EQUIDISTRIBUTION OF INTEGER POINTS ON A FAMILY OF HOMOGENEOUS VARIETIES: A PROBLEM OF LINNIK

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1. Introduction

Let f be a homogeneous polynomial in n variables with integer coefficients. For any integer m, consider the affine subvariety of $\mathbb{R}^n$ defined by

$$V_m = \{x \in \mathbb{R}^n : f(x) = m\}.$$

It is a classical problem in number theory to understand the distribution of the set $V_m(\mathbb{Z})$ of integer points in V_m.

Two basic types of questions have been studied in the literature. The first type of problem is perhaps more well-known. Here, for a fixed integer m, one sets

$$N(\Omega) = \#V_m(\mathbb{Z}) \cap \Omega \quad \text{for any nice compact subset } \Omega \text{ of } V_m.$$

One is then interested in the asymptotics of $N(\Omega_i)$ for a nice family of growing compact subsets $\Omega_i \subset V_m$, for $i = 1, 2, \cdots$. For example, one would like to show that

$$N(\Omega_i) \sim vol(\Omega_i) \quad \text{as } i \to \infty \tag{1.1}$$

for a suitably normalized measure on V_m. The second type of problem deals with a family of varieties instead of a single one. To be more precise, in order to compare $V_m(\mathbb{Z})$ for different positive integers m, one does a rescaling by radially projecting $V_m(\mathbb{Z})$ to a fixed (non-empty) variety, say V_1. Note that the radial projection π of V_m onto V_1 is given by $x \mapsto m^{-\frac{1}{d}} \cdot x$ where d is the degree of f. One can then ask whether the points $\pi(V_m(\mathbb{Z}))$ are equidistributed in V_1 as $m \to \infty$. In other words, for nice compact subsets Ω_1 and Ω_2 of V_1, one would like to show that

$$\frac{\#\pi(V_m(\mathbb{Z})) \cap \Omega_1}{\#\pi(V_m(\mathbb{Z})) \cap \Omega_2} \sim \frac{vol(\Omega_1)}{vol(\Omega_2)} \quad \text{as } m \to \infty.$$

This problem was raised by Linnik in the early sixties and particular examples were studied by Linnik and Skubenko (cf. [LS] and [Li]). Hence, following Sarnak [Sa], we shall call a question of this type Linnik's problem.

The only known general approach to these two types of problems is the Hardy-Littlewood circle method. However, this applies only when the number of variables involved is much larger compared to the degree of the homogeneous polynomial in question and in many interesting cases, this condition is not satisfied. When the Hardy-Littlewood method does not apply, both problems are hopeless ventures in the generality of the above setting. The expectation highlighted in [Sa] is that if one restricts attention to the case where the varieties in question are homogeneous varieties of a linear semisimple algebraic group, then both problems can be related to the harmonic analysis of the group, thus becoming more tractable. We refer the reader to [Sa] for references to earlier works which exploit this relation. Subsequent to the appearance of [Sa], this expectation was realized for the first type of problem by Duke, Rudnick and Sarnak in [DRS], where (1.1) was shown when V_m is an affine symmetric space. The method of [DRS] allows one in principle to obtain an estimate on the rate of convergence in (1.1). A simpler proof of the results in [DRS] was given by Eskin and McMullen in [EM], using the decay of matrix coefficients and some geometric property of affine symmetric spaces. A few years later, Eskin, Mozes and Shah extended these to a much greater generality, using Ratner's results on unipotent flows on homogeneous spaces. This method unfortunately does not provide information on rates of convergence.

The purpose of the present paper is the realization of the expectation expressed in [Sa] for Linnik's problem. For the sake of simplicity, we assume that the group in question is $\mathbb{Q}$-split in the introduction. Thus, let $\mathcal{G}$ be a connected reductive $\mathbb{Q}$-split algebraic group with absolutely simple derived group and one dimensional center. Set $G = \mathcal{G}(\mathbb{R})^0$ and $G_0 = [G, G]$. Let $\iota : \mathcal{G} \to \mathcal{GL}_n(\mathbb{C}) = \mathcal{GL}(V)$ be a rational representation defined over $\mathbb{Z}$ such that the identity component of the center of G acts by non-trivial scalars on V. Suppose that the polynomial f is a semi-invariant of $\mathcal{G}$, that is, for some non-trivial $\mathbb{Q}$-rational character χ of $\mathcal{G}$, $f(vg) = \chi(g)f(v)$ for any $v \in V$ and $g \in \mathcal{G}$. Then G_0 acts on each V_m. Let $v_0 \in V_1(\mathbb{Z})$ be such that the stabilizer of v_0 in G_0 does not possess any non-trivial $\mathbb{Q}$-rational character. Note that the $[\mathcal{G}, \mathcal{G}]$-orbit of v_0 in V may not be Zariski closed; so that the stabilizer of v_0 is not necessarily reductive (cf. [BH]). Then we have the following equidistribution statement, whose special case for $f = \det$ was proven by Linnik and Skubenko [Li, Thm. 1]:

Theorem 1.2. *Fix a compact subset $\Omega \subset v_0G_0$ and for any small $\epsilon > 0$, consider the standard division of $\mathbb{R}^n$ into ϵ-cubes. Then there exists an* **effective** *constant $m_{\Omega,\epsilon}$ such that for any positive integer $m > m_{\Omega,\epsilon}$, any ϵ-cube intersecting the interior of Ω contains at least one point in the radial projection of $V_{m^r}(\mathbb{Z})$ into V_1. Here r is an explicit positive integer which depends only on $\mathcal{G}$, ι and deg(f) and is given in (8.1).*

We emphasize that our proof of Theorem 1.2, essentially making use of harmonic analysis of G, does yield an effective estimate for the constant $m_{\Omega,\epsilon}$.

Corollary 1.3. *Given any open set $U \subset v_0G_0$, there exists a constant m_U such that U contains a point in the radial projection $V_{m^r}(\mathbb{Z})$ for any positive integer $m > m_U$. In particular, the subset $\bigcup_{m=1}^{\infty} \pi(V_m(\mathbb{Z}))$ is dense in v_0G_0.*

Remark

- As we explain in Section 9 (Exs. 2 and 3), there are fundamental obstructions to having such a theorem for any sequence m tending to infinity; so the restriction to a sub-sequence of an r-th power of m is not that surprising, and is even necessary.
- If V_1 is the union of finitely many G_0-orbits each of which possessing an integer point, then one can replace v_0G_0 by the whole variety V_1 in Theorem 1.2. For example, in the case of a regular prehomogeneous vector space with a unique semi-invariant, each V_m, for $m \neq 0$, is the union of finitely many G_0-orbits. Therefore the classification of $\mathbb{Q}$-split irreducible regular prehomogeneous vector spaces given by Sato and Kimura in [SK] provides explicit examples to which Theorem 1.2 applies, with v_0G_0 replaced by V_1, as long as each G_0-orbit contains an integer point.
- Assuming only that the derived group of $\mathcal{G}$ is $\mathbb{Q}$-isotropic, we obtain a slightly weaker version of Theorem 1.2 (see Theorem 5.1).

To prove Theorem 1.2, we introduce for each $m \in \mathbb{N}$ a subset $G[m]$ of $G_{\mathbb{Q}}$ as follows:

$$G[m] = \{g \in G_{\mathbb{Q}} : \iota(g) \in End(V_{\mathbb{Z}}) \text{ and } \chi_0(g) = m\}$$

where χ_0 denotes the basis element of the character group of $\mathcal{G}$ whose restriction to the center of $\mathcal{G}$ is a positive multiple of the central character of ι. If $G_{\mathbb{Z}}$ is the arithmetic subgroup of G associated to the Chevalley $\mathbb{Z}$-structure, each $G[m]$ is a (possibly empty) finite union of $G_{\mathbb{Z}}$-double cosets. For some fixed $r_0 \in \mathbb{N}$ (depending only on $\mathcal{G}$ and ι), we shall see that $\#G_{\mathbb{Z}}\backslash G[m^{r_0}] \geq m^{\beta \cdot r_0}$ for some fixed constant $\beta > 0$ independent of $m \in \mathbb{N}$ and further $v_0G[m^{r_0}] \subset V_{m^r}(\mathbb{Z})$. Thus the subsets $G[m^{r_0}]$ allow us to produce many integer points in V_{m^r} starting from $v_0 \in V_1(\mathbb{Z})$. There is of course no reason to expect that every point in $V_{m^r}(\mathbb{Z})$ is obtained in this way. In fact, in the general case, there will be primitive or new points in $V_{m^r}(\mathbb{Z})$ which do not arise from any lower stratum in this way. There is no doubt that these primitive points are the most interesting from the arithmetic point of view. However, for the purpose of Linnik's problem, and in particular for the proof of Theorem 1.2, there is no harm in discarding these points. Indeed, Theorem 1.2 is an immediate consequence of the following equidistribution statement, which is of independent interest:

Theorem 1.4. *Fix a nice (see Def. 4.5) compact subset Ω of v_0G_0 and $0 < \epsilon \ll 1$. Then there exists a constant $C_{\Omega,\epsilon}$ such that for any positive integer m,*

$$\left|\frac{1}{\#G_{\mathbb{Z}}\backslash G[m^{r_0}]} \cdot \left(\sum_{y \in G_{\mathbb{Z}}\backslash G[m^{r_0}]} \#v_0G_{\mathbb{Z}}y \cap \mathbb{R}_+\Omega\right) - vol(\Omega)\right| \leq C_{\Omega,\epsilon} \cdot m^{-r_0\kappa+\epsilon}.$$

Here the volume of Ω is with respect to a suitably normalized G_0-invariant measure on v_0G_0 and $\mathbb{R}_+\Omega = \{x \in \mathbb{R}^n : tx \in \Omega$ for some $t > 0\}$. Moreover, r_0 is an explicit positive integer depending only on $\mathcal{G}$ and ι (see (7.3)) and the exponent $\kappa > 0$ is independent of Ω and is explicitly computable (see (7.7)).

Remark

- Though in the above we have restricted ourselves to homogeneous varieties defined by a single polynomial, there are in fact no additional difficulties in dealing with a more general case, where the varieties are defined by several polynomials.
- When the stabilizer of v_0 in G_0 is trivial, Theorem 1.4 yields

$$\# v_0 G[m^{r_0}] \cap \mathbb{R}_+\Omega \sim \# G_{\mathbb{Z}} \backslash G[m^{r_0}] \cdot vol(\Omega) \quad \text{as } m \to \infty.$$

  It will be very interesting to know whether in general the asymptotic of above type exists. Some new results are obtained in this direction [EO].

The main tool in the proof of Theorem 1.4 is the use of Hecke operators. The relation of Hecke operators to Linnik's problem was first observed by Sarnak in [Sa]. Our starting point is then an equidistribution result for Hecke points in $Z\Gamma\backslash G$ where Z is the connected center of G and Γ is a congruence subgroup of G. This result was recently proved by Clozel, Oh and Ullmo in [COU] for simple and simply-connected algebraic groups over $\mathbb{Q}$ (not necessarily $\mathbb{Q}$-split). For our purpose, we need to extend this to a slightly more general class of algebraic groups. This extension is provided in Section 3 using a suitably modified definition of Hecke operators given in Section 2. Using this extension, we obtain in Section 4 an equidistribution result for Hecke points on homogeneous varieties of G with an estimate on the rate of convergence. The difficulties involved in passing from an equidistribution result on G to that on a homogeneous variety of G are analytic in nature and are addressed in Section 4. In Section 7, we deduce Theorem 7.6, which directly implies Theorem 1.4, from the (rate of) equidistribution of Hecke points on homogeneous varieties of G. To do so, we need to estimate the number of $G_{\mathbb{Z}}$-double cosets in $G[m^{r_0}]$ as well as the number of single $G_{\mathbb{Z}}$-cosets in each double coset. These are handled in Sections 6 and 7.

We conclude the introduction by discussing the classical example treated by Linnik and Skubenko in [LS] and [Li], and revisited by Sarnak in [Sa].

Example: Consider the action of GL_n on the space M_n of $n \times n$ matrices by right multiplication. The determinant map is a homogeneous polynomial on M_n of degree n. Then for any $n \geq 3$,

$$V_m(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : \det(A) = m\} = G[m].$$

Set $\|A\| = (\sum_{i,j} A_{ij}^2)^{\frac{1}{2}}$. Then, taking v_0 to be the identity matrix I_n, Theorem 1.4 implies that for any given $R > 0$ and $0 < \epsilon \ll 1$, as $m \to \infty$,

$$\begin{aligned} &\#\{A \in M_n(\mathbb{Z}) : \det(A) = m, \text{ and } \|A\| \leq m^{\frac{1}{n}} R\} \\ &= c_{n,R} \cdot \# SL_n(\mathbb{Z}) \backslash G[m] \cdot (1 + O_{R,\epsilon}(m^{-\frac{1}{2n^2+2}+\epsilon})) \end{aligned} \tag{1.5}$$

Here $c_{n,R}$ is the volume of the set $\{A \in SL_n(\mathbb{R}) : \|A\| \le R\}$ with respect to the Haar measure of $SL_n(\mathbb{R})$ giving $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ volume 1. For $n = 2$, the same asymptotic holds except that the error term should be replaced by $m^{-\frac{1}{20}+\epsilon}$.

Furthermore one can show that (cf. [COU])

$$\# SL_n(\mathbb{Z})\backslash G[m] \sim b_{m,n} \quad \text{as } m \to \infty$$

where

$$\begin{aligned} b_{m,n} &= [SL_n(\mathbb{Z}) : SL_n(\mathbb{Z}) \cap diag(m, 1, \cdots, 1) SL_n(\mathbb{Z}) diag(m^{-1}, 1, \cdots, 1)] \\ &= \prod_i \frac{(p_i^{k_i+1} - 1) \cdots (p_i^{k_i+n-1} - 1)}{(p_i - 1) \cdots (p_i^{n-1} - 1)} \end{aligned}$$

when $m = \prod_i p_i^{k_i}$ is the prime factorization of m.

The above example is deceptively simple because of the following reasons. Firstly, $r_0 = r = 1$ and every point in $V_m(\mathbb{Z})$ is obtained from v_0 via $G[m]$, i.e. $V_m(\mathbb{Z}) = v_0 G[m]$. Hence Theorem 1.4 gives a precise result for all integer points. As mentioned before, this is far from being true in general. Secondly, the stabilizer of v_0 in GL_n is trivial. This ensures that the sets $v_0 SL_n(\mathbb{Z}) y$ appearing in Theorem 1.4 are disjoint as y ranges over $SL_n(\mathbb{Z})\backslash G[m]$. When the stabilizer of v_0 is non-trivial, this will not be the case and Theorem 1.4 should be interpreted as an equidistribution theorem of integer points counted *with multiplicities* (see the remark following Theorem 1.4). In Section 8, we give a couple of examples which illustrate these phenomenons.

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Contents

2. **Hecke Operators**

We begin with some preliminaries on global and local Hecke operators. In particular, we shall give a modified definition of global Hecke operators, which possesses good localization properties.

Let $\mathcal{G}$ be a connected reductive linear algebraic group over $\mathbb{Q}$, and let $\mathcal{Z}$ be the identity component of the center of $\mathcal{G}$. Suppose that the algebraic group $\mathcal{Z}\backslash\mathcal{G}$ is absolutely simple with $\mathbb{Q}$-rank at least 1. This assumption is not strictly necessary but it results in cleaner statements for our main results. We set

$$\begin{cases} G = \mathcal{G}(\mathbb{R})^0; \\ Z = \mathcal{Z}(\mathbb{R})^0; \\ \overline{G} = Z\backslash G; \\ G_{\mathbb{Q}} = \mathcal{G}(\mathbb{Q}) \cap G. \end{cases}$$

For any subset $S \subset G$, $\overline{S}$ will denote the image of S in $\overline{G}$.

Let $\mathcal{G}(\mathbb{A}_f)$ be the group of finite adeles attached to $\mathcal{G}$. It is the restricted direct product over all primes of the groups $\mathcal{G}(\mathbb{Q}_p)$ with respect to some family of open compact subgroups $K_p \subset \mathcal{G}(\mathbb{Q}_p)$. For almost all p, K_p is a hyperspecial maximal compact subgroup of $\mathcal{G}(\mathbb{Q}_p)$. Without loss of generality, we may assume that for all p, K_p is a special maximal compact subgroup. The group $\mathcal{G}(\mathbb{A})$ of adeles attached to $\mathcal{G}$ is equal to $\mathcal{G}(\mathbb{R}) \times \mathcal{G}(\mathbb{A}_f)$.

Let $\Gamma \subset G_{\mathbb{Q}}$ be an arithmetic subgroup of G such that

$$\Gamma = G_{\mathbb{Q}} \cap U$$

for some open compact subgroup $U = \prod_p U_p$ of $\mathcal{G}(\mathbb{A}_f)$.

