi|_{H^0(\mathbb{A})} = V_\pi \oplus V_{\pi\circ\mathrm{Ad}(\boldsymbol{t})}$ (respectively $V_{\Pi'}|_{H^0(\mathbb{A}_E)} = V_{\pi'}$). Let $\dot\pi$ (respectively $\dot\pi'$) be the Jacquet–Langlands transfer of $\pi|_{D^\times(\mathbb{A}_K)}$ (respectively $\pi'|_{D^\times(\mathbb{A}_{E\otimes K})}$) to $\mathrm{GL}_2(\mathbb{A}_K)$ (respectively $\mathrm{GL}_2(\mathbb{A}_{E\otimes K})$). We define the adjoint $L$-functions of $\Pi$ and $\Pi'$ by

$$L(s, \Pi, \mathrm{Ad}) = L(s, \dot\pi, \mathrm{Ad}) \quad \text{and} \quad L(s, \Pi', \mathrm{Ad}) = L(s, \dot\pi', \mathrm{Ad}),$$

respectively. Note that $L(s, \Pi, \mathrm{Ad})$ does not depend on the choice of $\pi$. We define an $L$-function $L(s, \Pi \times \Pi')$ of degree eight over $K$ by

$$L(s, \Pi \times \Pi') = \prod_v L_v(s, \dot\pi_v \times \dot\pi'_v),$$

where $L_v(s, \dot{\pi}_v \times \dot{\pi}'_v)$ is the triple product $L$-factor associated to the Langlands parameters of $\dot{\pi}_v$ and $\dot{\pi}'_v$ and the eight-dimensional representation of ${}^L\mathrm{R}_{(K\times E\otimes K)/K}(\mathrm{GL}_2)$ defined in [**45**, § 0]. We remark that there is another definition of this $L$-factor à la Garrett [**10**], Piatetski-Shapiro and Rallis [**45**], and Ikeda [**22**] using local zeta integrals and these two definitions agree if $v$ is non-archimedean and $\dot{\pi}_v$ and $\dot{\pi}'_v$ are unramified, but we do not assume that they agree for all $v$ in this paper. The following lemma asserts that $L(s, \Pi \times \Pi')$ does not depend on the choice of $\pi$.

**Lemma 3.1.** *We have*

$$L_v(s, (\dot{\pi}_v \circ c) \times \dot{\pi}'_v) = L_v(s, \dot{\pi}_v \times \dot{\pi}'_v),$$

*where $c$ is the non-trivial automorphism of $K$ over $F$.*

**Proof.** We fix a place $v$ of $F$ and suppress it from the notation. Let $W_F$ be the Weil group of $F$ and $\mathcal{L}_F$ the Langlands group of $F$ given by

$$\mathcal{L}_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean,} \\ W_F & \text{if } F \text{ is archimedean.} \end{cases}$$

We only consider the case where $E$ and $K$ are quadratic extensions of $F$ and $E \neq K$; the other cases are similar. Then $EK \cong E \otimes K$ is a quartic extension of $F$. Let

$$\mathrm{BC}_{K/F} : {}^L\mathrm{GL}_4 \to {}^L\mathrm{R}_{K/F}(\mathrm{GL}_4) \quad \text{and} \quad \mathrm{BC}_{EK/E} : {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2) \to {}^L\mathrm{R}_{EK/F}(\mathrm{GL}_2)$$

be the base change $L$-homomorphisms. We define an $L$-homomorphism

$$\mathrm{Asai}_{E/F} : {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2) \to {}^L\mathrm{GL}_4$$

by

$$\mathrm{Asai}_{E/F}((g_1, g_2), 1) = (g_1 \otimes g_2, 1),$$
$$\mathrm{Asai}_{E/F}((1,1), w) = \begin{cases} (\mathrm{id}, w) & \text{if } w \in W_E, \\ (\mathrm{sw}, w) & \text{if } w \notin W_E, \end{cases}$$

for $g_1, g_2 \in \mathrm{GL}_2(\mathbb{C})$ and $w \in W_F$, where $\mathrm{sw} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is an isomorphism given by $\mathrm{sw}(x \otimes y) = y \otimes x$. Similarly, we define an $L$-homomorphism

$$\mathrm{Asai}_{EK/K} : {}^L\mathrm{R}_{EK/F}(\mathrm{GL}_2) \to {}^L\mathrm{R}_{K/F}(\mathrm{GL}_4).$$

Then we have

$$\mathrm{Asai}_{EK/K} \circ \mathrm{BC}_{EK/E} = \mathrm{BC}_{K/F} \circ \mathrm{Asai}_{E/F}.$$

Let $\phi : \mathcal{L}_F \to {}^L\mathrm{R}_{K/F}(\mathrm{GL}_2)$ and $\phi' : \mathcal{L}_F \to {}^L\mathrm{R}_{EK/F}(\mathrm{GL}_2)$ be the Langlands parameters of $\dot{\pi}$ and $\dot{\pi}'$, respectively. We identify $\phi$ and $\mathrm{Asai}_{EK/K} \circ \phi'$ with homomorphisms $\phi : \mathcal{L}_K \to \mathrm{GL}_2(\mathbb{C})$ and $\mathrm{Asai}_{EK/K} \circ \phi' : \mathcal{L}_K \to \mathrm{GL}_4(\mathbb{C})$, respectively. By definition, we have

$$L(s, \dot{\pi} \times \dot{\pi}') = L(s, \phi \otimes (\mathrm{Asai}_{EK/K} \circ \phi')).$$

By assumption on $\Pi'$, there exists a Langlands parameter $\phi'' : \mathcal{L}_F \to {}^L\mathrm{R}_{E/F}(\mathrm{GL}_2)$ such that $\phi' = \mathrm{BC}_{EK/E} \circ \phi''$. Hence we have

$$\begin{aligned} L(s, \phi \otimes (\mathrm{Asai}_{EK/K} \circ \phi')) &= L(s, \phi \otimes (\mathrm{Asai}_{EK/K} \circ \mathrm{BC}_{EK/E} \circ \phi'')) \\ &= L(s, \phi \otimes (\mathrm{BC}_{K/F} \circ \mathrm{Asai}_{E/F} \circ \phi'')). \end{aligned}$$

This completes the proof. □

Let $\mathcal{B}_\Pi : V_\Pi \otimes \bar{V}_\Pi \to \mathbb{C}$ and $\mathcal{B}_{\Pi'} : V_{\Pi'} \otimes \bar{V}_{\Pi'} \to \mathbb{C}$ be the Petersson pairings. We fix decompositions $\mathcal{B}_\Pi = \prod_v \mathcal{B}_{\Pi_v}$ and $\mathcal{B}_{\Pi'} = \prod_v \mathcal{B}_{\Pi'_v}$, where $\mathcal{B}_{\Pi_v} : \Pi_v \otimes \bar{\Pi}_v \to \mathbb{C}$ and $\mathcal{B}_{\Pi'_v} : \Pi'_v \otimes \bar{\Pi}'_v \to \mathbb{C}$ are pairings. Let $\mathrm{d}h$ be the Tamagawa measure on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$. We fix a decomposition $\mathrm{d}h = \prod_v \mathrm{d}h_v$, where $\mathrm{d}h_v$ is a Haar measure on $Z_{H,v}\backslash H_v$. We define an $H(\mathbb{A}) \times H(\mathbb{A})$-invariant functional

$$\mathcal{I} : (V_\Pi \boxtimes \bar{V}_\Pi) \otimes (V_{\Pi'} \boxtimes \bar{V}_{\Pi'}) \to \mathbb{C}$$

by

$$\mathcal{I}(\phi_1, \phi_2; \phi'_1, \phi'_2) = \left( \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h)\phi'_1(h)\, \mathrm{d}h \right) \left( \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \overline{\phi_2(h)\phi'_2(h)}\, \mathrm{d}h \right)$$

for $\phi_1, \phi_2 \in V_\Pi$ and $\phi'_1, \phi'_2 \in V_{\Pi'}$. For each place $v$ of $F$, we define an $H_v \times H_v$-invariant functional

$$\mathcal{I}^\natural_v : (\Pi_v \boxtimes \bar{\Pi}_v) \otimes (\Pi'_v \boxtimes \bar{\Pi}'_v) \to \mathbb{C}$$

by

$$\mathcal{I}^\natural_v(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) = \int_{Z_{H,v}\backslash H_v} \mathcal{B}_{\Pi_v}(\Pi_v(h_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\Pi'_v}(\Pi'_v(h_v)\phi'_{1,v}, \phi'_{2,v})\, \mathrm{d}h_v$$

for $\phi_{1,v}, \phi_{2,v} \in \Pi_v$ and $\phi'_{1,v}, \phi'_{2,v} \in \Pi'_v$. By [**20**, Lemma 2.1], this integral is absolutely convergent.

**Proposition 3.2.** *We have*

$$\mathcal{I} = 2^c \zeta_{E\otimes K}(2) \frac{L(\frac{1}{2}, \Pi \times \Pi')}{L(1, \Pi, \mathrm{Ad}) L(1, \Pi', \mathrm{Ad})} \prod_v \mathcal{I}_v$$

*as functionals on* $(V_\Pi \boxtimes \bar{V}_\Pi) \otimes (V_{\Pi'} \boxtimes \bar{V}_{\Pi'})$*. Here*

$$c = \begin{cases} -4 & \text{if } E = K = F \times F, \\ -1 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -3 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K, \end{cases}$$

*and*

$$\mathcal{I}_v = \frac{1}{\zeta_{E_v \otimes K_v}(2)} \frac{L_v(1, \Pi_v, \mathrm{Ad}) L_v(1, \Pi'_v, \mathrm{Ad})}{L_v(\frac{1}{2}, \Pi_v \times \Pi'_v)} \mathcal{I}^\natural_v.$$

The rest of this section is devoted to the proof of Proposition 3.2. Let $\mathfrak{S}$ be the set of places $v$ of $F$ such that $\Pi_v \otimes \mathrm{sgn} \cong \Pi_v$. Fix a sufficiently large finite set $S$ of places of $F$. Put

$$S' = S \smallsetminus (S \cap \mathfrak{S}), \qquad s = |S \cap \mathfrak{S}|, \qquad s' = |S'|.$$

We may assume that $\phi_1, \phi_2 \in V^1_{\Pi,S}$ and $\phi'_1, \phi'_2 \in V_{\Pi',S}$. Here $V^1_{\Pi,S}$ is the subspace of $V_\Pi$ given in § 2 and $V_{\Pi',S}$ is the subspace of $V_{\Pi'}$ consisting of $\prod_{v \notin S} H(\mathfrak{o}_{E_v})$-invariant elements.

**Lemma 3.3.** *We have*

$$\int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\phi'(h)\,\mathrm{d}h = \frac{1}{2^{s+s'}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi(h_0\epsilon)\phi'(h_0\epsilon)\,\mathrm{d}h_0$$

*for $\phi \in V^1_{\Pi,S}$ and $\phi' \in V_{\Pi',S}$, where $\mathrm{d}h_0$ is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H^0(\mathbb{A})$.*

**Proof.** As in the proof of Lemma 2.3, we have

$$\int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\phi'(h)\,\mathrm{d}h = \frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_S)} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} \phi(h_0\epsilon)\phi'(h_0\epsilon)\,\mathrm{d}h_0.$$

By Lemma 2.2, this integral is equal to

$$\frac{1}{2^{s+s'+1}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \int_{Z_H(\mathbb{A})H^0(F)\backslash H^0(\mathbb{A})} (\phi(h_0\epsilon)\phi'(h_0\epsilon) + \phi(h_0\epsilon\boldsymbol{t})\phi'(h_0\epsilon\boldsymbol{t}))\,\mathrm{d}h_0.$$

This completes the proof. □

We fix an isomorphism

$$\pi \cong \left(\bigotimes_{v \in \mathfrak{S}} \pi_v^+\right) \otimes \left(\bigotimes_{v \notin \mathfrak{S}} \pi_v\right)$$

and a decomposition

$$\mathcal{B}_\pi = \prod_{v \in \mathfrak{S}} \mathcal{B}_v^+ \cdot \prod_{v \notin \mathfrak{S}} \mathcal{B}_v$$

as in § 2. By Lemma 2.3, we may assume that $\mathcal{B}_{\Pi_v} = \mathcal{B}^\natural_{\Pi_v}$, where $\mathcal{B}^\natural_{\Pi_v} : \Pi_v \otimes \bar{\Pi}_v \to \mathbb{C}$ is the pairing given in § 2. We fix an isomorphism $\pi' \cong \bigotimes_v \pi'_v$. Let $\mathcal{B}^\flat_{\pi'_v} : \pi'_v \otimes \bar{\pi}'_v \to \mathbb{C}$ be the pairing given by $\mathcal{B}^\flat_{\pi'_v} = \mathcal{B}_{\Pi'_v}|_{\pi'_v \otimes \bar{\pi}'_v}$. By Lemma 2.1, we have

$$\mathcal{B}_{\pi'} = 2^\beta \prod_v \mathcal{B}^\flat_{\pi'_v}.$$

Here

$$\beta = \begin{cases} 2 & \text{if } E = F \times F, \\ 1 & \text{if } E \text{ is a quadratic extension of } F. \end{cases}$$

Let $\mathrm{d}h_{0,v}$ be the Haar measure on $Z_{H,v}\backslash H_v^0$ such that

$$\int_{Z_{H,v}\backslash H_v} f(h_v)\,\mathrm{d}h_v = \frac{1}{2}\sum_{\epsilon_v \in \boldsymbol{\mu}_2(F_v)} \int_{Z_{H,v}\backslash H_v^0} f(h_{0,v}\epsilon_v)\,\mathrm{d}h_{0,v}$$

for $f \in L^1(Z_{H,v}\backslash H_v)$. Then the product measure $\prod_v \mathrm{d}h_{0,v}$ is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H^0(\mathbb{A})$. We define an $H_v^0 \times H_v^0$-invariant functional

$$\mathcal{J}_v : (\pi_v^\bullet \boxtimes \bar{\pi}_v^\bullet) \otimes (\pi_v' \boxtimes \bar{\pi}_v') \to \mathbb{C}$$

by

$$\mathcal{J}_v(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v}) = \int_{Z_{H,v}\backslash H_v^0} \mathcal{B}_v^\bullet(\pi_v^\bullet(h_{0,v})\phi_{1,v}, \phi_{2,v})\mathcal{B}^\flat_{\pi'_v}(\pi'_v(h_{0,v})\phi'_{1,v}, \phi'_{2,v})\,\mathrm{d}h_{0,v}$$

for $\phi_{1,v}, \phi_{2,v} \in \pi_v^\bullet$ and $\phi'_{1,v}, \phi'_{2,v} \in \pi'_v$, where

$$\bullet = \begin{cases} + & \text{if } v \in \mathfrak{S}, \\ \varnothing & \text{if } v \notin \mathfrak{S}. \end{cases}$$

By [**20**, Theorem 1.1] and Lemma 3.3, $\mathcal{I}(\phi_1, \phi_2; \phi'_1, \phi'_2)$ is equal to

$$2^{\beta + c_0} \cdot \zeta^S_{E\otimes K}(2) \cdot \frac{L^S(\frac{1}{2}, \dot{\pi} \times \dot{\pi}')}{L^S(1, \dot{\pi}, \mathrm{Ad})L^S(1, \dot{\pi}', \mathrm{Ad})} \frac{1}{2^{2s+2s'}}$$
$$\times \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \sum_{\epsilon' \in \boldsymbol{\mu}_2(F_{S'})} \prod_{v \in S} \mathcal{J}_v(\mathit{\Pi}_v(\epsilon_v)\phi_{1,v}, \mathit{\Pi}_v(\epsilon'_v)\phi_{2,v}; \mathit{\Pi}'_v(\epsilon_v)\phi'_{1,v}, \mathit{\Pi}'_v(\epsilon'_v)\phi'_{2,v})$$

for $\phi_1 = \bigotimes_v \phi_{1,v}$, $\phi_2 = \bigotimes_v \phi_{2,v} \in V^1_{\mathit{\Pi},S}$ and $\phi'_1 = \bigotimes_v \phi'_{1,v}$, $\phi'_2 = \bigotimes_v \phi'_{2,v} \in V_{\mathit{\Pi}',S}$. Here

$$c_0 = \begin{cases} -6 & \text{if } E = K = F \times F, \\ -3 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -4 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -3 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K. \end{cases}$$

To finish the proof of Proposition 3.2, it remains to show the following lemma.

**Lemma 3.4.** *We have*

$$\frac{1}{2^{2s+2s'}} \sum_{\epsilon \in \boldsymbol{\mu}_2(F_{S'})} \sum_{\epsilon' \in \boldsymbol{\mu}_2(F_{S'})} \prod_{v \in S} \mathcal{J}_v(\mathit{\Pi}_v(\epsilon_v)\phi_{1,v}, \mathit{\Pi}_v(\epsilon'_v)\phi_{2,v}; \mathit{\Pi}'_v(\epsilon_v)\phi'_{1,v}, \mathit{\Pi}'_v(\epsilon'_v)\phi'_{2,v})$$
$$= \prod_{v \in S} \mathcal{I}_v^\natural(\phi_{1,v}, \phi_{2,v}; \phi'_{1,v}, \phi'_{2,v})$$

*for* $\phi_1 = \bigotimes_v \phi_{1,v}$, $\phi_2 = \bigotimes_v \phi_{2,v} \in V^1_{\mathit{\Pi},S}$ *and* $\phi'_1 = \bigotimes_v \phi'_{1,v}$, $\phi'_2 = \bigotimes_v \phi'_{2,v} \in V_{\mathit{\Pi}',S}$.

**Proof.** If $v \in \mathfrak{S}$, then

$$
\begin{aligned}
&\frac{1}{2^2}\mathcal{J}_v(\phi_{1,v},\phi_{2,v};\phi'_{1,v},\phi'_{2,v}) \\
&\qquad = \frac{1}{2}\int_{Z_{H,v}\backslash H_v^0} \mathcal{B}^{\natural}_{\Pi_v}(\Pi_v(h_{0,v})\phi_{1,v},\phi_{2,v})\mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v})\phi'_{1,v},\phi'_{2,v})\,\mathrm{d}h_{0,v} \\
&\qquad = \frac{1}{2}\sum_{\epsilon_v\in\boldsymbol{\mu}_2(F_v)}\int_{Z_{H,v}\backslash H_v^0} \mathcal{B}^{\natural}_{\Pi_v}(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v},\phi_{2,v})\mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v},\phi'_{2,v})\,\mathrm{d}h_{0,v} \\
&\qquad = \mathcal{I}^{\natural}_v(\phi_{1,v},\phi_{2,v};\phi'_{1,v},\phi'_{2,v}).
\end{aligned}
$$

If $v \notin \mathfrak{S}$, then

$$
\begin{aligned}
&\frac{1}{2^2}\sum_{\epsilon_v\in\boldsymbol{\mu}_2(F_v)}\sum_{\epsilon'_v\in\boldsymbol{\mu}_2(F_v)} \mathcal{J}_v(\Pi_v(\epsilon_v)\phi_{1,v},\Pi_v(\epsilon'_v)\phi_{2,v};\Pi'_v(\epsilon_v)\phi'_{1,v},\Pi'_v(\epsilon'_v)\phi'_{2,v}) \\
&\quad = \frac{1}{2^2}\sum_{\epsilon_v\in\boldsymbol{\mu}_2(F_v)}\sum_{\epsilon'_v\in\boldsymbol{\mu}_2(F_v)} \\
&\qquad \times \int_{Z_{H,v}\backslash H_v^0} \mathcal{B}^{\natural}_{\Pi_v}(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v},\Pi_v(\epsilon'_v)\phi_{2,v})\mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v},\Pi'_v(\epsilon'_v)\phi'_{2,v})\,\mathrm{d}h_{0,v} \\
&\quad = \frac{1}{2}\sum_{\epsilon_v\in\boldsymbol{\mu}_2(F_v)}\int_{Z_{H,v}\backslash H_v^0} \mathcal{B}^{\natural}_{\Pi_v}(\Pi_v(h_{0,v}\epsilon_v)\phi_{1,v},\phi_{2,v})\mathcal{B}_{\Pi'_v}(\Pi'_v(h_{0,v}\epsilon_v)\phi'_{1,v},\phi'_{2,v})\,\mathrm{d}h_{0,v} \\
&\quad = \mathcal{I}^{\natural}_v(\phi_{1,v},\phi_{2,v};\phi'_{1,v},\phi'_{2,v}).
\end{aligned}
$$

This completes the proof. □

This completes the proof of Proposition 3.2.

## 4. Local theta lifts from $\mathrm{GL}_2$ to $\mathrm{GO}(V)$

Let $F$ be a local field of characteristic zero. Let $W$ be a two-dimensional symplectic space over $F$ and $V$ a four-dimensional quadratic space over $F$. Set

$$
\begin{aligned}
G &= \mathrm{GSp}(W)(F) \cong \mathrm{GL}_2(F), & G_1 &= \mathrm{Sp}(W)(F) \cong \mathrm{SL}_2(F), \\
H &= \mathrm{GO}(V)(F), & H_1 &= \mathrm{O}(V)(F).
\end{aligned}
$$

Let

$$
R = \mathrm{G}(\mathrm{Sp}(W)\times\mathrm{O}(V))(F) = \{(g,h)\in G\times H \mid \nu(g)=\nu(h)\},
$$

where $\nu : G \to F^\times$ and $\nu : H \to F^\times$ are the similitude characters.

Fix a non-trivial additive character $\psi$ of $F$. Let $\omega$ denote the Weil representation of $G_1 \times H_1$ with respect to $\psi$. As in [**13**, § 5.1] and [**47**], we extend $\omega$ to a representation of $R$. Let

$$
\Omega = \text{c-ind}_R^{G^+\times H}(\omega),
$$

where $G^+ = \{g \in G \mid \nu(g) \in \nu(H)\}$. The induced Weil representation $\Omega$ depends only on the orbit of $\psi$ under the natural action of $\nu(H) \subset F^\times$. Let $\pi^+$ be an infinite-dimensional irreducible admissible representation of $G^+$. Then the maximal $(\pi^+)^\vee$-isotypic quotient of $\Omega$ is of the form

$$(\pi^+)^\vee \boxtimes \Theta(\pi^+),$$

where $(\pi^+)^\vee$ is the contragredient representation of $\pi^+$ and $\Theta(\pi^+)$ is a smooth representation of $H$. If the residual characteristic of $F$ is not two, then the Howe duality conjecture, which is a theorem of Howe [**18**] and Waldspurger [**56**], and a result of Roberts [**47**] assert that $\Theta(\pi^+)$ has a unique irreducible quotient $\theta(\pi^+)$. Even if the residual characteristic of $F$ is two, the same assertion follows from [**8**, Lemmas 4.1 and 5.4]. Thus, we obtain a unique (up to a scalar) $R$-equivariant surjective map

$$\theta : \omega \otimes \pi^+ \to \theta(\pi^+).$$

**Lemma 4.1.** *Assume that $G^+ \neq G$. Let $\pi$ be an infinite-dimensional irreducible admissible representation of $G$ such that $\pi|_{G^+}$ is reducible. Then we can write $\pi|_{G^+} = \pi^+ \oplus \pi^-$, where $\pi^\pm$ is an irreducible admissible representation of $G^+$ such that*

$$\theta(\pi^+) \neq 0, \qquad \theta(\pi^-) = 0.$$

**Proof.** If $F$ is a non-archimedean local field, then the assertion follows from [**8**, § 5]. If $F = \mathbb{R}$, see [**43**]. □

Let $\pi$ be an infinite-dimensional irreducible admissible representation of $G$. If $G^+ \neq G$, let

$$\theta(\pi) = \begin{cases} \theta(\pi|_{G^+}) & \text{if } \pi|_{G^+} \text{ is irreducible,} \\ \theta(\pi^+) & \text{if } \pi|_{G^+} \text{ is reducible,} \end{cases}$$

where $\pi^+$ is an irreducible subrepresentation of $\pi|_{G^+}$ as in Lemma 4.1. Thus, we obtain a unique (up to a scalar) $R$-equivariant surjective map

$$\theta : \omega \otimes \pi \to \theta(\pi).$$

## 5. Local theta lifts from GO($V$) to GSp$_4$

Let $F$ be a local field of characteristic zero. Let $V$ be a four-dimensional quadratic space over $F$ and $W$ a four-dimensional symplectic space over $F$. Set

$$H = \mathrm{GO}(V)(F), \qquad H_1 = \mathrm{O}(V)(F),$$
$$G = \mathrm{GSp}(W)(F) \cong \mathrm{GSp}_4(F), \qquad G_1 = \mathrm{Sp}(W)(F) \cong \mathrm{Sp}_4(F).$$

Let

$$R = \mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F) = \{(h, g) \in H \times G \mid \nu(h) = \nu(g)\},$$

where $\nu : H \to F^\times$ and $\nu : G \to F^\times$ are the similitude characters. Let $K$ be the discriminant algebra of $V$ and choose a quaternion algebra $D$ over $F$ associated to $V$ as in § 1.

Fix a non-trivial additive character $\psi$ of $F$. Let $\omega$ denote the Weil representation of $H_1 \times G_1$ with respect to $\psi$. As in [**13**, § 5.1] and [**47**], we extend $\omega$ to a representation of $R$. Let

$$\Omega = \text{c-ind}_R^{H\times G^+}(\omega),$$

where $G^+ = \{g \in G \mid \nu(g) \in \nu(H)\}$. The induced Weil representation $\Omega$ depends only on the orbit of $\psi$ under the natural action of $\nu(H) \subset F^\times$.

**Lemma 5.1.** *Assume that $G^+ \neq G$. Let $g_0 \in G \smallsetminus G^+$. Let $\sigma$ and $\pi^+$ be irreducible admissible representations of $H$ and $G^+$, respectively. If $\mathrm{Hom}_{H\times G^+}(\Omega, \sigma \boxtimes \pi^+) \neq 0$, then $\pi^+ \circ \mathrm{Ad}(g_0) \ncong \pi^+$.*

**Proof.** The assertion follows from [**49**, Lemmas 1.4 and 1.5] and the proof of [**49**, Theorem 1.8]. We remark that [**49**, Lemma 1.5] follows from [**43**] even if $\mathrm{disc}(V) \notin F^{\times,2}$. □

Let $\sigma$ be an irreducible unitary admissible representation of $H$. Then the maximal $\bar{\sigma}$-isotypic quotient of $\Omega$ is of the form

$$\bar{\sigma} \boxtimes \Theta(\sigma),$$

where $\bar{\sigma}$ is the complex conjugate representation of $\sigma$ and $\Theta(\sigma)$ is a smooth representation of $G^+$. We call $\Theta(\sigma)$ the big theta lift of $\sigma$ to $G^+$. If the residual characteristic of $F$ is not two, then the Howe duality conjecture, which is a theorem of Howe [**18**] and Waldspurger [**56**], and a result of Roberts [**47**] assert that $\Theta(\sigma)$ has a unique irreducible quotient $\theta(\sigma)$. Even if the residual characteristic of $F$ is two, the same assertion follows from Theorem A.1 in Appendix A. We call $\theta(\sigma)$ the theta lift of $\sigma$ to $G^+$. Thus, we obtain a unique (up to a scalar) $R$-equivariant surjective map

$$\theta : \omega \otimes \sigma \to \theta(\sigma).$$

By Lemma 5.1, we obtain the following lemma.

**Lemma 5.2.** *Assume that $\theta(\sigma)$ is non-zero and unitary. Let $\pi = \mathrm{ind}_{G^+}^G(\theta(\sigma))$. Then $\pi$ is irreducible. Moreover, we have*

$$\mathcal{B}_\pi(\pi(g)\phi_1, \phi_2) = 0$$

*for $g \in G \smallsetminus G^+$ and $\phi_1, \phi_2 \in \theta(\sigma)$. Here $\mathcal{B}_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$ is a pairing and we regard $\theta(\sigma)$ as a subrepresentation of $\pi|_{G^+}$.*

**Lemma 5.3.** *Assume that $\sigma \otimes \mathrm{sgn} \ncong \sigma$ and the theta lift of $\sigma$ to $\mathrm{GL}_2(F)^+$ is non-zero. Then we have*

$$\theta(\sigma \otimes \mathrm{sgn}) = 0.$$

**Proof.** By [**46**, p. 399], the theta lift of the sign character of $\mathrm{O}(V)(F)$ to $\mathrm{Sp}_{2n}(F)$ is zero unless $n \geqslant 4$. As in [**1**, Proposition 1.7], this yields the lemma. □

**Definition 5.4.** Set $H^0 = \mathrm{GSO}(V)(F)$. Let $\sigma_0$ be an irreducible admissible representation of $H^0$. We say that $\sigma_0$ is distinguished if

$$\sigma_0 \cong \varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$$

as representations of $D^\times(K) \times F^\times$ for some irreducible admissible representation $\varsigma$ of $\mathrm{GL}_2(F)$ with central character $\omega_\varsigma$. Here $\varsigma_K$ is the base change of $\varsigma$ to $\mathrm{GL}_2(K)$ and $\varsigma_K^D$ is the Jacquet–Langlands transfer of $\varsigma_K$ to $D^\times(K)$. Then we can write $\mathrm{ind}_{H^0}^H(\sigma_0) = \sigma_0^+ \oplus \sigma_0^-$, where $\sigma_0^\pm$ is an irreducible admissible representation of $H$ such that the theta lift of $\sigma_0^+$ to $\mathrm{GL}_2(F)^+$ is non-zero (and hence $\theta(\sigma_0^-) = 0$ by Lemma 5.3).

Let $\sigma$ be an irreducible unitary admissible representation of $H$. We assume that $\sigma$ is a local component of an irreducible unitary cuspidal automorphic representation as in § 7.2. In particular, if $\sigma \otimes \mathrm{sgn} \not\cong \sigma$, then $\sigma \not\cong \sigma_0^-$ for any distinguished representation $\sigma_0$ of $\mathrm{GSO}(V)(F)$. In § 7 below, we will show that $\theta(\sigma)$ is non-zero and unitary. By Theorem A.1 in Appendix A, we obtain the following proposition.

**Proposition 5.5.** *If $F$ is a non-archimedean local field, then the multiplicity of $\theta(\sigma)$ in $\Theta(\sigma)$ is one.*

Let $\bar{\Omega}$ be the complex conjugate representation of $\Omega$. Then we have

$$\bar{\Theta}(\bar{\sigma}) \cong \overline{\Theta(\sigma)}, \qquad \bar{\theta}(\bar{\sigma}) \cong \overline{\theta(\sigma)},$$

where $\bar{\Theta}(\bar{\sigma})$ (respectively $\bar{\theta}(\bar{\sigma})$) is the big theta lift (respectively the theta lift) of $\bar{\sigma}$ to $G^+$ with respect to $\bar{\Omega}$. Let

$$\theta : \omega \otimes \sigma \to \theta(\sigma), \qquad \bar{\theta} : \bar{\omega} \otimes \bar{\sigma} \to \overline{\theta(\sigma)}$$

be the unique (up to a scalar) $R$-equivariant surjective maps.

Let

$$\begin{aligned} \boldsymbol{H} &= \{\boldsymbol{h} = (h_1, h_2) \in H \times H \mid \nu(h_1) = \nu(h_2)\}, \\ \boldsymbol{R} &= \{(\boldsymbol{h}, g) \in \boldsymbol{H} \times G \mid \nu(\boldsymbol{h}) = \nu(g)\}. \end{aligned}$$

We define an $\boldsymbol{R}$-equivariant map

$$\mathcal{Z} : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \to \mathbb{C}$$

by

$$\mathcal{Z}(\varphi_1, \varphi_2; f_1, f_2) = \int_{H_1} \mathcal{B}_\omega(\omega(h_1)\varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(h_1) f_1, f_2) \,\mathrm{d}h_1$$

for $\varphi_1, \varphi_2 \in \omega$ and $f_1, f_2 \in \sigma$. Here $\mathcal{B}_\omega : \omega \otimes \bar{\omega} \to \mathbb{C}$ and $\mathcal{B}_\sigma : \sigma \otimes \bar{\sigma} \to \mathbb{C}$ are pairings. In § 7, we will show that this integral is absolutely convergent and $\mathcal{Z} \neq 0$.

**Lemma 5.6.** *If $F$ is a non-archimedean local field, then there exists a pairing $\mathcal{B}_{\theta(\sigma)} : \theta(\sigma) \otimes \overline{\theta(\sigma)} \to \mathbb{C}$ such that*

$$\mathcal{Z} = \mathcal{B}_{\theta(\sigma)} \circ (\theta \otimes \bar{\theta}).$$

**Proof.** It suffices to show that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\boldsymbol{R}}((\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) = 1.$$

We have

$$\begin{aligned}\operatorname{Hom}_{\boldsymbol{R}}((\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) &\cong \operatorname{Hom}_{H \times H \times G^+}((\Omega \boxtimes \bar{\Omega}) \otimes (\sigma \boxtimes \bar{\sigma}), \mathbb{C}) \\ &\cong \operatorname{Hom}_{G^+}(\Theta(\sigma) \otimes \overline{\Theta(\sigma)}, \mathbb{C}) \\ &\cong \operatorname{Hom}_{G^+}(\Theta(\sigma), \overline{\Theta(\sigma)}^\vee).\end{aligned}$$

Let $l : \Theta(\sigma) \to \overline{\Theta(\sigma)}^\vee$ be a non-zero $G^+$-equivariant map. Then the image of $l$ contains the unique irreducible subrepresentation $\overline{\theta(\sigma)}^\vee \cong \theta(\sigma)$ of $\overline{\Theta(\sigma)}^\vee$. By Proposition 5.5, $l$ factors through the quotient $\Theta(\sigma) \to \theta(\sigma)$. This yields the lemma. □

**Lemma 5.7.** *If $F$ is an archimedean local field, then there exists a pairing $\mathcal{B}_{\theta(\sigma)}$ : $\theta(\sigma) \otimes \overline{\theta(\sigma)} \to \mathbb{C}$ such that*

$$\mathcal{Z} = \mathcal{B}_{\theta(\sigma)} \circ (\theta \otimes \bar{\theta}).$$

**Proof.** We can write $\sigma|_{H_1} = \bigoplus_{i=1}^n \sigma_i$, where $n \leqslant 2$ and $\sigma_i$ is an irreducible unitary admissible representation of $H_1$. As in [**8**, Lemma 3.1], we have

$$\Theta(\sigma)|_{G_1} = \bigoplus_{i=1}^n \Theta(\sigma_i) \quad \text{and} \quad \theta(\sigma)|_{G_1} = \bigoplus_{i=1}^n \theta(\sigma_i).$$

Here $\Theta(\sigma_i)$ (respectively $\theta(\sigma_i)$) is the big theta lift (respectively the theta lift) of $\sigma_i$ to $G_1$. If $n = 2$, then $\sigma_1 \ncong \sigma_2$ and hence $\theta(\sigma_1) \ncong \theta(\sigma_2)$. We have

$$\begin{aligned}(\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma})|_{H_1 \times H_1 \times G_1 \times G_1} &= \bigoplus_{i=1}^n \bigoplus_{j=1}^n (\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j), \\ \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)}|_{G_1 \times G_1} &= \bigoplus_{i=1}^n \bigoplus_{j=1}^n \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)}, \\ \theta(\sigma) \boxtimes \overline{\theta(\sigma)}|_{G_1 \times G_1} &= \bigoplus_{i=1}^n \bigoplus_{j=1}^n \theta(\sigma_i) \boxtimes \overline{\theta(\sigma_j)}.\end{aligned}$$

Let

$$\begin{aligned}t &: (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \to \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)}, & p &: \Theta(\sigma) \boxtimes \overline{\Theta(\sigma)} \to \theta(\sigma) \boxtimes \overline{\theta(\sigma)}, \\ t_{ij} &: (\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j) \to \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)}, & p_{ij} &: \Theta(\sigma_i) \boxtimes \overline{\Theta(\sigma_j)} \to \theta(\sigma_i) \boxtimes \overline{\theta(\sigma_j)},\end{aligned}$$

be equivariant surjective maps. We may assume that

$$t = \bigoplus_{i=1}^n \bigoplus_{j=1}^n t_{ij} \quad \text{and} \quad p = \bigoplus_{i=1}^n \bigoplus_{j=1}^n p_{ij}.$$

In particular, we have

$$\ker(p) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} \ker(p_{ij}).$$

Since $\mathcal{Z}$ is an $\boldsymbol{R}$-equivariant map, there exists a $G^{+}$-invariant functional $l : \Theta(\sigma) \otimes \overline{\Theta(\sigma)} \to \mathbb{C}$ such that $\mathcal{Z} = l \circ t$. It remains to show that $\ker(p) \subset \ker(l)$. Let $\mathcal{Z}_{ij}$ (respectively $l_{ij}$) be the restriction of $\mathcal{Z}$ (respectively $l$) to $(\omega \boxtimes \bar{\omega}) \otimes (\sigma_i \boxtimes \bar{\sigma}_j)$ (respectively $\Theta(\sigma_i) \otimes \overline{\Theta(\sigma_j)}$). It follows from the definition of $\mathcal{Z}$ that $\mathcal{Z}_{ij} = 0$ if $i \neq j$, so that $l_{ij} = 0$ if $i \neq j$ and

$$\mathcal{Z} = \sum_{i=1}^{n} \mathcal{Z}_{ii} = \sum_{i=1}^{n} l_{ii} \circ t_{ii}.$$

By a result of He [**16**], the $G_1$-invariant functional $l_{ii} : \Theta(\sigma_i) \otimes \overline{\Theta(\sigma_i)} \to \mathbb{C}$ factors through $p_{ii}$, so that $\ker(p_{ii}) \subset \ker(l_{ii})$. Hence we have

$$\left( \bigoplus_{i=1}^{n} \ker(p_{ii}) \right) \oplus \left( \bigoplus_{i \neq j} \Theta(\sigma_i) \otimes \overline{\Theta(\sigma_j)} \right) \subset \ker(l).$$

This yields the lemma. □

## 6. Global theta lifts from $\mathbf{GL_2}$ to $\mathbf{GO}(\boldsymbol{V})$

Let $F$ be a totally real number field. Let $W$ be a two-dimensional symplectic space over $F$ and $V$ a four-dimensional quadratic space over $F$. Let $\tilde{W} = W \oplus (-W)$. Set

$$\begin{aligned} G &= \mathrm{GSp}(W) \cong \mathrm{GL}_2, & G_1 &= \mathrm{Sp}(W) \cong \mathrm{SL}_2, \\ \tilde{G} &= \mathrm{GSp}(\tilde{W}) \cong \mathrm{GSp}_4, & \tilde{G}_1 &= \mathrm{Sp}(\tilde{W}) \cong \mathrm{Sp}_4, \\ H &= \mathrm{GO}(V), & H_1 &= \mathrm{O}(V). \end{aligned}$$

Let

$$\boldsymbol{G} = \{\boldsymbol{g} = (g_1, g_2) \in G \times G \mid \nu(g_1) = \nu(g_2)\},$$

where $\nu : G \to \mathbb{G}_m$ is the similitude character. Let $\iota : \boldsymbol{G} \hookrightarrow \tilde{G}$ be the natural embedding. Let $K$ be the discriminant algebra of $V$ and choose a quaternion algebra $D$ over $F$ associated to $V$ as in § 1.