To define global Hecke operators with nice localization properties, we assume that

$$\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q}) \cdot G \cdot U; \tag{2.1}$$

$$\mathcal{Z}(\mathbb{A}) = \mathcal{Z}(\mathbb{Q}) \cdot Z \cdot (U \cap \mathcal{Z}(\mathbb{A}_f)). \tag{2.2}$$

Remark: We note that the above assumptions are satisfied in the following two cases.

- When $\mathcal{G}$ is simply connected and Γ is a congruence subgroup: noting that $\mathcal{G}(\mathbb{R})$ is connected, (2.1) is just a consequence of the strong approximation property. (2.2) trivially holds since $\mathcal{G}$ is then semisimple and hence $\mathcal{Z} = \{e\}$.
- When $\mathcal{G}$ is $\mathbb{Q}$-split and hence canonically defined over $\mathbb{Z}$ and $\Gamma = G \cap \mathcal{G}(\mathbb{Z})$: to see this, note that we have $U = \prod_p \mathcal{G}(\mathbb{Z}_p)$ and hence $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q}) \cdot \mathcal{G}(\mathbb{R}) \cdot U$ (see [PR, Pg. 486. Cor 2]). Moreover, it was proven by Matsumoto (cf. [BT, Thm. 14.4]) that $\mathcal{G}(\mathbb{R}) = G \cdot S(\mathbb{R})$ for any maximal $\mathbb{R}$-split torus S of $\mathcal{G}$. This implies that for a maximal $\mathbb{Q}$-split torus S defined over $\mathbb{Z}$, $S(\mathbb{Z})$ meets every connected component of $\mathcal{G}(\mathbb{R})$; hence so does $\mathcal{G}(\mathbb{Z})$, from which (2.1) follows. Since $U \cap \mathcal{Z}(\mathbb{A}_f) = \prod_p \mathcal{Z}(\mathbb{Z}_p)$, (2.2) follows from the well known fact that $\mathbb{Q}$ has class number 1.

Via the diagonal embedding, we may consider an element of $G_{\mathbb{Q}}$ as an element in $\mathcal{G}(\mathbb{A}_f)$. For each $a \in G_{\mathbb{Q}}$, we now set

$$G[a] = \{g \in G_{\mathbb{Q}} : g \in UaU\}.$$

If $\mathcal{G}$ is simply-connected, the strong approximation implies that $G[a] = \Gamma a \Gamma$. In general $G[a]$ is a union of Γ-double cosets of $G_{\mathbb{Q}}$.

Lemma 2.3. *The natural map from $\Gamma\backslash G[a]$ to $U\backslash UaU$ is a bijection.*

Proof. Denote this map by ι. It is clear that ι sends the set $\Gamma\backslash G[a]$ into the set $U\backslash UaU$. If b_1 and b_2 are elements of $G[a]$ such that $Ub_1 = Ub_2$, then $b_1 b_2^{-1} \in U \cap G_{\mathbb{Q}} = \Gamma$. Hence $\Gamma b_1 = \Gamma b_2$, and the map is injective. To show the surjectivity, let $x \in UaU$, and consider the element $(1, x^{-1}) \in G(\mathbb{A})$. Then by hypothesis (2.1), $(1, x^{-1}) = (br, bu)$ for some $b \in \mathcal{G}(\mathbb{Q})$, $r \in G$ and $u \in U$. Thus $b^{-1} = r \in G_{\mathbb{Q}}$; $b^{-1} = ux \in UaU$, hence $b^{-1} \in G[a]$. Therefore $x = u^{-1}b^{-1} \in UG[a]$. The surjectivity is proved. ∎

If we set

$$\begin{cases} \deg(a) = \#\Gamma\backslash G[a], \\ \deg_p(a) = \#U_p\backslash U_p a U_p. \end{cases}$$

Then Lemma 2.3 says that for any $a \in G_{\mathbb{Q}}$,

$$\deg(a) = \prod_p \deg_p(a); \tag{2.4}$$

in particular, $\deg(a) < \infty$.

Note that $\overline{\Gamma}$ is a lattice in $\overline{G}$ by the well-known theorem of Borel and Harish-Chandra [BH]. Denote by $\mu_{\bar{G}}$ the Haar measure on $\overline{G}$ with respect to which the quotient $\overline{\Gamma}\backslash\overline{G} \cong Z\Gamma\backslash G$ has volume 1. The hypotheses (2.1) and (2.2) imply that there is a G-equivariant bijection

$$\phi : \mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U \to \overline{\Gamma}\backslash\overline{G}.$$

This then defines a pullback map ϕ^* from functions on the space $\overline{\Gamma}\backslash\overline{G}$ to those on the space $\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U$. Naturally functions on $\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U$ can be considered as right U-invariant functions on $\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$. In what follows, we shall not distinguish these two spaces.

Let $C_c(\overline{\Gamma}\backslash\overline{G})$ denote the space of continuous functions with compact support on the real manifold $\overline{\Gamma}\backslash\overline{G}$, and $C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ the subspace of smooth functions. One also has the space $L^q(\overline{\Gamma}\backslash\overline{G})$ of L^q-integrable functions relative to the measure $\mu_{\bar{G}}$, with associated norm $\|\cdot\|_q$. We shall let $\langle -, - \rangle$ denote the natural sesquilinear pairing induced by $\mu_{\bar{G}}$ between $L^p(\overline{\Gamma}\backslash\overline{G})$ and $L^q(\overline{\Gamma}\backslash\overline{G})$ when $p^{-1} + q^{-1} = 1$.

If we give the locally compact group $\mathcal{Z}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)$ its unique Haar measure for which $(\mathcal{Z}(\mathbb{A}_f) \cap U)\backslash U$ has volume 1, this together with $\mu_{\bar{G}}$ defines a measure μ_0 on $\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$ which gives rise to a pairing $\langle -, - \rangle$ between $L^p(\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$

and $L^q(\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$ when $p^{-1}+q^{-1}=1$. Further, given $f_1 \in L^p(\overline{\Gamma}\backslash\overline{G})$ and $f_2 \in L^q(\overline{\Gamma}\backslash\overline{G})$, we have

$$\langle f_1, f_2\rangle = \langle \phi^*(f_1), \phi^*(f_2)\rangle. \tag{2.5}$$

Definition: Fix $a \in G_{\mathbb{Q}}$. For any function f on $\overline{\Gamma}\backslash\overline{G}$, set

$$T_a(f)(g) = \frac{1}{\deg(a)} \sum_{y \in \Gamma\backslash G[a]} f(yg).$$

Then $T_a(f)$ is also a function on $\overline{\Gamma}\backslash\overline{G}$, and is independent of the choice of representatives y of $\Gamma\backslash G[a]$ used. We call T_a the **Hecke operator** attached to a. Note that T_a preserves the spaces $C_c(\overline{\Gamma}\backslash\overline{G})$, $C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ and $L^2(\overline{\Gamma}\backslash\overline{G})$.

For each prime p, one can also define the local analog $T_{a(p)}$, which acts on functions f on $\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U$ as follows:

$$T_{a(p)}(f)(g) = \frac{1}{\deg_p(a)} \sum_{y \in U_p\backslash U_p a U_p} f(gy^{-1}).$$

The operators $T_{a(p)}$, for different primes p, commute with each other, and are equal to the identity operator for almost all p. Hence we obtain an operator

$$\hat{T}_a = \prod_p T_{a(p)}.$$

The following lemma relates the global Hecke operators to the local ones. Using Lemma 2.3, it can be proved in the same way as [COU, Thm. 2.3]:

Lemma 2.6. *Let $a \in G_{\mathbb{Q}}$. For any function f on $\overline{\Gamma}\backslash\overline{G}$, we have*

$$\phi^*(T_a(f)) = \hat{T}_a(\phi^*(f)).$$

It is clear from the definition of $\hat{T}_a$, considered as an operator on $L^2(\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/U)$, that $\|\hat{T}_a\| = 1$; hence by (2.5) and the above lemma, T_a also has norm 1 as an operator on $L^2(\overline{\Gamma}\backslash\overline{G})$.

Lemma 2.7. *Let $a \in G_{\mathbb{Q}}$.*
(i) We have $deg(a) = deg(a^{-1})$.
(ii) Whenever both sides in the following converge, we have:

$$\langle T_a f, \psi\rangle = \langle f, T_{a^{-1}}\psi\rangle.$$

Proof. For any function f on $\overline{\Gamma}\backslash\overline{G}$, set $f' = \phi^*(f)$ for simplicity. Also set $X = \mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$. By (2.5) and Lemma 2.6, it suffices to prove the lemma for $\hat{T}_a$.