### 6.1. Weil representations

Fix a non-trivial additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/F$. Let $W = X \oplus Y$ be a complete polarization and set

$$\mathbb{W} = V \otimes W, \qquad \mathbb{X} = V \otimes X, \qquad \mathbb{Y} = V \otimes Y.$$

Then $\mathbb{W}$ is a symplectic space over $F$ and $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a complete polarization. Let $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ denote the metaplectic extension of $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$. Let $\omega$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space $V_\omega = S(\mathbb{X}(\mathbb{A}))$ with respect to $\psi$ and $\mathcal{B}_\omega : V_\omega \otimes \bar{V}_\omega \to \mathbb{C}$

the canonical pairing given by

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) = \int_{\mathbb{X}(\mathbb{A})} \varphi_1(x)\overline{\varphi_2(x)}\,\mathrm{d}x$$

for $\varphi_1, \varphi_2 \in V_\omega$. Here $\mathrm{d}x$ is the Tamagawa measure on $\mathbb{X}(\mathbb{A})$. For each place $v$ of $F$, let $\mathrm{Mp}(\mathbb{W}_v)$ denote the metaplectic extension of $\mathrm{Sp}(\mathbb{W})(F_v)$. Let $\omega_v$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}_v)$ on the space $S(\mathbb{X}_v)$ with respect to $\psi_v$ and $\mathcal{B}_{\omega_v} : \omega_v \otimes \bar{\omega}_v \to \mathbb{C}$ the canonical pairing given by

$$\mathcal{B}_{\omega_v}(\varphi_{1,v}, \varphi_{2,v}) = \int_{\mathbb{X}_v} \varphi_{1,v}(x_v)\overline{\varphi_{2,v}(x_v)}\,\mathrm{d}x_v$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$. Here $\mathrm{d}x_v$ is the self-dual measure on $\mathbb{X}_v$ with respect to the Fourier transform determined by $\psi_v$. Then we have $\omega = \bigotimes_v \omega_v$ and $\mathcal{B}_\omega = \prod_v \mathcal{B}_{\omega_v}$. By [**30**], there exists a splitting

$$G_1(\mathbb{A}) \times H_1(\mathbb{A}) \to \mathrm{Mp}(\mathbb{W}(\mathbb{A})).$$

By [**13**, § 5.1] and [**47**], we can extend it to a splitting

$$\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A}) \to \mathrm{Mp}(\mathbb{W}(\mathbb{A})).$$

We regard $\omega$ as a representation of $\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$ via this splitting. Similarly, we may regard $\omega_v$ as a representation of $\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(F_v)$.

Let

$$\tilde{\mathbb{W}} = V \otimes \tilde{W}, \qquad \tilde{\mathbb{X}} = V \otimes (X \oplus (-X)), \qquad \tilde{\mathbb{Y}} = V \otimes (Y \oplus (-Y)).$$

Then $\tilde{\mathbb{W}}$ is a symplectic space over $F$ and $\tilde{\mathbb{W}} = \tilde{\mathbb{X}} \oplus \tilde{\mathbb{Y}}$ is a complete polarization. Let $\tilde{\omega}$ be the Weil representation of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ on $S(\tilde{\mathbb{X}}(\mathbb{A}))$ with respect to $\psi$. We may regard $\tilde{\omega}$ as a representation of $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$. We have a natural isomorphism

$$S(\tilde{\mathbb{X}}(\mathbb{A})) \cong V_\omega \otimes \bar{V}_\omega$$

as representations of $\mathrm{Mp}(\mathbb{W}(\mathbb{A})) \times \mathrm{Mp}(\mathbb{W}(\mathbb{A}))$. Let

$$W^\Delta = \{(x, x) \mid x \in W\}, \qquad \mathbb{W}^\Delta = V \otimes W^\Delta,$$
$$W^\nabla = \{(x, -x) \mid x \in W\}, \qquad \mathbb{W}^\nabla = V \otimes W^\nabla.$$

Then $\tilde{\mathbb{W}} = \mathbb{W}^\nabla \oplus \mathbb{W}^\Delta$ is a complete polarization. Hence we can realize the Weil representation $\tilde{\omega}$ on $S(\mathbb{W}^\nabla(\mathbb{A}))$. By [**40**, § 2], there exists an isomorphism

$$\delta : S(\tilde{\mathbb{X}}(\mathbb{A})) \to S(\mathbb{W}^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ such that

$$\delta(\varphi_1 \otimes \bar{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2)$$

for $\varphi_1, \varphi_2 \in V_\omega$.

### 6.2. Theta lifts

Let $\pi \cong \bigotimes_v \pi_v$ be an irreducible unitary cuspidal automorphic representation of $G(\mathbb{A})$ on the space $V_\pi$ with central character $\omega_\pi$. We assume the following.

- The base change $\pi_K$ of $\pi$ to $G(\mathbb{A}_K) \cong \mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- The Jacquet–Langlands transfer $\pi_K^D$ of $\pi_K$ to $D^\times(\mathbb{A}_K)$ exists.

**Lemma 6.1.** *The partial L-function* $L^S(s, \pi, \mathrm{Ad} \otimes \omega_{K/F})$ *is holomorphic and non-zero at* $s = 1$.

**Proof.** It is well known that $L^S(s, \pi, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$. If $K$ is a quadratic extension of $F$, then

$$L^S(s, \pi_K, \mathrm{Ad}) = L^S(s, \pi, \mathrm{Ad}) L^S(s, \pi, \mathrm{Ad} \otimes \omega_{K/F})$$

is also holomorphic and non-zero at $s = 1$ since $\pi_K$ is cuspidal. This yields the lemma. □

Let $\varphi \in V_\omega$. The theta function associated to $\varphi$ is given by

$$\theta(g, h; \varphi) = \sum_{x \in \mathbb{X}(F)} \omega(g, h)\varphi(x)$$

for $(g, h) \in \mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$. Let $f \in V_\pi$. For $h \in H(\mathbb{A})$, choose $g \in G(\mathbb{A})$ such that $\nu(g) = \nu(h)$, and put

$$\theta(h; \varphi, f) = \int_{G_1(F) \backslash G_1(\mathbb{A})} \theta(g_1 g, h; \varphi) f(g_1 g)\, \mathrm{d}g_1.$$

Here $\mathrm{d}g_1 = \prod_v \mathrm{d}g_{1,v}$ is the Tamagawa measure on $G_1(\mathbb{A})$. Note that $\mathrm{vol}(G_1(F) \backslash G_1(\mathbb{A})) = 1$ and we may assume that the volume of a hyperspecial maximal compact subgroup of $G_{1,v}$ with respect to $\mathrm{d}g_{1,v}$ is 1 for almost all $v$. This integral defines an automorphic form $\theta(\varphi, f)$ on $H(\mathbb{A})$. Let $\theta(\pi)$ be the automorphic representation of $H(\mathbb{A})$ on the space $V_{\theta(\pi)}$ generated by $\theta(\varphi, f)$ for all $\varphi \in V_\omega$ and $f \in V_\pi$. By assumption on $\pi$, $\theta(\pi)$ is cuspidal. In Lemma 6.9 below, we will show that $V_{\theta(\pi)} \neq 0$. In particular, $\theta(\pi_v) \neq 0$ for all $v$. Hence $\theta(\pi)$ is irreducible,

$$\theta(\pi) \cong \bigotimes_v \theta(\pi_v),$$

and $\theta(\pi_v)$ is unitary for all $v$. Thus, we obtain a $\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(\mathbb{A})$-equivariant surjective map

$$\theta : V_\omega \otimes V_\pi \to V_{\theta(\pi)}$$

and $\mathrm{G}(\mathrm{Sp}(W) \times \mathrm{O}(V))(F_v)$-equivariant surjective maps

$$\theta_v : \omega_v \otimes \pi_v \to \theta(\pi_v)$$

such that $\theta = \bigotimes_v \theta_v$. As in [**52**], we have

$$\theta(\pi)|_{D^\times(\mathbb{A}_K) \times \mathbb{A}^\times} \cong \pi_K^D \boxtimes \omega_\pi \omega_{K/F}$$

by the local unramified theta correspondence and the strong multiplicity one theorem. We should remark that the local theta correspondence for $\mathrm{GL}_2 \times \mathrm{GO}(V)$ has also been studied by Cognet [**6**, **7**] and Roberts [**48**].

### 6.3. Eisenstein series

Let $P$ be the parabolic subgroup of $\tilde{G}$ stabilizing $W^{\Delta}$ with modulus character $\delta_P$. We regard $\omega_{K/F}$ as a character of $P(\mathbb{A})$ via the natural homomorphism

$$P \to \mathrm{GL}(W^{\nabla}) \xrightarrow{\det} \mathbb{G}_m.$$

For $\nu \in \mathbb{G}_m$, we define an element $d(\nu)$ of $P$ by

$$d(\nu)|_{W^{\nabla}} = \mathrm{id}, \qquad d(\nu)|_{W^{\Delta}} = \nu \cdot \mathrm{id}.$$

We fix a maximal compact subgroup $\boldsymbol{K}$ of $\tilde{G}(\mathbb{A})$ such that $\tilde{G}(\mathbb{A}) = P(\mathbb{A})\boldsymbol{K}$.

Let $\boldsymbol{I}(s)$ denote the degenerate principal series representation of $\tilde{G}(\mathbb{A})$ given by

$$\boldsymbol{I}(s) = \mathrm{Ind}_{P(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(\omega_{K/F}\delta_P^{s/3}),$$

where Ind denotes the normalized induction. Given a holomorphic section $\Phi$ of $\boldsymbol{I}(s)$, we define an Eisenstein series $E(s,\Phi)$ on $\tilde{G}(\mathbb{A})$ by

$$E(g;s,\Phi) = \sum_{\gamma \in P(F)\backslash \tilde{G}(F)} \Phi(\gamma g, s)$$

for $\mathrm{Re}(s) \gg 0$. By [**34**, Theorem 1.1], $E(s,\Phi)$ has at most a simple pole at $s = \frac{1}{2}$. Let $P_1 = P \cap \tilde{G}_1$. Let $\boldsymbol{I}_1(s)$ denote the degenerate principal series representation of $\tilde{G}_1(\mathbb{A})$ given by

$$\boldsymbol{I}_1(s) = \mathrm{Ind}_{P_1(\mathbb{A})}^{\tilde{G}_1(\mathbb{A})}(\omega_{K/F}\delta_{P_1}^{s/3}).$$

If $\Phi_1$ is a holomorphic section of $\boldsymbol{I}_1(s)$, we similarly define an Eisenstein series $E(s,\Phi_1)$ on $\tilde{G}_1(\mathbb{A})$. If $\Phi$ is a holomorphic section of $\boldsymbol{I}(s)$, then $\Phi|_{\tilde{G}_1(\mathbb{A})}$ is a holomorphic section of $\boldsymbol{I}_1(s)$ and

$$E(s,\Phi)|_{\tilde{G}_1(\mathbb{A})} = E(s,\Phi|_{\tilde{G}_1(\mathbb{A})}).$$

We define a $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$-equivariant map

$$[\cdot] : S(\mathbb{W}^{\nabla}(\mathbb{A})) \to \boldsymbol{I}(\tfrac{1}{2})$$

by

$$[\varphi](g, \tfrac{1}{2}) = |\nu(g)|^{-2}\tilde{\omega}(d(\nu(g)^{-1})g)\varphi(0)$$

for $g \in \tilde{G}(\mathbb{A})$. Here $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$ acts on $\boldsymbol{I}(\frac{1}{2})$ via the projection $\mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A}) \to \tilde{G}(\mathbb{A})^+$. We extend $[\varphi]$ to a holomorphic section of $\boldsymbol{I}(s)$ such that its restriction to $\boldsymbol{K}$ is independent of $s$. Let

$$E(s,[\varphi]) = \sum_{d=-1}^{\infty} (s - \tfrac{1}{2})^d A_d(\varphi)$$

be the Laurent expansion of $E(s,[\varphi])$ at $s = \frac{1}{2}$.

### 6.4. Theta integrals

Let $r$ be the Witt index of $V$ and $V = X' \oplus V_0 \oplus Y'$ a Witt decomposition, where $V_0$ is an anisotropic quadratic space over $F$ of dimension $4 - 2r$. Let $\mathrm{d}h_1$ be the Tamagawa measure on $H_1(\mathbb{A})$ and note that $\mathrm{vol}(H_1(F)\backslash H_1(\mathbb{A})) = 1$. Let $\varphi \in S(\mathbb{W}^{\nabla}(\mathbb{A}))$. The theta function associated to $\varphi$ is given by

$$\theta(g, h; \varphi) = \sum_{x \in \mathbb{W}^{\nabla}(F)} \tilde{\omega}(g, h)\varphi(x)$$

for $(g, h) \in \mathrm{G}(\mathrm{Sp}(\tilde{W}) \times \mathrm{O}(V))(\mathbb{A})$. If $r = 0$, then the theta integral $I(\varphi)$ is given by

$$I(g_1; \varphi) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g_1, h_1; \varphi)\, \mathrm{d}h_1$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$.

Assume that $r > 0$. Let $P'$ be the parabolic subgroup of $H_1$ stabilizing $Y'$ with modulus character $\delta_{P'}$. We fix a maximal compact subgroup $\boldsymbol{K}'$ of $H_1(\mathbb{A})$ such that $H_1(\mathbb{A}) = P'(\mathbb{A})\boldsymbol{K}'$. Let $\mathrm{d}_l p'$ be the left-invariant Tamagawa measure on $P'(\mathbb{A})$ and $\mathrm{d}k'$ the Haar measure on $\boldsymbol{K}'$ such that $\mathrm{vol}(\boldsymbol{K}') = 1$. There exists a constant $\kappa$ such that

$$\int_{H_1(\mathbb{A})} f(h_1)\, \mathrm{d}h_1 = \kappa \int_{P'(\mathbb{A})} \int_{\boldsymbol{K}'} f(p'k')\, \mathrm{d}_l p'\, \mathrm{d}k'$$

for $f \in L^1(H_1(\mathbb{A}))$.

Put $\varrho' = \frac{1}{2}(3 - r)$. Let $\Phi'$ be the holomorphic section of $\mathrm{Ind}_{P'(\mathbb{A})}^{H_1(\mathbb{A})}(\delta_{P'}^{s/(3-r)})$ such that $\Phi'(k', s) = 1$ for all $k' \in \boldsymbol{K}'$. We define an Eisenstein series $\mathcal{E}(s)$ on $H_1(\mathbb{A})$ by

$$\mathcal{E}(h_1; s) = \sum_{\gamma \in P'(F)\backslash H_1(F)} \Phi'(\gamma h_1, s)$$

for $\mathrm{Re}(s) > \varrho'$. By [**36**, § 5] and [**23**, § 9], we have

$$\mathrm{Res}_{s=\varrho'}\, \mathcal{E}(h_1; s) = \kappa$$

for $h_1 \in H_1(\mathbb{A})$.

Let $z \in \mathfrak{z}(\tilde{\mathfrak{g}}_{1,v})$ be the regularizing differential operator as in [**35**, § 3.2] and [**34**, § 5], where $v$ is a real place of $F$. There exists a self-adjoint differential operator $z' \in \mathfrak{z}(\mathfrak{h}_{1,v})$ such that $\tilde{\omega}(z) = \tilde{\omega}(z')$. Then we have $z'\mathcal{E}(s) = p(s)\mathcal{E}(s)$ with some $p(s) \in \mathbb{C}[s]$. Following Kudla and Rallis [**34**, § 5], we define the regularized theta integral $I(s, \varphi)$ by

$$I(g_1; s, \varphi) = \frac{1}{\kappa p(s)} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g_1, h_1; z\varphi)\mathcal{E}(h_1; s)\, \mathrm{d}h_1$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$. By [**34**, Lemma 5.5.6], $I(s, \varphi)$ has at most a double pole at $s = \varrho'$. Let

$$I(s, \varphi) = \sum_{d=-2}^{\infty} (s - \varrho')^d B_d(\varphi)$$

be the Laurent expansion of $I(s, \varphi)$ at $s = \varrho'$.

### 6.5. The Siegel–Weil formula

Let $\mathcal{A}(\tilde{G}_1)$ denote the space of automorphic forms on $\tilde{G}_1(\mathbb{A})$ and $\mathcal{R}(\tilde{G}_1)$ the subspace of $\mathcal{A}(\tilde{G}_1)$ generated by $\mathrm{Res}_{s=1/2} E(s, \Phi_1)$ for all holomorphic sections $\Phi_1$ of $\boldsymbol{I}_1(s)$.

Let $\varphi \in S(\mathbb{W}^\nabla(\mathbb{A}))$. If $r = 0$, then the Siegel–Weil formula by Kudla and Rallis [**31**] asserts that

$$I(\varphi) = A_0(\varphi)|_{\tilde{G}_1(\mathbb{A})}.$$

If $r > 0$, then the Siegel–Weil formula (the second term identity) by Kudla *et al*. [**35**, § 6] asserts that

$$B_{-1}(\varphi) \equiv A_0(\varphi)|_{\tilde{G}_1(\mathbb{A})} \mod \mathcal{R}(\tilde{G}_1). \tag{6.1}$$

**Remark 6.2.** In [**35**, § 6], Kudla *et al*. proved (6.1) up to a scalar. Computing Fourier coefficients as in [**35**, Proposition 6.2], [**54**, Proposition 5.1.1] and [**19**, Proposition 6.2], we can determine the constant of proportionality.

### 6.6. The doubling method

Let $\mathcal{A}(\tilde{G})$ denote the space of automorphic forms on $\tilde{G}(\mathbb{A})$ and $\mathcal{R}(\tilde{G})$ the subspace of $\mathcal{A}(\tilde{G})$ generated by $\mathrm{Res}_{s=1/2} E(s, \Phi)$ for all holomorphic sections $\Phi$ of $\boldsymbol{I}(s)$. If $\mathcal{F} \in \mathcal{R}(\tilde{G})$, then $\mathcal{F}|_{\tilde{G}_1(\mathbb{A})} \in \mathcal{R}(\tilde{G}_1)$.

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\pi(f_1, f_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} f_1(g)\overline{f_2(g)}\, \mathrm{d}g$$

for $f_1, f_2 \in V_\pi$. Here $Z_G$ is the identity component of the centre of $G$ and $\mathrm{d}g$ is the Tamagawa measure on $Z_G(\mathbb{A})\backslash G(\mathbb{A})$. Note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})) = 2$. We fix a decomposition $\mathcal{B}_\pi = \prod_v \mathcal{B}_{\pi_v}$, where $\mathcal{B}_{\pi_v} : \pi_v \otimes \bar{\pi}_v \to \mathbb{C}$ is a pairing. Let

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \nu(g) \in \nu(H(\mathbb{A}))\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$. Put

$$\mathfrak{v} = \mathrm{vol}(Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

**Lemma 6.3.** *We have*

$$\int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} f_1(g)\overline{f_2(g)}\, \mathrm{d}g = \tfrac{1}{2}\mathfrak{v}\mathcal{B}_\pi(f_1, f_2)$$

*for* $f_1, f_2 \in V_\pi$.

**Proof.** We may assume that $K$ is a quadratic extension of $F$. Let

$$\mathcal{G} = Z_G(\mathbb{A})G(\mathbb{A})^+G(F).$$

Note that $|\mathcal{G}\backslash G(\mathbb{A})| = 2$. By assumption on $\pi$, the group

$$\{\omega \in (\mathcal{G}\backslash G(\mathbb{A}))^D \mid \pi \otimes \omega \cong \pi\}$$

is trivial and hence $\pi|_{\mathcal{G}}$ is irreducible. The restriction to $\mathcal{G}$ as functions induces an isomorphism

$$V_\pi \cong V_\pi|_{\mathcal{G}}$$

as representations of $\mathcal{G}$.

We define a $\mathcal{G}$-invariant pairing $\mathcal{B}_\pi^+ : V_\pi|_{\mathcal{G}} \otimes \bar{V}_\pi|_{\mathcal{G}} \to \mathbb{C}$ by

$$\mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}}) = \int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} f_1(g)\overline{f_2(g)}\,\mathrm{d}g$$

for $f_1, f_2 \in V_\pi$. As in the proof of Lemma 2.1, we have

$$\mathcal{B}_\pi^+(\pi(g_0)f_1|_{\mathcal{G}}, \pi(g_0)f_2|_{\mathcal{G}}) = \mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}})$$

for $g_0 \in G(\mathbb{A}) \smallsetminus \mathcal{G}$. Hence we have

$$\mathcal{B}_\pi(f_1, f_2) = \sum_{g_0 \in \mathcal{G}\backslash G(\mathbb{A})} \mathcal{B}_\pi^+(\pi(g_0)f_1|_{\mathcal{G}}, \pi(g_0)f_2|_{\mathcal{G}}) = 2\mathcal{B}_\pi^+(f_1|_{\mathcal{G}}, f_2|_{\mathcal{G}}).$$

□

Let

$$\boldsymbol{G}(\mathbb{A})^+ = \{\boldsymbol{g} \in \boldsymbol{G}(\mathbb{A}) \mid \nu(\boldsymbol{g}) \in \nu(H(\mathbb{A}))\}$$

and $\boldsymbol{G}(F)^+ = \boldsymbol{G}(F) \cap \boldsymbol{G}(\mathbb{A})^+$. For a holomorphic section $\Phi$ of $\boldsymbol{I}(s)$ and $f_1, f_2 \in V_\pi$, the zeta integral of Piatetski-Shapiro and Rallis [**44**] and [**12**, § 6.2] is given by

$$Z(s, \Phi, f_1, f_2) = \int_{Z_{\tilde{G}}(\mathbb{A})\boldsymbol{G}(F)^+\backslash \boldsymbol{G}(\mathbb{A})^+} E(\iota(g_1, g_2); s, \Phi) f_1(g_1)\overline{f_2(g_2)}\,\mathrm{d}\boldsymbol{g}.$$

Here $Z_{\tilde{G}}$ is the identity component of the centre of $\tilde{G}$ and $\mathrm{d}\boldsymbol{g}$ is the Tamagawa measure on $Z_{\tilde{G}}(\mathbb{A})\backslash\boldsymbol{G}(\mathbb{A})$. Note that $\mathrm{vol}(Z_{\tilde{G}}(\mathbb{A})\boldsymbol{G}(F)^+\backslash\boldsymbol{G}(\mathbb{A})^+) = \mathfrak{v}$. For each place $v$ of $F$, let

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) = \int_{G_{1,v}} \Phi_v(\iota(g_{1,v}, 1), s)\mathcal{B}_{\pi_v}(\pi_v(g_{1,v})f_{1,v}, f_{2,v})\,\mathrm{d}g_{1,v}.$$

**Lemma 6.4.** *For a holomorphic section* $\Phi = \bigotimes_v \Phi_v$ *of* $\boldsymbol{I}(s)$ *and* $f_1 = \bigotimes_v f_{1,v}, f_2 = \bigotimes_v f_{2,v} \in V_\pi$, *we have*

$$Z(s, \Phi, f_1, f_2) = \frac{\mathfrak{v}}{2} \cdot \frac{L^S(s + \frac{1}{2}, \pi, \mathrm{Ad} \otimes \omega_{K/F})}{L^S(s + \frac{3}{2}, \omega_{K/F})\zeta^S(2s+1)} \prod_{v \in S} Z_v(s, \Phi_v, f_{1,v}, f_{2,v}).$$

**Proof.** The assertion follows from the doubling method of [**44**] and [**12**, § 6.2] and from Lemma 6.3. □

### 6.7. Local zeta integrals

Let $\boldsymbol{I}_v(s) = \mathrm{Ind}_{P_v}^{\tilde{G}_v}(\omega_{K_v/F_v}\delta_{P_v}^{s/3})$ denote the degenerate principal series representation of $\tilde{G}_v$.

**Lemma 6.5.** *For a holomorphic section $\Phi_v$ of $\boldsymbol{I}_v(s)$ and $f_{1,v}, f_{2,v} \in \pi_v$, the integral $Z_v(s, \Phi_v, f_{1,v}, f_{2,v})$ is absolutely convergent at $s = \frac{1}{2}$.*

**Proof.** By [**44**, Proposition 6.4], the function $g_{1,v} \mapsto \Phi_v(\iota(g_{1,v}, 1), \frac{1}{2})$ belongs to $L^{1+\varepsilon}(G_{1,v})$ for any $\varepsilon > 0$. This yields the lemma. □

For $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$, we have

$$Z_v(\tfrac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}) = \int_{G_{1,v}} \mathcal{B}_{\omega_v}(\omega_v(g_{1,v})\varphi_{1,v}, \varphi_{2,v})\mathcal{B}_{\pi_v}(\pi_v(g_{1,v})f_{1,v}, f_{2,v})\, \mathrm{d}g_{1,v}.$$

**Lemma 6.6.** *There exist $\varphi_v \in S(\mathbb{W}_v^{\nabla})$ and $f_{1,v}, f_{2,v} \in \pi_v$ such that*

$$Z_v(\tfrac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}) \neq 0.$$

**Proof.** We fix a place $v$ of $F$ and suppress it from the notation. By [**34**, Proposition 7.2.1], there exist $\Phi \in \boldsymbol{I}(\frac{1}{2})$ and $f_1, f_2 \in \pi$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$. Let $R$ be the image of the equivariant map $S(\mathbb{W}^{\nabla}) \to \boldsymbol{I}(\frac{1}{2})$, where $\mathbb{W}^{\nabla} = V \otimes W^{\nabla}$. It suffices to show that there exist $\Phi \in R$ and $f_1, f_2 \in \pi$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$.

We first consider the case $K = F \times F$. If $D$ is split, then $\boldsymbol{I}(\frac{1}{2}) = R$ by [**33, 37**] and the assertion is obvious. We assume that $D$ is division and $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \pi$. If $F$ is archimedean, let $R_-$ be the image of the equivariant map $S(\mathbb{W}_-^{\nabla}) \to \boldsymbol{I}(\frac{1}{2})$, where $\mathbb{W}_-^{\nabla} = (-V) \otimes W^{\nabla}$. Let $V_0$ be the two-dimensional split quadratic space over $F$. Let $R_0$ be the image of the equivariant map $S(\mathbb{W}_0^{\nabla}) \to \boldsymbol{I}(-\frac{1}{2})$, where $\mathbb{W}_0^{\nabla} = V_0 \otimes W^{\nabla}$. By [**33, 37**], we have

$$\boldsymbol{I}(\tfrac{1}{2})/R \cong R_0 \quad \text{if } F \text{ is non-archimedean,}$$
$$\boldsymbol{I}(\tfrac{1}{2})/(R + R_-) \cong R_0 \quad \text{if } F \text{ is archimedean.}$$

Since $\pi \circ \mathrm{Ad}(g_0) \cong \pi$ for $g_0 \in G \smallsetminus G^+$, we have $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R_-$ and $f_1, f_2 \in \pi$ if $F$ is archimedean. Hence $Z(\frac{1}{2}, \Phi, f_1, f_2)$ defines a non-zero equivariant map

$$R_0 \otimes (\pi \boxtimes \bar{\pi}) \to \mathbb{C}.$$

As in [**14**, Proposition 3.1], this shows that the theta lift of $\pi$ to $\mathrm{GO}(V_0)(F)$ is non-zero. Hence $\pi$ is a principal series representation of $G$. This contradicts the assumption that the Jacquet–Langlands transfer $\pi^D$ of $\pi$ to $D^\times$ exists.

We next consider the case where $K$ is a quadratic extension of $F$. We assume that $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \pi$. Let $V_-$ be the four-dimensional quadratic space over $F$ such that $\mathrm{disc}(V_-) = \mathrm{disc}(V)$ and $V_- \not\cong V$. Let $R_-$ be the image of the equivariant map $S(\mathbb{W}_-^\nabla) \to \boldsymbol{I}(\frac{1}{2})$, where $\mathbb{W}_-^\nabla = V_- \otimes W^\nabla$. By [**33, 37**], we have

$$\boldsymbol{I}(\tfrac{1}{2}) = R + R_-.$$

Since $\pi \circ \mathrm{Ad}(g_0) \cong \pi$ for $g_0 \in G \smallsetminus G^+$, we have $Z(\frac{1}{2}, \varPhi, f_1, f_2) = 0$ for all $\varPhi \in R_-$ and $f_1, f_2 \in \pi$ and hence a contradiction. □

### 6.8. The Rallis inner product formula

**Lemma 6.7.** *For* $\mathcal{F} \in \mathcal{R}(\tilde{G}_1)$ *and* $f_1, f_2 \in V_\pi$, *we have*

$$\int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} \mathcal{F}(\iota(g_1, g_2)) f_1(g_1) \overline{f_2(g_2)} \, \mathrm{d}g_1 \, \mathrm{d}g_2 = 0.$$

**Proof.** The assertion follows from Lemmas 6.1, 6.4, and 6.5. Here we have used the version of Lemma 6.4 for isometry groups. □

Let $\mathbb{A}^{\times,+} = \nu(H(\mathbb{A}))$, $F^{\times,+} = F^\times \cap \mathbb{A}^{\times,+}$, and $\mathcal{C} = \mathbb{A}^{\times,2} F^{\times,+} \backslash \mathbb{A}^{\times,+}$. The similitude characters induce isomorphisms

$$Z_G(\mathbb{A}) G_1(\mathbb{A}) G(F)^+ \backslash G(\mathbb{A})^+ \cong \mathcal{C}, \qquad Z_H(\mathbb{A}) H_1(\mathbb{A}) H(F) \backslash H(\mathbb{A}) \cong \mathcal{C}.$$

Fix cross-sections $c \mapsto g_c$ and $c \mapsto h_c$ of $G(\mathbb{A})^+ \to \mathcal{C}$ and $H(\mathbb{A}) \to \mathcal{C}$, respectively. Let $\mathrm{d}h$ be the Tamagawa measure on $Z_H(\mathbb{A}) \backslash H(\mathbb{A})$ and note that $\mathrm{vol}(Z_H(\mathbb{A}) H(F) \backslash H(\mathbb{A})) = \mathfrak{v}$.

**Lemma 6.8.** *Let* $\varphi_1 = \bigotimes_v \varphi_{1,v}, \varphi_2 = \bigotimes_v \varphi_{2,v} \in V_\omega$ *and* $f_1 = \bigotimes_v f_{1,v}, f_2 = \bigotimes_v f_{2,v} \in V_\pi$. *Then we have*

$$\int_{Z_H(\mathbb{A}) H(F) \backslash H(\mathbb{A})} \theta(h; \varphi_1, f_1) \overline{\theta(h; \varphi_2, f_2)} \, \mathrm{d}h = \frac{\mathfrak{v}}{2} \cdot \frac{L^S(1, \pi, \mathrm{Ad} \otimes \omega_{K/F})}{\zeta_K^S(2)} \prod_{v \in S} Z_v(\tfrac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}).$$

**Proof.** Let $\varPhi \in \boldsymbol{I}(\frac{1}{2})$. We extend $\varPhi$ to a holomorphic section of $\boldsymbol{I}(s)$ such that its restriction to $\boldsymbol{K}$ is independent of $s$. Let

$$E(s, \varPhi) = \sum_{d=-1}^{\infty} (s - \tfrac{1}{2})^d E_d(\tfrac{1}{2}, \varPhi)$$

be the Laurent expansion of $E(s, \varPhi)$ at $s = \frac{1}{2}$. Then the map $\varPhi \mapsto E_0(\frac{1}{2}, \varPhi)$ induces a $\tilde{G}(\mathbb{A})$-equivariant map

$$\boldsymbol{I}(\tfrac{1}{2}) \to \mathcal{A}(\tilde{G}) / \mathcal{R}(\tilde{G}).$$

We only consider the case $r > 0$. We have

$$Z(s, [\varphi], f_1, f_2) = \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} E(\iota(g_1, g_2)\iota(g_c, g_c); s, [\varphi]) \times f_1(g_1 g_c) \overline{f_2(g_2 g_c)} \, \mathrm{d}g_1 \, \mathrm{d}g_2 \, \mathrm{d}c.$$

Here d$c$ is the Haar measure on $\mathcal{C}$ such that $\mathrm{vol}(\mathcal{C}) = 1$ and d$g_1$, d$g_2$ are the Tamagawa measures on $G_1(\mathbb{A})$. By Lemma 6.7, $Z(s, [\varphi], f_1, f_2)$ is holomorphic at $s = \frac{1}{2}$. We have

$$[\varphi](g\iota(g_c, g_c), \tfrac{1}{2}) = [\tilde{\omega}(\iota(g_c, g_c), h_c)\varphi](g, \tfrac{1}{2})$$

for $g \in \tilde{G}(\mathbb{A})$ and $c \in \mathcal{C}$. For each $c \in \mathcal{C}$, there exists $\mathcal{F}_c \in \mathcal{R}(\tilde{G})$ such that

$$A_0(g\iota(g_c, g_c); \varphi) = A_0(g; \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) + \mathcal{F}_c(g)$$

for $g \in \tilde{G}(\mathbb{A})$. By Lemma 6.7, $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\begin{aligned}\mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} & A_0(\iota(g_1, g_2)\iota(g_c, g_c); \varphi) \cdot f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}c \\ = \mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} & A_0(\iota(g_1, g_2); \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) \\ & \times f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}c.\end{aligned}$$

For each $c \in \mathcal{C}$, there exists $\mathcal{F}'_c \in \mathcal{R}(\tilde{G}_1)$ such that

$$A_0(g_1; \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) = B_{-1}(g_1; \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) + \mathcal{F}'_c(g_1)$$

for $g_1 \in \tilde{G}_1(\mathbb{A})$ by (6.1). By Lemma 6.7, $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} B_{-1}(\iota(g_1, g_2); \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) \cdot f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}c.$$

This integral is equal to the residue at $s = \varrho'$ of

$$\begin{aligned}\mathfrak{v} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} & I(\iota(g_1, g_2); s, \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) \cdot f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}c \\ = \frac{\mathfrak{v}}{\kappa p(s)} \int_{\mathcal{C}} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} & \theta(\iota(g_1, g_2), h_1; z\tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) \\ & \times \mathcal{E}(h_1; s) f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}h_1\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}c \\ = \frac{\mathfrak{v}}{\kappa} \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} & \theta(\iota(g_1, g_2), h_1; \tilde{\omega}(\iota(g_c, g_c), h_c)\varphi) \\ & \times \mathcal{E}(h_1; s) f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}h_1\, \mathrm{d}c.\end{aligned}$$

Hence $Z(\frac{1}{2}, [\varphi], f_1, f_2)$ is equal to

$$\begin{aligned}\mathfrak{v} \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} \int_{G_1(F)\backslash G_1(\mathbb{A})} & \theta(g_1 g_c, h_1 h_c; \varphi_1) \cdot \overline{\theta(g_2 g_c, h_1 h_c; \varphi_2)} \\ & \times f_1(g_1 g_c)\overline{f_2(g_2 g_c)}\, \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}h_1\, \mathrm{d}c \\ & = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \theta(h; \varphi_1, f_1)\overline{\theta(h; \varphi_2, f_2)}\, \mathrm{d}h.\end{aligned}$$

□

By Lemmas 6.1, 6.6, and 6.8, we obtain the following lemma.

**Lemma 6.9.** *We have*

$$V_{\theta(\pi)} \neq 0.$$

Let $\mathcal{B}_{\theta(\pi)} : V_{\theta(\pi)} \otimes \bar{V}_{\theta(\pi)} \to \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_{\theta(\pi)}(\phi_1, \phi_2) = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h)\overline{\phi_2(h)}\,\mathrm{d}h$$

for $\phi_1, \phi_2 \in V_{\theta(\pi)}$. For each place $v$ of $F$, we define an equivariant map

$$\mathcal{Z}_v^\sharp : (\omega_v \boxtimes \bar{\omega}_v) \otimes (\pi_v \boxtimes \bar{\pi}_v) \to \mathbb{C}$$

by

$$\mathcal{Z}_v^\sharp(\varphi_{1,v}, \varphi_{2,v}; f_{1,v}, f_{2,v}) = \zeta_{K_v}(2)L_v(1, \pi_v, \mathrm{Ad}\otimes\omega_{K_v/F_v})^{-1} Z_v(\tfrac{1}{2}, [\delta(\varphi_{1,v}\otimes\bar{\varphi}_{2,v})], f_{1,v}, f_{2,v})$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$ and $f_{1,v}, f_{2,v} \in \pi_v$. By Lemma 6.6, $\mathcal{Z}_v^\sharp \neq 0$. By Lemma 6.8, there exists a pairing $\mathcal{B}^\sharp_{\theta(\pi_v)} : \theta(\pi_v) \otimes \overline{\theta(\pi_v)} \to \mathbb{C}$ such that

$$\mathcal{Z}_v^\sharp = \mathcal{B}^\sharp_{\theta(\pi_v)} \circ (\theta_v \otimes \bar{\theta}_v).$$

By Lemma 6.8, we obtain the following proposition.