Observe that

$$\begin{aligned}
\deg(a)\cdot\langle \hat{T}_a f', \psi'\rangle &= \int_{X/U}\left(\sum_{x\in U\cap a^{-1}Ua\backslash U} f'(g(ax)^{-1})\right)\cdot\overline{\psi'(g)}\,d\mu_0(g)\\
&= \int_{X/U}\left(\sum_{x\in U/U\cap a^{-1}Ua} f'(gxa^{-1})\right)\cdot\overline{\psi'(g)}\,d\mu_0(g)\\
&= \int_{X/U\cap a^{-1}Ua} f'(ga^{-1})\cdot\overline{\psi'(g)}\,d\mu_0(g)\\
&= \int_{X/U\cap aUa^{-1}} f'(t)\cdot\overline{\psi'(ta)}\,d\mu_0(t)\\
&= \int_{X/U} f'(t)\cdot\left(\sum_{y\in U/U\cap aUa^{-1}} \overline{\psi'(tya)}\right)\,d\mu_0(t)\\
&= \int_{X/U} f'(t)\cdot\left(\sum_{y\in U\cap aUa^{-1}\backslash U} \overline{\psi'(t(a^{-1}y)^{-1})}\right)\,d\mu_0(t)\\
&= \int_{X/U} f'(t)\cdot\left(\sum_{y\in U\backslash Ua^{-1}U} \overline{\psi'(ty^{-1})}\right)\,d\mu_0(t)\\
&= \deg(a^{-1})\cdot\langle f', \hat{T}_{a^{-1}}\psi'\rangle
\end{aligned}$$

The above equality applied to constant functions f' and ψ' yields (i). (ii) then follows from (i) and the above equality. □

We illustrate the above discussion by considering the case when $\mathcal{G}$ is simply-connected; this is the case treated in [COU]. Then $G[a] = \Gamma a\Gamma$ is a single Γ-double coset and (2.4) holds (cf. [COU, Lemma 2.2]). In [COU], (2.4) and Lemma 2.6 allow one to reduce the global problem considered there to local harmonic analysis on $\mathcal{G}(\mathbb{Q}_p)$. When $\mathcal{G}$ is not simply-connected, the definition of T_a given above is designed so that the passage between local and global Hecke operators continues to hold.

3. Equidistribution of Hecke points on $Z\Gamma\backslash G$

The main result of [COU] is an equidistribution theorem for Hecke points on $\Gamma\backslash G$, where $\mathcal{G}$ is simple and simply-connected. In this section, we shall extend this theorem to the class of $\mathcal{G}$ considered in the previous section. To state the result, we need to introduce more notations.

For each prime p, let $\mathcal{A}_p$ be a maximal $\mathbb{Q}_p$-split torus of $\mathcal{G}$ such that K_p is good with respect to $\mathcal{A}_p$ in the sense of [Oh2, Prop. 2.1], and let $X^\bullet(\mathcal{A}_p)$ and $X_\bullet(\mathcal{A}_p)$ be the character and cocharacter groups respectively. Let $\Phi_p \subset X^\bullet(\mathcal{A}_p)$ be the set of non-multipliable roots in the root system $\Phi(\mathcal{G}, \mathcal{A}_p)$. We fix a system of positive roots Φ_p^+. Denoting by $\langle -, - \rangle$ the canonical perfect pairing between $X_\bullet(\mathcal{A}_p)$ and $X^\bullet(\mathcal{A}_p)$, we now set

$$P_p^+ = \{\lambda \in X_\bullet(\mathcal{A}_p) : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi_p^+\}.$$

Then there exists a finite set Ω_p contained in the centralizer of $\mathcal{A}_p(\mathbb{Q}_p)$ in $\mathcal{G}(\mathbb{Q}_p)$ such that

$$\mathcal{G}(\mathbb{Q}_p) = \bigcup_{\lambda \in P_p^+} \bigcup_{\varpi \in \Omega_p} K_p \lambda(p) \varpi K_p,$$

where the union above is disjoint (cf. [Si]). Using this decomposition, we regard a element $\alpha \in X^\bullet(\mathcal{A}_p)$ as a bi-K_p-invariant function on $\mathcal{G}(\mathbb{Q}_p)$. More precisely, if $g = k_1 \lambda(p) \varpi k_2$, then we set

$$\alpha(g) = p^{\langle \lambda, \alpha \rangle} \in \mathbb{Q}^\times.$$

Let $\mathcal{S}_p \subset \Phi_p^+$ be a maximal strongly orthogonal system of positive roots in the sense of [Oh1]. Such a system is not uniquely determined. However, the element

$$\eta_p := \sum_{\alpha \in \mathcal{S}_p} \alpha \in X^\bullet(\mathcal{A}_p)$$

is independent of the choice of $\mathcal{S}_p$ and has been determined in [Oh1]. Following [Oh2], we set

$$\xi_{\mathcal{S}_p}(g) = \prod_{\alpha \in \mathcal{S}_p} \Xi_p \begin{pmatrix} \alpha(g) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for each } g \in \mathcal{G}(\mathbb{Q}_p).$$

Here, Ξ_p is the Harish-Chandra function of $PGL_2(\mathbb{Q}_p)$; it is bi-invariant under $PGL_2(\mathbb{Z}_p)$, and is uniquely determined by:

$$\Xi_p \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = p^{-\frac{|ord_p(x)|}{2}} \cdot \left(\frac{|ord_p(x)| \cdot (p-1) + (p+1)}{p+1} \right) \quad \text{for any } x \in \mathbb{Q}_p^\times. \tag{3.1}$$

In the above, ord_p denotes the valuation on $\mathbb{Q}_p$ such that $ord_p(p) = 1$ with associated absolute value $|\cdot|_p$. We refer the reader to [Oh2] for more details on the properties of the function $\xi_{\mathcal{S}_p}$ and remark only that $0 < \xi_{\mathcal{S}_p}(g) \leq 1$ and for any $\epsilon > 0$, there exists a constant $C_{p,\epsilon} > 0$ such that

$$|\eta_p(g)|_p^{\frac{1}{2}} \leq \xi_{\mathcal{S}_p}(g) \leq C_{p,\epsilon} \cdot |\eta_p(g)|_p^{\frac{1}{2}-\epsilon} \text{ for any } g \in \mathcal{G}(\mathbb{Q}_p). \tag{3.2}$$

From the explicit formula given in (3.1), it is not difficult to see that for a fixed $\epsilon > 0$, there exists a constant $N_\epsilon > 0$ such that the constant $C_{p,\epsilon}$ can be chosen to be 1 for each prime $p > N_\epsilon$.

Henceforth, for each prime p, *we fix a maximal strongly orthogonal system* $\mathcal{S}_p$. Setting

$$R_1 = \{\text{primes } p : \mathbb{Q}_p\text{-rank of } \mathcal{Z}\backslash\mathcal{G} = 1\};$$
$$R_2 = \{\text{primes } p : \mathbb{Q}_p\text{-rank of } \mathcal{Z}\backslash\mathcal{G} \geq 2\},$$

we define a real-valued function ξ on $G_{\mathbb{Q}}$ by

$$\xi(g) = \prod_{p\in R_1} \xi_{\mathcal{S}_p}(g)^{\frac{1}{2}} \cdot \prod_{p\in R_2} \xi_{\mathcal{S}_p}(g). \tag{3.3}$$

Note that almost all terms in the above product is 1. Since almost all the constants $C_{p,\epsilon}$ can be taken to be 1 in (3.2), we deduce:

Lemma 3.4. *Given* $\epsilon > 0$*, there exists a constant* $C_\epsilon > 0$ *such that*

$$\xi(g) \leq C_\epsilon \prod_{p\in R_1} |\eta_p(g)|_p^{\frac{1}{4}-\epsilon} \cdot \prod_{p\in R_2} |\eta_p(g)|_p^{\frac{1}{2}-\epsilon}$$

for any $g \in G_{\mathbb{Q}}$.