**Proposition 6.10.** *We have*

$$\mathcal{B}_{\theta(\pi)} = 2^\beta \frac{L(1, \pi, \mathrm{Ad} \otimes \omega_{K/F})}{\zeta_K(2)} \prod_v \mathcal{B}^\sharp_{\theta(\pi_v)}.$$

*Here*

$$\beta = \begin{cases} 0 & \text{if } K = F \times F, \\ -1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

## 7. Global theta lifts from GO($V$) to GSp$_4$

Let $F$ be a number field. Let $V$ be a four-dimensional quadratic space over $F$ and $W$ a four-dimensional symplectic space over $F$. Let $\tilde{V} = V \oplus (-V)$. Set

$$\begin{aligned} H &= \mathrm{GO}(V), & H_1 &= \mathrm{O}(V), \\ \tilde{H} &= \mathrm{GO}(\tilde{V}) \cong \mathrm{GO}_8, & \tilde{H}_1 &= \mathrm{O}(\tilde{V}) \cong \mathrm{O}_8, \\ G &= \mathrm{GSp}(W) \cong \mathrm{GSp}_4, & G_1 &= \mathrm{Sp}(W) \cong \mathrm{Sp}_4. \end{aligned}$$

Let

$$\boldsymbol{H} = \{\boldsymbol{h} = (h_1, h_2) \in H \times H \mid \nu(h_1) = \nu(h_2)\},$$

where $\nu : H \to \mathbb{G}_m$ is the similitude character. Let $\iota : \boldsymbol{H} \hookrightarrow \tilde{H}$ be the natural embedding. Let $K$ be the discriminant algebra of $V$ and choose a quaternion algebra $D$ over $F$ associated to $V$ as in § 1.

### 7.1. Weil representations

Fix a non-trivial additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/F$. We may assume that $\psi_v(x) = \exp(2\pi\sqrt{-1}\,\mathrm{tr}_{F_v/\mathbb{R}}(x))$ for $x \in F_v$ if $v$ is archimedean (see [**49**, Lemma 5.1]). Let $W = X \oplus Y$ be a complete polarization and set

$$\mathbb{W} = W \otimes V, \qquad \mathbb{X} = X \otimes V, \qquad \mathbb{Y} = Y \otimes V.$$

Then $\mathbb{W}$ is a symplectic space over $F$ and $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a complete polarization. Let $\omega$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space $V_\omega = S(\mathbb{X}(\mathbb{A}))$ with respect to $\psi$ and $\mathcal{B}_\omega : V_\omega \otimes \bar{V}_\omega \to \mathbb{C}$ the canonical pairing. For each place $v$ of $F$, let $\omega_v$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}_v)$ on the space $S(\mathbb{X}_v)$ with respect to $\psi_v$ and $\mathcal{B}_{\omega_v} : \omega_v \otimes \bar{\omega}_v \to \mathbb{C}$ the canonical pairing. Then we have $\omega = \bigotimes_v \omega_v$ and $\mathcal{B}_\omega = \prod_v \mathcal{B}_{\omega_v}$. By [**30**], [**13**, § 5.1], and [**47**], we may regard $\omega$ (respectively $\omega_v$) as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$ (respectively $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F_v)$).

Let

$$\tilde{\mathbb{W}} = W \otimes \tilde{V}, \qquad \tilde{\mathbb{X}} = X \otimes \tilde{V}, \qquad \tilde{\mathbb{Y}} = Y \otimes \tilde{V}.$$

Then $\tilde{\mathbb{W}}$ is a symplectic space over $F$ and $\tilde{\mathbb{W}} = \tilde{\mathbb{X}} \oplus \tilde{\mathbb{Y}}$ is a complete polarization. Let $\tilde{\omega}$ be the Weil representation of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ on $S(\tilde{\mathbb{X}}(\mathbb{A}))$ with respect to $\psi$. We may regard $\tilde{\omega}$ as a representation of $\mathrm{G}(\mathrm{O}(\tilde{V}) \times \mathrm{Sp}(W))(\mathbb{A})$. We have a natural isomorphism

$$S(\tilde{\mathbb{X}}(\mathbb{A})) \cong V_\omega \otimes \bar{V}_\omega$$

as representations of $\mathrm{Mp}(\mathbb{W}(\mathbb{A})) \times \mathrm{Mp}(\mathbb{W}(\mathbb{A}))$. Let

$$V^\Delta = \{(x,x) \mid x \in V\}, \qquad \mathbb{W}^\Delta = W \otimes V^\Delta,$$
$$V^\nabla = \{(x,-x) \mid x \in V\}, \qquad \mathbb{W}^\nabla = W \otimes V^\nabla.$$

Then $\tilde{\mathbb{W}} = \mathbb{W}^\nabla \oplus \mathbb{W}^\Delta$ is a complete polarization. Hence we can realize the Weil representation $\tilde{\omega}$ on $S(\mathbb{W}^\nabla(\mathbb{A}))$. By [**40**, § 2], there exists an isomorphism

$$\delta : S(\tilde{\mathbb{X}}(\mathbb{A})) \to S(\mathbb{W}^\nabla(\mathbb{A}))$$

as representations of $\mathrm{Mp}(\tilde{\mathbb{W}}(\mathbb{A}))$ such that

$$\delta(\varphi_1 \otimes \bar{\varphi}_2)(0) = \mathcal{B}_\omega(\varphi_1, \varphi_2)$$

for $\varphi_1, \varphi_2 \in V_\omega$.

More generally, let $\mathcal{V} = F^{2n}$ be the space of row vectors equipped with a non-degenerate symmetric bilinear form $(x,y) = xJ\,{}^t y$ for $x, y \in \mathcal{V}$, where

$$J = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

We identify $\mathrm{GO}(\mathcal{V})$ with

$$\mathrm{GO}_{2n} = \{h \in \mathrm{GL}_{2n} \mid hJ\,{}^t h = \nu(h)J,\ \nu(h) \in \mathbb{G}_m\}.$$

Let $\mathcal{W} = F^{2r}$ be the space of column vectors equipped with a non-degenerate antisymmetric bilinear form $\langle x, y\rangle = {}^{\mathrm{t}}xJ'y$ for $x, y \in \mathcal{W}$, where

$$J' = \begin{pmatrix} 0 & \mathbf{1}_r \\ -\mathbf{1}_r & 0 \end{pmatrix}.$$

We identify $\mathrm{GSp}(\mathcal{W})$ with

$$\mathrm{GSp}_{2r} = \{g \in \mathrm{GL}_{2r} \mid {}^{\mathrm{t}}gJ'g = \nu(g)J',\ \nu(g) \in \mathbb{G}_m\}.$$

Let $\boldsymbol{W} = \mathcal{W} \otimes \mathcal{V}$,

$$\begin{aligned} \mathcal{X} &= \{(x,0) \in F^{2n} \mid x \in F^n\}, \qquad & \boldsymbol{X} &= \mathcal{W} \otimes \mathcal{X}, \\ \mathcal{Y} &= \{(0,y) \in F^{2n} \mid y \in F^n\}, \qquad & \boldsymbol{Y} &= \mathcal{W} \otimes \mathcal{Y}. \end{aligned}$$

Then $\boldsymbol{W}$ is a symplectic space over $F$ and $\boldsymbol{W} = \boldsymbol{X} \oplus \boldsymbol{Y}$ is a complete polarization. We identify $\boldsymbol{X}$ with $\mathrm{M}_{2r,n}(F)$. Let $\boldsymbol{\omega}$ be the Weil representation of $\mathrm{Mp}(\boldsymbol{W}(\mathbb{A}))$ on the space $S(\mathrm{M}_{2r,n}(\mathbb{A}))$ with respect to $\psi$. We may regard $\boldsymbol{\omega}$ as a representation of $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$.

Choosing bases, we fix an isomorphism

$$S(\mathbb{W}^{\nabla}(\mathbb{A})) \cong S(\mathrm{M}_{4,4}(\mathbb{A}))$$

as representations of $\mathrm{G}(\mathrm{O}(\tilde{V}) \times \mathrm{Sp}(W))(\mathbb{A}) \cong \mathrm{G}(\mathrm{O}_8 \times \mathrm{Sp}_4)(\mathbb{A})$.

### 7.2. Theta lifts

Let $\sigma \cong \bigotimes_v \sigma_v$ be an irreducible unitary cuspidal automorphic representation of $H(\mathbb{A})$ on the space $V_\sigma$. We assume the following.

- The Jacquet–Langlands transfer of $\sigma|_{D^\times(\mathbb{A}_K)}$ to $\mathrm{GL}_2(\mathbb{A}_K)$ is cuspidal.
- $\sigma_v \otimes \mathrm{sgn} \cong \sigma_v$ for some place $v$ of $F$.
- If $\sigma_v \otimes \mathrm{sgn} \ncong \sigma_v$, then $\sigma_v \ncong \sigma_{0,v}^-$ for any distinguished representation $\sigma_{0,v}$ of $\mathrm{GSO}(V)(F_v)$.

**Lemma 7.1.** *The partial L-function* $L^S(s, \sigma, \mathrm{std})$ *is holomorphic and non-zero at* $s = 1$.

**Proof.** We first consider the case $K = F \times F$. We have $\sigma|_{D^\times(\mathbb{A}_K)} \cong \tau_1^D \boxtimes \tau_2^D$ with an irreducible unitary cuspidal automorphic representation $\tau_i$ of $\mathrm{GL}_2(\mathbb{A})$ such that $\tau_1 \ncong \tau_2$ and $\omega_{\tau_1} = \omega_{\tau_2}$. Here $\tau_i^D$ is the Jacquet–Langlands transfer of $\tau_i$ to $D^\times(\mathbb{A})$. Then we have

$$L^S(s, \sigma, \mathrm{std}) = L^S(s, \tau_1 \times \tau_2^\vee)$$

and the assertion is well known.

We next consider the case where $K$ is a quadratic extension of $F$. We have

$$\sigma|_{D^\times(\mathbb{A}_K) \times \mathbb{A}^\times} \cong \tau^D \boxtimes \chi$$

with an irreducible unitary cuspidal automorphic representation $\tau$ of $\mathrm{GL}_2(\mathbb{A}_K)$ and a Hecke character $\chi$ of $\mathbb{A}^\times$ such that $\tau^c \not\cong \tau$ and $\omega_\tau = \chi \circ \mathrm{N}_{K/F}$. Here $\tau^D$ is the Jacquet–Langlands transfer of $\tau$ to $D^\times(\mathbb{A}_K)$. Let $\mu$ be a Hecke character of $\mathbb{A}_K^\times$ such that $\mu|_{\mathbb{A}^\times} = \chi^{-1}$. Then we have

$$L^S(s, \sigma, \mathrm{std}) = L^S(s, \tau \otimes \mu, \mathrm{Asai}).$$

Since $(\tau^c \otimes \mu^c)^\vee \not\cong \tau \otimes \mu$ and $L_v(s, \tau_v \otimes \mu_v, \mathrm{Asai})$ is holomorphic and non-zero at $s = 1$ for all $v$, the assertion follows from [**27**, Proposition 5.3]. □

Let

$$G(\mathbb{A})^+ = \{g \in G(\mathbb{A}) \mid \nu(g) \in \nu(H(\mathbb{A}))\}$$

and $G(F)^+ = G(F) \cap G(\mathbb{A})^+$. Let $\varphi \in V_\omega$. The theta function associated to $\varphi$ is given by

$$\theta(h, g; \varphi) = \sum_{x \in \mathbb{X}(F)} \omega(h, g)\varphi(x)$$

for $(h, g) \in \mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$. Let $f \in V_\sigma$. For $g \in G(\mathbb{A})^+$, choose $h \in H(\mathbb{A})$ such that $\nu(h) = \nu(g)$, and put

$$\theta(g; \varphi, f) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(h_1 h, g; \varphi) f(h_1 h)\, \mathrm{d}h_1.$$

Here

$$\mathrm{d}h_1 = \prod_v \mathrm{d}h_{1,v}$$

is the Tamagawa measure on $H_1(\mathbb{A})$. Note that $\mathrm{vol}(H_1(F)\backslash H_1(\mathbb{A})) = 1$ and we may assume that the volume of a hyperspecial maximal compact subgroup of $H_{1,v}$ with respect to $\mathrm{d}h_{1,v}$ is 1 for almost all $v$. This integral defines an automorphic form $\theta(\varphi, f)$ on $G(\mathbb{A})^+$. We extend $\theta(\varphi, f)$ to an automorphic form on $G(\mathbb{A})$ by the natural embedding

$$G(F)^+\backslash G(\mathbb{A})^+ \hookrightarrow G(F)\backslash G(\mathbb{A})$$

and extension by zero. Let $\theta(\sigma)$ be the automorphic representation of $G(\mathbb{A})^+$ on the space $V_{\theta(\sigma)}$ generated by $\theta(\varphi, f)$ for all $\varphi \in V_\omega$ and $f \in V_\sigma$. By assumption on $\sigma$, $\theta(\sigma)$ is cuspidal. In Lemma 7.12 below, we will show that $V_{\theta(\sigma)} \neq 0$. In particular, $\theta(\sigma_v) \neq 0$ for all $v$. Hence $\theta(\sigma)$ is irreducible,

$$\theta(\sigma) \cong \bigotimes_v \theta(\sigma_v),$$

and $\theta(\sigma_v)$ is unitary for all $v$. Thus, we obtain a $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$-equivariant surjective map

$$\theta : V_\omega \otimes V_\sigma \to V_{\theta(\sigma)}$$

and $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F_v)$-equivariant surjective maps

$$\theta_v : \omega_v \otimes \sigma_v \to \theta(\sigma_v)$$

such that $\theta = \bigotimes_v \theta_v$.

Let $\pi$ be the automorphic representation of $G(\mathbb{A})$ on the space $V_\pi$ generated by $V_{\theta(\sigma)}$. For each place $v$ of $F$, let $\pi_v = \mathrm{ind}_{G_v^+}^{G_v}(\theta(\sigma_v))$. By Lemma 5.2, $\pi_v$ is irreducible.

**Lemma 7.2.** *We have*

$$\pi \cong \bigotimes_v \pi_v.$$

**Proof.** Since $G(\mathbb{A})^+$ is an open subgroup of $G(\mathbb{A})$, we have a natural $G(\mathbb{A})$-equivariant map

$$\text{c-ind}_{G(\mathbb{A})^+}^{G(\mathbb{A})}(V_{\theta(\sigma)}) \to V_\pi. \tag{7.1}$$

By definition, (7.1) is surjective. Since

$$\text{c-ind}_{G(\mathbb{A})^+}^{G(\mathbb{A})}(\theta(\sigma)) \cong \bigotimes_v \pi_v$$

is irreducible, (7.1) is injective. □

### 7.3. Eisenstein series

For each $r \in \mathbb{N}$ with $r \leqslant n$, we define a parabolic subgroup $P_{n,r}$ of $\mathrm{GO}_{2n}$ by

$$P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu(h')^{\mathrm{t}}a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in \mathrm{GO}_{2n} \;\middle|\; a \in \mathrm{GL}_r,\ h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GO}_{2n-2r} \right\}.$$

Let $\delta_{P_{n,r}}$ be the modulus character of $P_{n,r}(\mathbb{A})$. For $\nu \in \mathbb{G}_m$, let

$$d(\nu) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \nu\mathbf{1}_n \end{pmatrix}.$$

We define a maximal compact subgroup $\boldsymbol{K} = \prod_v \boldsymbol{K}_v$ of $\mathrm{GO}_{2n}(\mathbb{A})$ by

$$\boldsymbol{K}_v = \begin{cases} \mathrm{GO}_{2n}(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GO}_{2n}(F_v) \cap \mathrm{O}(2n) & \text{if } v \text{ is real,} \\ \mathrm{GO}_{2n}(F_v) \cap \mathrm{U}(2n) & \text{if } v \text{ is complex.} \end{cases}$$

Then we have $\mathrm{GO}_{2n}(\mathbb{A}) = P_{n,r}(\mathbb{A})\boldsymbol{K}$.

Let $\boldsymbol{I}^{(n,r)}(s)$ denote the degenerate principal series representation of $\mathrm{GO}_{2n}(\mathbb{A})$ given by

$$\boldsymbol{I}^{(n,r)}(s) = \mathrm{Ind}_{P_{n,r}(\mathbb{A})}^{\mathrm{GO}_{2n}(\mathbb{A})}(\delta_{P_{n,r}}^{s/(2n-r-1)}),$$

where Ind denotes the normalized induction. Given a holomorphic section $\Phi$ of $\boldsymbol{I}^{(n,r)}(s)$, we define an Eisenstein series $E^{(n,r)}(s, \Phi)$ on $\mathrm{GO}_{2n}(\mathbb{A})$ by

$$E^{(n,r)}(h; s, \Phi) = \sum_{\gamma \in P_{n,r}(F)\backslash \mathrm{GO}_{2n}(F)} \Phi(\gamma h, s)$$

for $\mathrm{Re}(s) \gg 0$. If $\Phi^o$ is the holomorphic section of $\boldsymbol{I}^{(n,r)}(s)$ such that $\Phi^o(k,s) = 1$ for all $k \in \boldsymbol{K}$, we write $E^{(n,r)}(s) = E^{(n,r)}(s, \Phi^o)$. For each $s_0 \in \mathbb{C}$, let

$$E^{(n,r)}(s) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(n,r)}(s_0)$$

be the Laurent expansion of $E^{(n,r)}(s)$ at $s = s_0$.

We define a $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$-equivariant map

$$[\cdot] : S(\mathrm{M}_{2r,n}(\mathbb{A})) \to \boldsymbol{I}^{(n,n)}(r - \tfrac{1}{2}(n-1))$$

by

$$[\varphi](h, r - \tfrac{1}{2}(n-1)) = |\nu(h)|^{-nr/2} \boldsymbol{\omega}(d(\nu(h)^{-1})h)\varphi(0)$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$. Here $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$ acts on $\boldsymbol{I}^{(n,n)}(r - \frac{1}{2}(n-1))$ via the projection $\mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A}) \to \mathrm{GO}_{2n}(\mathbb{A})$. We extend $[\varphi]$ to a holomorphic section of $\boldsymbol{I}^{(n,n)}(s)$ such that its restriction to $\boldsymbol{K}$ is independent of $s$.

### 7.4. Theta integrals

We define a parabolic subgroup $P'$ of $\mathrm{GSp}_{2r}$ by

$$P' = \left\{ \begin{pmatrix} a & * \\ 0 & \nu\, {}^{\mathrm{t}}a^{-1} \end{pmatrix} \in \mathrm{GSp}_{2r} \;\middle|\; a \in \mathrm{GL}_r,\ \nu \in \mathbb{G}_m \right\}.$$

Let $\delta_{P'}$ be the modulus character of $P'(\mathbb{A})$. For $\nu \in \mathbb{G}_m$, let

$$d(\nu) = \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & \nu \mathbf{1}_r \end{pmatrix}.$$

We define a maximal compact subgroup $\boldsymbol{K}' = \prod_v \boldsymbol{K}'_v$ of $\mathrm{GSp}_{2r}(\mathbb{A})$ by

$$\boldsymbol{K}'_v = \begin{cases} \mathrm{Ad}(d(\varpi_v^{\mathfrak{c}_v}))(\mathrm{GSp}_{2r}(\mathfrak{o}_v)) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GSp}_{2r}(F_v) \cap \mathrm{O}(2r) & \text{if } v \text{ is real,} \\ \mathrm{GSp}_{2r}(F_v) \cap \mathrm{U}(2r) & \text{if } v \text{ is complex.} \end{cases}$$

Here $\varpi_v$ is a uniformizer of $F_v$ and $\mathfrak{c}_v$ is the largest integer such that $\psi_v$ is trivial on $\varpi_v^{-\mathfrak{c}_v}\mathfrak{o}_v$. Then we have $\mathrm{GSp}_{2r}(\mathbb{A}) = P'(\mathbb{A})\boldsymbol{K}'$. Let $P'_1 = P' \cap \mathrm{Sp}_{2r}$ and $\boldsymbol{K}'_1 = \boldsymbol{K}' \cap \mathrm{Sp}_{2r}(\mathbb{A})$ so that $\mathrm{Sp}_{2r}(\mathbb{A}) = P'_1(\mathbb{A})\boldsymbol{K}'_1$. Let $\mathrm{d}g_1$ be the Tamagawa measure on $\mathrm{Sp}_{2r}(\mathbb{A})$ and note that $\mathrm{vol}(\mathrm{Sp}_{2r}(F)\backslash \mathrm{Sp}_{2r}(\mathbb{A})) = 1$. Let $\mathrm{d}_l p'$ be the left-invariant Tamagawa measure on $P'_1(\mathbb{A})$ and $\mathrm{d}k'$ the Haar measure on $\boldsymbol{K}'_1$ such that $\mathrm{vol}(\boldsymbol{K}'_1) = 1$. There exists a constant $\kappa$ such that

$$\int_{\mathrm{Sp}_{2r}(\mathbb{A})} f(g_1)\, \mathrm{d}g_1 = \kappa \int_{P'_1(\mathbb{A})} \int_{\boldsymbol{K}'_1} f(p'k')\, \mathrm{d}_l p'\, \mathrm{d}k'$$

for $f \in L^1(\mathrm{Sp}_{2r}(\mathbb{A}))$.

Put $\varrho' = \frac{1}{2}(r+1)$. Let $\Phi'$ be the holomorphic section of $\mathrm{Ind}_{P'(\mathbb{A})}^{\mathrm{GSp}_{2r}(\mathbb{A})}(\delta_{P'}^{s/(r+1)})$ such that $\Phi'(k', s) = 1$ for all $k' \in \boldsymbol{K}'$. We define an Eisenstein series $\mathcal{E}(s)$ on $\mathrm{GSp}_{2r}(\mathbb{A})$ by

$$\mathcal{E}(g; s) = \sum_{\gamma \in P'(F)\backslash \mathrm{GSp}_{2r}(F)} \Phi'(\gamma g, s)$$

for $\mathrm{Re}(s) > \varrho'$. By [**36**, § 5] and [**19**, Lemma 9.1], we have

$$\mathrm{Res}_{s=\varrho'} \mathcal{E}(g; s) = \kappa$$

for $g \in \mathrm{GSp}_{2r}(\mathbb{A})$.

Let $\varphi \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$. The theta function associated to $\varphi$ is given by

$$\theta(h, g; \varphi) = \sum_{x \in \mathrm{M}_{2r,n}(F)} \boldsymbol{\omega}(h, g)\varphi(x)$$

for $(h, g) \in \mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$. Let $z \in C_c^\infty(\mathrm{O}_{2n}(F_v)//\mathrm{O}_{2n}(\mathfrak{o}_v))$ be the regularizing Hecke operator as in [**34**, § 5] and [**25**, § 2.3], where $v$ is a certain non-archimedean place of $F$ depending on $\varphi$. There exists a self-adjoint Hecke operator $z' \in C_c^\infty(\mathrm{Sp}_{2r}(F_v)//\mathrm{Sp}_{2r}(\mathfrak{o}_v))$ such that $\boldsymbol{\omega}(z) = \boldsymbol{\omega}(z')$. Then we have $z'\mathcal{E}(s) = p(s)\mathcal{E}(s)$ with some $p(s) \in \mathbb{C}[q_v^s, q_v^{-s}]$. Here $q_v$ is the cardinality of $\mathfrak{o}_v/\varpi_v\mathfrak{o}_v$. Following Kudla and Rallis [**34**, § 5], we define the regularized theta integral $I^{(n,r)}(s, \varphi)$ by

$$I^{(n,r)}(h; s, \varphi) = \frac{1}{\kappa p(s)} \int_{\mathrm{Sp}_{2r}(F)\backslash \mathrm{Sp}_{2r}(\mathbb{A})} \theta(h, g_1 g; z\varphi)\mathcal{E}(g_1 g; s)\, \mathrm{d}g_1$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$, where $g \in \mathrm{GSp}_{2r}(\mathbb{A})$ such that $\nu(g) = \nu(h)$. By [**34**, Lemma 5.5.6], $I^{(n,r)}(s, \varphi)$ has at most a double (respectively simple) pole at $s = \varrho'$ if $r \leqslant n-1 < 2r$ (respectively $2r \leqslant n-1$).

Let $\hat{\varphi} \in S(\mathrm{M}_{r,2n}(\mathbb{A}))$ be the partial Fourier transform of $\varphi \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$ defined by

$$\hat{\varphi}(u, v) = \int_{\mathrm{M}_{r,n}(\mathbb{A})} \varphi\begin{pmatrix} x \\ u \end{pmatrix} \psi(\mathrm{tr}(v\,{}^{\mathrm{t}}x))\, \mathrm{d}x$$

for $u, v \in \mathrm{M}_{r,n}(\mathbb{A})$, where $\mathrm{d}x$ is the Tamagawa measure on $\mathrm{M}_{r,n}(\mathbb{A})$. For $h \in \mathrm{GO}_{2n}(\mathbb{A})$, choose $g \in \mathrm{GSp}_{2r}(\mathbb{A})$ such that $\nu(g) = \nu(h)$, and put

$$\Psi(\varphi)(h, s) = \int_{\mathrm{GL}_r(\mathbb{A})} \int_{\boldsymbol{K}_1'} \widehat{(\boldsymbol{\omega}(h, k'g)\varphi)}(\mathbf{0}_{r,n}, {}^{\mathrm{t}}a, \mathbf{0}_{r,n-r})\Phi'(k'g, s)|\det(a)|^{s+n-\varrho'}\, \mathrm{d}k'\, \mathrm{d}^\times a.$$

Here $d^\times a$ is the Tamagawa measure on $\mathrm{GL}_r(\mathbb{A})$. By [**34**, Lemma 5.5.2], $\Psi(\varphi)$ is a meromorphic section of $\boldsymbol{I}^{(n,r)}(s)$ and is holomorphic for $\mathrm{Re}(s) > -\frac{1}{2}(2n-3r-1)$. By [**34**, § 5.5], we have

$$I^{(n,r)}(s, \varphi) = E^{(n,r)}(s, \Psi(\varphi)).$$

We now consider the spherical case. Let $\xi(s) = \mathfrak{D}^{s/2}\zeta(s)$, where $\mathfrak{D}$ is the absolute value of the discriminant of $F$ and $\zeta(s)$ is the zeta function of $F$ including archimedean factors. Put $\rho = \mathrm{Res}_{s=1}\, \xi(s)$. We define $\varphi_r = \bigotimes_v \varphi_{r,v} \in S(\mathrm{M}_{r,r}(\mathbb{A}))$ as follows.

- If $v$ is non-archimedean, then $\varphi_{r,v}$ is the characteristic function of $\mathrm{M}_{r,r}(\mathfrak{o}_v)$.
- If $v$ is real, then $\varphi_{r,v}(x) = \exp(-\pi \operatorname{tr}({}^{\mathrm{t}}xx))$ for $x \in \mathrm{M}_{r,r}(F_v)$.
- If $v$ is complex, then $\varphi_{r,v}(x) = \exp(-2\pi \operatorname{tr}({}^{\mathrm{t}}\bar{x}x))$ for $x \in \mathrm{M}_{r,r}(F_v)$.

Put

$$\Xi_r(s) = \int_{\mathrm{GL}_r(\mathbb{A})} \varphi_r(a)|\det(a)|^s \, \mathrm{d}^\times a,$$

where $\mathrm{d}^\times a$ is the Tamagawa measure on $\mathrm{GL}_r(\mathbb{A})$.

**Lemma 7.3.** *We have*

$$\Xi_r(s) = \mathfrak{D}^{-rs/2}\rho^{-1}\prod_{i=2}^{r}\xi(i)^{-1}\cdot\prod_{j=0}^{r-1}\xi(s-j).$$

*Here we omit* $\prod_{i=2}^{r}\xi(i)^{-1}$ *if* $r=1$.

**Proof.** If $r=1$, then we have $\Xi_1(s) = \rho^{-1}\zeta(s) = \mathfrak{D}^{-s/2}\rho^{-1}\xi(s)$. Assume that $r \geqslant 2$. Let $T$ (respectively $U$) be the subgroup of $\mathrm{GL}_r$ consisting of diagonal matrices (respectively unipotent upper triangular matrices). We define a maximal compact subgroup $\mathscr{K} = \prod_v \mathscr{K}_v$ of $\mathrm{GL}_r(\mathbb{A})$ by

$$\mathscr{K}_v = \begin{cases} \mathrm{GL}_r(\mathfrak{o}_v) & \text{if } v \text{ is non-archimedean,} \\ \mathrm{GL}_r(F_v)\cap \mathrm{O}(r) & \text{if } v \text{ is real,} \\ \mathrm{GL}_r(F_v)\cap \mathrm{U}(r) & \text{if } v \text{ is complex.} \end{cases}$$

Let $\mathrm{d}^\times t$ (respectively $\mathrm{d}u$) be the Tamagawa measure on $T(\mathbb{A})$ (respectively $U(\mathbb{A})$) and $\mathrm{d}k$ the Haar measure on $\mathscr{K}$ such that $\operatorname{vol}(\mathscr{K}) = 1$. By [**36**, § 5], we have

$$\int_{\mathrm{GL}_r(\mathbb{A})} f(a)\,\mathrm{d}^\times a = \varkappa \int_{T(\mathbb{A})}\int_{U(\mathbb{A})}\int_{\mathscr{K}} f(tuk)\,\mathrm{d}^\times t\,\mathrm{d}u\,\mathrm{d}k$$

for $f \in L^1(\mathrm{GL}_r(\mathbb{A}))$, where $\varkappa = \rho^{r-1}\prod_{i=2}^{r}\xi(i)^{-1}$. Hence we have

$$\Xi_r(s) = \varkappa \int_{T(\mathbb{A})}\int_{U(\mathbb{A})} \varphi_r(tu)|\det(t)|^s\,\mathrm{d}^\times t\,\mathrm{d}u.$$

Changing variables, we have

$$\Xi_r(s) = \varkappa \prod_{j=0}^{r-1}\Xi_1(s-j)\cdot\left(\int_{\mathbb{A}}\varphi_1(x)\,\mathrm{d}x\right)^{r(r-1)/2},$$

where $\mathrm{d}x$ is the Tamagawa measure on $\mathbb{A}$. Since

$$\int_{\mathbb{A}}\varphi_1(x)\,\mathrm{d}x = \mathfrak{D}^{-1/2},$$

the assertion follows. □

We define *the spherical Schwartz function* $\varphi^o = \bigotimes_v \varphi_v^o \in S(\mathrm{M}_{2r,n}(\mathbb{A}))$ as follows.

- If $v$ is non-archimedean, then

$$\varphi_v^o\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} q_v^{-\mathfrak{c}_v nr/2} & \text{if } x \in \mathrm{M}_{r,n}(\varpi_v^{-\mathfrak{c}_v}\mathfrak{o}_v) \text{ and } y \in \mathrm{M}_{r,n}(\mathfrak{o}_v), \\ 0 & \text{otherwise.} \end{cases}$$

- If $v$ is real, then $\varphi_v^o(x) = \exp(-\pi\,\mathrm{tr}({}^{\mathrm{t}}xx))$ for $x \in \mathrm{M}_{2r,n}(F_v)$.
- If $v$ is complex, then $\varphi_v^o(x) = \exp(-2\pi\,\mathrm{tr}({}^{\mathrm{t}}\bar{x}x))$ for $x \in \mathrm{M}_{2r,n}(F_v)$.

Here $q_v$ is the cardinality of $\mathfrak{o}_v/\varpi_v\mathfrak{o}_v$ and $\mathfrak{c}_v$ is the largest integer such that $\psi_v$ is trivial on $\varpi_v^{-\mathfrak{c}_v}\mathfrak{o}_v$. Then we have $\hat{\varphi}^o = \bigotimes_v \hat{\varphi}_v^o$, where

- $\hat{\varphi}_v^o$ is the characteristic function of $\mathrm{M}_{r,2n}(\mathfrak{o}_v)$ if $v$ is non-archimedean,
- $\hat{\varphi}_v^o(x) = \exp(-\pi\,\mathrm{tr}({}^{\mathrm{t}}xx))$ for $x \in \mathrm{M}_{r,2n}(F_v)$ if $v$ is real,
- $\hat{\varphi}_v^o(x) = \exp(-2\pi\,\mathrm{tr}({}^{\mathrm{t}}\bar{x}x))$ for $x \in \mathrm{M}_{r,2n}(F_v)$ if $v$ is complex.

Moreover, we have

$$\boldsymbol{\omega}(k,k')\varphi^o = \varphi^o$$

for $(k,k') \in (\boldsymbol{K} \times \boldsymbol{K}') \cap \mathrm{G}(\mathrm{O}_{2n} \times \mathrm{Sp}_{2r})(\mathbb{A})$. Hence we have

$$E^{(n,n)}(s,[\varphi^o]) = \mathfrak{D}^{-nr/2}E^{(n,n)}(s). \tag{7.2}$$

**Lemma 7.4.**

$$I^{(n,r)}(s,\varphi^o) = \mathfrak{D}^{-r(s+n-\varrho')/2}\rho^{-1}\prod_{i=2}^{n}\xi(i)^{-1}\cdot\prod_{j=0}^{r-1}\xi(s+n-\varrho'-j)\cdot E^{(n,r)}(s).$$

**Proof.** By Lemma 7.3, we have

$$\Psi(\varphi^o)(1,s) = \mathfrak{D}^{-r(s+n-\varrho')/2}\rho^{-1}\prod_{i=2}^{n}\xi(i)^{-1}\cdot\prod_{j=0}^{r-1}\xi(s+n-\varrho'-j).$$

Since $\Psi(\varphi^o)$ is $\boldsymbol{K}$-invariant, the assertion follows. □

### 7.5. The Siegel–Weil formula

We identify $\mathrm{GO}_8$ and $P_{4,4}$ with $\tilde{H}$ and the parabolic subgroup of $\tilde{H}$ stabilizing $V^\Delta$, respectively. Let $\mathcal{A}(\tilde{H})$ denote the space of automorphic forms on $\tilde{H}(\mathbb{A})$ and $\mathcal{R}(\tilde{H})$ the subspace of $\mathcal{A}(\tilde{H})$ generated by $\mathrm{Res}_{s=1/2}\,E^{(4,4)}(s,\Phi)$ for all holomorphic sections $\Phi$ of $\boldsymbol{I}^{(4,4)}(s)$. We remark that $E^{(4,4)}(s,\Phi)$ has at most a simple pole at $s=\frac{1}{2}$ by [**32**, Theorem 1.0.1].

Let $\varphi^o \in S(\mathrm{M}_{4,4}(\mathbb{A}))$ be the spherical Schwartz function as defined on p. 273. Let $S(\mathrm{M}_{4,4}(\mathbb{A}))^o$ be the subspace of $S(\mathrm{M}_{4,4}(\mathbb{A}))$ generated by $\boldsymbol{\omega}(h_1)\varphi^o$ for all $h_1 \in \tilde{H}_1(\mathbb{A})$. Let

$$\begin{aligned} A_0 &: S(\mathrm{M}_{4,4}(\mathbb{A})) \to \mathcal{A}(\tilde{H})/\mathcal{R}(\tilde{H}), \\ B_{-1} &: S(\mathrm{M}_{4,4}(\mathbb{A})) \to \mathcal{A}(\tilde{H}), \\ C_0 &: S(\mathrm{M}_{4,4}(\mathbb{A})) \to \mathcal{A}(\tilde{H}), \end{aligned}$$

be the $\tilde{H}_1(\mathbb{A})$-equivariant maps defined by

$$\begin{aligned} E^{(4,4)}(s,[\varphi]) &= \sum_{d=-1}^{\infty} (s-\tfrac{1}{2})^d A_d(\varphi), \\ I^{(4,2)}(s,\varphi) &= \sum_{d=-2}^{\infty} (s-\tfrac{3}{2})^d B_d(\varphi), \\ I^{(4,1)}(s,\mathrm{pr}(\varphi)) &= \sum_{d=-1}^{\infty} (s-1)^d C_d(\varphi), \end{aligned}$$

for $\varphi \in S(\mathrm{M}_{4,4}(\mathbb{A}))$. Here $\mathrm{pr} : S(\mathrm{M}_{4,4}(\mathbb{A})) \to S(\mathrm{M}_{2,4}(\mathbb{A}))$ is the $\tilde{H}_1(\mathbb{A})$-equivariant map defined by

$$\mathrm{pr}(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} = \int_{\mathbb{A}^4} \varphi \begin{pmatrix} u \\ x \\ 0 \\ y \end{pmatrix} \mathrm{d}u$$

for $x, y \in \mathbb{A}^4$, where $\mathrm{d}u$ is the Tamagawa measure on $\mathbb{A}^4$.

**Proposition 7.5.** *We have*

$$B_{-1}(\varphi) \equiv A_0(\varphi) + \mathfrak{D}^{-2}\rho\xi(4)^{-1}C_0(\varphi) \mod \mathcal{R}(\tilde{H})$$

*for* $\varphi \in S(\mathrm{M}_{4,4}(\mathbb{A}))^o$.

**Proof.** Let $\varphi^o \in S(\mathrm{M}_{4,4}(\mathbb{A}))$ be the spherical Schwartz function. Then $\mathrm{pr}(\varphi^o) \in S(\mathrm{M}_{2,4}(\mathbb{A}))$ is also the spherical Schwartz function. By (7.2), we have

$$A_0(\varphi^o) = \mathfrak{D}^{-4} E_0^{(4,4)}(\tfrac{1}{2}).$$

By Lemma 7.4 and Lemma B.6 in Appendix B, we have

$$B_{-1}(\varphi^o) \equiv \mathfrak{D}^{-4}\rho^{-1}\xi(2)^{-1}\xi(3)\xi(4)E_{-1}^{(4,2)}(\tfrac{3}{2}) \mod \mathcal{R}(\tilde{H}).$$

By Lemma 7.4 and Lemma B.7 in Appendix B, we have

$$C_0(\varphi^o) \equiv \mathfrak{D}^{-2}\rho^{-1}\xi(4)E_0^{(4,1)}(1) \mod \mathcal{R}(\tilde{H}).$$

Hence we have

$$\mathfrak{D}^4 B_{-1}(\varphi^o) \equiv \mathfrak{D}^4 A_0(\varphi^o) + \mathfrak{D}^2\rho\xi(4)^{-1}C_0(\varphi^o) \mod \mathcal{R}(\tilde{H})$$

by Proposition B.8 in Appendix B. This completes the proof. □

### 7.6. The doubling method

Let $\mathcal{B}_\sigma : V_\sigma \otimes \bar{V}_\sigma \to \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\sigma(f_1, f_2) = \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} f_1(h)\overline{f_2(h)}\,\mathrm{d}h$$

for $f_1, f_2 \in V_\sigma$. Here $Z_H$ is the identity component of the centre of $H$ and $\mathrm{d}h$ is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$. Put

$$\mathfrak{v} = \mathrm{vol}(Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

We fix a decomposition $\mathcal{B}_\sigma = \prod_v \mathcal{B}_{\sigma_v}$, where $\mathcal{B}_{\sigma_v} : \sigma_v \otimes \bar{\sigma}_v \to \mathbb{C}$ is a pairing.

For a holomorphic section $\Phi$ of $\boldsymbol{I}^{(4,4)}(s)$ and $f_1, f_2 \in V_\sigma$, the zeta integral of Piatetski-Shapiro and Rallis [**44**] and [**12**, § 6.2] is given by

$$Z(s, \Phi, f_1, f_2) = \int_{Z_{\tilde{H}}(\mathbb{A})\boldsymbol{H}(F)\backslash \boldsymbol{H}(\mathbb{A})} E^{(4,4)}(\iota(h_1, h_2); s, \Phi) f_1(h_1)\overline{f_2(h_2)}\,\mathrm{d}\boldsymbol{h}.$$

Here $Z_{\tilde{H}}$ is the identity component of the centre of $\tilde{H}$ and $\mathrm{d}\boldsymbol{h}$ is the Tamagawa measure on $Z_{\tilde{H}}(\mathbb{A})\backslash \boldsymbol{H}(\mathbb{A})$. Note that $\mathrm{vol}(Z_{\tilde{H}}(\mathbb{A})\boldsymbol{H}(F)\backslash \boldsymbol{H}(\mathbb{A})) = \mathfrak{v}$. For each place $v$ of $F$, let

$$Z_v(s, \Phi_v, f_{1,v}, f_{2,v}) = \int_{H_{1,v}} \Phi_v(\iota(h_{1,v}, 1), s)\mathcal{B}_{\sigma_v}(\sigma_v(h_{1,v})f_{1,v}, f_{2,v})\,\mathrm{d}h_{1,v}.$$

**Lemma 7.6.** *For a holomorphic section $\Phi = \bigotimes_v \Phi_v$ of $\boldsymbol{I}^{(4,4)}(s)$ and $f_1 = \bigotimes_v f_{1,v}, f_2 = \bigotimes_v f_{2,v} \in V_\sigma$, we have*

$$Z(s, \Phi, f_1, f_2) = \frac{L^S(s + \frac{1}{2}, \sigma, \mathrm{std})}{\zeta^S(2s+1)\zeta^S(2s+3)} \prod_{v \in S} Z_v(s, \Phi_v, f_{1,v}, f_{2,v}).$$

**Proof.** The assertion follows from the doubling method of [**44**] and [**12**, § 6.2]. □

### 7.7. Local zeta integrals

Let $\boldsymbol{I}_v^{(4,4)}(s) = \mathrm{Ind}_{P_{4,4,v}}^{\tilde{H}_v}(\delta_{P_{4,4,v}}^{s/3})$ denote the degenerate principal series representation of $\tilde{H}_v$.

**Lemma 7.7.** *For a holomorphic section $\Phi_v$ of $\boldsymbol{I}_v^{(4,4)}(s)$ and $f_{1,v}, f_{2,v} \in \sigma_v$, the integral $Z_v(s, \Phi_v, f_{1,v}, f_{2,v})$ is absolutely convergent at $s = \frac{1}{2}$.*

**Proof.** By [**3**, Proposition 3.3], [**41**, Appendix], and [**38**], there exist $\varphi_v, \varphi_v' \in S(\mathrm{M}_{4,4}(F_v))$ such that

$$\Phi_v(h_v, \tfrac{1}{2}) = [\varphi_v](h_v, \tfrac{1}{2}) + [\varphi_v'](h_v, \tfrac{1}{2})\,\mathrm{sgn}(h_v)$$

for $h_v \in \tilde{H}_v$. Hence we may assume that $\Phi_v = [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})]$ with $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$. But then we have

$$\Phi_v(\iota(h_{1,v}, 1), \tfrac{1}{2}) = \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v})$$

for $h_{1,v} \in H_{1,v}$. By [**39**, Theorem 3.2], the function $h_{1,v} \mapsto \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v})$ belongs to $L^{1+\varepsilon}(H_{1,v})$ for any $\varepsilon > 0$. This yields the lemma. □

For $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$, we have

$$Z_v(\tfrac{1}{2}, [\delta(\varphi_{1,v} \otimes \bar{\varphi}_{2,v})], f_{1,v}, f_{2,v}) \\ = \int_{H_{1,v}} \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v}) \mathcal{B}_{\sigma_v}(\sigma_v(h_{1,v}) f_{1,v}, f_{2,v}) \, \mathrm{d}h_{1,v}.$$

Let $S(\mathrm{M}_{4,4}(F_v))^o$ be the subspace of $S(\mathrm{M}_{4,4}(F_v))$ generated by $\boldsymbol{\omega}(h_{1,v})\varphi_v^o$ for all $h_{1,v} \in \tilde{H}_{1,v}$.

**Lemma 7.8.** *There exist $\varphi_v \in S(\mathrm{M}_{4,4}(F_v))^o$ and $f_{1,v}, f_{2,v} \in \sigma_v$ such that*

$$Z_v(\tfrac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}) \neq 0.$$

**Proof.** We fix a place $v$ of $F$ and suppress it from the notation. By [**32**], there exist $\Phi \in \boldsymbol{I}^{(4,4)}(\frac{1}{2})$ and $f_1, f_2 \in \sigma$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$. Let $R$ (respectively $R_0$) be the image of the equivariant map $S(\mathrm{M}_{4,4}(F)) \to \boldsymbol{I}^{(4,4)}(\frac{1}{2})$ (respectively $S(\mathrm{M}_{2,4}(F)) \to \boldsymbol{I}^{(4,4)}(-\frac{1}{2})$). By [**3**, Proposition 3.3], [**41**, Appendix], and [**38**], we have

$$\boldsymbol{I}^{(4,4)}(\tfrac{1}{2}) = R + R \otimes \mathrm{sgn}, \qquad \boldsymbol{I}^{(4,4)}(\tfrac{1}{2})/R \cong R_0 \otimes \mathrm{sgn}.$$

Moreover, $R$ is generated by a $\boldsymbol{K}$-invariant element of $\boldsymbol{I}^{(4,4)}(\frac{1}{2})$. It suffices to show that there exist $\Phi \in R$ and $f_1, f_2 \in \sigma$ such that $Z(\frac{1}{2}, \Phi, f_1, f_2) \neq 0$.

We assume that $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R$ and $f_1, f_2 \in \sigma$. If $\sigma \otimes \mathrm{sgn} \cong \sigma$, then we have $Z(\frac{1}{2}, \Phi, f_1, f_2) = 0$ for all $\Phi \in R \otimes \mathrm{sgn}$ and $f_1, f_2 \in \sigma$ and hence a contradiction. If $\sigma \otimes \mathrm{sgn} \not\cong \sigma$, then $Z(\frac{1}{2}, \Phi, f_1, f_2)$ defines a non-zero equivariant map

$$(R_0 \otimes \mathrm{sgn}) \otimes (\sigma \boxtimes \bar{\sigma}) \to \mathbb{C}.$$

As in [**14**, Proposition 3.1], this shows that the theta lift of $\sigma \otimes \mathrm{sgn}$ to $\mathrm{GL}_2(F)^+$ is non-zero. By [**8**, Lemmas 4.1 and 5.4], [**43**], and [**1**], we have

$$\sigma|_{D^\times(K) \times F^\times} \cong (\sigma \otimes \mathrm{sgn})|_{D^\times(K) \times F^\times} \cong \varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$$

for some irreducible admissible representation $\varsigma$ of $\mathrm{GL}_2(F)$ with central character $\omega_\varsigma$. Thus, $\sigma \cong (\varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F})^-$ for a distinguished representation $\varsigma_K^D \boxtimes \omega_\varsigma \omega_{K/F}$ of $\mathrm{GSO}(V)(F)$. This contradicts the assumption on $\sigma$. □

## 7.8. The Rallis inner product formula

**Lemma 7.9.** *For $\mathcal{F} \in \mathcal{R}(\tilde{H})$ and $f_1, f_2 \in V_\sigma$, we have*

$$\int_{Z_{\tilde{H}}(\mathbb{A}) \boldsymbol{H}(F) \backslash \boldsymbol{H}(\mathbb{A})} \mathcal{F}(\iota(h_1, h_2)) f_1(h_1) \overline{f_2(h_2)} \, \mathrm{d}\boldsymbol{h} = 0.$$

**Proof.** The assertion follows from Lemmas 7.1, 7.6, and 7.7. □

**Lemma 7.10.** *For $\varphi \in S(\mathrm{M}_{4,4}(\mathbb{A}))$ and $f_1, f_2 \in V_\sigma$, we have*

$$\int_{Z_{\tilde{H}}(\mathbb{A}) \boldsymbol{H}(F) \backslash \boldsymbol{H}(\mathbb{A})} C_0(\iota(h_1, h_2); \varphi) f_1(h_1) \overline{f_2(h_2)} \, \mathrm{d}\boldsymbol{h} = 0.$$

**Proof.** Let $W'$ be a two-dimensional symplectic space over $F$ and set $G' = \mathrm{GSp}(W') \cong \mathrm{GL}_2$. Let $W' = X' \oplus Y'$ be a complete polarization and set

$$\mathbb{W}' = W' \otimes V, \qquad \mathbb{X}' = X' \otimes V, \qquad \mathbb{Y}' = Y' \otimes V.$$

Then $\mathbb{W}'$ is a symplectic space over $F$ and $\mathbb{W}' = \mathbb{X}' \oplus \mathbb{Y}'$ is a complete polarization. Let $\omega'$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}'(\mathbb{A}))$ on the space $V_{\omega'} = S(\mathbb{X}'(\mathbb{A}))$ with respect to $\psi$. We may regard $\omega'$ as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W'))(\mathbb{A})$. By [**40**, § 2], there exists a natural isomorphism

$$\delta' : V_{\omega'} \otimes \bar{V}_{\omega'} \to S(\mathrm{M}_{2,4}(\mathbb{A})).$$

Let $\vartheta(\sigma)$ be the theta lift of $\sigma$ to $G'(\mathbb{A})$. Then $\vartheta(\sigma) = 0$. Indeed, it is easy to see that $\vartheta(\sigma)$ is cuspidal, and if $\vartheta(\sigma) \neq 0$, then $\sigma_v \otimes \mathrm{sgn} \ncong \sigma_v$ for all $v$ by the local unramified theta correspondence and the strong multiplicity one theorem. This contradicts the assumption on $\sigma$. For $\varphi'_1, \varphi'_2 \in V_{\omega'}$ and $f_1, f_2 \in V_\sigma$, we have

$$\begin{aligned}\int_{Z_{\tilde{H}}(\mathbb{A})\boldsymbol{H}(F)\backslash \boldsymbol{H}(\mathbb{A})} & I^{(4,1)}(\iota(h_1,h_2); s, \delta'(\varphi'_1 \otimes \bar{\varphi}'_2)) \cdot f_1(h_1)\overline{f_2(h_2)}\, \mathrm{d}\boldsymbol{h} \\ &= \frac{1}{\kappa} \int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g';\varphi'_1, f_1)\overline{\theta(g';\varphi'_2, f_2)}\mathcal{E}(g';s)\, \mathrm{d}g'.\end{aligned}$$

(See also the proof of Lemma 7.11 below.) Since $\vartheta(\sigma) = 0$, this integral is zero. This completes the proof. □

Let $\mathbb{A}^{\times,+} = \nu(H(\mathbb{A}))$, $F^{\times,+} = F^\times \cap \mathbb{A}^{\times,+}$ and $\mathcal{C} = \mathbb{A}^{\times,2}F^{\times,+}\backslash \mathbb{A}^{\times,+}$. The similitude characters induce isomorphisms

$$Z_G(\mathbb{A})G_1(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+ \cong \mathcal{C}, \qquad Z_H(\mathbb{A})H_1(\mathbb{A})H(F)\backslash H(\mathbb{A}) \cong \mathcal{C}.$$

Fix cross-sections $c \mapsto g_c$ and $c \mapsto h_c$ of $G(\mathbb{A})^+ \to \mathcal{C}$ and $H(\mathbb{A}) \to \mathcal{C}$, respectively. Let $\mathrm{d}g$ be the Tamagawa measure on $Z_G(\mathbb{A})\backslash G(\mathbb{A})$ and note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})) = 2$.

**Lemma 7.11.** *Let* $\varphi = \bigotimes_v \varphi_v \in S(\mathrm{M}_{4,4}(\mathbb{A}))^o$ *and* $f_1 = \bigotimes_v f_{1,v}, f_2 = \bigotimes_v f_{2,v} \in V_\sigma$. *We write* $\varphi = \sum_i \delta(\varphi_{1,i} \otimes \bar{\varphi}_{2,i})$, *where* $\varphi_{1,i}, \varphi_{2,i} \in V_\omega$. *Then we have*

$$\sum_i \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta(g;\varphi_{1,i}, f_1)\overline{\theta(g;\varphi_{2,i}, f_2)}\, \mathrm{d}g = \frac{L^S(1,\sigma,\mathrm{std})}{\zeta^S(2)\zeta^S(4)} \prod_{v \in S} Z_v(\tfrac{1}{2}, [\varphi_v], f_{1,v}, f_{2,v}).$$

**Proof.** By Lemma 7.9, $Z(s,[\varphi], f_1, f_2)$ is holomorphic at $s = \frac{1}{2}$. By Proposition 7.5 and Lemmas 7.9 and 7.10, we have

$$Z(\tfrac{1}{2}, [\varphi], f_1, f_2) = \int_{Z_{\tilde{H}}(\mathbb{A})\boldsymbol{H}(F)\backslash \boldsymbol{H}(\mathbb{A})} B_{-1}(\iota(h_1,h_2);\varphi) f_1(h_1)\overline{f_2(h_2)}\, \mathrm{d}\boldsymbol{h}.$$

This integral is equal to the residue at $s = \frac{3}{2}$ of

$$\mathfrak{v}\int_{\mathcal{C}}\int_{H_1(F)\backslash H_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})} I^{(4,2)}(\iota(h_1h_c,h_2h_c);s,\varphi)\cdot f_1(h_1h_c)\overline{f_2(h_2h_c)}\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}c$$

$$= \frac{\mathfrak{v}}{\kappa p(s)}\int_{\mathcal{C}}\int_{H_1(F)\backslash H_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})}\int_{G_1(F)\backslash G_1(\mathbb{A})} \theta(\iota(h_1h_c,h_2h_c),g_1g_c;z\varphi)$$
$$\times\,\mathcal{E}(g_1g_c;s)f_1(h_1h_c)\overline{f_2(h_2h_c)}\,\mathrm{d}g_1\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}c$$

$$= \frac{\mathfrak{v}}{\kappa}\int_{\mathcal{C}}\int_{G_1(F)\backslash G_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(\iota(h_1h_c,h_2h_c),g_1g_c;\varphi)$$
$$\times\,\mathcal{E}(g_1g_c;s)f_1(h_1h_c)\overline{f_2(h_2h_c)}\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}g_1\,\mathrm{d}c.$$

Here $\mathrm{d}c$ is the Haar measure on $\mathcal{C}$ such that $\mathrm{vol}(\mathcal{C}) = 1$ and $\mathrm{d}h_1$, $\mathrm{d}h_2$ are the Tamagawa measures on $H_1(\mathbb{A})$. Hence $Z(\frac{1}{2},[\varphi],f_1,f_2)$ is equal to

$$\mathfrak{v}\sum_i\int_{\mathcal{C}}\int_{G_1(F)\backslash G_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})}\int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(h_1h_c,g_1g_c;\varphi_{1,i})\cdot\overline{\theta(h_2h_c,g_1g_c;\varphi_{2,i})}$$
$$\times\, f_1(h_1h_c)\overline{f_2(h_2h_c)}\,\mathrm{d}h_1\,\mathrm{d}h_2\,\mathrm{d}g_1\,\mathrm{d}c$$
$$= \sum_i\int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} \theta(g;\varphi_{1,i},f_1)\overline{\theta(g;\varphi_{2,i},f_2)}\,\mathrm{d}g.$$

Note that $\mathrm{vol}(Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+) = \mathfrak{v}$. Since the supports of $\theta(\varphi_{1,i},f_1)$ and $\theta(\varphi_{2,i},f_2)$ are contained in $G(F)G(\mathbb{A})^+$, we have

$$\int_{Z_G(\mathbb{A})G(F)^+\backslash G(\mathbb{A})^+} \theta(g;\varphi_{1,i},f_1)\overline{\theta(g;\varphi_{2,i},f_2)}\,\mathrm{d}g$$
$$= \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta(g;\varphi_{1,i},f_1)\overline{\theta(g;\varphi_{2,i},f_2)}\,\mathrm{d}g.$$

This completes the proof. □

By Lemmas 7.1, 7.8, and 7.11, we obtain the following lemma.

**Lemma 7.12.** *We have*

$$V_{\theta(\sigma)} \neq 0.$$

Let $\mathcal{B}_\pi : V_\pi \otimes \bar{V}_\pi \to \mathbb{C}$ be the Petersson pairing given by

$$\mathcal{B}_\pi(\phi_1,\phi_2) = \int_{Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2(g)}\,\mathrm{d}g$$

for $\phi_1,\phi_2 \in V_\pi$. For each place $v$ of $F$, we define an equivariant map

$$\mathcal{Z}_v^\sharp : (\omega_v \boxtimes \bar{\omega}_v)\otimes(\sigma_v\boxtimes\bar{\sigma}_v) \to \mathbb{C}$$

by

$$\mathcal{Z}_v^\sharp(\varphi_{1,v},\varphi_{2,v};f_{1,v},f_{2,v}) = \zeta_v(2)\zeta_v(4)L_v(1,\sigma_v,\mathrm{std})^{-1}Z_v(\tfrac{1}{2},[\delta(\varphi_{1,v}\otimes\bar{\varphi}_{2,v})],f_{1,v},f_{2,v})$$

for $\varphi_{1,v}, \varphi_{2,v} \in S(\mathbb{X}_v)$ and $f_{1,v}, f_{2,v} \in \sigma_v$. By Lemma 7.8, $\mathcal{Z}_v^\sharp \neq 0$. By Lemmas 5.6 and 5.7, there exists a non-zero $G_v^+$-invariant pairing $\mathcal{B}^\sharp_{\theta(\sigma_v)} : \theta(\sigma_v) \otimes \overline{\theta(\sigma_v)} \to \mathbb{C}$ such that

$$\mathcal{Z}_v^\sharp = \mathcal{B}^\sharp_{\theta(\sigma_v)} \circ (\theta_v \otimes \bar{\theta}_v).$$

We extend $\mathcal{B}^\sharp_{\theta(\sigma_v)}$ uniquely to a $G_v$-invariant pairing $\mathcal{B}^\sharp_{\pi_v} : \pi_v \otimes \bar{\pi}_v \to \mathbb{C}$ such that $\mathcal{B}^\sharp_{\pi_v}|_{\theta(\sigma_v)\otimes\overline{\theta(\sigma_v)}} = \mathcal{B}^\sharp_{\theta(\sigma_v)}$. By Lemma 7.11, we obtain the following proposition.

**Proposition 7.13.** *We have*

$$\mathcal{B}_\pi = \frac{L(1, \sigma, \mathrm{std})}{\zeta(2)\zeta(4)} \prod_v \mathcal{B}^\sharp_{\pi_v}.$$

## 8. Tamagawa measures

Let $F$ be a totally real number field and $E$ a totally real étale quadratic algebra over $F$. Let $W_0$ be a two-dimensional symplectic space over $E$ and $V$ a four-dimensional quadratic space over $F$. Set

$$G' = \{g' \in \mathrm{R}_{E/F}(\mathrm{GSp}(W_0)) \mid \nu(g') \in \mathbb{G}_m\}, \qquad G'_1 = \mathrm{R}_{E/F}(\mathrm{Sp}(W_0)) \cong \mathrm{R}_{E/F}(\mathrm{SL}_2),$$
$$H = \mathrm{GO}(V), \qquad H_1 = \mathrm{O}(V).$$

Let $Z_{G'}$ and $Z_H$ be the identity component of the centre of $G'$ and $H$, respectively. We have isogenies

$$\mathrm{pr} : G'_1 \to Z_{G'}\backslash G', \qquad \mathrm{pr} : H_1 \to Z_H\backslash H.$$

Let $\omega_{G'}$ and $\omega_H$ be non-zero invariant differential forms of top degree on $Z_{G'}\backslash G'$ and $Z_H\backslash H$ over $F$, respectively. Then $\omega_{G'_1} = \mathrm{pr}^*(\omega_{G'})$ and $\omega_{H_1} = \mathrm{pr}^*(\omega_H)$ are also non-zero invariant differential forms of top degree on $G'_1$ and $H_1$ over $F$, respectively. Fix a non-trivial additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/F$. For each place $v$ of $F$, let $\mathrm{d}x_v$ be the self-dual Haar measure on $F_v$ with respect to $\psi_v$. Let $\mathrm{d}g'_{1,v}$, $\mathrm{d}g'_v$, $\mathrm{d}h^0_{1,v}$, $\mathrm{d}h^0_v$ be the Haar measures on

$$G'_{1,v}, \qquad Z_{G',v}\backslash G'_v, \qquad H_{1,v}, \qquad Z_{H,v}\backslash H_v,$$

determined by $\mathrm{d}x_v$ and $\omega_{G'_1}$, $\omega_{G'}$, $\omega_{H_1}$, $\omega_H$, respectively. Let $\mathrm{d}h_{1,v} = \frac{1}{2}\,\mathrm{d}h^0_{1,v}$ and $\mathrm{d}h_v = \frac{1}{2}\,\mathrm{d}h^0_v$. Let $F_v^{\times,+} = \nu(H_v)$. Then we have

$$\int_{Z_{G',v}\backslash G'_v} f(g'_v)\,\mathrm{d}g'_v = \frac{1}{2} \sum_{c \in F_v^{\times,2}\backslash F_v^\times} \int_{G'_{1,v}} f(g'_{1,v} g'_c)\,\mathrm{d}g'_{1,v},$$
$$\int_{Z_{H,v}\backslash H_v} \phi(h_v)\,\mathrm{d}h_v = \frac{1}{2} \sum_{c \in F_v^{\times,2}\backslash F_v^{\times,+}} \int_{H_{1,v}} \phi(h_{1,v} h_c)\,\mathrm{d}h_{1,v},$$

for $f \in L^1(Z_{G',v}\backslash G'_v)$ and $\phi \in L^1(Z_{H,v}\backslash H_v)$. Here $g'_c \in Z_{G',v}\backslash G'_v$ with $\nu(g'_c) = c$ and $h_c \in Z_{H,v}\backslash H_v$ with $\nu(h_c) = c$. By definition, the product measures

$$\prod_v \mathrm{d}g'_{1,v}, \qquad \prod_v \mathrm{d}g'_v, \qquad \prod_v \mathrm{d}h_{1,v}, \qquad \prod_v \mathrm{d}h_v$$

are the Tamagawa measures on

$$G_1'(\mathbb{A}), \qquad Z_{G'}(\mathbb{A})\backslash G'(\mathbb{A}), \qquad H_1(\mathbb{A}), \qquad Z_H(\mathbb{A})\backslash H(\mathbb{A}),$$

respectively. We remark that the products of these measures are convergent, although the volumes of hyperspecial maximal compact subgroups are not 1 for almost all $v$.

## 9. The explicit local seesaw identity

We retain the notation of §§ 1 and 8. We fix a place $v$ of $F$ and suppress it from the notation. Let $\mathbb{W} = V \otimes_F W = \mathrm{R}_{E/F}(W_0 \otimes_E V_E)$, where $V_E = V \otimes_F E$. Let $\omega$ be the Weil representation of $\mathrm{Mp}(\mathbb{W})$ with respect to $\psi$. We may regard $\omega$ as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(F)$ or that of $\mathrm{G}(\mathrm{Sp}(W_0) \times \mathrm{O}(V_E))(E)$. In particular, we have a seesaw diagram:

$$\begin{array}{ccc} G = \mathrm{GSp}(W)(F) & & H' = \mathrm{GO}(V_E)(E)' \\ \Big| & \times & \Big| \\ G' = \mathrm{GSp}(W_0)(E)' & & H = \mathrm{GO}(V)(F) \end{array}$$

The goal of this section is to establish an *explicit* seesaw identity for this seesaw diagram.

Recall that $\sigma$ is an irreducible unitary admissible representation of $H$ and $\tau$ is an irreducible unitary admissible representation of $\mathrm{GSp}(W_0)(E)$ containing an irreducible subrepresentation $\pi'$ of $G'$. Let $\mathcal{B}_\sigma : \sigma \otimes \bar{\sigma} \to \mathbb{C}$ and $\mathcal{B}_\tau : \tau \otimes \bar{\tau} \to \mathbb{C}$ be pairings. Let $\mathcal{B}^\flat_{\pi'} : \pi' \otimes \bar{\pi}' \to \mathbb{C}$ be the pairing given by $\mathcal{B}^\flat_{\pi'} = \mathcal{B}_\tau|_{\pi' \otimes \bar{\pi}'}$. Let $\mathcal{C} = F^{\times,2}\backslash F^{\times,+}$. Put

$$\begin{aligned}\mathcal{Q}&(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2) \\ &= \sum_{c \in \mathcal{C}} \int_{G_1'} \int_{H_1} \mathcal{B}_\omega(\omega(g_1' g_c', h_1 h_c)\varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(h_1 h_c)\phi_1, \phi_2) \mathcal{B}_\tau(\tau(g_1' g_c')f_1, f_2) \, \mathrm{d}h_1 \, \mathrm{d}g_1'\end{aligned}$$

for $\varphi_1, \varphi_2 \in \omega$, $\phi_1, \phi_2 \in \sigma$, and $f_1, f_2 \in \pi'$. Here $\mathrm{d}h_1$ and $\mathrm{d}g_1'$ are the Haar measures on $H_1$ and $G_1'$ given in § 8, respectively.

**Lemma 9.1.** *The integral* $\mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2)$ *is absolutely convergent.*

**Proof.** We only consider the case $E = K = F \times F$ and $D = \mathrm{M}_{2,2}(F)$; the other cases are similar. It suffices to show that the integral

$$\begin{aligned}\int_{(F^+)^4} & \mathcal{B}_\omega(\omega(t(a_1, a_2), t(a_3, a_4))\varphi_1, \varphi_2) \mathcal{B}_\sigma(\sigma(t(a_3, a_4))\phi_1, \phi_2) \\ & \times \mathcal{B}_\tau(\tau(t(a_1, a_2))f_1, f_2) |a_1^{-2} a_2^{-2} a_3^{-2} a_4^{-2}| \, \mathrm{d}^\times a_1 \, \mathrm{d}^\times a_2 \, \mathrm{d}^\times a_3 \, \mathrm{d}^\times a_4\end{aligned}$$

is absolutely convergent. Here $F^+ = \{a \in F^\times \mid |a| \leqslant 1\}$,

$$t(a, a') = \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & a'^{-1} \end{pmatrix} \right),$$

and we regard $t(a, a')$ as an element of $G'$ or that of $H$. By the Kim–Shahidi estimate [**26**, **28**], there exist $\frac{7}{9} < \lambda_1, \lambda_2, \lambda_3, \lambda_4 \leqslant 1$ such that

$$|\mathcal{B}_\tau(\tau(t(a_1, a_2))f_1, f_2)| \leqslant C|a_1|^{\lambda_1}|a_2|^{\lambda_2},$$
$$|\mathcal{B}_\sigma(\sigma(t(a_3, a_4))\phi_1, \phi_2)| \leqslant C|a_3|^{\lambda_3}|a_4|^{\lambda_4},$$

for $a_1, a_2, a_3, a_4 \in F^+$ with some constant $C$. We define a function $\Upsilon$ on $F^\times$ by

$$\Upsilon(a) = \begin{cases} 1 & \text{if } a \in F^+, \\ |a|^{-1} & \text{otherwise.} \end{cases}$$

For $\phi, \phi' \in S(F)$, we have

$$\left| \int_F \phi(ax)\overline{\phi'(x)}\, \mathrm{d}x \right| \leqslant C'\Upsilon(a)$$

for $a \in F^\times$ with some constant $C'$. Realizing the Weil representation $\omega$ on $S(\mathrm{M}_{2,2}(F)^2)$, we have

$$\omega(t(a_1, a_2), t(a_3, a_4))\varphi_1(x, y) = |a_1^2 a_2^2|\varphi_1\left( \begin{pmatrix} a_1a_3^{-1}a_4x_1 & a_1a_3^{-1}a_4^{-1}x_2 \\ a_1a_3a_4x_3 & a_1a_3a_4^{-1}x_4 \end{pmatrix}, \begin{pmatrix} a_2a_3^{-1}a_4y_1 & a_2a_3^{-1}a_4^{-1}y_2 \\ a_2a_3a_4y_3 & a_2a_3a_4^{-1}y_4 \end{pmatrix} \right)$$

for

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}.$$

Hence we have

$$\begin{aligned} &|\mathcal{B}_\omega(\omega(t(a_1, a_2), t(a_3, a_4))\varphi_1, \varphi_2)| \\ &\quad\leqslant C''|a_1^2a_2^2|\Upsilon(a_1a_3^{-1}a_4)\Upsilon(a_1a_3^{-1}a_4^{-1})\Upsilon(a_1a_3a_4)\Upsilon(a_1a_3a_4^{-1}) \\ &\qquad\times \Upsilon(a_2a_3^{-1}a_4)\Upsilon(a_2a_3^{-1}a_4^{-1})\Upsilon(a_2a_3a_4)\Upsilon(a_2a_3a_4^{-1}) \end{aligned}$$

with some constant $C''$. By symmetry, it suffices to show that the integral

$$\int_{|a_1|\leqslant|a_2|\leqslant 1} \int_{|a_3|\leqslant|a_4|\leqslant 1} \Upsilon(a_1a_3^{-1}a_4)\Upsilon(a_1a_3^{-1}a_4^{-1})\Upsilon(a_2a_3^{-1}a_4)\Upsilon(a_2a_3^{-1}a_4^{-1}) \times |a_1^{\lambda_1}a_2^{\lambda_2}a_3^{\lambda_3-2}a_4^{\lambda_4-2}|\,\mathrm{d}^\times a_1\,\mathrm{d}^\times a_2\,\mathrm{d}^\times a_3\,\mathrm{d}^\times a_4$$

is convergent. We change variables $b_1 = a_3a_4$ and $b_2 = a_3a_4^{-1}$. Let $\mu_1 = \frac{1}{2}(\lambda_3 + \lambda_4)$ and $\mu_2 = \frac{1}{2}(\lambda_3 - \lambda_4)$. Note that $\frac{7}{9} < \mu_1 \leqslant 1$ and $|\mu_2| < \frac{1}{9}$. It remains to show that the integral

$$\int_{|a_1|\leqslant|a_2|\leqslant 1} \int_{|b_1|\leqslant|b_2|\leqslant 1} \Upsilon(a_1b_1^{-1})\Upsilon(a_1b_2^{-1})\Upsilon(a_2b_1^{-1})\Upsilon(a_2b_2^{-1}) \times |a_1^{\lambda_1}a_2^{\lambda_2}b_1^{\mu_1-2}b_2^{\mu_2}|\,\mathrm{d}^\times a_1\,\mathrm{d}^\times a_2\,\mathrm{d}^\times b_1\,\mathrm{d}^\times b_2$$

is convergent. The integrand is equal to

$$\begin{array}{ll} |a_1^{\lambda_1} a_2^{\lambda_2} b_1^{\mu_1-2} b_2^{\mu_2}| & \text{if } |a_1| \leqslant |a_2| \leqslant |b_1| \leqslant |b_2| \leqslant 1, \\ |a_1^{\lambda_1} a_2^{\lambda_2-1} b_1^{\mu_1-1} b_2^{\mu_2}| & \text{if } |a_1| \leqslant |b_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ |a_1^{\lambda_1} a_2^{\lambda_2-2} b_1^{\mu_1-1} b_2^{\mu_2+1}| & \text{if } |a_1| \leqslant |b_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ |a_1^{\lambda_1-1} a_2^{\lambda_2-1} b_1^{\mu_1} b_2^{\mu_2}| & \text{if } |b_1| \leqslant |a_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ |a_1^{\lambda_1-1} a_2^{\lambda_2-2} b_1^{\mu_1} b_2^{\mu_2+1}| & \text{if } |b_1| \leqslant |a_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ |a_1^{\lambda_1-2} a_2^{\lambda_2-2} b_1^{\mu_1} b_2^{\mu_2+2}| & \text{if } |b_1| \leqslant |b_2| \leqslant |a_1| \leqslant |a_2| \leqslant 1. \end{array}$$

We change variables:

$$\begin{array}{ll} a_1 = t_1t_2t_3t_4,\ a_2 = t_1t_2t_3,\ b_1 = t_1t_2,\ b_2 = t_1 & \text{if } |a_1| \leqslant |a_2| \leqslant |b_1| \leqslant |b_2| \leqslant 1, \\ a_1 = t_1t_2t_3t_4,\ b_1 = t_1t_2t_3,\ a_2 = t_1t_2,\ b_2 = t_1 & \text{if } |a_1| \leqslant |b_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ a_1 = t_1t_2t_3t_4,\ b_1 = t_1t_2t_3,\ b_2 = t_1t_2,\ a_2 = t_1 & \text{if } |a_1| \leqslant |b_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ b_1 = t_1t_2t_3t_4,\ a_1 = t_1t_2t_3,\ a_2 = t_1t_2,\ b_2 = t_1 & \text{if } |b_1| \leqslant |a_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ b_1 = t_1t_2t_3t_4,\ a_1 = t_1t_2t_3,\ b_2 = t_1t_2,\ a_2 = t_1 & \text{if } |b_1| \leqslant |a_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ b_1 = t_1t_2t_3t_4,\ b_2 = t_1t_2t_3,\ a_1 = t_1t_2,\ a_2 = t_1 & \text{if } |b_1| \leqslant |b_2| \leqslant |a_1| \leqslant |a_2| \leqslant 1, \end{array}$$

where $t_1, t_2, t_3, t_4 \in F^+$. Then the integrand is equal to

$$\begin{array}{ll} |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\lambda_2} t_4^{\lambda_1}| & \text{if } |a_1| \leqslant |a_2| \leqslant |b_1| \leqslant |b_2| \leqslant 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\mu_1-1} t_4^{\lambda_1}| & \text{if } |a_1| \leqslant |b_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\lambda_1+\mu_1-1} t_4^{\lambda_1}| & \text{if } |a_1| \leqslant |b_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\lambda_2+\mu_1-2} t_3^{\lambda_1+\mu_1-1} t_4^{\mu_1}| & \text{if } |b_1| \leqslant |a_1| \leqslant |a_2| \leqslant |b_2| \leqslant 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\lambda_1+\mu_1-1} t_4^{\mu_1}| & \text{if } |b_1| \leqslant |a_1| \leqslant |b_2| \leqslant |a_2| \leqslant 1, \\ |t_1^{\lambda_1+\lambda_2+\mu_1+\mu_2-2} t_2^{\lambda_1+\mu_1+\mu_2} t_3^{\mu_1+\mu_2+2} t_4^{\mu_1}| & \text{if } |b_1| \leqslant |b_2| \leqslant |a_1| \leqslant |a_2| \leqslant 1. \end{array}$$

It is easy to check that the integral in each of the six ranges is convergent. This completes the proof of the lemma. □

Let $\theta(\tau)$ be the theta lift of $\tau$ to $H(E)$. Let $\theta : \omega \otimes \tau \to \theta(\tau)$ be an equivariant surjective map. Let $\theta(\pi')$ be the image of $\omega \otimes \pi'$ in $\theta(\tau)$. Let

$$\mathcal{T} : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') \to (\theta(\sigma) \boxtimes \overline{\theta(\sigma)}) \otimes (\pi' \boxtimes \bar{\pi}'),$$
$$\mathcal{T}' : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}') \to (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\pi') \boxtimes \overline{\theta(\pi')})$$

be equivariant surjective maps induced by

$$\theta \otimes \bar{\theta} : (\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \to \theta(\sigma) \boxtimes \overline{\theta(\sigma)},$$
$$\theta \otimes \bar{\theta} : (\omega \boxtimes \bar{\omega}) \otimes (\pi' \boxtimes \bar{\pi}') \to \theta(\pi') \boxtimes \overline{\theta(\pi')},$$

respectively. Let $\mathcal{B}^\sharp_\pi : \pi \otimes \bar{\pi} \to \mathbb{C}$ and $\mathcal{B}^\sharp_{\theta(\tau)} : \theta(\tau) \otimes \overline{\theta(\tau)} \to \mathbb{C}$ be the pairings given by

$$\mathcal{B}^\sharp_\pi(\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2)) = \frac{\zeta_{E\otimes K}(2)}{L(1, \tau, \mathrm{Ad} \otimes \omega_{E\otimes K/E})} \int_{H_1} \mathcal{B}_\omega(\omega(h_1)\varphi_1, \varphi_2)\mathcal{B}_\sigma(\sigma(h_1)\phi_1, \phi_2)\, \mathrm{d}h_1 \quad (9.1)$$

and

$$\mathcal{B}^\sharp_{\theta(\tau)}(\theta(\varphi_1, f_1), \theta(\varphi_2, f_2)) = \frac{\zeta(2)\zeta(4)}{L(1, \sigma, \mathrm{std})} \int_{G'_1} \mathcal{B}_\omega(\omega(g'_1)\varphi_1, \varphi_2)\mathcal{B}_\tau(\tau(g'_1)f_1, f_2)\, \mathrm{d}g'_1 \quad (9.2)$$

for $\varphi_1, \varphi_2 \in \omega$, $\phi_1, \phi_2 \in \sigma$, and $f_1, f_2 \in \tau$, respectively. Here $\mathrm{d}h_1$ and $\mathrm{d}g'_1$ are the Haar measures on $H_1$ and $G'_1$ given in § 8, respectively. We remark that the normalizing factors in the front of the integrals are introduced to ensure that (9.1) and (9.2) are 1 for unramified data if we take Haar measures such that the volumes of hyperspecial maximal compact subgroups are 1. (Notice that such measures do not agree with the measures given in § 8.) We define a $G' \times G'$-invariant functional

$$\mathcal{P}^\sharp : (\pi \boxtimes \bar{\pi}) \otimes (\pi' \boxtimes \bar{\pi}') \to \mathbb{C}$$

by

$$\mathcal{P}^\sharp(\phi_1, \phi_2; f_1, f_2) = \frac{1}{\zeta(2)\zeta(4)} \frac{L(1, \sigma, \mathrm{std})L(1, \sigma, \mathrm{Ad})L(1, \tau, \mathrm{Ad})}{L(\frac{1}{2}, \sigma \times \theta(\tau))} \times \int_{Z_{G'}\backslash G'} \mathcal{B}^\sharp_\pi(\pi(g')\phi_1, \phi_2)\mathcal{B}^\flat_{\pi'}(\pi'(g')f_1, f_2)\, \mathrm{d}g' \quad (9.3)$$

for $\phi_1, \phi_2 \in \pi$ and $f_1, f_2 \in \pi'$. Here $\mathrm{d}g'$ is the Haar measure on $Z_{G'}\backslash G'$ given in § 8. By Lemma 9.1, this integral is absolutely convergent. We define an $H \times H$-invariant functional

$$\mathcal{I}^\sharp : (\sigma \boxtimes \bar{\sigma}) \otimes (\theta(\tau) \boxtimes \overline{\theta(\tau)}) \to \mathbb{C}$$

by

$$\mathcal{I}^\sharp(\phi_1, \phi_2; f_1, f_2) = \frac{1}{\zeta_{E\otimes K}(2)} \frac{L(1, \sigma, \mathrm{Ad})L(1, \tau_K, \mathrm{Ad})}{L(\frac{1}{2}, \sigma \times \theta(\tau))} \times \int_{Z_H\backslash H} \mathcal{B}_\sigma(\sigma(h)\phi_1, \phi_2)\mathcal{B}^\sharp_{\theta(\tau)}(\theta(\tau)(h)f_1, f_2)\, \mathrm{d}h \quad (9.4)$$

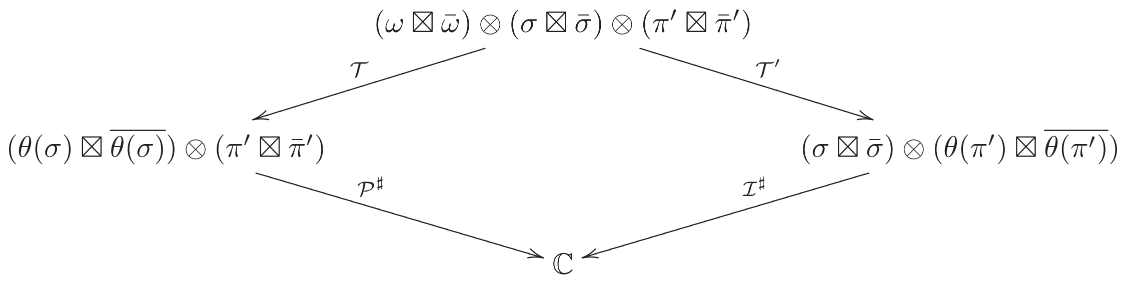
for $\phi_1, \phi_2 \in \sigma$ and $f_1, f_2 \in \theta(\tau)$. Here $\mathrm{d}h$ is the Haar measure on $Z_H\backslash H$ given in § 8.