We now introduce the Sobolev norm on $C^\infty(\overline{\Gamma}\backslash\overline{G})$ and our exposition below follows [BR, Appendix B] closely. Let $\mathfrak{g}$ be the Lie algebra of $\overline{G}$ and fix a basis $\{X_i\}$ of $\mathfrak{g}$. Each X_i acts on each $f \in C^\infty(\overline{\Gamma}\backslash\overline{G})$ by infinitesimal right translation and we set

$$S_k(f) = \left(\sum \|X_\alpha f\|_2^2\right)^{\frac{1}{2}}$$

where the sum is taken over all monomials $X_\alpha = X_{i_1}X_{i_2}...X_{i_n}$ of order $\leq k$ in the universal enveloping algebra of $\mathfrak{g}$. Note that if $f \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$, then $S_k(f) < \infty$. Then S_k is called the k-th Sobolev norm on $C^\infty(\overline{\Gamma}\backslash\overline{G})$. Henceforth, set

$$k = \text{ the smallest integer} > \frac{1}{2} \cdot dim(\overline{G}).$$

If we fix a closed embedding $\iota : \mathcal{Z}\backslash\mathcal{G} \hookrightarrow GL_n$, we obtain a norm function $\|\cdot\|$ on $\overline{G}$ by setting

$$\|g\| = \max_{i,j}\{|\iota(g)_{ij}|, |\iota(g^{-1})_{ij}|\}.$$

Let

$$B = \{g \in \overline{G} : \|g\| \leq 2\}.$$

Then B is a symmetric compact neighbourhood of the identity element in $\overline{G}$. For each $x \in \overline{G}$, we set

$$w(x) = (\text{the maximal cardinality of the fibers of } p_x)^{\frac{1}{2}}$$

where the map $p_x : B \to \overline{\Gamma}\backslash\overline{G}$ is given by $g \mapsto xg$. This defines a function w on $\overline{G}$ which is left-invariant under $\overline{\Gamma}$. The following lemma gives the key property of w we need:

Lemma 3.5. *There exists constants $C > 0$ and $r \geq 1$ such that*

$$w(g) \leq C \cdot \|g\|^r \quad \text{for any } g \in \overline{G}.$$

Proof. This is not difficult to prove using reduction theory. See for example [MW, Lem. I.2.4(a), Pg. 25-26]. □

The importance of the function w lies in its role in the following Sobolev type inequality [BR, Prop. B.2, Pg. 349]:

Proposition 3.6. *There exists a constant $C > 0$ such that*

$$|f(x)| \leq C \cdot w(x) \cdot S_k(f)$$

for any $f \in C^\infty(\overline{\Gamma}\backslash\overline{G})$ and $x \in \overline{\Gamma}\backslash\overline{G}$.

Having introduced the necessary notations, we can now state the main result of this section, which is an extension of [COU, Thm. 1.1 and Thm. 1.7] to a more general class of groups:

Theorem 3.7. *(i) There exists a constant $C > 0$ such that for any $f \in L^2(\overline{\Gamma}\backslash\overline{G})$ and any $a \in G_{\mathbb{Q}}$,*

$$\|T_a(f) - \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_{\bar{G}}(g)\|_2 \leq C \cdot \|f\|_2 \cdot \xi(a).$$

(ii) There exists a constant $C > 0$ such that for any $f \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$, $x \in \overline{\Gamma}\backslash\overline{G}$ and $a \in G_{\mathbb{Q}}$,

$$|T_a(f)(x) - \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_{\bar{G}}(g)| \leq C \cdot w(x) \cdot S_k(f) \cdot \xi(a).$$

(iii) For any $f \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ and $x \in \overline{\Gamma}\backslash\overline{G}$, we have

$$\lim_{deg(a)\to\infty} T_a(f)(x) = \int_{\overline{\Gamma}\backslash\overline{G}} f(g)\, d\mu_{\bar{G}}(g).$$

The rest of the section is devoted to the proof of the above theorem. Using the definition of the Hecke operator T_a given in the previous section, the proof of (i) is virtually identical to that of [COU, Thm. 1.1]. Hence, we shall only give a brief sketch of the proof.

- The main point of the proof is to give an upper bound for the operator norm $||T_a||$ for the action of T_a on the subspace L_0 of functions in $L^2(\overline{\Gamma}\backslash\overline{G})$ which are orthogonal to the constant functions. Using Lemma 2.6 and the fact that ϕ^* is an isometry, we are reduced to estimating the operator norm of $\hat{T}_a$ on the subspace of U-invariant functions in the orthogonal complement $\hat{L}_0$ of the constant functions in $L^2(\mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$.

- By an elementary but crucial observation [COU, Prop. 2.6], this is equivalent to estimating the size of matrix coefficients for the unitary representations of $\mathcal{G}(\mathbb{Q}_p)$ intervening in the representation $\hat{L}_0$ of $\mathcal{G}(\mathbb{A})$. This observation puts us in a position to apply [Oh2, Thm. 1.1], at least when the $\mathbb{Q}_p$-rank of $\mathcal{Z}\backslash\mathcal{G}$ is ≥ 2.
- To be able to apply [Oh2, Thm. 1.1], it is necessary to show the following lemma:

Lemma 3.8. *Fix a direct integral decomposition of $\hat{L}_0$ into irreducible unitary representations of $\mathcal{G}(\mathbb{A})$. Let $\mathcal{A}$ be the set of irreducible unitary representation $\pi = \hat{\otimes}_v \pi_v$ of $\mathcal{G}(\mathbb{A})$ occurring in the direct integral decomposition of $\hat{L}_0$ such that*

- *π has a non-zero U-invariant vector;*
- *for some finite prime p, π_p is 1-dimensional.*

Then the set $\mathcal{A}$ has measure zero with respect to the measure giving the direct integral decomposition of $\hat{L}_0$.

Proof. Let us decompose the unitary representation $\hat{L}_0$ into the direct sum of its continuous and discrete spectrum. The continuous spectrum has been described by Langlands in terms of the discrete spectrum of Levi subgroups, using his theory of Eisenstein series (cf. [MW]). One sees directly from this description that the set of irreducible representations $\pi \in \mathcal{A}$ which occur in the continuous spectrum indeed has measure zero. Thus it remains to deal with the discrete spectrum.

Suppose V is a subspace of $\hat{L}_0$ which affords the irreducible unitary representation π. Let $\tilde{\mathcal{G}}$ be the simply-connected cover of the derived group of $\mathcal{G}$. Then there is a natural projection map

$$\tilde{\mathcal{G}}(\mathbb{Q})\backslash\tilde{\mathcal{G}}(\mathbb{A}) \to \mathcal{Z}(\mathbb{A})\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$$

and using this, we can pull back a function $f \in V$ to obtain a function $\tilde{f}$ on $\tilde{\mathcal{G}}(\mathbb{Q})\backslash\tilde{\mathcal{G}}(\mathbb{A})$. Suppose that there is a finite prime p such that $\tilde{f}$ is right-invariant under $\tilde{\mathcal{G}}(\mathbb{Q}_p)$. Suppose that f is continuous; hence so is $\tilde{f}$. Since $\tilde{f}$ is left $\tilde{\mathcal{G}}(\mathbb{Q})$-invariant and right $\tilde{\mathcal{G}}(\mathbb{Q}_p)$-invariant, it follows by the strong approximation theorem for $\tilde{\mathcal{G}}$ that $\tilde{f}$ is constant. Since the continuous functions in V are dense in V, we deduce that all functions in V are fixed by the image of $\tilde{\mathcal{G}}(\mathbb{A})$ in $\mathcal{G}(\mathbb{A})$, as well as by $\mathcal{Z}(\mathbb{A})$. This implies that V is a 1-dimensional space spanned by a unitary character χ of $\mathcal{G}(\mathbb{A})$, which is trivial on $\mathcal{G}(\mathbb{Q})$ and $\mathcal{Z}(\mathbb{A})$. Since $V \subset \hat{L}_0$, χ is a non-trivial character.

Now suppose further that χ is trivial on U. Then it follows by the hypothesis (2.1) that χ gives rise to a non-trivial character on $\overline{G}$. This is a contradiction, since $\overline{G}$ is a connected semisimple real Lie group and has no non-trivial character. The lemma is proved. □

With the lemma, we can now apply [Oh2, Thm. 1.1] to obtain the desired bound on matrix coefficients, when the $\mathbb{Q}_p$-rank of $\mathcal{Z}\backslash\mathcal{G}$ is ≥ 2. On the other hand, if the $\mathbb{Q}_p$-rank of $\mathcal{Z}\backslash\mathcal{G}$ is equal to 1, one appeals to [CU, Thm. 5.1], which is a p-adic analogue of the results of Burger and Sarnak [BS], and the Gelbart-Jacquet bound towards the Ramanujan conjecture of GL_2, as in the proof of [COU, Thm. 1.1]. This completes a sketch of the proof of (i) in the theorem.

The statement (ii) now follows from (i), using Prop. 3.6 and the fact that the Hecke operator T_a commutes with all infinitesimal right translations. (iii) is a direct consequence of (ii), since $\xi(a) \to 0$ as $\deg(a) \to \infty$.

This completes the proof of Theorem 3.7.