Now we have the following lemma, which should be thought of as an explicit local seesaw identity.

**Lemma 9.2.** *We have*

$$\mathcal{P}^\sharp \circ \mathcal{T} = \mathcal{I}^\sharp \circ \mathcal{T}'$$

*as functionals on* $(\omega \boxtimes \bar{\omega}) \otimes (\sigma \boxtimes \bar{\sigma}) \otimes (\pi' \boxtimes \bar{\pi}')$. *Namely, the diagram*




*is commutative.*

**Proof.** Given the absolute convergence of Lemma 9.1, the commutativity of the diagram is essentially a consequence of Fubini's theorem. More precisely, we have

$$\begin{aligned}\zeta_{E\otimes K}(2)L(1, \tau, \mathrm{Ad} \otimes \omega_{E\otimes K/E})^{-1} \mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2)\\ = \sum_{c\in\mathcal{C}} \int_{H_1} \mathcal{B}_\sigma(\sigma(h_1h_c)\phi_1, \phi_2)\mathcal{B}^\sharp_{\theta(\tau)}(\theta(\tau)(h_1h_c)\theta(\varphi_1, f_1), \theta(\varphi_2, f_2))\, \mathrm{d}h_1\\ = 2 \int_{Z_H\backslash H} \mathcal{B}_\sigma(\sigma(h)\phi_1, \phi_2)\mathcal{B}^\sharp_{\theta(\tau)}(\theta(\tau)(h)\theta(\varphi_1, f_1), \theta(\varphi_2, f_2))\, \mathrm{d}h.\end{aligned}$$

Also, we have

$$\begin{aligned}\zeta(2)\zeta(4)L(1, \sigma, \mathrm{std})^{-1} \mathcal{Q}(\varphi_1, \varphi_2; \phi_1, \phi_2; f_1, f_2)\\ = \sum_{c\in\mathcal{C}} \int_{G'_1} \mathcal{B}^\sharp_\pi(\pi(g'_1g'_c)\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2))\mathcal{B}^\flat_{\pi'}(\pi'(g'_1g'_c)f_1, f_2)\, \mathrm{d}g'_1\\ = 2 \int_{Z_{G'}\backslash G'^+} \mathcal{B}^\sharp_\pi(\pi(g')\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2))\mathcal{B}^\flat_{\pi'}(\pi'(g')f_1, f_2)\, \mathrm{d}g',\end{aligned}$$

where $G'^+ = \{g' \in G' \mid \nu(g') \in \nu(H)\}$. By Lemma 5.2, this integral is equal to

$$2 \int_{Z_{G'}\backslash G'} \mathcal{B}^\sharp_\pi(\pi(g')\theta(\varphi_1, \phi_1), \theta(\varphi_2, \phi_2))\mathcal{B}^\flat_{\pi'}(\pi'(g')f_1, f_2)\, \mathrm{d}g'.$$

This completes the proof. □

## 10. Proof of Theorem 1.1

We retain the notation of §§ 1 and 8. Let $\mathbb{W} = V \otimes_F W = \mathrm{R}_{E/F}(W_0 \otimes_E V_E)$, where $V_E = V \otimes_F E$. Let $\omega$ be the Weil representation of $\mathrm{Mp}(\mathbb{W}(\mathbb{A}))$ on the space $V_\omega$ with respect to $\psi$. We may regard $\omega$ as a representation of $\mathrm{G}(\mathrm{O}(V) \times \mathrm{Sp}(W))(\mathbb{A})$ or that of $\mathrm{G}(\mathrm{Sp}(W_0) \times \mathrm{O}(V_E))(\mathbb{A}_E)$.

We fix a subspace $\tilde{V}_{\pi'}$ of $V^1_\tau$ such that the restriction to $G'(\mathbb{A})$ as functions induces an isomorphism

$$\tilde{V}_{\pi'} \cong V_{\pi'}$$

as representations of $G'(\mathbb{A})$. Let $\mathcal{B}_\tau : V_\tau \otimes \bar{V}_\tau \to \mathbb{C}$ and $\mathcal{B}_{\pi'} : V_{\pi'} \otimes \bar{V}_{\pi'} \to \mathbb{C}$ be the Petersson pairings. We fix a decomposition $\mathcal{B}_\tau = \prod_v \mathcal{B}_{\tau_v}$, where $\mathcal{B}_{\tau_v} : \tau_v \otimes \bar{\tau}_v \to \mathbb{C}$ is a pairing. Let $\mathcal{B}^\flat_{\pi'_v} : \pi'_v \otimes \bar{\pi}'_v \to \mathbb{C}$ be the pairing given by $\mathcal{B}^\flat_{\pi'_v} = \mathcal{B}_{\tau_v}|_{\pi'_v \otimes \bar{\pi}'_v}$.

**Lemma 10.1.** *We have*

$$\mathcal{B}_{\pi'} = 2^{\beta'} |\mathfrak{X}_\tau| \prod_v \mathcal{B}^\flat_{\pi'_v}.$$

*Here*

$$\beta' = \begin{cases} -1 & \text{if } E = F \times F, \\ 0 & \text{if } E \text{ is a quadratic extension of } F. \end{cases}$$

**Proof.** By [**17**, Remark 4.20], we have

$$\mathcal{B}_{\pi'}(\tilde{f}_1|_{G'(\mathbb{A})}, \tilde{f}_2|_{G'(\mathbb{A})}) = |\mathfrak{X}_\tau| \frac{\mathrm{vol}(Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A}))}{\mathrm{vol}(Z_{\tilde{G}'}(\mathbb{A})\tilde{G}'(F)\backslash \tilde{G}'(\mathbb{A}))} \mathcal{B}_\tau(\tilde{f}_1, \tilde{f}_2)$$

for $\tilde{f}_1, \tilde{f}_2 \in \tilde{V}_{\pi'}$. Here $Z_{G'}$ and $Z_{\tilde{G}'}$ are the identity components of the centres of $G'$ and $\tilde{G}'$, respectively. Since $\mathrm{vol}(Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})) = 2$ and

$$\mathrm{vol}(Z_{\tilde{G}'}(\mathbb{A})\tilde{G}'(F)\backslash \tilde{G}'(\mathbb{A})) = \begin{cases} 4 & \text{if } E = F \times F, \\ 2 & \text{if } E \text{ is a quadratic extension of } F, \end{cases}$$

the assertion follows. □

Let $\theta(\tau)$ be the theta lift of $\tau$ to $H(\mathbb{A}_E)$ on the space $V_{\theta(\tau)}$. Let

$$\theta : V_\omega \otimes V_\tau \to V_{\theta(\tau)}$$

and

$$\theta_v : \omega_v \otimes \tau_v \to \theta(\tau_v)$$

be equivariant surjective maps such that $\theta = \bigotimes_v \theta_v$. Let $V_{\theta(\pi')}$ and $\theta(\pi'_v)$ be the images of $V_\omega \otimes \tilde{V}_{\pi'}$ and $\omega_v \otimes \pi'_v$ in $V_{\theta(\tau)}$ and $\theta(\tau_v)$, respectively. Let

$$\begin{aligned} \mathcal{T} : (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) &\to (V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \\ &\to (V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}) \otimes (V_{\pi'} \boxtimes \bar{V}_{\pi'}), \end{aligned}$$

$$\mathcal{T}' : (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \to (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (V_{\theta(\pi')} \boxtimes \bar{V}_{\theta(\pi')}),$$

be equivariant surjective maps induced by

$$\begin{aligned} \theta \otimes \bar{\theta} &: (V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \to V_{\theta(\sigma)} \boxtimes \bar{V}_{\theta(\sigma)}, \\ \theta \otimes \bar{\theta} &: (V_\omega \boxtimes \bar{V}_\omega) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'}) \to V_{\theta(\pi')} \boxtimes \bar{V}_{\theta(\pi')}, \end{aligned}$$

respectively. Let

$$\begin{aligned} \mathcal{T}_v &: (\omega_v \boxtimes \bar{\omega}_v) \otimes (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \to (\theta(\sigma_v) \boxtimes \overline{\theta(\sigma_v)}) \otimes (\pi'_v \boxtimes \bar{\pi}'_v), \\ \mathcal{T}'_v &: (\omega_v \boxtimes \bar{\omega}_v) \otimes (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\pi'_v \boxtimes \bar{\pi}'_v) \to (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\pi'_v) \boxtimes \overline{\theta(\pi'_v)}), \end{aligned}$$

be equivariant surjective maps such that $\mathcal{T} = \bigotimes_v \mathcal{T}_v$ and $\mathcal{T}' = \bigotimes_v \mathcal{T}'_v$. We define an $H(\mathbb{A}) \times H(\mathbb{A})$-invariant functional

$$\mathcal{I} : (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (V_{\theta(\tau)} \boxtimes \bar{V}_{\theta(\tau)}) \to \mathbb{C}$$

by

$$\mathcal{I}(\phi_1, \phi_2; f_1, f_2) = \left( \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi_1(h) f_1(h)\, \mathrm{d}h \right) \left( \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \overline{\phi_2(h) f_2(h)}\, \mathrm{d}h \right)$$

for $\phi_1, \phi_2 \in V_\sigma$ and $f_1, f_2 \in V_{\theta(\tau)}$. Here $\mathrm{d}h = \prod_v \mathrm{d}h_v$ is the Tamagawa measure on $Z_H(\mathbb{A})\backslash H(\mathbb{A})$. Put

$$\mathfrak{v} = \mathrm{vol}(Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})) = \begin{cases} 2 & \text{if } K = F \times F, \\ 1 & \text{if } K \text{ is a quadratic extension of } F. \end{cases}$$

Now we have the global seesaw identity.

**Lemma 10.2.** *We have*

$$\mathcal{P} \circ \mathcal{T} = \mathcal{I} \circ \mathcal{T}'$$

*as functionals on* $(V_\omega \boxtimes \bar{V}_\omega) \otimes (V_\sigma \boxtimes \bar{V}_\sigma) \otimes (\tilde{V}_{\pi'} \boxtimes \bar{\tilde{V}}_{\pi'})$.

**Proof.** Let $\mathcal{C} = \mathbb{A}^{\times,2} F^{\times,+} \backslash \mathbb{A}^{\times,+}$. Put

$$\mathfrak{Q}(\varphi, \phi, \tilde{f}) = \int_{\mathcal{C}} \int_{G'_1(F)\backslash G'_1(\mathbb{A})} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g'_1 g'_c, h_1 h_c; \varphi) \phi(h_1 h_c) \tilde{f}(g'_1 g'_c)\, \mathrm{d}h_1\, \mathrm{d}g'_1\, \mathrm{d}c$$

for $\varphi \in V_\omega$, $\phi \in V_\sigma$, and $\tilde{f} \in \tilde{V}_{\pi'}$. Here $\mathrm{d}c$ is the Haar measure on $\mathcal{C}$ such that $\mathrm{vol}(\mathcal{C}) = 1$, $\mathrm{d}g'_1$ is the Tamagawa measure on $G'_1(\mathbb{A})$, and $\mathrm{d}h_1$ is the Tamagawa measure on $H_1(\mathbb{A})$. We have

$$\begin{aligned} \mathfrak{Q}(\varphi, \phi, \tilde{f}) &= \int_{\mathcal{C}} \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(h_1 h_c; \varphi, \tilde{f}) \phi(h_1 h_c)\, \mathrm{d}h_1\, \mathrm{d}c \\ &= \frac{1}{\mathfrak{v}} \int_{Z_H(\mathbb{A})H(F)\backslash H(\mathbb{A})} \theta(h; \varphi, \tilde{f}) \phi(h)\, \mathrm{d}h. \end{aligned}$$

Also, we have

$$\begin{aligned} \mathfrak{Q}(\varphi, \phi, \tilde{f}) &= \int_{\mathcal{C}} \int_{G'_1(F)\backslash G'_1(\mathbb{A})} \theta(g'_1 g'_c; \varphi, \phi) f(g'_1 g'_c)\, \mathrm{d}g'_1\, \mathrm{d}c \\ &= \frac{1}{\mathfrak{v}} \int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g'; \varphi, \phi) f(g')\, \mathrm{d}g', \end{aligned}$$

where $f = \tilde{f}|_{G'(\mathbb{A})}$. Note that $\mathrm{vol}(Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+) = \mathfrak{v}$. Since the support of $\theta(\varphi, \phi)$ is contained in $G(F)G(\mathbb{A})^+$, we have

$$\int_{Z_{G'}(\mathbb{A})G'(F)^+\backslash G'(\mathbb{A})^+} \theta(g'; \varphi, \phi) f(g')\, \mathrm{d}g' = \int_{Z_{G'}(\mathbb{A})G'(F)\backslash G'(\mathbb{A})} \theta(g'; \varphi, \phi) f(g')\, \mathrm{d}g'.$$

This completes the proof. $\square$

Let

$$\mathcal{B}_\sigma : V_\sigma \otimes \bar{V}_\sigma \to \mathbb{C} \quad \text{and} \quad \mathcal{B}_{\theta(\tau)} : V_{\theta(\tau)} \otimes \bar{V}_{\theta(\tau)} \to \mathbb{C}$$

be the Petersson pairings. We fix decompositions $\mathcal{B}_\sigma = \prod_v \mathcal{B}_{\sigma_v}$ and $\mathcal{B}_{\theta(\tau)} = \prod_v \mathcal{B}_{\theta(\tau_v)}$, where $\mathcal{B}_{\sigma_v} : \sigma_v \otimes \bar{\sigma}_v \to \mathbb{C}$ and $\mathcal{B}_{\theta(\tau_v)} : \theta(\tau_v) \otimes \overline{\theta(\tau_v)} \to \mathbb{C}$ are pairings. For each place $v$ of $F$, we define an $H_v \times H_v$-invariant functional

$$\mathcal{I}_v^\natural : (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\tau_v) \boxtimes \overline{\theta(\tau_v)}) \to \mathbb{C}$$

by

$$\mathcal{I}_v^\natural(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) = \int_{Z_{H,v} \backslash H_v} \mathcal{B}_{\sigma_v}(\sigma_v(h_v)\phi_{1,v}, \phi_{2,v}) \mathcal{B}_{\theta(\tau_v)}(\theta(\tau_v)(h_v) f_{1,v}, f_{2,v}) \, \mathrm{d}h_v$$

for $\phi_{1,v}, \phi_{2,v} \in \sigma_v$ and $f_{1,v}, f_{2,v} \in \theta(\tau_v)$. By Proposition 3.2, we have

$$\mathcal{I} = 2^c \zeta_{E \otimes K}(2) \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad}) L(1, \tau_K, \mathrm{Ad})} \prod_v \mathcal{I}_v.$$

Here

$$c = \begin{cases} -4 & \text{if } E = K = F \times F, \\ -1 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ -3 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ -2 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K \end{cases}$$

and

$$\mathcal{I}_v = \frac{1}{\zeta_{E_v \otimes K_v}(2)} \frac{L_v(1, \sigma_v, \mathrm{Ad}) L_v(1, \tau_{K,v}, \mathrm{Ad})}{L_v(\frac{1}{2}, \sigma_v \times \theta(\tau_v))} \mathcal{I}_v^\natural.$$

By Lemma 10.2, we have

$$\mathcal{P} \circ \mathcal{T} = 2^c \zeta_{E \otimes K}(2) \frac{L(\frac{1}{2}, \sigma \times \theta(\tau))}{L(1, \sigma, \mathrm{Ad}) L(1, \tau_K, \mathrm{Ad})} \prod_v \mathcal{I}_v \circ \mathcal{T}_v'.$$

Let $\mathcal{B}_{\theta(\tau_v)}^\sharp : \theta(\tau_v) \otimes \overline{\theta(\tau_v)} \to \mathbb{C}$ be the pairing defined by (9.2) and

$$\mathcal{I}_v^\sharp : (\sigma_v \boxtimes \bar{\sigma}_v) \otimes (\theta(\tau_v) \boxtimes \overline{\theta(\tau_v)}) \to \mathbb{C}$$

the $H_v \times H_v$-invariant functional defined by (9.4). By Proposition 6.10, we have

$$\mathcal{B}_{\theta(\tau)} = 2^{\beta''} \frac{L(1, \tau, \mathrm{Ad} \otimes \omega_{E \otimes K/E})}{\zeta_{E \otimes K}(2)} \prod_v \mathcal{B}_{\theta(\tau_v)}^\sharp.$$

Here

$$\beta'' = \begin{cases} 0 & \text{if } E = K = F \times F, \\ -2 & \text{if } E = F \times F \text{ and } K \text{ is a quadratic extension of } F, \\ 0 & \text{if } E \text{ is a quadratic extension of } F \text{ and } K = F \times F, \\ 0 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E = K, \\ -1 & \text{if } E \text{ and } K \text{ are quadratic extensions of } F \text{ and } E \neq K. \end{cases}$$

Hence we have

$$\mathcal{P}\circ\mathcal{T} = 2^{\beta''+c}\frac{L(\frac{1}{2},\sigma\times\theta(\tau))}{L(1,\sigma,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\prod_v \mathcal{I}_v^\sharp\circ\mathcal{T}'_v.$$

Let $\mathcal{B}^\sharp_{\pi_v}:\pi_v\otimes\bar{\pi}_v\to\mathbb{C}$ be the pairing defined by (9.1) and let

$$\mathcal{P}_v^\sharp : (\pi_v\boxtimes\bar{\pi}_v)\otimes(\pi'_v\boxtimes\bar{\pi}'_v)\to\mathbb{C}$$

be the $G'_v\times G'_v$-invariant functional defined by (9.3). By Lemma 9.2, we have

$$\mathcal{P}\circ\mathcal{T} = 2^{\beta''+c}\frac{L(\frac{1}{2},\sigma\times\theta(\tau))}{L(1,\sigma,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\prod_v \mathcal{P}_v^\sharp\circ\mathcal{T}_v.$$

By Proposition 7.13 and Lemma 10.1, we have

$$\mathcal{P}\circ\mathcal{T} = \frac{2^{\beta''+c}}{2^{\beta'}|\mathfrak{X}_\tau|}\frac{L(\frac{1}{2},\sigma\times\theta(\tau))}{L(1,\sigma,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\frac{\zeta(2)\zeta(4)}{L(1,\sigma,\mathrm{std})}\prod_v \mathcal{P}_v\circ\mathcal{T}_v.$$

This shows the desired identity of invariant functionals on the image of $\mathcal{T}$, which is the subspace $(V_{\theta(\sigma)}\boxtimes\bar{V}_{\theta(\sigma)})\otimes(V_{\pi'}\boxtimes\bar{V}_{\pi'})$ of $(V_\pi\boxtimes\bar{V}_\pi)\otimes(V_{\pi'}\boxtimes\bar{V}_{\pi'})$. Since $G(\mathbb{A}) = G'(\mathbb{A})G(\mathbb{A})^+$ and $V_\pi$ is generated by $V_{\theta(\sigma)}$, we have

$$\mathcal{P} = \frac{\zeta(2)\zeta(4)}{2^{\beta'-\beta''-c}|\mathfrak{X}_\tau|}\frac{L(\frac{1}{2},\sigma\times\theta(\tau))}{L(1,\sigma,\mathrm{std})L(1,\sigma,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\prod_v \mathcal{P}_v.$$

This completes the proof of Theorem 1.1.

## Appendix A. Explicit local theta correspondence for $\mathbf{GO}(V)\times\mathbf{GSp}_4$

In this appendix, let $F$ be a non-archimedean local field of characteristic zero and residual characteristic $p$. We consider an arbitrary four-dimensional quadratic space $V$ and a four-dimensional symplectic space $W$ over $F$. The discriminant algebra of $V$ is an étale quadratic $F$-algebra $K$. Let $\nu_V$ and $\nu_W$ denote the similitude characters of the corresponding similitude groups $\mathrm{GO}(V)$ and $\mathrm{GSp}(W)$, respectively. The image of $\nu_V$ is the subgroup $\mathrm{N}_{K/F}(K^\times)\subset F^\times$ and we set

$$\mathrm{GSp}(W)^+ = \{g\in\mathrm{GSp}(W)\mid \nu_W(g)\in\mathrm{N}_{K/F}(K^\times)\}.$$

For a non-trivial additive character $\psi$ of $F$, one has an induced Weil representation $\Omega_\psi$ of the similitude dual pair $\mathrm{GO}(V)\times\mathrm{GSp}(W)^+$. If $K$ is split, then $\Omega_\psi$ is independent of $\psi$, and in general, $\Omega_\psi$ depends only on the orbit of $\psi$ under the natural action of $\mathrm{N}_{K/F}(K^\times)$. Thus, when $K$ is a field, there are two such induced Weil representations. Henceforth, we shall fix an orbit of $\psi$ and write $\Omega$ for $\Omega_\psi$, suppressing $\psi$ from the notation. However, because we shall be dealing with various different dual pairs, we shall sometimes write $\Omega_{V,W}$ to indicate the particular dual pair we are considering.

If $\sigma$ is an irreducible representation of $\mathrm{GO}(V)$, then the maximal $\sigma$-isotypic quotient of $\Omega$ is of the form $\sigma\boxtimes\Theta(\sigma)$ for some smooth representation $\Theta(\sigma)$ (the big theta lift of $\sigma$)

of $\mathrm{GSp}(W)^+$. One knows that $\Theta(\sigma)$ is a representation of finite length. We let $\theta(\sigma)$ (the small theta lift of $\sigma$) denote the maximal semisimple quotient of $\Theta(\sigma)$. Moreover, we set

$$\tilde{\Theta}(\sigma) = \mathrm{ind}_{\mathrm{GSp}(W)^+}^{\mathrm{GSp}(W)}(\Theta(\sigma)) \quad \text{and} \quad \tilde{\theta}(\sigma) = \mathrm{ind}_{\mathrm{GSp}(W)^+}^{\mathrm{GSp}(W)}(\theta(\sigma)).$$

Again, we shall sometimes write $\Theta_{V,W}(\sigma)$, $\theta_{V,W}(\sigma)$ and so on if there is a need to be specific about the dual pair we are considering.

We should remark that the definition of the induced Weil representation $\Omega$ used in this appendix is as given in [**9**], which is slightly different from that given in [**47**] (and used in the main body of this paper). Moreover, our definition of $\Theta(\sigma)$ is slightly different from that given in the main body of this paper. The upshot is that these two changes cancel each other, so that the local theta correspondence defined in this appendix agrees with that defined in the main body of this paper. In particular, the local theta correspondence preserves central characters.

The main result of this appendix is the following.

**Theorem A.1.** *Let* $\sigma$ *be an irreducible representation of* $\mathrm{GO}(V)$*.*

(i) $\Theta(\sigma)$ *is multiplicity free (possibly zero) and has a unique irreducible quotient* $\theta(\sigma)$*.*

(ii) $\theta(\sigma)$ *can be precisely determined in terms of* $\sigma$ *(in terms of the local Langlands correspondence for* $\mathrm{GSp}_4$ *established in [**9**]).*

For the purpose of this paper, we really only need part (i) of the theorem, but we find it useful to include part (ii) as well. Part (ii) of the theorem will be stated in full details in the respective cases later on. In order to do that, we first need to introduce some notation for representations of $\mathrm{GO}(V)$ and $\mathrm{GSp}(W)^+$.

### A.1. Principal series representations of $\mathbf{GSp_4}$

We have a Witt decomposition $W = Y^* \oplus Y$ with a two-dimensional isotropic space $Y$. We can write

$$Y^* = Fe_1 \oplus Fe_2 \quad \text{and} \quad Y = Ff_1 \oplus Ff_2$$

with $\langle e_i, f_j \rangle = \delta_{ij}$ and consider the decomposition $W = Fe_1 \oplus W' \oplus Ff_1$, where $W' = Fe_2 \oplus Ff_2$. Let $Q(Z) = L(Z)U(Z)$ be the parabolic subgroup stabilizing $Z = Ff_1$, so that

$$L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$$

and $U(Z)$ is a Heisenberg group:

$$1 \to \mathrm{Sym}^2 Z \to U(Z) \to \mathrm{Hom}(W', Z) \to 1.$$

This is typically called the Klingen or Heisenberg parabolic subgroup. An irreducible representation of $L(Z)$ is thus of the form $\chi \boxtimes \tau$ with a character $\chi$ of $F^\times$ and an irreducible representation $\tau$ of $\mathrm{GSp}(W') \cong \mathrm{GL}_2$. We denote the corresponding normalized induced representation by $I_{Q(Z)}(\chi, \tau)$. If $I_{Q(Z)}(\chi, \tau)$ is a standard module, then it has a unique irreducible quotient (the Langlands quotient), which we shall denote by $J_{Q(Z)}(\chi,\tau)$. The same notation applies to other principal series representations to be introduced later.

The module structure of $I_{Q(Z)}(\chi,\tau)$ is known by Sally and Tadić [**51**] and a convenient reference is [**50**]. In particular, we note the following.

**Lemma A.2.**

(a) *Suppose that $\tau$ is a supercuspidal representation of* $\mathrm{GL}_2$. *Then $I_{Q(Z)}(\chi,\tau)$ is reducible if and only if one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = \chi_0|\cdot|^{\pm 1}$ *with a non-trivial quadratic character $\chi_0$ such that $\tau\chi_0 = \tau$.*

*In case (i), $I_{Q(Z)}(1,\tau)$ is the direct sum of two irreducible quasi-tempered representations, exactly one of which is generic. In case (ii), assuming without loss of generality that $\chi = \chi_0|\cdot|$, one has a non-split short exact sequence:*

$$0 \to \mathrm{St}(\chi_0,\tau_0) \to I_{Q(Z)}(\chi_0|\cdot|,\tau_0|\cdot|^{-1/2}) \to \mathrm{Sp}(\chi_0,\tau_0) \to 0.$$

*Here $\mathrm{St}(\chi_0,\tau_0)$ is a generic discrete series representation and the Langlands quotient $\mathrm{Sp}(\chi_0,\tau_0)$ is non-generic.*

(b) *Suppose that $\tau$ is a twisted Steinberg representation of* $\mathrm{GL}_2$. *Then $I_{Q(Z)}(\chi,\tau)$ is reducible if and only if one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = |\cdot|^{\pm 2}$.

*In case (i), $I_{Q(Z)}(1,\mathrm{st}_\mu)$ is the direct sum of two irreducible quasi-tempered representations, exactly one of which is generic. In case (ii), $I_{Q(Z)}(|\cdot|^2,\mathrm{st}_\mu|\cdot|^{-1})$ has the twisted Steinberg representation $\mathrm{St}_{\mathrm{PGSp}_4}\cdot\mu$ as its unique irreducible submodule.*

(c) *There is a standard intertwining operator*

$$I_{Q(Z)}(\chi^{-1},\tau\chi) \to I_{Q(Z)}(\chi,\tau),$$

*which is an isomorphism if $I_{Q(Z)}(\chi,\tau)$ is irreducible. If $I_{Q(Z)}(\chi^{-1},\tau\chi)$ is a standard module, then the image of this operator is the unique irreducible submodule of $I_{Q(Z)}(\chi,\tau)$.*

Let $P(Y) = M(Y)N(Y)$ be the Siegel parabolic subgroup stabilizing $Y$, so that

$$M(Y) \cong \mathrm{GL}(Y)\times\mathbb{G}_m \quad\text{and}\quad N(Y)\cong \mathrm{Sym}^2 Y.$$

An irreducible representation of $M(Y)$ is thus of the form $\tau\boxtimes\mu$ with an irreducible representation $\tau$ of $\mathrm{GL}(Y)\cong\mathrm{GL}_2$ and a character $\mu$ of $F^\times$. We denote the corresponding normalized induced representation by $I_{P(Y)}(\tau,\mu)$. As before, the module structure of $I_{P(Y)}(\tau,\mu)$ is completely known by [**51**] and a convenient reference is [**50**]. In particular, we note the following.

**Lemma A.3.**

(a) *Suppose that* $\tau$ *is a supercuspidal representation of* $\mathrm{GL}_2$. *Then* $I_{P(Y)}(\tau, \mu)$ *is reducible if and only if* $\tau = \tau_0|\cdot|^{\pm 1/2}$ *with* $\tau_0$ *having trivial central character. In this case, one has a non-split short exact sequence:*

$$0 \to \mathrm{St}(\tau_0, \mu_0) \to I_{P(Y)}(\tau_0|\cdot|^{1/2}, \mu_0|\cdot|^{-1/2}) \to \mathrm{Sp}(\tau_0, \mu_0) \to 0.$$

*Here* $\mathrm{St}(\tau_0, \mu_0)$ *is a generic discrete series representation and the Langlands quotient* $\mathrm{Sp}(\tau_0, \mu_0)$ *is non-generic.*

(b) *Suppose that* $\tau$ *is a twisted Steinberg representation of* $\mathrm{GL}_2$. *Then* $I_{P(Y)}(\tau, \mu)$ *is reducible if and only if one of the following holds.*

(i) $\tau = \mathrm{st}\,|\cdot|^{\pm 1/2}$; *in this case,* $I_{P(Y)}(\mathrm{st}\,|\cdot|^{1/2}, \mu|\cdot|^{-1/2})$ *has a unique irreducible Langlands quotient and a unique irreducible quasi-tempered submodule, which is the unique generic constituent of* $I_{Q(Z)}(1, \mathrm{st}_\mu)$.

(ii) $\tau = \mathrm{st}_\chi|\cdot|^{\pm 1/2}$ *with a non-trivial quadratic character* $\chi$; *in this case,* $I_{P(Y)}(\mathrm{st}_\chi|\cdot|^{1/2}, \mu_0|\cdot|^{-1/2})$ *has a unique irreducible Langlands quotient and a unique irreducible submodule, which is a generic discrete series representation* $\mathrm{St}(\mathrm{st}_\chi, \mu_0)$. *Moreover,* $\mathrm{St}(\mathrm{st}_\chi, \chi\mu_0) = \mathrm{St}(\mathrm{st}_\chi, \mu_0)$.

(iii) $\tau = \mathrm{st}\,|\cdot|^{\pm 3/2}$; *in this case,* $I_{P(Y)}(\mathrm{st}\,|\cdot|^{3/2}, \mu|\cdot|^{-3/2})$ *has the twisted Steinberg representation* $\mathrm{St}_{\mathrm{PGSp}_4}\mu$ *as its unique irreducible submodule.*

(c) *There is a standard intertwining operator*

$$I_{P(Y)}(\tau, \mu) \to I_{P(Y)}(\tau^\vee, \mu\omega_\tau),$$

*which is an isomorphism if* $I_{P(Y)}(\tau, \mu)$ *is irreducible. If* $I_{P(Y)}(\tau, \mu)$ *is a standard module, then the image of this operator is the unique irreducible submodule of* $I_{P(Y)}(\tau^\vee, \mu\omega_\tau)$.

Finally, let $B = P(Y) \cap Q(Z) = TU$ be a Borel subgroup of $\mathrm{GSp}(W)$, so that

$$T \cong (\mathrm{GL}(Ff_1) \times \mathrm{GL}(Ff_2)) \times \mathbb{G}_m.$$

In particular, for characters $\chi_1$, $\chi_2$, and $\chi$ of $F^\times$, we let $I_B(\chi_1, \chi_2; \chi)$ denote the normalized induced representation. Again, we refer the reader to [**50**] for the reducibility points and module structure of $I_B(\chi_1, \chi_2; \chi)$. We simply note here that $I_B(\chi_1, \chi_2; \chi)$ is multiplicity free, and if $\chi_1$ and $\chi_2$ are unitary, then $I_B(\chi_1, \chi_2; \chi)$ is irreducible.

### A.2. Representations of GO($V$)

Now we come to representations of $\mathrm{GO}(V)$. We consider the various cases separately.

*K is split*

In this case, we have

$$V = (D, -\mathrm{N}_D),$$

where $D$ is a quaternion $F$-algebra (possibly split) with reduced norm $\mathrm{N}_D$. We have the identification

$$\mathrm{GSO}(V) \cong (D^\times \times D^\times)/\{(z, z^{-1}) \mid z \in F^\times\}$$

via

$$(g_1, g_2) : x \mapsto g_1 x \bar{g}_2.$$

Moreover, the main involution $x \mapsto \bar{x}$ on $D$ gives an order two element $\boldsymbol{t}$ of $\mathrm{O}(V)$ with determinant $-1$, so that $\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \langle \boldsymbol{t} \rangle$. The conjugation of $\boldsymbol{t}$ on $\mathrm{GSO}(V)$ is given by $(g_1, g_2) \mapsto (g_2, g_1)$. Thus, an irreducible representation of $\mathrm{GSO}(V)$ is of the form $\tau_1 \boxtimes \tau_2$ with an irreducible representation $\tau_i$ of $D^\times$ such that $\omega_{\tau_1} = \omega_{\tau_2}$. Moreover, the action of $\boldsymbol{t}$ sends $\tau_1 \boxtimes \tau_2$ to $\tau_2 \boxtimes \tau_1$.

In particular, if $\tau_1 = \tau_2 = \tau$, then there are two extensions of $\tau \boxtimes \tau$ to $\mathrm{GO}(V)$, which we denote by $(\tau \boxtimes \tau)^\pm$. To distinguish these two extensions, we note that exactly one of them participates in the theta correspondence with $\mathrm{GSp}(W') \cong \mathrm{GL}_2$, and we denote this distinguished extension by $(\tau \boxtimes \tau)^+$.

On the other hand, if $\tau_1 \neq \tau_2$, then

$$\mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau_1 \boxtimes \tau_2) = \mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau_2 \boxtimes \tau_1)$$

is irreducible, in which case we denote this irreducible representation by $(\tau_1 \boxtimes \tau_2)^+ = (\tau_1 \boxtimes \tau_2)^-$.

When $D$ is split, the quadratic space $V$ is split and we have a Witt decomposition $V = X \oplus X^*$ with a two-dimensional isotropic space $X$. Let $P(X) = M(X)N(X)$ be the parabolic subgroup stabilizing $X$, so that

$$M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m \quad \text{and} \quad N(X) \cong \wedge^2 X.$$

For an irreducible representation $\tau \boxtimes \chi$ of $\mathrm{GL}(X) \times \mathbb{G}_m \cong \mathrm{GL}_2 \times F^\times$, we let $I_{P(X)}(\tau, \chi)$ denote the normalized induced representation. The following lemma is easy to check.

**Lemma A.4.** *Under the identification* $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$, *we have*

$$\pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee \chi_1, \chi_2) = I_{P(X)}(\tau \chi_2^{-1}, \chi_2).$$

*K is a field*

In this case, we have two quadratic spaces

$$V^+ = \mathbb{H} \oplus V_K^+ \quad \text{and} \quad V^- = \mathbb{H} \oplus V_K^-,$$

where $\mathbb{H}$ is the hyperbolic plane and

$$V_K^+ = (K, \mathrm{N}_{K/F}) \quad \text{and} \quad V_K^- = (K, \delta \mathrm{N}_{K/F})$$

with $\delta \in F^\times \smallsetminus \mathrm{N}_{K/F}(K^\times)$. One can realize these quadratic spaces on the space $V$ of $2 \times 2$ Hermitian matrices with entries in $K$. The determinant map defines a quadratic form on $V$ and we have

$$V^+ = (V, -\det) \quad \text{and} \quad V^- = (V, -\delta \det).$$

The similitude groups of $V = V^\pm$ are isomorphic:

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \langle \boldsymbol{t} \rangle$$

with

$$\mathrm{GSO}(V) \cong (\mathrm{GL}_2(K) \times F^\times)/\{(z, \mathrm{N}_{K/F}(z)^{-1}) \mid z \in K^\times\}$$

acting via

$$(g, \lambda) : x \mapsto \lambda g x\, {}^{\mathrm{t}}g^c,$$

and $\boldsymbol{t} \in \mathrm{O}(V)$ with determinant $-1$ acting via

$$\boldsymbol{t} : x \mapsto x^c,$$

where $c$ is the non-trivial element of $\mathrm{Gal}(K/F)$. Without loss of generality, we shall henceforth fix

$$V = V^+.$$

The conjugation of $\boldsymbol{t}$ on $\mathrm{GSO}(V)$ is given by $(g, \lambda) \mapsto (g^c, \lambda)$. Moreover, we let

$$\mathrm{sgn} : \mathrm{GO}(V) \to \{\pm 1\}$$

be the unique non-trivial quadratic character of $\mathrm{GO}(V)$ trivial on $\mathrm{GSO}(V)$.

Thus, an irreducible representation of $\mathrm{GSO}(V)$ is of the form $\tau \boxtimes \chi$ with an irreducible representation $\tau$ of $\mathrm{GL}_2(K)$ and a character $\chi$ of $F^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}_{K/F}$. Such a representation is invariant under the action of $\boldsymbol{t}$ if and only if $\tau$ is obtained by base change from $\mathrm{GL}_2(F)$, in which case there are two extensions of $\tau \boxtimes \chi$ to $\mathrm{GO}(V)$, which we denote by $(\tau \boxtimes \chi)^\pm$. How can one distinguish between these two extensions of $\tau \boxtimes \chi$? As we now explain, one can do this using the Whittaker model when $\tau$ is generic.

More precisely, if $U_0$ is the unipotent radical of a $\boldsymbol{t}$-stable Borel subgroup of $\mathrm{GSO}(V)$, let $\psi_0$ be a generic character of $U_0$ which is fixed by the action of the outer automorphism $\boldsymbol{t}$. Then, if $\tau$ is invariant and generic, $\boldsymbol{t}$ acts on the one-dimensional space $\mathrm{Hom}_{U_0}((\tau \boxtimes \chi)^\pm, \mathbb{C}_{\psi_0})$ with $\boldsymbol{t}^2 = 1$. Then $(\tau \boxtimes \chi)^+$ is the extension of $\tau \boxtimes \chi$ such that $\boldsymbol{t}$ acts by $+1$ on $\mathrm{Hom}_{U_0}((\tau \boxtimes \chi)^+, \mathbb{C}_{\psi_0})$. Note that this characterization is independent of the choice of the generic character $\psi_0$ which is fixed by $\boldsymbol{t}$.

There is another way of specifying the two extensions of $\tau \boxtimes \chi$ in the invariant case, using their behaviour under the theta correspondence. Following Roberts [**49**], we distinguish two mutually exclusive scenarios in the invariant case.

**Invariant and distinguished representations:** these are the representations $\tau \boxtimes \chi$, where $\tau$ is the base change of an irreducible representation $\tau_F$ of $\mathrm{GL}_2(F)$ with central character $\chi \omega_{K/F}$. In this case, by [**49**, Theorem 3.4], one of the extensions $(\tau \boxtimes \chi)^+$ of

$\tau \boxtimes \chi$ to $\mathrm{GO}(V)$ participates in the theta correspondence with $\mathrm{GSp}(W')^+ \cong \mathrm{GL}_2^+$ (and hence with $\mathrm{GSp}_4^+$), whereas the other extension $(\tau \boxtimes \chi)^-$ does not participate in the theta correspondence with $\mathrm{GSp}_4^+$. When $\tau$ is generic, it follows from Corollary A.17 below that the extension $(\tau \boxtimes \chi)^+$ is the same as the one defined above using the Whittaker model.

**Invariant but not distinguished representations:** these are the remaining invariant representations $\tau \boxtimes \chi$. In this case, [**49**, Theorem 3.4] says that neither of the two extensions $(\tau \boxtimes \chi)^\pm$ participates in the theta correspondence with $\mathrm{GSp}(W')^+ \cong \mathrm{GL}_2^+$, but both participate in the theta correspondence with $\mathrm{GSp}_4^+$. Thus, the theta correspondence does not allow one to distinguish between the two extensions. When $\tau$ is generic, it follows from Corollary A.17 below that the theta lift of exactly one of the extensions, namely the extension $(\tau \boxtimes \chi)^+$ defined above, is a generic representation of $\mathrm{GSp}_4^+$.

We should remark that Roberts's definition of distinguished representations uses the existence of $\mathrm{SO}(2,1)$-invariant functionals. In [**48**], he showed that his definition agrees with the one above when the residual characteristic of $F$ is $p \neq 2$. It is not difficult to prove the same assertion for all $p$ by computing the theta correspondence for $\mathrm{GL}_2^+ \times \mathrm{GSO}(V)$ and the Whittaker modules of the induced Weil representations.

On the other hand, if $\tau \boxtimes \chi$ is not invariant, then

$$\mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau \boxtimes \chi) = \mathrm{ind}_{\mathrm{GSO}(V)}^{\mathrm{GO}(V)}(\tau^c \boxtimes \chi)$$

is irreducible, in which case we denote this irreducible representation by $(\tau \boxtimes \chi)^+ = (\tau \boxtimes \chi)^-$.

Now we describe principal series representations of $\mathrm{GO}(V)$. We have a Witt decomposition $V = J \oplus V_K \oplus J^*$ with an isotropic line $J$. Let $Q(J) = L(J)U(J)$ be the parabolic subgroup stabilizing $J$, so that

$$L(J) \cong \mathrm{GL}(J) \times \mathrm{GO}(V_K) \quad \text{and} \quad U(J) \cong \mathrm{Hom}(V_K, J).$$

We set $Q(J)^+ = Q(J) \cap \mathrm{GSO}(V)$.

Let $\chi \boxtimes \mu$ be an irreducible representation of $\mathrm{GL}(J) \times \mathrm{GSO}(V_K) \cong F^\times \times K^\times$. If $\mu$ is invariant under the Galois action, then $\mu$ has two extensions $\mu^\pm$ to $\mathrm{GO}(V_K)$, whereas if $\mu$ is not invariant under the Galois action, then we set

$$\mu^+ = \mu^- = \mathrm{ind}_{\mathrm{GSO}(V_K)}^{\mathrm{GO}(V_K)}(\mu).$$

We consider the normalized induced representations $I_{Q(J)}(\chi, \mu^\pm)$ and $I_{Q(J)^+}(\chi, \mu)$ of $\mathrm{GO}(V)$ and $\mathrm{GSO}(V)$, respectively. If we take $J$ to be the isotropic line spanned by the matrix $\mathrm{diag}(1,0) \in V$, then the following lemma is easy to check.

**Lemma A.5.** *Under the identification* $\mathrm{GSO}(V) \cong (\mathrm{GL}_2(K) \times F^\times)/K^\times$, *we have*

$$I_{Q(J)}(\chi, \mu^\pm) = (\pi((\chi \circ \mathrm{N}_{K/F})\mu, \mu^c) \boxtimes (\chi \cdot \mu|_{F^\times}))^\pm.$$

From this lemma, it is not difficult to deduce the following.

**Lemma A.6.**

(i) $I_{Q(J)}(\chi, \mu^{\pm})$ *is reducible if and only if one of the following holds:*

- $\chi \circ \mathrm{N}_{K/F} = \mu^c/\mu \cdot |\cdot|_K^{\pm 1}$*; in this case,* $I_{Q(J)^+}(\chi, \mu)$ *is reducible;*
- $\mu^c \neq \mu$ *and* $\chi = 1$ *or* $\omega_{K/F}$*; in this case,* $I_{Q(J)^+}(\chi, \mu)$ *is irreducible, but*

$$I_{Q(J)}(\chi, \mu^{\pm}) = \sigma \oplus \sigma \cdot \mathrm{sgn}$$

*for some irreducible representation* $\sigma$ *of* $\mathrm{GO}(V)$*.*

(ii) $I_{Q(J)^+}(\chi, \mu)$ *is invariant if and only if one of the following mutually exclusive conditions holds:*

- $\mu^c = \mu$*;*
- $\mu^c \neq \mu$ *and* $\chi = 1$ *or* $\omega_{K/F}$*.*

*In this case, it is distinguished unless* $\mu^c \neq \mu$ *and* $\chi = \omega_{K/F}$*.*

### A.3. Theta lifts from $\mathrm{GO}(V_K)$

Before coming to our main results, let us recall the theta lifts from $\mathrm{GO}(V_K)$ to $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ and $\mathrm{GSp}(W) \cong \mathrm{GSp}_4$. The following proposition is well known.

**Proposition A.7.** *Let* $\mu$ *be an irreducible representation of* $\mathrm{GSO}(V_K) \cong K^{\times}$*.*

(i) *If* $\mu$ *is not Galois invariant (so that* $\mu^+ = \mu^-$*), then*

$$\Theta(\mu^{\pm}) = \theta(\mu^{\pm})$$

*is a non-zero irreducible supercuspidal representation of* $\mathrm{GL}_2^+$ *such that*

$$\pi(\mu) := \mathrm{ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2}(\Theta(\mu^{\pm}))$$

*is irreducible supercuspidal. These are precisely the supercuspidal representations of* $\mathrm{GL}_2$ *which are dihedral with respect to* $K$*.*

(ii) *If* $\mu$ *is Galois invariant so that* $\mu = \mu_F \circ \mathrm{N}_{K/F}$ *for some* $\mu_F$*, then*

$$\Theta(\mu^+) = \theta(\mu^+)$$

*is a non-zero irreducible representation of* $\mathrm{GL}_2^+$ *such that*

$$\pi(\mu) := \mathrm{ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2}(\Theta(\mu^+)) = \pi(\mu_F, \mu_F\omega_{K/F}).$$

*Moreover,*

$$\Theta(\mu^-) = 0.$$

**Proposition A.8.** *Let $\mu$ be an irreducible representation of* $\mathrm{GSO}(V_K) \cong K^\times$.

(i) *If $\mu$ is not Galois invariant (so that $\mu^+ = \mu^-$), then*

$$\Theta(\mu^\pm) = \theta(\mu^\pm)$$

*is a non-zero irreducible representation of* $\mathrm{GSp}_4^+$ *such that*

$$\tilde{\Theta}(\mu^\pm) := \mathrm{ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4}(\Theta(\mu^\pm)) = J_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2}).$$

(ii) *If $\mu$ is Galois invariant so that $\mu = \mu_F \circ \mathrm{N}_{K/F}$ for some $\mu_F$, then*

$$\Theta(\mu^+) = \theta(\mu^+)$$

*is a non-zero irreducible representation of* $\mathrm{GSp}_4^+$ *such that*

$$\tilde{\Theta}(\mu^+) := \mathrm{ind}_{\mathrm{GSp}_4^+}^{\mathrm{GSp}_4}(\Theta(\mu^+))$$

*is the unique irreducible quotient of*

$$I_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2}) = I_B(\omega_{K/F}|\cdot|, \omega_{K/F}; \mu_F|\cdot|^{-1/2}).$$

*On the other hand,*

$$\Theta(\mu^-) = \theta(\mu^-)$$

*is a non-zero irreducible representation of* $\mathrm{GSp}_4^+$ *such that $\tilde{\Theta}(\mu^-)$ is the irreducible non-generic supercuspidal representation of* $\mathrm{GSp}_4$ *with $L$-parameter $(\mu_F \boxtimes S_2) \oplus (\mu_F\omega_{K/F} \boxtimes S_2)$ and similitude character $\mu_F^2$. Note that the $L$-parameter is a representation of the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ and $S_2$ is the irreducible two-dimensional representation of* $\mathrm{SL}_2(\mathbb{C})$.

**Proof.** We shall only give a sketch of the proof. Applying the normalized Jacquet module functor $R_{Q(Z)}$ to the induced Weil representation $\Omega_{V_K,W}$ of $\mathrm{GO}(V_K) \times \mathrm{GSp}(W)^+$, one sees that there is a $\mathrm{GL}(Z) \times (\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+)$-equivariant surjective map

$$R_{Q(Z)}(\Omega_{V_K,W}) \to \omega_{K/F}|\cdot|^{-1} \boxtimes (\Omega_{V_K,W'}|\mathrm{det}_{W'}|^{1/2}).$$

By the previous proposition, one has a $\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+$-equivariant surjective map

$$\Omega_{V_K,W'}|\mathrm{det}_{W'}|^{1/2} \to \mu^+ \boxtimes (\Theta_{V_K,W'}(\mu^+)|\cdot|^{1/2}).$$

Frobenius reciprocity shows that there is a non-zero equivariant map

$$\Omega_{V_K,W} \to \mu^+ \boxtimes I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \Theta_{V_K,W'}(\mu^+)|\cdot|^{1/2}).$$

Since $\Theta_{V_K,W}(\mu^+)$ is an irreducible representation (as $\mathrm{O}(V_K)$ is anisotropic), we have

$$\Theta_{V_K,W}(\mu^+) \hookrightarrow I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \Theta_{V_K,W'}(\mu^+)|\cdot|^{1/2}),$$

so that

$$\tilde{\Theta}_{V_K,W}(\mu^+) \hookrightarrow I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}).$$

The latter representation has $J_{Q(Z)}(\omega_{K/F}|\cdot|, \pi(\mu)|\cdot|^{-1/2})$ as its unique irreducible submodule and this proves the proposition for $\mu^+$.

To complete the proof of the proposition, we need to show the claim in (ii) that $\tilde{\Theta}(\mu^-)$ is the irreducible non-generic supercuspidal representation with the desired $L$-parameter. Proposition A.7 shows that $\tilde{\Theta}(\mu^-)$ is supercuspidal and it is non-zero since we are in the stable range. Moreover, it is not difficult to show that the Whittaker module of $\Omega_{V_K,W}$ is zero, so that $\tilde{\Theta}(\mu^-)$ is non-generic. Now by [**9**], $\tilde{\Theta}(\mu^-)$ has a non-zero theta lift to the anisotropic group $\mathrm{GSO}(D, -\mathrm{N}_D)$, where $D$ is the quaternion division $F$-algebra. We need to show that its theta lift to $\mathrm{GSO}(D, -\mathrm{N}_D)$ is $\mu_F \boxtimes \mu_F\omega_{K/F}$.

For this, we resort to a global argument.

- Choose number fields $\mathbb{F} \subset \mathbb{K}$ such that for some place $v$ of $\mathbb{F}$, one has $\mathbb{K}_v/\mathbb{F}_v = K/F$.
- Choose a quaternion $\mathbb{F}$-algebra $\mathbb{D}$ such that $\mathbb{D}_v = D$ and $\mathbb{K} \subset \mathbb{D}$.
- Let $\Xi$ be a Hecke character of $\mathbb{A}_\mathbb{F}^\times$ such that $\Xi_v = \mu_F$.
- One has the automorphic representation $\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}}$ of $\mathrm{GSO}(\mathbb{D}, -\mathrm{N}_\mathbb{D})(\mathbb{A}_\mathbb{F})$ and one may consider its theta lift $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ to $\mathrm{GSp}_4$.

We claim that this global theta lift is non-zero. To see this, one computes a Fourier coefficient of this theta lift along the Siegel parabolic subgroup $P(Y)$. More precisely, the generic $M(Y)$-orbits of Fourier coefficients are naturally parametrized by étale quadratic $\mathbb{F}$-algebras. If one takes a character $\Psi$ of $N(Y)$ corresponding to $\mathbb{K}$, then the identity component of the stabilizer of $\Psi$ in $M(Y)$ is isomorphic to $\mathrm{GSO}(V_\mathbb{K}) \cong \mathbb{K}^\times$. One can then compute the Bessel period of the theta lift defined by the character $\Psi$ of $N(Y)(\mathbb{A}_\mathbb{F})$ and the character $\Xi \circ \mathrm{N}_{\mathbb{K}/\mathbb{F}}$ of $\mathbb{A}_\mathbb{K}^\times$. By a standard computation, one sees that this Bessel period is non-zero precisely when both the representations $\Xi$ and $\Xi\omega_{\mathbb{K}/\mathbb{F}}$ have non-zero period integrals over the torus $\mathbb{A}_\mathbb{K}^\times$ against the character $\Xi^{-1} \circ \mathrm{N}_{\mathbb{K}/\mathbb{F}}$. Since this last condition evidently holds, we conclude that $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is non-zero.

In addition, we know that the theta lift $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is irreducible, and its local component at $v$ is non-generic supercuspidal with $L$-parameter $(\mu_F \boxtimes S_2) \oplus (\mu_F\omega_{K/F} \boxtimes S_2)$. Moreover, $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is nearly equivalent to the global theta lift $\Theta(\Xi \circ \mathrm{N}_{\mathbb{K}/\mathbb{F}})$ of $\Xi \circ \mathrm{N}_{\mathbb{K}/\mathbb{F}}$ from $\mathrm{GSO}(V_\mathbb{K})$ to $\mathrm{GSp}_4^+$. In particular, it is CAP with respect to the Borel subgroup of $\mathrm{GSp}_4$.

According to a result of Soudry [**53**], all such CAP representations are obtained by theta lifts from $\mathrm{GO}(V_\mathbb{K})$ and so one concludes that $\Theta(\Xi \boxtimes \Xi\omega_{\mathbb{K}/\mathbb{F}})$ is an irreducible constituent of $\Theta(\Xi \circ \mathrm{N}_{\mathbb{K}/\mathbb{F}})$. By extracting the local component at $v$, one concludes that

$$\Theta_{D,W}(\mu_F \boxtimes \mu_F\omega_{K/F}) = \tilde{\Theta}_{V_K,W}(\mu^+) \text{ or } \tilde{\Theta}_{V_K,W}(\mu^-).$$

Since we have already seen that $\tilde{\Theta}_{V_K,W}(\mu^+)$ is non-supercuspidal, we must have

$$\Theta_{D,W}(\mu_F \boxtimes \mu_F\omega_{K/F}) = \tilde{\Theta}_{V_K,W}(\mu^-).$$

This completes the proof of the proposition. □

### A.4. Explicit determination of local theta lifts

Now we can state the main results of this appendix. These are contained in the following three theorems. Together, they imply Theorem A.1.

**Theorem A.9.** *Let $V$ be the anisotropic quadratic space and let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of* $\mathrm{GSO}(V) = \mathrm{GSO}(4)$.

- $\Theta((\tau_1 \boxtimes \tau_2)^{\pm})$ *is either zero or an irreducible representation of* $\mathrm{GSp}_4$.
- *If $\tau_1 = \tau_2 = \tau$, then*
$$\Theta((\tau \boxtimes \tau)^{+}) = \pi_{\mathrm{ng}}(\mathrm{JL}(\tau)),$$
*which is the unique non-generic constituent of $I_{Q(Z)}(1, \mathrm{JL}(\tau))$, whereas*
$$\Theta((\tau \boxtimes \tau)^{-}) = 0.$$
- *If $\tau_1 \neq \tau_2$, then*
$$\Theta((\tau_1 \boxtimes \tau_2)^{\pm}) = \Theta((\tau_2 \boxtimes \tau_1)^{\pm})$$
*is the irreducible non-generic supercuspidal representation of $\mathrm{GSp}_4$ with $L$-parameter $\phi_{\tau_1} \oplus \phi_{\tau_2}$ and similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.*

**Theorem A.10.** *Let $V$ be the split quadratic space and let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of* $\mathrm{GSO}(V) = \mathrm{GSO}(2,2)$.

- *If $\tau_1 = \tau_2 = \tau$ is a discrete series representation, then*
$$\Theta((\tau \boxtimes \tau)^{+}) = \theta((\tau \boxtimes \tau)^{+}) = \pi_{\mathrm{gen}}(\tau),$$
*which is the unique generic constituent of $I_{Q(Z)}(1, \tau)$, whereas*
$$\Theta((\tau \boxtimes \tau)^{-}) = 0.$$
- *If $\tau_1 \neq \tau_2$ are both supercuspidal, then $\Theta((\tau_1 \boxtimes \tau_2)^{\pm}) = \theta((\tau_1 \boxtimes \tau_2)^{\pm})$ is the irreducible generic supercuspidal representation of $\mathrm{GSp}_4$ with $L$-parameter $\phi_{\tau_1} \oplus \phi_{\tau_2}$ and similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.*
- *If $\tau_1$ is supercuspidal and $\tau_2 = \mathrm{st}_{\chi}$, then*
$$\Theta((\tau_1 \boxtimes \tau_2)^{+}) = \theta((\tau_1 \boxtimes \tau_2)^{+}) = \mathrm{St}(\tau_1 \chi^{-1}, \chi).$$
- *Suppose that $\tau_1 = \mathrm{st}_{\chi_1}$ and $\tau_2 = \mathrm{st}_{\chi_2}$ with $\chi_1 \neq \chi_2$ but $\chi_1^2 = \chi_2^2$. Then*
$$\Theta((\tau_1 \boxtimes \tau_2)^{+}) = \theta((\tau_1 \boxtimes \tau_2)^{+}) = \mathrm{St}(\mathrm{st}_{\chi_1/\chi_2}, \chi_2) = \mathrm{St}(\mathrm{st}_{\chi_2/\chi_1}, \chi_1).$$
- *Suppose that $\tau_1$ is a discrete series representation and $\tau_2 \hookrightarrow \pi(\chi, \chi')$ with $|\chi/\chi'| = |\cdot|^{-s_0}$ and $s_0 \geqslant 0$, so that $\tau_2$ is a non-discrete series representation. Then*
$$I_{P(Y)}(\tau_1 \chi^{-1}, \chi) \twoheadrightarrow \Theta((\tau_1 \boxtimes \tau_2)^{+}),$$
*so that the latter representation is multiplicity free and*
$$\theta((\tau_1 \boxtimes \tau_2)^{+}) = J_{P(Y)}(\tau_1 \chi^{-1}, \chi).$$

- *Suppose that*

$$\tau_1 \hookrightarrow \pi(\chi_1, \chi_1') \quad \text{and} \quad \tau_2 \hookrightarrow \pi(\chi_2, \chi_2')$$

*with* $|\chi_i/\chi_i'| = |\cdot|^{-s_i}$ *and* $s_1 \geqslant s_2 \geqslant 0$*. Then*

$$I_{P(Y)}(\pi(\chi_2', \chi_2)\chi_1^{-1}, \chi_1) = I_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow \Theta((\tau_1 \boxtimes \tau_2)^+),$$

*so that the latter representation is multiplicity free and*

$$\theta((\tau_1 \boxtimes \tau_2)^+) = J_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1).$$

*If* $\tau_1 = \tau_2 = \tau$*, then*

$$\Theta((\tau \boxtimes \tau)^-) = 0.$$

**Theorem A.11.** *Suppose that the discriminant algebra* $K$ *of* $V$ *is a field. Let* $\tau \boxtimes \chi$ *be an irreducible representation of*

$$\mathrm{GSO}(V) = \mathrm{GSO}(3,1) \cong (\mathrm{GL}_2(K) \times F^\times)/K^\times,$$

*so that* $\omega_\tau = \chi \circ \mathrm{N}_{K/F}$*.*

(i) *If* $\sigma$ *is an irreducible representation of* $\mathrm{GO}(V)$*, then* $\Theta(\sigma) = 0$ *if and only if* $\sigma = (\tau \boxtimes \chi)^-$ *for an invariant and distinguished* $\tau \boxtimes \chi$*; we shall say that such a* $\sigma$ *is of* forbidden type*. If* $\sigma$ *is not of forbidden type, then* $\theta(\sigma)$ *is an irreducible representation of* $\mathrm{GSp}_4^+$ *such that* $\tilde{\theta}(\sigma)$ *is irreducible.*

(ii) *Suppose that* $\tau$ *is supercuspidal. Then we have the following situations.*

- *(Non-invariant case.) If* $\tau^c \neq \tau$*, then* $\Theta((\tau \boxtimes \chi)^+) = \theta((\tau \boxtimes \chi)^+)$ *is generic supercuspidal.*
- *(Invariant and distinguished case.) Suppose that* $\tau^c = \tau$ *and* $\tau$ *is obtained by base change of some supercuspidal representation* $\tau_F$ *of* $\mathrm{GL}_2(F)$ *and* $\chi = \omega_{\tau_F}\omega_{K/F}$*. Then*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{E/F}, \tau_F)$$

  *with L-parameter* $\phi_{\tau_F} \oplus \phi_{\tau_F}\omega_{K/F}$ *and similitude character* $\omega_{\tau_F}\omega_{K/F}$*.*
- *(Invariant but not distinguished case.) Suppose that* $\tau^c = \tau$ *and* $\tau$ *is obtained by base change of some supercuspidal representation* $\tau_F$ *of* $\mathrm{GL}_2(F)$ *but* $\chi = \omega_{\tau_F}$*. Then*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) \quad \text{and} \quad \tilde{\Theta}((\tau \boxtimes \chi)^-)$$

  *are both irreducible supercuspidal with L-parameter* $\phi_{\tau_F} \oplus \phi_{\tau_F}\omega_{K/F}$ *and similitude character* $\omega_{\tau_F}$*. Exactly one of them, namely* $\tilde{\Theta}((\tau \boxtimes \chi)^+)$*, is generic.*

(iii) *Suppose that* $\tau = \mathrm{St}_\mu$ *is a twisted Steinberg representation so that* $\mu^2 = \chi \circ \mathrm{N}_{K/F}$. *Then there is a quadratic character* $\eta$ *(possibly trivial) of* $F^\times$ *such that* $\mu^c/\mu = \eta \circ \mathrm{N}_{K/F}$ *and* $\chi = \eta \cdot \mu|_{F^\times}$, *so that*

$$\mathrm{St}_\mu \boxtimes \chi \hookrightarrow I_{Q(J)^+}(\eta|\cdot|, \mu^c|\cdot|_K^{-1/2}).$$

*Then we have the following situations.*

- *(Non-invariant case.) If* $\mu^c \neq \mu$, *then* $\eta \neq 1$ *or* $\omega_{K/F}$, *and*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = \mathrm{St}(\eta\omega_{K/F}, \pi(\mu)).$$

- *(Invariant and distinguished case.) In this case, we have* $\eta = \omega_{K/F}$, $\mu = \mu_F \circ \mathrm{N}_{K/F}$, *and* $\chi = \mu_F^2 \omega_{K/F}$. *Then*

$$\mathrm{St}_\mu \boxtimes \chi \hookrightarrow I_{Q(J)^+}(\omega_{K/F}|\cdot|, \mu|\cdot|_K^{-1/2})$$

  *and*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \tilde{\theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{K/F}, \mathrm{st}_{\mu_F}).$$

- *(Invariant but not distinguished case.) In this case, we have* $\eta = 1$, $\mu = \mu_F \circ \mathrm{N}_{K/F}$, *and* $\chi = \mu_F^2$. *Then*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) \quad \text{and} \quad \tilde{\Theta}((\tau \boxtimes \chi)^-)$$

  *are both irreducible discrete series representations of* $\mathrm{GSp}_4$ *with L-parameter* $(\mu_F \boxtimes S_2) \oplus (\mu_F\omega_{K/F} \boxtimes S_2)$ *and similitude character* $\mu_F^2$. *In particular,*

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = \mathrm{St}(\mathrm{st}_{\omega_{K/F}}, \mu_F)$$

  *is generic, whereas* $\tilde{\Theta}((\tau \boxtimes \chi)^-)$ *is non-generic supercuspidal.*

(iv) *Suppose that* $\sigma$ *is a non-discrete series representation of* $\mathrm{GO}(V)$ *which is not of forbidden type, so that*

$$\sigma \hookrightarrow I_{Q(J)}(\chi, \mu^+)$$

*with* $|\chi| = |\cdot|^{-s_0}$ *and* $s_0 \geqslant 0$. *Then we have the following situations.*

- *(Non-invariant or invariant and distinguished case.) In this case, we have*

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu)\chi) \twoheadrightarrow \tilde{\Theta}(\sigma),$$

  *where* $\pi(\mu)$ *is as given in Proposition A.7. In particular,* $\tilde{\Theta}(\sigma)$ *is multiplicity free and has a unique irreducible quotient.*

- *(Invariant but not distinguished case.) In this case,*

$$\tilde{\Theta}(\sigma) \quad \text{and} \quad \tilde{\Theta}(\sigma \cdot \mathrm{sgn})$$

  *are the two irreducible constituents of* $I_{Q(Z)}(1, \pi(\mu))$.

## A.5. Jacquet and Whittaker modules

Because Theorem A.11 is the most subtle part of the three theorems, we shall give its proof in detail here and then give a sketch of the proof of Theorem A.10 later. (Theorem A.9 is the easiest part and its proof will be omitted.) Hence we shall assume that $K$ is a field until § A.7. A key step in the proof of Theorem A.11 is the computation of normalized Jacquet modules of the induced Weil representation $\Omega$ with respect to $Q(J)$ and $Q(Z)$. This is a by-now-standard computation, following the lines of [**29**], and we shall simply state the results below.

**Proposition A.12.** *Let $R_{Q(J)}(\Omega)$ denote the normalized Jacquet module of $\Omega$ along $Q(J)$. Then we have a short exact sequence of $L(J) \times \mathrm{GSp}(W)^+$-modules:*

$$0 \to A \to R_{Q(J)}(\Omega) \to B \to 0.$$

*Here, as $\mathrm{GL}(J) \times (\mathrm{GO}(V_K) \times \mathrm{GSp}(W)^+)$-modules,*

$$B \cong |\det_J| \boxtimes (\Omega_{V_K,W} \otimes |\nu_{V_K}|^{-1/2}),$$

*where $\Omega_{V_K,W}$ is the induced Weil representation of $\mathrm{GO}(V_K) \times \mathrm{GSp}(W)^+$, and*

$$A \cong I_{Q(Z)^+}(S(F^\times) \otimes \Omega_{V_K,W'} \otimes |\det_J| \cdot |\det_Z|^{-1}(\omega_{K/F} \circ \det_Z)|\nu_{V_K}|^{-1}),$$

*where the action of $(\mathrm{GL}(J) \times \mathrm{GO}(V_K)) \times (\mathrm{GL}(Z) \times \mathrm{GSp}(W')^+)$ on $S(F^\times)$ is given by*

$$(((a,h),(b,g))f)(x) = f(b^{-1}xa\nu_{W'}(g))$$

*and $\Omega_{V_K,W'}$ is the induced Weil representation of $\mathrm{GO}(V_K) \times \mathrm{GSp}(W')^+$.*

**Corollary A.13.** *Let $\chi \boxtimes \mu$ be an irreducible representation of $\mathrm{GL}(J) \times \mathrm{GSO}(V_K)$.*

(i) *We have*

$$\mathrm{Hom}_{\mathrm{GL}(J)\times\mathrm{GO}(V_K)}(B, \chi \boxtimes \mu^{\pm}) \neq 0$$

*if and only if $\chi = |\cdot|$, in which case*

$$\mathrm{Hom}_{\mathrm{GL}(J)\times\mathrm{GO}(V_K)}(B, \chi \boxtimes \mu^{\epsilon}) = (\Theta_{V_K,W}(\mu^\epsilon)|\nu_W|^{1/2})^*.$$

*In particular,*

$$\mathrm{Hom}_{\mathrm{GL}(J)\times\mathrm{GO}(V_K)}(B, \chi \boxtimes \mu^{+}) = J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \Theta_{V_K,W'}(\mu^+))^*.$$

(ii) *We have*

$$\mathrm{Hom}_{\mathrm{GL}(J)\times\mathrm{GO}(V_K)}(A, \chi \boxtimes \mu^{\epsilon}) = 0$$

*if and only if $\mu$ is invariant and $\epsilon = -$. Outside of this case, we have*

$$\mathrm{Hom}_{\mathrm{GL}(J)\times\mathrm{GO}(V_K)}(A, \chi \boxtimes \mu^{+}) = I_{Q(Z)^+}(\chi^{-1}\omega_{K/F}, \Theta_{V_K,W'}(\mu^+)\cdot\chi)^*.$$

(iii) *In particular, if* $\chi \neq |\cdot|$, *then*

$$\mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(\chi, \mu^+)) = I_{Q(Z)^+}(\chi^{-1}\omega_{K/F}, \Theta_{V_K,W'}(\mu^+)\cdot\chi)^*.$$

**Proposition A.14.** *Let* $R_{Q(Z)^+}(\Omega)$ *denote the normalized Jacquet module of* $\Omega$ *along* $Q(Z)^+$. *Then we have a short exact sequence of* $\mathrm{GO}(V)\times L(Z)^+$*-modules:*

$$0 \to A' \to R_{Q(Z)^+}(\Omega) \to B' \to 0.$$

*Here, as* $\mathrm{GL}(Z)\times(\mathrm{GO}(V)\times\mathrm{GSp}(W')^+)$*-modules,*

$$B' \cong (\omega_{K/F}\circ\det_Z)\boxtimes\Omega_{V,W'}$$

*where* $\Omega_{V,W'}$ *is the induced Weil representation of* $\mathrm{GO}(V)\times\mathrm{GSp}(W')^+$, *and*

$$A' \cong I_{Q(J)}(S(F^\times)\otimes\Omega_{V_K,W'}\otimes(\omega_{K/F}\circ\det_Z)|\nu_{V_K}|^{-1}|\nu_{W'}|^{-1}),$$

*where the action of* $(\mathrm{GL}(J)\times\mathrm{GO}(V_K))\times(\mathrm{GL}(Z)\times\mathrm{GSp}(W')^+)$ *on* $S(F^\times)$ *is given by*

$$(((a,h),(b,g))f)(x) = f(a^{-1}\nu_{W'}(g)^{-1}xb)$$

*and* $\Omega_{V_K,W'}$ *is the induced Weil representation of* $\mathrm{GO}(V_K)\times\mathrm{GSp}(W')^+$.

**Corollary A.15.**

(i) *Suppose that* $\tau\boxtimes\chi$ *is an irreducible representation of* $\mathrm{GSO}(V)\cong(\mathrm{GL}_2(K)\times F^\times)/K^\times$ *which is invariant and distinguished, so that* $(\tau\boxtimes\chi)^+$ *participates in the theta correspondence with* $\mathrm{GSp}(W')^+\cong\mathrm{GL}_2^+$. *If the small theta lift of* $(\tau\boxtimes\chi)^+$ *to* $\mathrm{GL}_2^+$ *is denoted by* $\tau_F^+$, *then*

$$\mathrm{Hom}_{\mathrm{GO}(V)\times\mathrm{GSp}(W)^+}(\Omega, (\tau\boxtimes\chi)^+\boxtimes I_{Q(Z)^+}(\omega_{K/F},\tau_F^+))\neq 0.$$

(ii) *We have*

$$\mathrm{Hom}_{\mathrm{GL}(Z)\times\mathrm{GSp}(W')^+}(A', \chi\boxtimes\theta_{V_K,W'}(\mu^+)) = I_{Q(J)}(\chi^{-1}\omega_{K/F}, \mu^+(\chi\circ\nu_{V_K}))^*.$$

We also need the computation of the Whittaker module of the induced Weil representation $\Omega$. This is given by the following.