4. Equidistribution of Hecke points on $ZH\backslash G$

In this section, we extend the equidistribution result in Theorem 3.7 to homogeneous varieties of G. Keeping the same notation of the previous sections, we further let $\mathcal{H} \subset \mathcal{G}$ be a $\mathbb{Q}$-algebraic subgroup and let $H = \mathcal{H}(\mathbb{R}) \cap G$. Assume that $\overline{\Gamma} \cap \overline{H}$ is a lattice in $\overline{H}$, or equivalently that the identity component of $(\mathcal{Z} \cap \mathcal{H})\backslash\mathcal{H}$ does not possess any non-trivial $\mathbb{Q}$-rational character. Let $\mu_{\bar{H}}$ be the right H-invariant measure on $\overline{H}$ which gives $(\overline{\Gamma} \cap \overline{H})\backslash\overline{H}$ volume 1. The measures $\mu_{\bar{G}}$ and $\mu_{\bar{H}}$ induce a unique G-invariant measure μ on the homogeneous space $\overline{H}\backslash\overline{G} \cong ZH\backslash G$. Given a measurable subset Ω of $\overline{H}\backslash\overline{G}$, we shall write $vol(\Omega)$ for its measure with respect to μ.

Given an integrable function f with compact support on $\overline{H}\backslash\overline{G}$, we define a function F on $\overline{\Gamma}\backslash\overline{G}$ by:

$$F(g) = \sum_{\gamma \in (\overline{\Gamma} \cap \overline{H})\backslash\overline{\Gamma}} f(\gamma g). \tag{4.1}$$

Observe that:

- the support of F is compact if and only if $\overline{\Gamma} \cap \overline{H}$ is cocompact in $\overline{H}$;
- F is an integrable function on $\overline{\Gamma}\backslash\overline{G}$:

$$\int_{\overline{\Gamma}\backslash\overline{G}} F(g) d\mu_{\bar{G}}(g) = \mu(f) := \int_{\overline{H}\backslash\overline{G}} f(g) d\mu(g).$$

We would like to show that for any $x \in \overline{\Gamma}\backslash\overline{G}$,

$$T_a(F)(x) \to \mu(f) \quad \text{as } \deg(a) \to \infty.$$

This is not a consequence of Theorem 3.7, since we do not know that F is smooth of compact support, or even square-integrable. Nevertheless, the following theorem says that the above limit holds in the weak sense and further that the rate of convergence can be estimated.

Theorem 4.2. *Let f be an integrable function of compact support on $\overline{H}\backslash\overline{G}$, and let F be constructed from f as in (4.1).*

(i) For any $\psi \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$,

$$\langle T_a F, \psi \rangle \to \langle \mu(f), \psi \rangle \quad \text{as } \deg(a) \to \infty.$$

(ii) For any $\psi \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ *and* $a \in G_{\mathbb{Q}}$,

$$|\langle T_a(F) - \mu(f), \psi\rangle| \leq C_f \cdot C_\psi \cdot \xi(a^{-1})^\delta$$

where $C_f > 0$ *is a constant depending on* f,

$$C_\psi = \begin{cases} S_k(\psi), & \text{if } \overline{\Gamma}\cap\overline{H} \text{ is cocompact in } \overline{H};\\ S_k(\psi) + \|\psi\|_1 + \|\psi\|_\infty, & \text{otherwise,}\end{cases}$$

and δ *is a positive constant* ≤ 1, *with equality when* $\overline{\Gamma}\cap\overline{H}$ *is cocompact in* $\overline{H}$.

Proof. By Lemma 2.7(ii), we have

$$\langle T_a(F), \psi\rangle = \langle F, T_{a^{-1}}\psi\rangle.$$

Note also that

$$\langle \mu(f), \psi\rangle = \langle F, \mu_{\bar{G}}(\psi)\rangle.$$

Hence

$$\langle T_a(F) - \mu(f), \psi\rangle = \langle F, T_{a^{-1}}\psi - \mu_{\bar{G}}(\psi)\rangle.$$

This latter integral can be written as:

$$\int_{\overline{H}\backslash\overline{G}} f(g)\cdot\left(\int_{(\overline{\Gamma}\cap\overline{H})\backslash\overline{H}} (T_{a^{-1}}(\psi)(hg) - \mu_{\bar{G}}(\psi))d\mu_{\bar{H}}(h)\right) d\mu(g).$$

Statement (i) now follows by Theorem 3.7(iii), applied to ψ, and the dominated convergence theorem. Similarly, the cocompact case in statement (ii) follows immediately by Theorem 3.7(ii), using the fact that the function w is bounded on compact subsets of $\overline{\Gamma}\backslash\overline{G}$.

It remains to consider the case when $\overline{\Gamma}\cap\overline{H}$ is not cocompact in $\overline{H}$, which is much more involved. Let us fix a compact subset $\tilde{\Omega} \subset \overline{G}$ which maps bijectively to the closure of the support of f. We first obtain a bound for the integral

$$\Phi(g) := \int_{(\overline{\Gamma}\cap\overline{H})\backslash\overline{H}} |T_{a^{-1}}(\psi)(hg) - \mu_{\bar{G}}(\psi)| d\mu_{\bar{H}}(h),$$

as g varies over $\tilde{\Omega}$. This is done by dividing the domain of the integration, using Siegel sets and reduction theory, as we now explain.

We first recall what a Siegel set is. Let $\mathcal{L} \ltimes \mathcal{U}$ be a Levi decomposition of the (possibly disconnected) algebraic group $(\mathcal{Z}\cap\mathcal{H})\backslash\mathcal{H}$ with $\mathcal{U}$ its unipotent radical. Choose a maximal $\mathbb{Q}$-split torus $\mathcal{A}$ of $\mathcal{L}$ and let $\mathcal{P}$ be a minimal parabolic subgroup of $\mathcal{L}^0$ containing $\mathcal{A}$, with unipotent radical $\mathcal{N}$. Then the Levi subgroup of $\mathcal{P}$ containing $\mathcal{A}$ is an almost direct product $\mathcal{M}\cdot\mathcal{A}$, where $\mathcal{M}$ is an anisotropic reductive algebraic group over $\mathbb{Q}$:

$$\mathcal{P} = \mathcal{N}\mathcal{M}\mathcal{A}.$$

The choice of $\mathcal{P}$ determines a system of simple roots Δ for $\mathcal{L}$ relative to $\mathcal{A}$, and we set:

$$A_t = \{a \in \mathcal{A}(\mathbb{R})^0 : \alpha(a) \geq t \text{ for all } \alpha \in \Delta\}.$$

Choose compact subsets $\omega_1 \subset (\mathcal{N}\cdot\mathcal{M})(\mathbb{R})$ and $\omega_2 \subset \mathcal{U}(\mathbb{R})$. Then for a suitable maximal compact subgroup K of $\mathcal{L}(\mathbb{R})$, the subset

$$\omega_1 A_t K \omega_2 \subset ((\mathcal{Z} \cap \mathcal{H})\backslash\mathcal{H})(\mathbb{R})$$

is called a Siegel set.

The natural map $\overline{H} \to (\mathcal{Z} \cap \mathcal{H}\backslash\mathcal{H})(\mathbb{R})$ has finite kernel and cokernel. Using this, we let $\Sigma \subset \overline{H}$ be the inverse image of $\omega_1 A_t K \omega_2$. Reduction theory (cf. [Bo] and [PR, Ch. 4]) says that for some $t < 1$, ω_1 and ω_2 which will be fixed henceforth, there exists a finite number $h_1, ..., h_r$ of elements in $\overline{H}$ such that $\bigcup_i \Sigma \cdot h_i$ is a fundamental set for the quotient $(\overline{\Gamma} \cap \overline{H})\backslash\overline{H}$.