**Proposition A.16.** *Let* $U$ *be the unipotent radical of the Borel subgroup* $B = P(Y)\cap Q(Z)$ *of* $\mathrm{GSp}_4$. *Let* $\psi$ *and* $\psi'$ *be representatives of the two orbits of generic characters of* $U$ *under the action of* $B^+$. *Similarly, let* $U_0$ *be the unipotent radical of a* $\boldsymbol{t}$*-stable Borel subgroup of* $\mathrm{GO}(V)$ *and* $\psi_0$ *a generic character of* $U_0$ *which is fixed by* $\boldsymbol{t}$. *Then (perhaps after relabelling* $\psi$ *and* $\psi'$*)*

$$(\Omega_{V,W})_{U,\psi}\cong \text{c-ind}_{U_0\rtimes\langle\boldsymbol{t}\rangle}^{\mathrm{GO}(V)}(\psi_0\boxtimes\mathbf{1}),$$

*whereas*

$$(\Omega_{V,W})_{U,\psi'} = 0.$$

**Corollary A.17.** *Let* $\tau\boxtimes\chi$ *be an irreducible representation of* $\mathrm{GSO}(V)$ *for a generic* $\tau$. *Then* $\Theta_{V,W}((\tau\boxtimes\chi)^+)$ *is* $\psi$*-generic, whereas in the invariant case,* $\Theta_{V,W}((\tau\boxtimes\chi)^-)$ *is non-generic with respect to any generic character.*

## A.6. Proof of Theorem A.11

We are now ready to give the proof of Theorem A.11. Suppose that $\sigma$ is an irreducible representation of $\mathrm{GO}(V)$. We first note the following.

- If $\sigma$ is infinite dimensional, then [**49**, Theorem 3.4] says that $\Theta(\sigma) = 0$ if and only if $\sigma = (\tau \boxtimes \chi)^-$ for an invariant and distinguished $\tau \boxtimes \chi$. The case where $\sigma$ is finite dimensional can be established in the course of the proof below, but since it is not relevant to the application to this paper, we shall omit the details.
- By a result of Muić [**42**, Theorem 6.2], $\Theta(\sigma)$ is irreducible or zero if $\sigma$ is a discrete series representation, at least when the residual characteristic of $F$ is $p \neq 2$. The reason for this restriction on $p$ in [**42**] is that the Howe duality conjecture on the irreducibility of $\theta(\sigma)$ is known to hold in general for $p \neq 2$ but not for $p = 2$. However, our proof below actually verifies the Howe duality conjecture for all $p$, so that [**42**, Theorem 6.2] holds without restriction on residual characteristic, at least for the dual pair considered here. Note, however, that this information is not necessary for Theorem A.1.

In view of the above, we may assume henceforth that $\sigma$ is not of forbidden type. We now consider the various cases in Theorem A.11 in turn.

*Non-discrete series representations*

Let $\sigma$ be a non-discrete series representation of $\mathrm{GO}(V)$ which is not of forbidden type. Then, as in Theorem A.11 (iv), we have

$$\sigma \hookrightarrow I_{Q(J)}(\chi, \mu^+)$$

with $|\chi| = |\cdot|^{-s_0}$ and $s_0 \geqslant 0$. By Frobenius reciprocity, one has

$$\begin{aligned}\Theta(\sigma)^* &= \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, \sigma) \hookrightarrow \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(\chi, \mu^+)) \\ &= \mathrm{Hom}_{\mathrm{GL}(J) \times \mathrm{GO}(V_K)}(R_{Q(J)}(\Omega), \chi \boxtimes \mu^+).\end{aligned}$$

By Corollary A.13, we see that

$$\mathrm{Hom}_{\mathrm{GL}(J) \times \mathrm{GO}(V_K)}(R_{Q(J)}(\Omega), \chi \boxtimes \mu^+) = I_{Q(Z)^+}(\chi^{-1}\omega_{K/F}, \theta_{V_K,W'}(\mu^+)\chi)^*$$

and hence

$$I_{Q(Z)^+}(\chi^{-1}\omega_{K/F}, \theta_{V_K,W'}(\mu^+)\chi) \twoheadrightarrow \Theta(\sigma),$$

so that

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu)\chi) \twoheadrightarrow \tilde{\Theta}(\sigma).$$

We now examine the various cases in Theorem A.11 (iv).

- (Non-invariant case.) In this case, $\mu^c \neq \mu$ and $\chi \neq 1$ or $\omega_{K/F}$ and $\theta_{V_K,W'}(\mu^+)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu)\chi)$ is either irreducible or of length two with a unique irreducible quotient. This shows that $\tilde{\Theta}(\sigma)$ is multiplicity free and $\tilde{\theta}(\sigma)$ is irreducible, as desired.

- (Invariant and distinguished case.) In this case, either $\mu^c = \mu$ or $\mu^c \neq \mu$ and $\chi = 1$. When $\mu^c = \mu$ so that $\mu = \mu_F \circ \mathrm{N}_{K/F}$, we have $\pi(\mu) = \pi(\mu_F, \mu_F\omega_{K/F})$ and

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu_F, \mu_F\omega_{K/F})\chi) \twoheadrightarrow \tilde{\Theta}(\sigma).$$

Moreover,

$$I_{Q(Z)}(\chi^{-1}\omega_{K/F}, \pi(\mu_F, \mu_F\omega_{K/F})\chi) = I_B(\chi^{-1}\omega_{K/F}, \omega_{K/F}; \chi\mu_F).$$

This is irreducible unless $\chi = |\cdot|^{-1}$ or $\omega_{K/F}|\cdot|^{-1}$. In any case, it is multiplicity free and has a unique irreducible quotient (see [**50**, Table A.1, Type V, p. 270]).

When $\mu^c \neq \mu$ and $\chi = 1$, we have

$$I_{Q(J)}(1, \mu^+) = \sigma \oplus \sigma \cdot \mathrm{sgn}.$$

Hence we have

$$\Theta(\sigma) \oplus \Theta(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)^+}(\omega_{K/F}, \theta_{V_K,W'}(\mu^+)),$$

so that

$$\tilde{\Theta}(\sigma) \oplus \tilde{\Theta}(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)}(\omega_{K/F}, \pi(\mu)),$$

where $\pi(\mu)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(\omega_{K/F}, \pi(\mu))$ is irreducible. Moreover, $\sigma \cdot \mathrm{sgn}$ is of forbidden type, so that $\Theta(\sigma \cdot \mathrm{sgn}) = 0$. Hence we conclude that

$$\tilde{\Theta}(\sigma) = I_{Q(Z)}(\omega_{K/F}, \pi(\mu)).$$

- (Invariant but not distinguished case.) In this case, $\mu^c \neq \mu$ but $\chi = \omega_{K/F}$. Then

$$I_{Q(J)}(\omega_{K/F}, \mu^+) = \sigma \oplus \sigma \cdot \mathrm{sgn}.$$

Hence we have

$$\Theta(\sigma) \oplus \Theta(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)^+}(1, \theta_{V_K,W'}(\mu^+)),$$

so that

$$\tilde{\Theta}(\sigma) \oplus \tilde{\Theta}(\sigma \cdot \mathrm{sgn}) = I_{Q(Z)}(1, \pi(\mu)),$$

where $\pi(\mu)$ is supercuspidal. By Lemma A.2, $I_{Q(Z)}(1, \pi(\mu))$ is the direct sum of a generic representation and a non-generic one, which constitute an $L$-packet of size two.

*Twisted Steinberg representations*

Let $\tau = \mathrm{St}_\mu$ be a twisted Steinberg representation so that $\mu^2 = \chi \circ \mathrm{N}_{K/F}$ and $\sigma = (\mathrm{St}_\mu \boxtimes \chi)^\pm$. It is easy to see that there is a quadratic character $\eta$ (possibly trivial) of $F^\times$ such that $\mu^c/\mu = \eta \circ \mathrm{N}_{K/F}$ and $\chi = \eta \cdot \mu|_{F^\times}$. The representation $\mathrm{St}_\mu \boxtimes \chi$ is invariant but not distinguished if and only if $\eta$ is trivial. We now examine the various cases in Theorem A.11 (iii).

- (Non-invariant or invariant and distinguished case.) In this case, $\eta$ is non-trivial and we have
$$\sigma \hookrightarrow I_{Q(J)}(\eta|\cdot|, \mu^+|\nu_{V_K}|^{-1/2}).$$
By Corollary A.13, we have
$$I_{Q(Z)^+}(\eta\omega_{K/F}|\cdot|^{-1}, \theta_{V_K,W'}(\mu^+)|\cdot|^{1/2}) \twoheadrightarrow \Theta(\sigma),$$
so that
$$I_{Q(Z)}(\eta\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}) \twoheadrightarrow \tilde{\Theta}(\sigma).$$
By Lemma A.2 and [**50**, Table A.1, Type III, p. 270], $I_{Q(Z)}(\eta\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2})$ is multiplicity free with a unique irreducible quotient
$$\begin{array}{ll} \mathrm{St}(\eta\omega_{K/F}, \pi(\mu)) & \text{if } \eta \neq \omega_{K/F} \text{ (non-invariant)}, \\ I_{Q(Z)}(\omega_{K/F}, \mathrm{st}_{\mu_F}) & \text{if } \eta = \omega_{K/F} \text{ (invariant and distinguished)}, \end{array}$$
where $\mu_F$ is a character of $F^\times$ such that $\mu = \mu_F \circ \mathrm{N}_{K/F}$ in the invariant and distinguished case. This verifies Theorem A.1 in this case. It does not quite show that $\Theta(\sigma)$ is irreducible, but as we explained above, this follows from a result of Muić [**42**, Theorem 6.2].

- (Invariant but not distinguished case.) To a certain extent, this is the most non-trivial case of Theorem A.11. If $\sigma = (\mathrm{St}_\mu \boxtimes \mu_F^2)^-$, then
$$\sigma \hookrightarrow I_{Q(J)}(|\cdot|, \mu^-|\nu_{V_K}|^{-1/2}).$$
By Corollary A.13 (i), (ii) and Proposition A.8 (ii) we deduce that
$$\tilde{\Theta}(\sigma) = \tilde{\Theta}_{V_K,W}(\mu^-)$$
is the non-generic supercuspidal representation with the desired $L$-parameter. On the other hand, if $\sigma = (\mathrm{St}_\mu \boxtimes \mu_F^2)^+$, then
$$\sigma \hookrightarrow I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2}).$$
In this case, Corollary A.13 implies that one has an exact sequence:
$$\begin{aligned} 0 &\to J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \theta_{V_K,W'}(\mu^+)|\cdot|^{-1/2})^* \\ &\to \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})) \\ &\xrightarrow{\delta} I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K,W'}(\mu^+)|\cdot|^{1/2})^*. \end{aligned}$$
Since
$$i : \Theta(\sigma)^* \hookrightarrow \mathrm{Hom}_{\mathrm{GO}(V)}(\Omega, I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})),$$
we obtain by composition with $\delta$ a map
$$\delta \circ i : \Theta(\sigma)^* \to I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K,W'}(\mu^+)|\cdot|^{1/2})^*.$$

We claim that this map is still injective; this will be sufficient to establish the desired result in this case. Indeed, it will give

$$I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}) \twoheadrightarrow \tilde{\Theta}(\sigma),$$

and one knows by [**51**] (see also [**50**, Table A.1, Type V, p. 270]) that

$$I_{Q(Z)}(\omega_{K/F}|\cdot|^{-1}, \pi(\mu)|\cdot|^{1/2}) = I_B(\omega_{K/F}|\cdot|^{-1}, \omega_{K/F}; \mu_F|\cdot|^{1/2})$$

is multiplicity free with a unique irreducible quotient $\mathrm{St}(\mathrm{st}_{\omega_{K/F}}, \mu_F)$. This verifies Theorem A.1 in this case, and together with [**42**, Theorem 6.2], one has

$$\tilde{\Theta}(\sigma) = \mathrm{St}(\mathrm{st}_{\omega_{K/F}}, \mu_F).$$

It remains to show that $\delta \circ i$ is injective. Suppose on the contrary that it is not. Then we would have a non-zero equivariant map

$$\Omega \to \sigma \boxtimes J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \theta_{V_K,W'}(\mu^+)|\cdot|^{-1/2}),$$

so that

$$\begin{aligned}\sigma^* &\hookrightarrow \mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, J_{Q(Z)^+}(\omega_{K/F}|\cdot|, \theta_{V_K,W'}(\mu^+)|\cdot|^{-1/2})) \\ &\hookrightarrow \mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K,W'}(\mu^+)|\cdot|^{1/2})).\end{aligned}$$

Now we compute the latter Hom space using Corollary A.15 (ii). We conclude that

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q(Z)^+}(\omega_{K/F}|\cdot|^{-1}, \theta_{V_K,W'}(\mu^+)|\cdot|^{1/2})) = I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})^*,$$

so that

$$I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2}) \twoheadrightarrow \sigma.$$

This is a contradiction, since $I_{Q(J)}(|\cdot|, \mu^+|\nu_{V_K}|^{-1/2})$ has $\sigma$ as a submodule but not a quotient.

*Supercuspidal representations*

Let $\tau \boxtimes \chi$ be a supercuspidal representation of $\mathrm{GSO}(V)$. Finally, we examine the various cases in Theorem A.11 (ii).

- (Non-invariant case.) In this case, one knows by [**49**, Theorem 3.4] that $\Theta((\tau \boxtimes \chi)^+)$ is non-zero and irreducible supercuspidal. Moreover, the $L$-parameter of $\tilde{\Theta}((\tau \boxtimes \chi)^+)$ is identified in [**9**, § 11].

- (Invariant and distinguished case.) In this case, $\tau$ is the base change of some supercuspidal representation $\tau_F$ of $\mathrm{GL}_2(F)$ and $\chi = \omega_{\tau_F}\omega_{K/F}$. Moreover, one knows that the extension $(\tau \boxtimes \chi)^+$ participates in the theta correspondence with $\mathrm{GSp}(W')^+ \cong \mathrm{GL}_2^+$ and its theta lift to $\mathrm{GL}_2^+$ is a constituent $\tau_F^+$ of $\tau_F|_{\mathrm{GL}_2^+}$. By Corollary A.15, we deduce that

$$\Theta((\tau \boxtimes \chi)^+) = I_{Q(Z)^+}(\omega_{K/F}, \tau_F^+),$$

and hence

$$\tilde{\Theta}((\tau \boxtimes \chi)^+) = I_{Q(Z)}(\omega_{K/F}, \tau_F),$$

which is irreducible.

- (Invariant but not distinguished case.) In this case, $\tau$ is the base change of some supercuspidal representation $\tau_F$ of $\mathrm{GL}_2(F)$ but $\chi = \omega_{\tau_F}$. One knows by [**49**, Theorem 3.4] that both extensions $(\tau \boxtimes \chi)^{\pm}$ have non-zero big theta lifts to $\mathrm{GSp}_4^+$ and $\tilde{\Theta}((\tau \boxtimes \chi)^{\pm})$ is irreducible supercuspidal. It remains to show that these two supercuspidal representations of $\mathrm{GSp}_4$ make up an $L$-packet with $L$-parameter $\phi_{\tau_F} \oplus \phi_{\tau_F}\omega_{K/F}$ and similitude character $\omega_{\tau_F}$. Note that $\tau_F$ is necessarily non-dihedral with respect to $K$, so that $\tau_F\omega_{K/F} \neq \tau_F$.

  For this, we consider the representations

$$\begin{aligned} &\tau_F \boxtimes \tau_F\omega_{K/F} && \text{of } \mathrm{GSO}(2,2), \\ &\mathrm{JL}(\tau_F) \boxtimes \mathrm{JL}(\tau_F)\omega_{K/F} && \text{of } \mathrm{GSO}(4) \end{aligned}$$

  and their theta lifts to $\mathrm{GSp}_4$. Then we are required to show that

$$\begin{aligned} \tilde{\Theta}((\tau \boxtimes \chi)^+) &= \Theta(\tau_F \boxtimes \tau_F\omega_{K/F}), \\ \tilde{\Theta}((\tau \boxtimes \chi)^-) &= \Theta(\mathrm{JL}(\tau_F) \boxtimes \mathrm{JL}(\tau_F)\omega_{K/F}). \end{aligned}$$

  We achieve this by using a global argument.

  - Choose a totally real number field $\mathbb{F}$ such that for two places $v$ and $v'$ of $\mathbb{F}$, one has $\mathbb{F}_v = \mathbb{F}_{v'} = F$.
  - Choose a totally real quadratic extension $\mathbb{K}$ of $\mathbb{F}$ such that $\mathbb{K}_v = \mathbb{K}_{v'} = K$.
  - Let $\Sigma$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{F})$ such that $\Sigma_v = \Sigma_{v'} = \tau_F$ and the archimedean component $\Sigma_\infty$ of $\Sigma$ is a discrete series representation. This can be achieved by using a simple trace formula. By [**4**], such a $\Sigma$ is tempered. Then $\Sigma \boxtimes \Sigma\omega_{\mathbb{K}/\mathbb{F}}$ is a tempered cuspidal representation of $\mathrm{GSO}(2,2)(\mathbb{A}_\mathbb{F})$.
  - Consider the global theta lift

$$\Pi := \Theta(\Sigma \boxtimes \Sigma\omega_{\mathbb{K}/\mathbb{F}})$$

    of $\Sigma \boxtimes \Sigma\omega_{\mathbb{K}/\mathbb{F}}$ to $\mathrm{GSp}_4$. It is an irreducible globally generic cuspidal representation.

  On the other hand, we may consider the base change $\mathrm{BC}(\Sigma)$ of $\Sigma$ to $\mathrm{GL}_2(\mathbb{A}_\mathbb{K})$, so that $\mathrm{BC}(\Sigma) \boxtimes \omega_\Sigma$ is a globally generic tempered cuspidal representation of $\mathrm{GSO}(\mathbb{V})(\mathbb{A}_\mathbb{F}) \cong (\mathrm{GL}_2(\mathbb{A}_\mathbb{K}) \times \mathbb{A}_\mathbb{F}^\times)/\mathbb{A}_\mathbb{K}^\times$, where $\mathbb{V}$ is the quadratic space $\mathbb{H} \oplus (\mathbb{K}, \mathrm{N}_{\mathbb{K}/\mathbb{F}})$. Observe that $\mathrm{BC}(\Sigma) \boxtimes \omega_\Sigma$ is a globally invariant representation and almost all of its local components are distinguished, but its local components at $v$ and $v'$ are isomorphic to $\tau \boxtimes \chi$ which is not distinguished.

Because $\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma$ is globally invariant, it can be abstractly extended to an irreducible representation of $\mathrm{GO}(\mathbb{V})(\mathbb{A}_\mathbb{F})$ in infinitely many ways; more precisely, at each place of $\mathbb{F}$, one has two possible extensions. One knows that at least half of these extensions occur in the space of cusp forms on $\mathrm{GO}(\mathbb{V})(\mathbb{A}_\mathbb{F})$. This is because at least one of these extensions is automorphic, and one can twist an automorphic extension by an automorphic sign character of $\mathrm{GO}(\mathbb{V})(\mathbb{A}_\mathbb{F})$. In particular, one can find an automorphic extension of $\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma$ whose local component at $v$ is any one of the two extensions $(\tau\boxtimes\chi)^\pm$. We denote one such automorphic extension by $(\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma)^\pm$. By using the place $v'$, one can further ensure that at any place $w$ such that $\mathrm{BC}(\Sigma_w)\boxtimes\omega_{\Sigma_w}$ is distinguished, the local component of $(\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma)^\pm$ is the $+$-extension

$$(\mathrm{BC}(\Sigma_w)\boxtimes\omega_{\Sigma_w})^+.$$

Thus, we may ensure that all the local components of $(\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma)^\pm$ have non-zero local theta lifts to $\mathrm{GSp}_4^+$.

Now by [**49**, Theorem 8.3], the global theta lift of $(\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma)^\pm$ to $\mathrm{GSp}_4$ is non-zero and irreducible cuspidal. Thus, we obtain an irreducible cuspidal representation

$$\Pi^\pm := \tilde{\Theta}((\mathrm{BC}(\Sigma)\boxtimes\omega_\Sigma)^\pm)$$

of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{F})$. By the local unramified theta correspondence, one sees that $\Pi^\pm$ is nearly equivalent to $\Pi$, so that the partial standard $L$-function $L^S(s,\Pi^\pm,\mathrm{std})$ of degree five has a pole at $s=1$. By a result of Kudla and Rallis [**34**], this implies that $\Pi^\pm$ has a non-zero global theta lift to an inner form of $\mathrm{GSO}(2,2)$. Such an inner form is associated to a quaternion $\mathbb{F}$-algebra $\mathbb{D}_\pm$ (possibly split) and is isomorphic to $(\mathbb{D}_\pm^\times\times\mathbb{D}_\pm^\times)/\mathbb{F}^\times$. If we denote the theta lift of $\Pi^\pm$ to such an inner form by $\Theta_{\mathbb{D}_\pm}(\Pi^\pm)$, then $\Theta_{\mathbb{D}_\pm}(\Pi^\pm)$ is a cuspidal representation which is nearly equivalent to $\Sigma\boxtimes\Sigma\omega_{\mathbb{K}/\mathbb{F}}$. Thus, $\Sigma\boxtimes\Sigma\omega_{\mathbb{K}/\mathbb{F}}$ must be the Jacquet–Langlands transfer of $\Theta_{\mathbb{D}_\pm}(\Pi^\pm)$. Note that at the place $v$, we necessarily have $(\mathbb{D}_+)_v\neq(\mathbb{D}_-)_v$. By extracting the local component at $v$, we conclude that

$$\{\tilde{\Theta}((\tau\boxtimes\chi)^+),\tilde{\Theta}((\tau\boxtimes\chi)^-)\}=\{\Theta(\tau_F\boxtimes\tau_F\omega_{K/F}),\Theta(\mathrm{JL}(\tau_F)\boxtimes\mathrm{JL}(\tau_F)\omega_{K/F})\}.$$

Since we know that $\tilde{\Theta}((\tau\boxtimes\chi)^+)$ and $\Theta(\tau_F\boxtimes\tau_F\omega_{K/F})$ are generic and the other two representations are not, we obtain the desired result.

This completes the proof of Theorem A.11.

## A.7. Proof of Theorem A.10

For the sake of completeness, we shall give a sketch of the proof of Theorem A.10. As before, a key step is the computation of normalized Jacquet modules of the induced Weil representation $\Omega_{V,W}$, where $V$ is now the split four-dimensional quadratic space. Before coming to this computation, we first introduce some more notation.

Recall that $V=X\oplus X^*$, where $X$ is a two-dimensional isotropic space. We can write

$$X=Fu_1\oplus Fu_2 \quad\text{and}\quad X^*=Fv_1\oplus Fv_2$$

with $(u_i, v_j) = \delta_{ij}$. Let $P(X)$ be the parabolic subgroup of $\mathrm{GSO}(V)$ stabilizing $X$ with Levi factor

$$M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m.$$

Let $J = Fu_1$ be the isotropic line spanned by $u_1$ in $X$ and let $B(J)$ be the stabilizer of $J$ in $M(X)$; it is also the stabilizer of the isotropic line spanned by $v_2$ in $X^*$. With respect to the basis $\{u_1, u_2\}$ of $X$, $B(J)$ is the group of upper triangular matrices in $M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m$. We write

$$(t(a,b), \lambda) = \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \lambda \right) \in B(J) \subset M(X).$$

Similarly, recall that $W = Y^* \oplus Y$,

$$Y^* = Fe_1 \oplus Fe_2 \quad \text{and} \quad Y = Ff_1 \oplus Ff_2$$

with $\langle e_i, f_j \rangle = \delta_{ij}$. The stabilizer of $Y$ in $\mathrm{GSp}(W)$ is the Siegel parabolic subgroup $P(Y)$ with Levi factor

$$M(Y) \cong \mathrm{GL}(Y) \times \mathbb{G}_m$$

and the stabilizer of $Z = Ff_1$ in $\mathrm{GSp}(W)$ is the Klingen parabolic subgroup $Q(Z)$ with Levi factor

$$L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W'),$$

where $W' = Fe_2 \oplus Ff_2$.

With the above notation, we have the following.

**Proposition A.18.** *The normalized Jacquet module* $R_{P(X)}(\Omega_{V,W})$ *of* $\Omega_{V,W}$ *along* $P(X)$ *has a natural three step filtration as an* $M(X) \times \mathrm{GSp}(W)$*-module whose successive quotients are described as follows.*

(i) *The top quotient is*

$$C \cong S(F^\times).$$

*Here the action of* $(m, \lambda) \in M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m$ *on* $S(F^\times)$ *is given by*

$$((m,\lambda)f)(t) = |\det_X(m)|^{3/2} |\lambda|^{-3/2} f(\lambda t).$$

(ii) *The middle subquotient is*

$$B \cong I_{B(J)\times Q(Z)}(S(F^\times) \otimes S(F^\times v_2 \otimes f_1)).$$

*Here the action of* $(t(a,b),\lambda) \in B(J)$ *on* $S(F^\times) \otimes S(F^\times v_2 \otimes f_1)$ *is given by*

$$((t(a,b),\lambda)f)(t,x) = |a|\,|b|^2 |\lambda|^{-3/2} f(\lambda t, bx),$$

*whereas the action of* $(\alpha, g) \in L(Z) \cong \mathrm{GL}(Z) \times \mathrm{GSp}(W')$ *is given by*

$$((\alpha,g)f)(t,x) = |\alpha|^{-2} |\nu_{W'}(g)| f(\nu_{W'}(g)t, \alpha^{-1}\nu_{W'}(g)x).$$

(iii) *Finally, the submodule is*

$$A \cong I_{P(Y)}(S(F^\times) \otimes S(\mathrm{Isom}(X,Y))),$$

*where* $\mathrm{Isom}(X,Y)$ *is the set of isomorphisms from* $X$ *to* $Y$ *as vector spaces (which is a torsor for* $\mathrm{GL}(X)$ *as well as for* $\mathrm{GL}(Y)$*). Here the action of* $(m,\lambda) \in M(X) \cong \mathrm{GL}(X) \times \mathbb{G}_m$ *on* $S(F^\times) \otimes S(\mathrm{Isom}(X,Y))$ *is given by*

$$((m,\lambda)f)(t,h) = |\det_X(m)|^{3/2}|\lambda|^{-3/2} f(\lambda t, h \circ m),$$

*whereas the action of* $(m',\lambda') \in M(Y) \cong \mathrm{GL}(Y) \times \mathbb{G}_m$ *is given by*

$$((m',\lambda')f)(t,h) = |\lambda'|^{3/2}|\det_Y(m')|^{-3/2} f(\lambda' t, \lambda' m'^{-1} \circ h).$$

**Corollary A.19.** *Let* $\sigma = \pi(\chi_1,\chi_2) \boxtimes \tau$ *be a representation of* $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$ *such that* $\tau$ *is irreducible but* $\pi(\chi_1,\chi_2)$ *may be reducible, so that* $\omega_\tau = \chi_1\chi_2$ *and*

$$\sigma = I_{P(X)}(\tau^\vee \chi_1, \chi_2).$$

*Then*

$$\mathrm{Hom}_{\mathrm{GSO}(V)}(\Omega,\sigma) = \mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee\chi_1 \boxtimes \chi_2).$$

(i) *If* $\chi_1/\chi_2 \neq |\cdot|^3$*, then*

$$\mathrm{Hom}_{M(X)}(C, \tau^\vee\chi_1 \boxtimes \chi_2) = 0.$$

(ii) *If* $R_B(\tau)$ *does not have* $\chi_1|\cdot|^{-1} \boxtimes \eta$ *as a subquotient for any character* $\eta$*, then*

$$\mathrm{Hom}_{M(X)}(B, \tau^\vee\chi_1 \boxtimes \chi_2) = 0.$$

(iii) *If the conditions in (i) and (ii) hold, then*

$$\mathrm{Hom}_{M(X)}(R_{P(X)}(\Omega), \tau^\vee\chi_1 \boxtimes \chi_2) \subset \mathrm{Hom}_{M(X)}(A, \tau^\vee\chi_1 \boxtimes \chi_2) = I_{P(Y)}(\tau\chi_1^{-1}, \chi_1)^*.$$

**Proposition A.20.** *Let* $U$ *be the unipotent radical of the Borel subgroup* $P(Y) \cap Q(Z)$ *of* $\mathrm{GSp}_4$ *and* $\psi$ *a generic character of* $U$*. Similarly, let* $U_0$ *be the unipotent radical of a Borel subgroup of* $\mathrm{GSO}(V)$ *and* $\psi_0$ *a generic character of* $U_0$*. Then*

$$(\Omega_{V,W})_{U,\psi} \cong \text{c-ind}_{U_0}^{\mathrm{GSO}(V)}(\psi_0).$$

*In particular, if* $\sigma$ *is an irreducible generic representation of* $\mathrm{GSO}(V)$*, then its big theta lift* $\Theta(\sigma)$ *to* $\mathrm{GSp}_4$ *is generic and hence non-zero.*

We are now ready to give the proof of Theorem A.10. Let $\tau_1 \boxtimes \tau_2$ be an irreducible representation of $\mathrm{GSO}(V) \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/F^\times$. Then one knows by results of Roberts that

$$\Theta(\tau_1 \boxtimes \tau_2) = \Theta(\tau_2 \boxtimes \tau_1) \neq 0.$$

We now consider the various cases in Theorem A.10 in turn.

*Supercuspidal representations*

Suppose that $\tau_1 \boxtimes \tau_2$ is supercuspidal. Then one knows that $\Theta(\tau_1 \boxtimes \tau_2) = \theta(\tau_1 \boxtimes \tau_2)$ is non-zero and irreducible. Moreover, if $\tau_1 \neq \tau_2$, then the theta lift of $\tau_1 \boxtimes \tau_2$ to $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ is zero and hence $\theta(\tau_1 \boxtimes \tau_2)$ is supercuspidal. By definition, the $L$-parameter of $\theta(\tau_1 \boxtimes \tau_2)$ is $\phi_{\tau_1} \oplus \phi_{\tau_2}$ with similitude character $\omega_{\tau_1} = \omega_{\tau_2}$.

On the other hand, if $\tau_1 = \tau_2 = \tau$, then $\tau \boxtimes \tau$ participates in the theta correspondence with $\mathrm{GSp}(W') \cong \mathrm{GL}_2$ and its big theta lift to $\mathrm{GL}_2$ is $\tau$. By an analogue of Proposition A.14 for the split $V$, there is a $\mathrm{GSO}(V) \times L(Z)$-equivariant surjective map

$$R_{Q(Z)}(\Omega_{V,W}) \to \Omega_{V,W'}.$$

By Frobenius reciprocity, one has a non-zero $\mathrm{GSO}(V) \times \mathrm{GSp}(W)$-equivariant map

$$\Omega_{V,W} \to (\tau \boxtimes \tau) \boxtimes I_{Q(Z)}(1, \tau).$$

Thus, we see that

$$\Theta(\tau \boxtimes \tau) \hookrightarrow I_{Q(Z)}(1, \tau).$$

We know that $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible constituents with a unique generic constituent $\pi_{\mathrm{gen}}(\tau)$. It follows from Proposition A.20 that

$$\Theta(\tau \boxtimes \tau) = \pi_{\mathrm{gen}}(\tau).$$

*Discrete series representations*

Suppose that $\sigma = \mathrm{st}_\chi \boxtimes \tau$, where $\mathrm{st}_\chi$ is a twisted Steinberg representation and $\tau$ is a discrete series representation so that $\omega_\tau = \chi^2$. Note that $\tau$ is either supercuspidal or equal to $\mathrm{st}_\mu$. Then

$$\sigma \hookrightarrow \pi(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}) \boxtimes \tau = I_{P(X)}(\tau^\vee \chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}).$$

We would like to apply Corollary A.19 (iii) and so we need to verify that the conditions in Corollary A.19 (i), (ii) hold. The condition in Corollary A.19 (i) obviously holds, and that in Corollary A.19 (ii) holds when $\tau$ is supercuspidal. If $\tau = \mathrm{st}_\mu$ is a twisted Steinberg representation (so that $\chi^2 = \mu^2$), then

$$R_B(\tau) = \mu|\cdot|^{1/2} \boxtimes \mu|\cdot|^{-1/2} \neq \chi|\cdot|^{-1/2} \boxtimes \eta$$

for any character $\eta$. Hence the condition in Corollary A.19 (ii) also holds when $\tau$ is a twisted Steinberg representation. In particular, we conclude by Corollary A.19 (iii) that

$$I_{P(Y)}(\tau\chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{1/2}) \twoheadrightarrow \Theta(\sigma).$$

By Lemma A.3, the above induced representation is multiplicity free and of length two with a unique irreducible quotient, so that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma)$ is irreducible. Moreover,

$$\theta(\sigma) = \begin{cases} \mathrm{St}(\tau\chi^{-1}, \chi) & \text{if } \tau \neq \mathrm{st}_\chi, \\ \pi_{\mathrm{gen}}(\tau) & \text{if } \tau = \mathrm{st}_\chi. \end{cases}$$

There remains the issue of whether $\Theta(\sigma) = \theta(\sigma)$. This follows from a result of Muić [**42**], but we can also give a brief sketch of the proof. Suppose on the contrary that $\Theta(\sigma) = I_{P(Y)}(\tau\chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{1/2})$. Then we would have

$$\sigma^* \hookrightarrow \mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, I_{P(Y)}(\tau\chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{1/2})).$$

Now one compute the latter Hom space, which amounts to the computation of the normalized Jacquet module $R_{P(Y)}(\Omega_{V,W})$. A short computation shows that

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\Omega_{V,W}, I_{P(Y)}(\tau\chi^{-1}|\cdot|^{-1/2}, \chi|\cdot|^{1/2})) = I_{P(X)}(\tau^\vee\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})^*,$$

so that

$$I_{P(X)}(\tau^\vee\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}) \twoheadrightarrow \sigma.$$

This is a contradiction, since $I_{P(X)}(\tau^\vee\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ has $\sigma$ as a submodule but not a quotient. Thus, we conclude that $\Theta(\sigma) = \theta(\sigma)$ is irreducible.

*Non-discrete series representations. I*

Suppose that

$$\sigma \hookrightarrow \pi(\chi_1, \chi_2) \boxtimes \tau = I_{P(X)}(\tau^\vee\chi_1, \chi_2),$$

where $\tau$ is a discrete series representation with $\omega_\tau = \chi_1\chi_2$, $|\chi_1/\chi_2| = |\cdot|^{-s_0}$, and $s_0 \geqslant 0$. Again, we would like to apply Corollary A.19 (iii) and so we need to verify the conditions there. As before, the only issue is the condition in Corollary A.19 (ii) when $\tau = \mathrm{st}_\chi$ is a twisted Steinberg representation, in which case

$$R_B(\tau) = \chi|\cdot|^{1/2} \boxtimes \chi|\cdot|^{-1/2}$$

and we need to show that this is different from $\chi_1|\cdot|^{-1} \boxtimes \eta$ for any character $\eta$. In other words, we need to show that $\chi/\chi_1 \neq |\cdot|^{-3/2}$. But observe that

$$|\chi|^2 = |\chi_1\chi_2| = |\chi_1|^2|\chi_2/\chi_1| = |\chi_1|^2|\cdot|^{s_0},$$

so that

$$|\chi/\chi_1| = |\cdot|^{s_0/2} \neq |\cdot|^{-3/2}.$$

This verifies that the conditions in Corollary A.19 (i), (ii) hold, so that we conclude that

$$I_{P(Y)}(\tau\chi_1^{-1}, \chi_1) \twoheadrightarrow \Theta(\sigma).$$

Since the above induced representation is multiplicity free with a unique irreducible quotient, we conclude that $\Theta(\sigma)$ is multiplicity free and $\theta(\sigma) = J_{P(Y)}(\tau\chi_1^{-1}, \chi_1)$ is irreducible.

*Non-discrete series representations. II*

Finally, we consider the case where

$$\sigma \hookrightarrow \pi(\chi_1, \chi_1') \boxtimes \pi(\chi_2, \chi_2')$$

with $\chi_1\chi_1' = \chi_2\chi_2'$, $|\chi_i/\chi_i'| = |\cdot|^{-s_i}$, and $s_1 \geqslant s_2 \geqslant 0$. We consider two subcases.

(a) $\chi_2/\chi_2' \neq |\cdot|^{-1}$; in this case $\pi(\chi_2, \chi_2') = \pi(\chi_2', \chi_2)$ is irreducible and

$$\sigma \hookrightarrow I_{P(X)}(\pi(\chi_2, \chi_2')^\vee \chi_1, \chi_1').$$

Again, to apply Corollary A.19 (iii), we need to verify the conditions there, and in particular the condition in Corollary A.19 (ii). We have

$$R_B(\pi(\chi_2, \chi_2')) = (\chi_2 \boxtimes \chi_2') \oplus (\chi_2' \boxtimes \chi_2)$$

up to semisimplification and so we need to verify that $\chi_2 \neq \chi_1|\cdot|^{-1}$ and $\chi_2' \neq \chi_1|\cdot|^{-1}$. To see these, we argue by contradiction. If $\chi_2 = \chi_1|\cdot|^{-1}$, then $\chi_2' = \chi_1'|\cdot|$, so that

$$|\cdot|^{-s_2} = |\chi_2/\chi_2'| = |\chi_1/\chi_1'|\,|\cdot|^{-2} = |\cdot|^{-s_1-2}.$$

This would give $s_2 = s_1 + 2$, which contradicts $s_1 \geqslant s_2$. On the other hand, if $\chi_2' = \chi_1|\cdot|^{-1}$, then $\chi_2 = \chi_1'|\cdot|$, so that

$$|\cdot|^{s_2} = |\chi_2'/\chi_2| = |\chi_1/\chi_1'|\,|\cdot|^{-2} = |\cdot|^{-s_1-2}.$$

This would give $s_2 = -s_1 - 2 < 0$, which is a contradiction. Thus, we may apply Corollary A.19 (iii) to conclude that

$$I_{P(Y)}(\pi(\chi_2', \chi_2)\chi_1^{-1}, \chi_1) = I_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow \Theta(\sigma).$$

This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1).$$

(b) $\chi_2/\chi_2' = |\cdot|^{-1}$; in this case, $\pi(\chi_2, \chi_2')$ is reducible and has the one-dimensional representation $\chi_2|\cdot|^{1/2}$ as its unique irreducible submodule. Then

$$\sigma \hookrightarrow \pi(\chi_1, \chi_1') \boxtimes \chi_2|\cdot|^{1/2} = I_{P(X)}(\chi_1\chi_2^{-1}|\cdot|^{-1/2}, \chi_1').$$

Applying Corollary A.19 (iii) (we leave the verification of the conditions there to the reader), we conclude that

$$I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{1/2}, \chi_1) \twoheadrightarrow \Theta(\sigma).$$

Observe that

$$I_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1) \twoheadrightarrow I_{P(Y)}(\chi_1^{-1}\chi_2|\cdot|^{1/2}, \chi_1)$$

and the former induced representation is a standard module. This shows that $\Theta(\sigma)$ is multiplicity free with a unique irreducible quotient

$$\theta(\sigma) = J_B(\chi_2'/\chi_1, \chi_2/\chi_1; \chi_1).$$

This completes the proof of Theorem A.10.