Now for $R \geq 1 > t$, we set

$$A_{t,R} = \{a \in \mathcal{A}(\mathbb{R})^0 : t \leq \alpha(a) \leq R \text{ for all } \alpha \in \Delta\},$$

which is a compact subset of A_t and let $\Sigma_{\leq R} \subset \overline{H}$ be the inverse image of $\omega_1 A_{t,R} K \omega_2$. Setting $\Sigma_{>R} = \Sigma \setminus \Sigma_{\leq R}$, we deduce from the above that

$$\Phi(g) \leq \Phi_{\leq R}(g) + \Phi_{>R}(g),$$

where

$$\Phi_{\leq R}(g) = \sum_i \int_{\Sigma_{\leq R}} |T_{a^{-1}}\psi(hh_i g) - \mu_{\bar{G}}(\psi)| d\mu_{\bar{H}}(h);$$

$$\Phi_{>R}(g) = \sum_i \int_{\Sigma_{>R}} |T_{a^{-1}}\psi(hh_i g) - \mu_{\bar{G}}(\psi)| d\mu_{\bar{H}}(h).$$

We first give a bound for $\Phi_{>R}(g)$. Using standard integration formulas [PR, Pg. 213], it is not difficult to check that

$$\int_{\Sigma_{>R}} d\mu_{\bar{H}} \leq C_1 \cdot R^{-n},$$

for some constants C_1 and n. Together with the fact that $\|T_{a^{-1}}(\psi)\|_\infty \leq \|\psi\|_\infty$, one sees that for some constant $C_1' > 0$

$$\Phi_{>R}(g) \leq C_1' \cdot (\|\psi\|_\infty + \|\psi\|_1) \cdot R^{-n} \quad \text{for any } g \in \overline{G}.$$

It remains to estimate $\Phi_{\leq R}(g)$, as g varies over $\tilde{\Omega}$. Applying Theorem 3.7(ii), we deduce that

$$\Phi_{\leq R}(g) \leq C_f \cdot S_k(\psi) \cdot \xi(a^{-1}) \cdot \sup_{h \in \Sigma_{\leq R}} w(h)$$

for some constant $C_f > 0$ depending on the support of f. By Lemma 3.5, it is not difficult to check that

$$\sup_{h \in \Sigma_{\leq R}} w(h) \leq C_2 \cdot R^m,$$

for some positive constants $C_2 > 0$ and $m \geq 1$.

In conclusion, we have shown that for any $g \in \tilde{\Omega}$,

$$|\Phi(g)| \leq C'_f \cdot (S_k(\psi) \cdot \xi(a^{-1}) \cdot R^m + (\|\psi\|_1 + \|\psi\|_\infty) \cdot R^{-n})$$

for some constant $C'_f > 0$ depending on f and some constants $m \geq 1$ and $n \geq 1$ (independent of f). Putting

$$R = \xi(a^{-1})^{-\frac{1}{m+n}} \geq 1 > t,$$

we have:

$$\Phi(g) \leq C''_f \cdot (S_k(\psi) + \|\psi\|_1 + \|\psi\|_\infty) \cdot \xi(a^{-1})^\delta,$$

where $\delta = \frac{n}{m+n} < 1$. Since

$$|\langle T_a(F) - \mu(f), \psi\rangle| \leq \int_{\overline{H}\backslash\overline{G}} |f(g)| \cdot \Phi(g) d\mu(g),$$

the desired result follows and Theorem 4.2 is proved completely. □

We shall henceforth specialize to the case where f is the characteristic function of a compact subset Ω of $\overline{H}\backslash\overline{G}$ so that $\mu(f) = vol(\Omega)$. The function F_Ω constructed from f by (4.1) satisfies:

$$F_\Omega(g) = \#\Omega g^{-1} \cap v_0\Gamma,$$

where v_0 denotes the coset of the identity element in $\overline{H}\backslash\overline{G}$. Further, for any $a \in G_{\mathbb{Q}}$, we have:

$$T_a(F_\Omega)(g) = \frac{1}{\deg(a)} \cdot \sum_{y \in \Gamma\backslash G[a]} \#\Omega g^{-1} \cap v_0\Gamma y. \tag{4.3}$$

We should remark here that the subsets $v_0\Gamma y$ of $\overline{H}\backslash\overline{G}$ need not be disjoint as y ranges over $\Gamma\backslash G[a]$, though they are disjoint if H is trivial. The goal of this section is to use Theorem 4.2 to obtain an asymptotic formula for $T_a(F_\Omega)(1)$ as $\deg(a) \to \infty$. To convert the weak convergence of Theorem 4.2 to a pointwise convergence, we shall need to restrict the class of compact subsets to consider.

Recall that we have chosen a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g}$ of $\overline{G}$. This induces a Euclidean metric on $\mathfrak{g}$, and a $\overline{G}$-invariant metric on $\overline{G}$. For a sufficiently small $\epsilon > 0$, and $D_\epsilon = \{x \in \mathfrak{g} : |x| < \epsilon\}$, we set

$$U_\epsilon := exp(D_\epsilon) \subset \overline{G},$$

and call this the ϵ-neighborhood of the identity element in $\overline{G}$.

Lemma 4.4. *For a sufficiently small $\epsilon > 0$, there exists a non-negative function $\psi_\epsilon \in C_c^\infty(\overline{\Gamma}\backslash\overline{G})$ which is supported on the image of U_ϵ in $\overline{\Gamma}\backslash\overline{G}$ and which satisfies:*

$$\begin{cases} \|\psi_\epsilon\|_1 = 1, \\ \|\psi_\epsilon\|_\infty \leq C \cdot \epsilon^{-d}; \\ S_k(\psi_\epsilon) \leq C \cdot \epsilon^{-d-1}, \end{cases}$$

where C is a constant independent of $d = dim(\overline{G})$ and ϵ.

Proof. For a sufficiently small ϵ_0, the natural map $\varphi : D_{\epsilon_0} \to \overline{\Gamma}\backslash\overline{G}$ is a local diffeomorphism and thus provides a local chart at the identity coset. So we are reduced to the question of finding a function ψ_ϵ supported on the disc D_ϵ in $\mathfrak{g}$ with suitable properties, for all $\epsilon \leq \epsilon_0$ say. Let f be a bump function on D_1, i.e. a non-negative smooth function supported on D_1 with integral 1. Now set

$$f_\epsilon(x) = \frac{1}{\epsilon^d} f(\frac{x}{\epsilon}),$$

which is supported on D_ϵ. Then it suffices to take ψ_ϵ to be the multiple of $f_\epsilon \circ \varphi^{-1}$ with L^1-norm 1. □

We now make the following definition:

Definition 4.5. *A compact subset $\Omega \subset \overline{H}\backslash\overline{G}$ is nice if for all sufficiently small $\epsilon > 0$ (depending on Ω),*

$$vol(\partial\Omega \cdot U_\epsilon) < C_\Omega \cdot \epsilon$$

for some constant $C_\Omega > 0$ depending on Ω. Here $\partial\Omega$ denotes the boundary of Ω.

Note that this definition is independent of the choice of the metric on $\mathfrak{g}$. A compact subset being nice is a very mild condition. Any compact subset of the manifold $\overline{H}\backslash\overline{G}$ with piecewise smooth boundary is nice in the above sense. In particular, any point x of $\overline{H}\backslash\overline{G}$ has a basis of neighborhoods consisting of nice compact subsets. The main property of nice compact subsets we need is contained in the following lemma (cf. [EM, Prop. 3.3]):

Lemma 4.6. *Let Ω be a nice compact subset of $\overline{H}\backslash\overline{G}$. For any sufficiently small $\epsilon > 0$, we have*

$$vol(\Omega_{\epsilon+}) - C_\Omega \cdot \epsilon \leq vol(\Omega) \leq vol(\Omega_{\epsilon-}) + C_\Omega \cdot \epsilon,$$

where $\Omega_{\epsilon-} = \cap_{u \in U_\epsilon} \Omega u$ and $\Omega_{\epsilon+} = \cup_{u \in U_\epsilon} \Omega u$.

We can now prove the main result of this section:

Theorem 4.7. *Let Ω be a nice compact subset of $\overline{H}\backslash\overline{G}$. There exists a constant C_Ω (depending only on Ω) such that for any $a \in G_{\mathbb{Q}}$, we have*

$$|\frac{1}{deg(a)} \cdot \left(\sum_{y \in \Gamma\backslash G[a]} \#v_0\Gamma y \cap \Omega \right) - vol(\Omega)| \leq C_\Omega \cdot \xi(a^{-1})^{\frac{\delta}{d+2}},$$

where $0 < \delta \leq 1$ is the exponent appearing in Theorem 4.2(ii) and $d = dim(\overline{G})$.

Proof. Fix a sufficiently small $\epsilon > 0$; we will specify its value later. Let ψ_ϵ be the function supported on U_ϵ furnished by Lemma 4.4. For any $g \in U_\epsilon$, it is clear that

$$\Omega_{\epsilon -} g^{-1} \subset \Omega \subset \Omega_{\epsilon +} g^{-1}$$

and hence, by virtue of (4.3), we have

$$T_a F_{\Omega_{\epsilon -}}(g) \leq T_a F_\Omega(1) \leq T_a F_{\Omega_{\epsilon +}}(g).$$

Since $\int \psi_\epsilon = 1$, we see that

$$\langle T_a F_{\Omega_{\epsilon -}}, \psi_\epsilon \rangle \leq T_a F_\Omega(1) \leq \langle T_a F_{\Omega_{\epsilon +}}, \psi_\epsilon \rangle.$$

On the other hand, by Theorem 4.2(ii) and Lemma 4.4, we have

$$|\langle T_a F_{\Omega_{\epsilon \pm}}, \psi_\epsilon \rangle - vol(\Omega_{\epsilon \pm})| \leq C_\Omega \cdot \xi(a^{-1})^\delta \cdot \epsilon^{-d-1}, \tag{4.8}$$

for some constant C_Ω, and some $0 < \delta \leq 1$.