## Appendix B. Spherical Eisenstein series on $\mathrm{GO}_{2n}$

Let $F$ be a number field. Let $\xi(s) = \mathfrak{D}^{s/2}\zeta(s)$, where $\mathfrak{D}$ is the absolute value of the discriminant of $F$ and $\zeta(s)$ is the zeta function of $F$ including archimedean factors. Then the functional equation $\xi(1-s) = \xi(s)$ holds. We write

$$\xi(s) = \frac{\rho}{s-1} + \gamma + O(s-1).$$

For each $s \in \mathbb{C}$, let

$$[s](a) = |a|^s, \qquad [s]'(a) = |a|^s \log |a|,$$

for $a \in \mathbb{A}^\times$. For an automorphic form $\phi$ on $\mathrm{GO}_{2n}(\mathbb{A})$, let

$$\phi[s](h) = \phi(h)[s](\nu(h)), \qquad \phi[s]'(h) = \phi(h)[s]'(\nu(h)),$$

for $h \in \mathrm{GO}_{2n}(\mathbb{A})$. Let $\mathbf{1}$ denote the constant function on $\mathrm{GO}_{2n}(\mathbb{A})$.

For each $r \in \mathbb{N}$ with $r \leqslant n$, let $P_{n,r}$ be the parabolic subgroup of $\mathrm{GO}_{2n}$ and $E^{(n,r)}(s)$ the spherical Eisenstein series given in § 7.3. Note that $E^{(n,0)}(s) = \mathbf{1}$. For each $s_0 \in \mathbb{C}$, let

$$E^{(n,r)}(s) = \sum_{d \gg -\infty} (s-s_0)^d E_d^{(n,r)}(s_0)$$

be the Laurent expansion of $E^{(n,r)}(s)$ at $s = s_0$.

Let $Q = P_{n,1}$. For an automorphic form $\phi$ on $\mathrm{GO}_{2n}(\mathbb{A})$, let $\phi_Q$ denote the constant term of $\phi$ along $Q$. We regard $\phi_Q$ as an automorphic form on $\mathbb{A}^\times \times \mathrm{GO}_{2n-2}(\mathbb{A})$ via the embedding

$$(a, h') \mapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & b' \\ 0 & 0 & \nu(h')a^{-1} & 0 \\ 0 & c' & 0 & d' \end{pmatrix},$$

where

$$h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

**Lemma B.1.** *Let $\phi$ be a $\boldsymbol{K}$-invariant automorphic form on $\mathrm{GO}_{2n}(\mathbb{A})$. Assume that $\phi$ is concentrated on the Borel subgroup. If $\phi_Q = 0$, then*

$$\phi = 0.$$

**Proof.** The assertion follows from the Langlands lemma (see [**24**, Corollary 3.1]). □

We have a double coset decomposition

$$\mathrm{GO}_{2n} = \begin{cases} P_{n,r}Q \cup P_{n,r}w_1^{(n,r)}Q \cup P_{n,r}w_2^{(n,r)}Q & \text{if } 1 \leqslant r < n, \\ P_{n,r}Q \cup P_{n,r}w_2^{(n,r)}Q & \text{if } r = n, \end{cases}$$

where

$$w_1^{(n,r)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1}_{r-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-r-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{r-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{n-r-1} \end{pmatrix}$$

and

$$w_2^{(n,r)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mathbf{1}_{n-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-1} \end{pmatrix}.$$

As in [**24**, Proposition 2.6], a routine calculation shows the following proposition.

**Proposition B.2.** *If* $1 \leqslant r < n$*, then* $E^{(n,r)}(s)_Q$ *is equal to*

$$\begin{aligned}
&[s+\tfrac{1}{2}(2n-r-1)] \otimes E^{(n-1,r-1)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s-\tfrac{1}{4}(2n-r-1)] \\
&\qquad + \frac{\xi(s+\frac{1}{2}(2n-3r-1))}{\xi(s+\frac{1}{2}(2n-r-1))}[r] \otimes E^{(n-1,r)}(s)[-\tfrac{1}{2}r] \\
&\qquad\qquad + \frac{\xi(s+\frac{1}{2}(r-1))\xi(s-\frac{1}{2}(2n-3r-1))\xi(2s)}{\xi(s+\frac{1}{2}(r+1))\xi(s+\frac{1}{2}(2n-r-1))\xi(2s+r-1)} \\
&\qquad\qquad\quad \times [-s+\tfrac{1}{2}(2n-r-1)] \otimes E^{(n-1,r-1)}(s-\tfrac{1}{2})[\tfrac{1}{2}s-\tfrac{1}{4}(2n-r-1)].
\end{aligned}$$

*If* $r = n$*, then* $E^{(n,n)}(s)_Q$ *is equal to*

$$\begin{aligned}
&[s+\tfrac{1}{2}(n-1)] \otimes E^{(n-1,n-1)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s-\tfrac{1}{4}(n-1)] \\
&\qquad + \frac{\xi(2s)}{\xi(2s+n-1)}[-s+\tfrac{1}{2}(n-1)] \otimes E^{(n-1,n-1)}(s-\tfrac{1}{2})[\tfrac{1}{2}s-\tfrac{1}{4}(n-1)].
\end{aligned}$$

*The case* $n = 1$

Obviously, $E^{(1,1)}(s)$ is entire. We have

$$E_0^{(1,1)}(0) = 2 \cdot \mathbf{1}, \qquad E_1^{(1,1)}(0) = 0.$$

*The case* $n = 2$

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned}
E^{(2,2)}(s)_Q = [s+\tfrac{1}{2}] \otimes E^{(1,1)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s-\tfrac{1}{4}] \\
+ \frac{\xi(2s)}{\xi(2s+1)}[-s+\tfrac{1}{2}] \otimes E^{(1,1)}(s-\tfrac{1}{2})[\tfrac{1}{2}s-\tfrac{1}{4}].
\end{aligned}$$

Hence $E^{(2,2)}(s)$ has a simple pole at $s = \frac{1}{2}$ and is holomorphic at $s = \frac{3}{2}$. Also, the functional equation

$$\xi(2s)E^{(2,2)}(-s) = \xi(2s+1)E^{(2,2)}(s)$$

holds. We have

$$E^{(2,2)}_{-1}(\tfrac{1}{2}) = \frac{\rho}{\xi(2)}\mathbf{1}.$$

Let $r = 1$. By Proposition B.2, we have

$$E^{(2,1)}(s)_Q = [s+1] \otimes \mathbf{1}[-\tfrac{1}{2}s - \tfrac{1}{2}] + \frac{\xi(s)}{\xi(s+1)}[1] \otimes E^{(1,1)}(s)[-\tfrac{1}{2}] + \frac{\xi(s)^2}{\xi(s+1)^2}[-s+1] \otimes \mathbf{1}[\tfrac{1}{2}s - \tfrac{1}{2}].$$

Hence $E^{(2,1)}(s)$ has a double pole at $s = 1$ and is holomorphic at $s = 2$. We have

$$E^{(2,1)}_{-2}(1) = \frac{\rho^2}{\xi(2)^2}\mathbf{1}.$$

**Lemma B.3.**

$$E^{(2,1)}_{-1}(1) = \frac{\rho}{\xi(2)}E^{(2,2)}_{0}(\tfrac{1}{2}).$$

**Proof.** We have

$$\begin{aligned}
E^{(2,2)}_{0}(\tfrac{1}{2})_Q &= [1] \otimes E^{(1,1)}_{0}(1)[-\tfrac{1}{2}] \\
&\quad + \frac{\rho}{2\xi(2)}(-[0]' \otimes E^{(1,1)}_{0}(0)[0] + \tfrac{1}{2}[0] \otimes E^{(1,1)}_{0}(0)[0]' + [0] \otimes E^{(1,1)}_{1}(0)[0]) \\
&\quad + \frac{\rho}{\xi(2)}\left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)}\right)[0] \otimes E^{(1,1)}_{0}(0)[0] \\
&= [1] \otimes E^{(1,1)}_{0}(1)[-\tfrac{1}{2}] - \frac{\rho}{\xi(2)}([0]' \otimes \mathbf{1}[0] - \tfrac{1}{2}[0] \otimes \mathbf{1}[0]') \\
&\quad + \frac{2\rho}{\xi(2)}\left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)}\right)[0] \otimes \mathbf{1}[0].
\end{aligned}$$

On the other hand,

$$E^{(2,1)}_{-1}(1)_Q = \frac{\rho}{\xi(2)}[1] \otimes E^{(1,1)}_{0}(1)[-\tfrac{1}{2}] + \frac{\rho^2}{\xi(2)^2}(-[0]' \otimes \mathbf{1}[0] + \tfrac{1}{2}[0] \otimes \mathbf{1}[0]') + \frac{2\rho^2}{\xi(2)^2}\left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)}\right)[0] \otimes \mathbf{1}[0].$$

By Lemma B.1, this yields the lemma. □

*The case* $n = 3$

Let $r = 3$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,3)}(s)_Q &= [s+1] \otimes E^{(2,2)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s - \tfrac{1}{2}] \\ &\qquad + \frac{\xi(2s)}{\xi(2s+2)}[-s+1] \otimes E^{(2,2)}(s-\tfrac{1}{2})[\tfrac{1}{2}s - \tfrac{1}{2}] \\ &= [s+1] \otimes E^{(2,2)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s - \tfrac{1}{2}] \\ &\qquad + \frac{\xi(2s-1)}{\xi(2s+2)}[-s+1] \otimes E^{(2,2)}(-s+\tfrac{1}{2})[\tfrac{1}{2}s - \tfrac{1}{2}]. \end{aligned}$$

Hence $E^{(3,3)}(s)$ has a simple pole at $s = 1$ and is holomorphic at $s = 0$. Also, the functional equation

$$\xi(2s-1)E^{(3,3)}(-s) = \xi(2s+2)E^{(3,3)}(s)$$

holds. We have

$$E_1^{(3,3)}(0) = -\frac{2\xi'(2)}{\xi(2)}E_0^{(3,3)}(0), \qquad E_{-1}^{(3,3)}(1) = \frac{\rho}{\xi(4)}\mathbf{1}.$$

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,2)}(s)_Q &= [s+\tfrac{3}{2}] \otimes E^{(2,1)}(s+\tfrac{1}{2})[-\tfrac{1}{2}s - \tfrac{3}{4}] + \frac{\xi(s-\frac{1}{2})}{\xi(s+\frac{3}{2})}[2] \otimes E^{(2,2)}(s)[1] \\ &\qquad + \frac{\xi(s+\frac{1}{2})^2\xi(2s)}{\xi(s+\frac{3}{2})^2\xi(2s+1)}[-s+\tfrac{3}{2}] \otimes E^{(2,1)}(s-\tfrac{1}{2})[\tfrac{1}{2}s - \tfrac{3}{4}]. \end{aligned}$$

Hence $E^{(3,2)}(s)$ has a double pole at $s = \frac{3}{2}$. We have

$$E_{-2}^{(3,2)}(\tfrac{3}{2}) = \frac{\rho^2}{\xi(3)\xi(4)}\mathbf{1}.$$

Let $r = 1$. By Proposition B.2, we have

$$\begin{aligned} E^{(3,1)}(s)_Q &= [s+2] \otimes \mathbf{1}[-\tfrac{1}{2}s - 1] + \frac{\xi(s+1)}{\xi(s+2)}[1] \otimes E^{(2,1)}(s)[-\tfrac{1}{2}] \\ &\qquad + \frac{\xi(s)\xi(s-1)}{\xi(s+1)\xi(s+2)}[-s+2] \otimes \mathbf{1}[\tfrac{1}{2}s - 1]. \end{aligned}$$

Hence $E^{(3,1)}(s)$ has a simple pole at $s = 2$ and has a simple pole at $s = 1$. We have

$$E_{-1}^{(3,1)}(2) = \frac{\rho\xi(2)}{\xi(3)\xi(4)}\mathbf{1}.$$

**Lemma B.4.**

$$E_{-1}^{(3,2)}(\tfrac{3}{2}) = \frac{\rho}{\xi(3)}E_0^{(3,3)}(1).$$

**Proof.** We have

$$\begin{aligned}
E_0^{(3,3)}(1)_Q &= [2]\otimes E_0^{(2,2)}(\tfrac{3}{2})[1] \\
&\quad + \frac{\xi(2)}{\xi(4)}(-[0]'\otimes E_{-1}^{(2,2)}(\tfrac12)[0] + \tfrac12[0]\otimes E_{-1}^{(2,2)}(\tfrac12)[0]' + [0]\otimes E_0^{(2,2)}(\tfrac12)[0]) \\
&\quad + \frac{2\xi(2)}{\xi(4)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[0]\otimes E_{-1}^{(2,2)}(\tfrac12)[0] \\
&= [2]\otimes E_0^{(2,2)}(\tfrac32)[1] - \frac{\rho}{\xi(4)}([0]'\otimes \mathbf{1}[0] - \tfrac12[0]\otimes\mathbf{1}[0]') \\
&\quad + \frac{\xi(2)}{\xi(4)}[0]\otimes E_0^{(2,2)}(\tfrac12)[0] + \frac{2\rho}{\xi(4)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[0]\otimes\mathbf{1}[0].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E_{-1}^{(3,2)}(\tfrac32)_Q &= \frac{\rho}{\xi(3)}[2]\otimes E_0^{(2,2)}(\tfrac32)[1] \\
&\quad + \frac{\xi(2)^2}{\xi(3)\xi(4)}(-[0]'\otimes E_{-2}^{(2,1)}(1)[0] + \tfrac12[0]\otimes E_{-2}^{(2,1)}(1)[0]' + [0]\otimes E_{-1}^{(2,1)}(1)[0]) \\
&\quad + \frac{2\xi(2)^2}{\xi(3)\xi(4)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[0]\otimes E_{-2}^{(2,1)}(1)[0] \\
&= \frac{\rho}{\xi(3)}[2]\otimes E_0^{(2,2)}(\tfrac32)[1] - \frac{\rho^2}{\xi(3)\xi(4)}([0]'\otimes\mathbf{1}[0] - \tfrac12[0]\otimes\mathbf{1}[0]') \\
&\quad + \frac{\xi(2)^2}{\xi(3)\xi(4)}[0]\otimes E_{-1}^{(2,1)}(1)[0] + \frac{2\rho^2}{\xi(3)\xi(4)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[0]\otimes\mathbf{1}[0].
\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.3. □

**Lemma B.5.**

$$E_{-1}^{(3,1)}(1) = \frac{\rho}{2\xi(3)}E_0^{(3,3)}(0).$$

**Proof.** We have

$$\begin{aligned}
E_0^{(3,3)}(0)_Q &= [1]'\otimes E_{-1}^{(2,2)}(\tfrac12)[-\tfrac12] - \tfrac12[1]\otimes E_{-1}^{(2,2)}(\tfrac12)[-\tfrac12]' + [1]\otimes E_0^{(2,2)}(\tfrac12)[-\tfrac12] \\
&\quad + [1]'\otimes E_{-1}^{(2,2)}(\tfrac12)[-\tfrac12] - \tfrac12[1]\otimes E_{-1}^{(2,2)}(\tfrac12)[-\tfrac12]' + [1]\otimes E_0^{(2,2)}(\tfrac12)[-\tfrac12] \\
&\quad + \frac{4\xi'(2)}{\xi(2)}[1]\otimes E_{-1}^{(2,2)}(\tfrac12)[-\tfrac12] \\
&= \frac{2\rho}{\xi(2)}([1]'\otimes\mathbf{1}[-\tfrac12] - \tfrac12[1]\otimes\mathbf{1}[-\tfrac12]') + 2[1]\otimes E_0^{(2,2)}(\tfrac12)[-\tfrac12] \\
&\quad + \frac{4\rho\xi'(2)}{\xi(2)^2}[1]\otimes\mathbf{1}[-\tfrac12].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
E^{(3,1)}_{-1}(1)_Q &= \frac{\xi(2)}{\xi(3)}[1] \otimes E^{(2,1)}_{-1}(1)[-\tfrac{1}{2}] + \frac{\xi(2)}{\xi(3)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(3)}{\xi(3)}\right)[1] \otimes E^{(2,1)}_{-2}(1)[-\tfrac{1}{2}] \\
&\quad - \frac{\rho^2}{\xi(2)\xi(3)}(-[1]' \otimes \mathbf{1}[-\tfrac{1}{2}] + \tfrac{1}{2}[1] \otimes \mathbf{1}[-\tfrac{1}{2}]') \\
&\quad + \frac{\rho^2}{\xi(2)\xi(3)}\left(\frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)}\right)[1] \otimes \mathbf{1}[-\tfrac{1}{2}] \\
&= \frac{\xi(2)}{\xi(3)}[1] \otimes E^{(2,1)}_{-1}(1)[-\tfrac{1}{2}] + \frac{2\rho^2\xi'(2)}{\xi(2)^2\xi(3)}[1] \otimes \mathbf{1}[-\tfrac{1}{2}] \\
&\quad + \frac{\rho^2}{\xi(2)\xi(3)}([1]' \otimes \mathbf{1}[-\tfrac{1}{2}] - \tfrac{1}{2}[1] \otimes \mathbf{1}[-\tfrac{1}{2}]').
\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.3. □

*The case* $n = 4$

Let $r = 4$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,4)}(s)_Q = [s + \tfrac{3}{2}] \otimes E^{(3,3)}(s + \tfrac{1}{2})[-\tfrac{1}{2}s - \tfrac{3}{4}] \\
+ \frac{\xi(2s)}{\xi(2s+3)}[-s + \tfrac{3}{2}] \otimes E^{(3,3)}(s - \tfrac{1}{2})[\tfrac{1}{2}s - \tfrac{3}{4}].
\end{aligned}$$

Hence $E^{(4,4)}(s)$ has a simple pole at $s = \frac{1}{2}$.

Let $r = 2$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,2)}(s)_Q = [s + \tfrac{5}{2}] \otimes E^{(3,1)}(s + \tfrac{1}{2})[-\tfrac{1}{2}s - \tfrac{5}{4}] + \frac{\xi(s+\frac{1}{2})}{\xi(s+\frac{5}{2})}[2] \otimes E^{(3,2)}(s)[-1] \\
+ \frac{\xi(s+\frac{1}{2})\xi(s-\frac{1}{2})\xi(2s)}{\xi(s+\frac{5}{2})\xi(s+\frac{3}{2})\xi(2s+1)}[-s + \tfrac{5}{2}] \otimes E^{(3,1)}(s - \tfrac{1}{2})[\tfrac{1}{2}s - \tfrac{5}{4}].
\end{aligned}$$

Hence $E^{(4,2)}(s)$ has a double pole at $s = \frac{3}{2}$.

Let $r = 1$. By Proposition B.2, we have

$$\begin{aligned}
E^{(4,1)}(s)_Q = [s + 3] \otimes \mathbf{1}[-\tfrac{1}{2}s - \tfrac{3}{2}] + \frac{\xi(s+2)}{\xi(s+3)}[1] \otimes E^{(3,1)}(s)[-\tfrac{1}{2}] \\
+ \frac{\xi(s)\xi(s-2)}{\xi(s+1)\xi(s+3)}[-s + 3] \otimes \mathbf{1}[\tfrac{1}{2}s - \tfrac{3}{2}].
\end{aligned}$$

Hence $E^{(4,1)}(s)$ has a simple pole at $s = 1$.

**Lemma B.6.**

$$E^{(4,2)}_{-2}(\tfrac{3}{2}) = \frac{\rho\xi(2)}{\xi(3)\xi(4)}E^{(4,4)}_{-1}(\tfrac{1}{2}).$$

**Proof.** We have

$$\begin{aligned}E_{-1}^{(4,4)}(\tfrac{1}{2})_Q &= [2]\otimes E_{-1}^{(3,3)}(1)[-1] + \frac{\rho}{2\xi(4)}[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}]\\ &= \frac{\rho}{\xi(4)}[2]\otimes \mathbf{1}[-1] + \frac{\rho}{2\xi(4)}[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}].\end{aligned}$$

On the other hand,

$$\begin{aligned}E_{-2}^{(4,2)}(\tfrac{3}{2})_Q &= \frac{\xi(2)}{\xi(4)}[2]\otimes E_{-2}^{(3,2)}(\tfrac{3}{2})[-1] + \frac{\rho\xi(2)}{\xi(4)^2}[1]\otimes E_{-1}^{(3,1)}(1)[-\tfrac{1}{2}]\\ &= \frac{\rho^2\xi(2)}{\xi(3)\xi(4)^2}[2]\otimes \mathbf{1}[-1] + \frac{\rho\xi(2)}{\xi(4)^2}[1]\otimes E_{-1}^{(3,1)}(1)[-\tfrac{1}{2}].\end{aligned}$$

Hence the assertion follows from Lemmas B.1 and B.5. □

**Lemma B.7.**

$$E_{-1}^{(4,1)}(1) = E_{-1}^{(4,4)}(\tfrac{1}{2}).$$

**Proof.** We have

$$E_{-1}^{(4,1)}(1)_Q = \frac{\xi(3)}{\xi(4)}[1]\otimes E_{-1}^{(3,1)}(1)[-\tfrac{1}{2}] + \frac{\rho}{\xi(4)}[2]\otimes \mathbf{1}[-1].$$

Hence the assertion follows from Lemmas B.1 and B.5. □

**Proposition B.8.**

$$\left(\frac{\rho\xi(2)}{\xi(3)\xi(4)}\right)^{-1} E_{-1}^{(4,2)}(\tfrac{3}{2}) = E_0^{(4,4)}(\tfrac{1}{2}) + E_0^{(4,1)}(1) + \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)}\right)E_{-1}^{(4,4)}(\tfrac{1}{2}).$$

**Proof.** We have

$$\begin{aligned}E_0^{(4,4)}(\tfrac{1}{2})_Q &= [2]'\otimes E_{-1}^{(3,3)}(1)[-1] - \tfrac{1}{2}[2]\otimes E_{-1}^{(3,3)}(1)[-1]' + [2]\otimes E_0^{(3,3)}(1)[-1]\\ &\quad + \frac{\rho}{2\xi(4)}(-[1]'\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}] + \tfrac{1}{2}[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}]' + [1]\otimes E_1^{(3,3)}(0)[-\tfrac{1}{2}])\\ &\quad + \frac{\rho}{\xi(4)}\left(\frac{\gamma}{\rho} - \frac{\xi'(4)}{\xi(4)}\right)[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}]\\ &= \frac{\rho}{\xi(4)}([2]'\otimes \mathbf{1}[-1] - \tfrac{1}{2}[2]\otimes \mathbf{1}[-1]') + [2]\otimes E_0^{(3,3)}(1)[-1]\\ &\quad - \frac{\rho}{2\xi(4)}([1]'\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}] - \tfrac{1}{2}[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}]')\\ &\quad + \frac{\rho}{\xi(4)}\left(\frac{\gamma}{\rho} - \frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[1]\otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}].\end{aligned}$$

By Lemmas B.4 and B.5, we have

$$\begin{aligned}
E^{(4,2)}_{-1}(\tfrac{3}{2})_Q &= [4]\otimes E^{(3,1)}_{-1}(2)[-2] \\
&\quad + \frac{\xi(2)}{\xi(4)}[2]\otimes E^{(3,2)}_{-1}(\tfrac{3}{2})[-1] \\
&\quad + \frac{\xi(2)}{\xi(4)}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[2]\otimes E^{(3,2)}_{-2}(\tfrac{3}{2})[-1] \\
&\quad + \frac{\rho\xi(2)}{\xi(4)^2}(-[1]'\otimes E^{(3,1)}_{-1}(1)[-\tfrac{1}{2}] + \tfrac{1}{2}[1]\otimes E^{(3,1)}_{-1}(1)[-\tfrac{1}{2}]' + [1]\otimes E^{(3,1)}_{0}(1)[-\tfrac{1}{2}]) \\
&\quad + \frac{\rho\xi(2)}{\xi(4)^2}\left(\frac{\gamma}{\rho} + \frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} - \frac{3\xi'(4)}{\xi(4)}\right)[1]\otimes E^{(3,1)}_{-1}(1)[-\tfrac{1}{2}] \\
&= \frac{\rho\xi(2)}{\xi(3)\xi(4)}[4]\otimes \mathbf{1}[-2] + \frac{\rho\xi(2)}{\xi(3)\xi(4)}[2]\otimes E^{(3,3)}_{0}(1)[-1] \\
&\quad + \frac{\rho^2\xi(2)}{\xi(3)\xi(4)^2}\left(\frac{\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[2]\otimes \mathbf{1}[-1] \\
&\quad - \frac{\rho^2\xi(2)}{2\xi(3)\xi(4)^2}([1]'\otimes E^{(3,3)}_{0}(0)[-\tfrac{1}{2}] - \tfrac{1}{2}[1]\otimes E^{(3,3)}_{0}(0)[-\tfrac{1}{2}]') \\
&\quad + \frac{\rho\xi(2)}{\xi(4)^2}[1]\otimes E^{(3,1)}_{0}(1)[-\tfrac{1}{2}] \\
&\quad + \frac{\rho^2\xi(2)}{2\xi(3)\xi(4)^2}\left(\frac{\gamma}{\rho} + \frac{\xi'(2)}{\xi(2)} + \frac{\xi'(3)}{\xi(3)} - \frac{3\xi'(4)}{\xi(4)}\right)[1]\otimes E^{(3,3)}_{0}(0)[-\tfrac{1}{2}]
\end{aligned}$$

and

$$\begin{aligned}
E^{(4,1)}_{0}(1)_Q &= [4]\otimes \mathbf{1}[-2] \\
&\quad + \frac{\xi(3)}{\xi(4)}[1]\otimes E^{(3,1)}_{0}(1)[-\tfrac{1}{2}] \\
&\quad + \frac{\xi(3)}{\xi(4)}\left(\frac{\xi'(3)}{\xi(3)} - \frac{\xi'(4)}{\xi(4)}\right)[1]\otimes E^{(3,1)}_{-1}(1)[-\tfrac{1}{2}] \\
&\quad + \frac{\rho}{\xi(4)}(-[2]'\otimes \mathbf{1}[-1] + \tfrac{1}{2}[2]\otimes \mathbf{1}[-1]') \\
&\quad + \frac{\rho}{\xi(4)}\left(\frac{\gamma}{\rho} - \frac{2\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[2]\otimes \mathbf{1}[-1] \\
&= [4]\otimes \mathbf{1}[-2] + \frac{\xi(3)}{\xi(4)}[1]\otimes E^{(3,1)}_{0}(1)[-\tfrac{1}{2}] \\
&\quad + \frac{\rho}{2\xi(4)}\left(\frac{\xi'(3)}{\xi(3)} - \frac{\xi'(4)}{\xi(4)}\right)[1]\otimes E^{(3,3)}_{0}(0)[-\tfrac{1}{2}] \\
&\quad - \frac{\rho}{\xi(4)}([2]'\otimes \mathbf{1}[-1] - \tfrac{1}{2}[2]\otimes \mathbf{1}[-1]') \\
&\quad + \frac{\rho}{\xi(4)}\left(\frac{\gamma}{\rho} - \frac{2\xi'(2)}{\xi(2)} - \frac{\xi'(4)}{\xi(4)}\right)[2]\otimes \mathbf{1}[-1].
\end{aligned}$$

Hence we have

$$\left(\frac{\rho\xi(2)}{\xi(3)\xi(4)}\right)^{-1} E_{-1}^{(4,2)}(\tfrac{3}{2})_Q - E_0^{(4,4)}(\tfrac{1}{2})_Q - E_0^{(4,1)}(1)_Q$$

$$= \frac{\rho}{\xi(4)}\left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)}\right)[2] \otimes \mathbf{1}[-1] + \frac{\rho}{2\xi(4)}\left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)}\right)[1] \otimes E_0^{(3,3)}(0)[-\tfrac{1}{2}]$$

$$= \left(-\frac{\gamma}{\rho} + \frac{3\xi'(2)}{\xi(2)}\right) E_{-1}^{(4,4)}(\tfrac{1}{2})_Q.$$

By Lemma B.1, this yields the proposition. □

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## References


1. J. Adams and D. Barbasch, Reductive dual pair correspondence for complex groups, *J. Funct. Analysis* **132** (1995), 1–42.
2. J. Arthur, Unipotent automorphic representations: conjectures, *Astérisque* **171–172** (1989), 13–71.
3. D. Ban and C. Jantzen, Degenerate principal series for even-orthogonal groups, *Represent. Theory* **7** (2003), 440–480.
4. D. Blasius and J. D. Rogawski, Motives for Hilbert modular forms, *Invent. Math.* **114** (1993), 55–87.
5. S. Böcherer, M. Furusawa and R. Schulze-Pillot, On the global Gross–Prasad conjecture for Yoshida liftings, in *Contributions to automorphic forms, geometry, and number theory*, pp. 105–130 (Johns Hopkins University Press, Baltimore, MD, 2004).
6. M. Cognet, Représentation de Weil et changement de base quadratique, *Bull. Soc. Math. France* **113** (1985), 403–457.
7. M. Cognet, Représentation de Weil et changement de base quadratique dans le cas archimédien, II, *Bull. Soc. Math. France* **114** (1986), 325–354.
8. W. T. Gan and N. Gurevich, Restriction of Saito–Kurokawa representations (with an appendix by G. Savin), in *Automorphic forms and L-functions, I, Global aspects*, Contemporary Mathematics, Volume 488, pp. 95–124 (American Mathematical Society, Providence, RI, 2009).
9. W. T. Gan and S. Takeda, The local Langlands conjecture for GSp(4), *Annals Math.*, in press.
10. P. B. Garrett, Decomposition of Eisenstein series: Rankin triple products, *Annals Math.* **125** (1987), 209–235.
11. B. H. Gross and D. Prasad, On the decomposition of a representation of $SO_n$ when restricted to $SO_{n-1}$, *Can. J. Math.* **44** (1992), 974–1002.
12. M. Harris, *L*-functions of $2 \times 2$ unitary groups and factorization of periods of Hilbert modular forms, *J. Am. Math. Soc.* **6** (1993), 637–719.
13. M. Harris and S. S. Kudla, Arithmetic automorphic forms for the nonholomorphic discrete series of GSp(2), *Duke Math. J.* **66** (1992), 59–121.
14. M. Harris, S. S. Kudla and W. J. Sweet Jr, Theta dichotomy for unitary groups, *J. Am. Math. Soc.* **9** (1996), 941–1004.
15. M. Harris, D. Soudry and R. Taylor, *l*-adic representations associated to modular forms over imaginary quadratic fields, I, Lifting to $GSp_4(\boldsymbol{Q})$, *Invent. Math.* **112** (1993), 377–411.

16. H. He, Theta correspondence, I, Semistable range: construction and irreducibility, *Commun. Contemp. Math.* **2** (2000), 255–283.
17. K. Hiraga and H. Saito, On $L$-packets for inner forms of $\mathrm{SL}_n$, preprint.
18. R. Howe, Transcending classical invariant theory, *J. Am. Math. Soc.* **2** (1989), 535–552.
19. A. Ichino, A regularized Siegel–Weil formula for unitary groups, *Math. Z.* **247** (2004), 241–277.
20. A. Ichino, Trilinear forms and the central values of triple product $L$-functions, *Duke Math. J.* **145** (2008), 281–307.
21. A. Ichino and T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture, *Geom. Funct. Analysis* **19** (2010), 1378–1425.
22. T. Ikeda, On the location of poles of the triple $L$-functions, *Compositio Math.* **83** (1992), 187–237.
23. T. Ikeda, On the residue of the Eisenstein series and the Siegel–Weil formula, *Compositio Math.* **103** (1996), 183–218.
24. D. Jiang, The first term identities for Eisenstein series, *J. Number Theory* **70** (1998), 67–98.
25. D. Jiang and D. Soudry, On the genericity of cuspidal automorphic forms of $\mathrm{SO}(2n+1)$, II, *Compositio Math.* **143** (2007), 721–748.
26. H. H. Kim, On local $L$-functions and normalized intertwining operators, *Can. J. Math.* **57** (2005), 535–597.
27. H. H. Kim and M. Krishnamurthy, Stable base change lift from unitary groups to $\mathrm{GL}_n$, *Int. Math. Res. Pap.* **1** (2005), 1–52.
28. H. H. Kim and F. Shahidi, Cuspidality of symmetric powers with applications, *Duke Math. J.* **112** (2002), 177–197.
29. S. S. Kudla, On the local theta-correspondence, *Invent. Math.* **83** (1986), 229–255.
30. S. S. Kudla, Splitting metaplectic covers of dual reductive pairs, *Israel J. Math.* **87** (1994), 361–401.
31. S. S. Kudla and S. Rallis, On the Weil–Siegel formula, *J. Reine Angew. Math.* **387** (1988), 1–68.
32. S. S. Kudla and S. Rallis, Poles of Eisenstein series and $L$-functions, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, Part II, Israel Mathematical Conference Proceedings, Volume 3, pp. 81–110 (Weizmann, Jerusalem, 1990).
33. S. S. Kudla and S. Rallis, Ramified degenerate principal series representations for $\mathrm{Sp}(n)$, *Israel J. Math.* **78** (1992), 209–256.
34. S. S. Kudla and S. Rallis, A regularized Siegel–Weil formula: the first term identity, *Annals Math.* **140** (1994), 1–80.
35. S. S. Kudla, S. Rallis and D. Soudry, On the degree 5 $L$-function for $\mathrm{Sp}(2)$, *Invent. Math.* **107** (1992), 483–541.
36. K. F. Lai, Tamagawa number of reductive algebraic groups, *Compositio Math.* **41** (1980), 153–188.
37. S. T. Lee and C.-B. Zhu, Degenerate principal series and local theta correspondence, II, *Israel J. Math.* **100** (1997), 29–59.
38. S. T. Lee and C.-B. Zhu, Degenerate principal series and local theta correspondence, III, The case of complex groups, *J. Alg.* **319** (2008), 336–359.
39. J.-S. Li, Singular unitary representations of classical groups, *Invent. Math.* **97** (1989), 237–255.
40. J.-S. Li, Nonvanishing theorems for the cohomology of certain arithmetic quotients, *J. Reine Angew. Math.* **428** (1992), 177–217.
41. H. Y. Loke, Howe quotients of unitary characters and unitary lowest weight modules (with an appendix by S. T. Lee), *Represent. Theory* **10** (2006), 21–47.
42. G. Muić, On the structure of theta lifts of discrete series for dual pairs $(\mathrm{Sp}(n), \mathrm{O}(V))$, *Israel J. Math.* **164** (2008), 87–124.

43. A. Paul, On the Howe correspondence for symplectic-orthogonal dual pairs, *J. Funct. Analysis* **228** (2005), 270–310.
44. I. I. Piatetski-Shapiro and S. Rallis, $L$-functions for the classical groups, in *Explicit constructions of automorphic L-functions*, Lecture Notes in Mathematics, Volume 1254, pp. 1–52 (Springer, 1987).
45. I. I. Piatetski-Shapiro and S. Rallis, Rankin triple $L$ functions, *Compositio Math.* **64** (1987), 31–115.
46. S. Rallis, On the Howe duality conjecture, *Compositio Math.* **51** (1984), 333–399.
47. B. Roberts, The theta correspondence for similitudes, *Israel J. Math.* **94** (1996), 285–317.
48. B. Roberts, The non-Archimedean theta correspondence for GSp(2) and GO(4), *Trans. Am. Math. Soc.* **351** (1999), 781–811.
49. B. Roberts, Global $L$-packets for GSp(2) and theta lifts, *Documenta Math.* **6** (2001), 247–314.
50. B. Roberts and R. Schmidt, *Local newforms for* GSp(4), Lecture Notes in Mathematics, Volume 1918 (Springer, 2007).
51. P. J. Sally Jr and M. Tadić, Induced representations and classifications for $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$, *Mém. Soc. Math. France (N.S.)* **52** (1993), 75–133.
52. H. Shimizu, Theta series and automorphic forms on $\mathrm{GL}_2$, *J. Math. Soc. Jpn* **24** (1972), 638–683.
53. D. Soudry, The CAP representations of $\mathrm{GSp}(4, \boldsymbol{A})$, *J. Reine Angew. Math.* **383** (1988), 87–108.
54. V. Tan, A regularized Siegel–Weil formula on $\mathrm{U}(2, 2)$ and $\mathrm{U}(3)$, *Duke Math. J.* **94** (1998), 341–378.
55. J.-L. Waldspurger, Sur les valeurs de certaines fonctions $L$ automorphes en leur centre de symétrie, *Compositio Math.* **54** (1985), 173–242.
56. J.-L. Waldspurger, Démonstration d'une conjecture de dualité de Howe dans le cas $p$-adique, $p \neq 2$, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, Part I, Israel Mathematical Conference Proceedings, Volume 2, pp. 267–324 (Weizmann, Jerusalem, 1990).
57. H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, *Invent. Math.* **60** (1980), 193–248.