Now using (4.8) and Lemma 4.6, there is a constant C'_Ω such that

$$|T_a(F_\Omega)(1) - vol(\Omega)| \leq C'_\Omega \cdot (\epsilon + \epsilon^{-d-1} \cdot \xi(a^{-1})^\delta),$$

for all sufficiently small $\epsilon > 0$. Now take

$$\epsilon = \epsilon_0 \cdot \xi(a^{-1})^{\frac{\delta}{d+2}}$$

for a sufficiently small ϵ_0 (independent of a). Then we conclude that for some constant $C''_\Omega > 0$

$$|T_a F_\Omega(1) - vol(\Omega)| \leq C''_\Omega \cdot \xi(a^{-1})^{\frac{\delta}{d+2}},$$

as required. □

The following is an immediate corollary of Theorem 4.7, though it can also be directly deduced from (4.8) and Theorem 4.2(i).

Corollary 4.9. *Let Ω be a nice compact subset of $\overline{H}\backslash\overline{G}$. Then*

$$\lim_{deg(a) \to \infty} \frac{1}{deg(a)} \cdot \sum_{y \in \Gamma\backslash G[a]} \#v_0\Gamma y \cap \Omega = vol(\Omega).$$

5. **Integer points on homogeneous varieties**

With the results of the previous section, we now give a proof of a somewhat weakened version of Theorem 1.2 for any $\mathbb{Q}$-isotropic G. Let us briefly recall the setting. Suppose that f is a homogeneous polynomial of degree d with integer coefficients on $V = \mathbb{R}^n$. Let

$$V_m(\mathbb{Z}) = \{x \in \mathbb{Z}^n : f(x) = m\} \quad \text{for each } m \in \mathbb{Z}$$

and fix $v_0 \in V_1(\mathbb{Z})$. Let $\mathcal{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$ with one-dimensional center and $\iota : \mathcal{G} \to \mathcal{GL}(V)$ a $\mathbb{Q}$-rational representation with respect to which f is a semi-invariant. Assume that $\mathcal{G}_0 := [\mathcal{G}, \mathcal{G}]$ is absolutely simple and $\mathbb{Q}$-isotropic. Assume that the identity component of the stabilizer $\mathcal{H}$ of v_0 in $\mathcal{G}_0$ has no non-trivial $\mathbb{Q}$-rational character.

Since $v_0\mathcal{G}_0(\mathbb{R})^0 = v_0\mathcal{G}_0^{sc}(\mathbb{R})$, there is no loss of generality in assuming that $\mathcal{G}_0$ is simply connected. In this case, $G_0 := \mathcal{G}_0(\mathbb{R})$ is connected and all the assumptions we made in Section 2 hold (cf. the remark following (2.2)), with Γ the stabilizer in G_0 of the lattice $\mathbb{Z}^n$. Under these conditions, we shall show the following:

Theorem 5.1. *Fix a compact subset $\Omega \subset v_0G_0$ and for any small $\epsilon > 0$, consider the standard division of $\mathbb{R}^n$ into ϵ-cubes. Then there exists an effective constant $m_{\Omega,\epsilon}$ such that for any positive integer $m > m_{\Omega,\epsilon}$, any ϵ-cube intersecting the interior of Ω contains at least one point in the radial projection of $V_{Nm^r}(\mathbb{Z})$ into V_1. Here N and r are explicit positive integers which depend only on $\mathcal{G}$, ι and deg(f).*

The rest of the section is devoted to the proof of the theorem. Since $\mathcal{G}_0$ is $\mathbb{Q}$-isotropic, there is a non-trivial $\mathbb{Q}$-rational one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathcal{G}_0$. There exist an element $h \in \mathcal{GL}_n(\mathbb{Q})$ and integers $k_1, ..., k_n$ (depending only on λ and ι and not all zero) such that

$$\lambda(t) = h \cdot \mathrm{diag}(t^{k_1}, ..., t^{k_n}) \cdot h^{-1} \quad \text{for all } t \in \mathbb{G}_m.$$

Now let us set $a_m = \lambda(m) \in G_{\mathbb{Q}}$ for each positive integer m. Setting

$$r = -\mathrm{min}_{1\le i\le n}\{k_i\} > 0,$$

it is easy to see that there is a positive integer N (depending only on λ and ι) such that $Nm^r a_m$ preserves the lattice $\mathbb{Z}^n$. Indeed, we see that

$$v_0\Gamma(Nm^r a_m)\Gamma \subset V_{N^dm^{rd}}(\mathbb{Z})$$

and thus the radial projection of these points onto V_1 are precisely the points $v_0\Gamma a_m\Gamma$.

For each open ϵ-cube B intersecting the interior $Int(\Omega)$ of Ω, fix a nice compact subset $\omega_B \subset B \cap Int(\Omega)$; as we remarked after the definition of nice compact sets in the previous section, this is possible since every point in v_0G has a basis of (compact) neighbourhoods which are nice compact sets. Since the number of ϵ-cubes B intersecting $Int(\Omega)$ is finite, there is a constant $\delta > 0$ such that $\mathrm{vol}(\omega_B) > \delta$ for each such B.

As an immediate consequence of Corollary 4.9 and the fact that $\xi(a_m) \to 0$ as $m \to \infty$, we see that the ineffective version of the theorem (i.e. for which one has no control on the constant $m_{\Omega,\epsilon}$) holds for the sequence $\{N^d m^{dr} : m \in \mathbb{Z}_{>0}\}$.

To obtain the effective version of Theorem 5.1, we apply Theorem 4.7 to the element a_m:

$$|\frac{1}{\deg(a_m)} \cdot \left(\sum_{y \in \Gamma \backslash G[a_m]} \# v_0 \Gamma y \cap \Omega \right) - vol(\Omega)| \leq C_\Omega \cdot \xi(a_m^{-1})^{\frac{\delta}{d+2}}.$$

It remains to analyze $\xi(a_m^{-1})$ more carefully and to show that there is an effectively computable constant $\kappa > 0$ such that for any $\epsilon > 0$,

$$\xi(a_m) \leq C_\epsilon \cdot m^{-\kappa+\epsilon} \quad \text{for some constant } C_\epsilon.$$

Recall that the function ξ is the product of the local functions $\xi_{\mathcal{S}_p}$ or $\xi_{\mathcal{S}_p}^{\frac{1}{2}}$ (cf. 3.3). Further, the function ξ_p is defined using a maximal $\mathbb{Q}_p$-split torus $\mathcal{A}_p$ and a special maximal compact subgroup K_p which is good with respect to $\mathcal{A}_p$. The sequence $\{K_p\}$ is furnished by the $\mathbb{Q}$-structure of $\mathcal{G}_0$; for almost all p, it can be taken to be the stabilizer of $\mathbb{Z}^n$ in $\mathcal{G}_0(\mathbb{Q}_p)$ under the representation ρ. However, we are allowed to modify K_p (and correspondingly $\mathcal{A}_p$) as we wish for any given finite set of primes. We now note:

Lemma 5.2. *Let $\mathcal{T}$ be any $\mathbb{Q}$-split torus contained in $\mathcal{G}_0$. For all sufficiently large p, there exists a maximal $\mathbb{Q}_p$-split torus A_p such that*

- *$\mathcal{T} \subset \mathcal{A}_p$;*
- *K_p is good with respect to $\mathcal{A}_p$.*

Proof. Let $\mathcal{C}$ be the centralizer of $\mathcal{T}$ in $\mathcal{G}_0$. For p sufficiently large, we have:

- K_p is a hyperspecial maximal compact subgroup;
- $\mathcal{T}(\mathbb{Q}_p) \cap K_p$ is the (unique) maximal compact subgroup T_0 of $\mathcal{T}(\mathbb{Q}_p)$.

For such primes p, if v_p is the unique vertex fixed by K_p in the Bruhat-Tits building $B(\mathcal{G}_0, \mathbb{Q}_p)$ of $\mathcal{G}_0(\mathbb{Q}_p)$, then v_p is fixed by T_0.

To prove the lemma, we need to show that v_p in fact lies in the subset $B(\mathcal{C}, \mathbb{Q}_p) \subset B(\mathcal{G}_0, \mathbb{Q}_p)$ for almost all p. Indeed, if this is the case, then v_p lies in some apartment of $B(\mathcal{C}, \mathbb{Q}_p)$. This apartment corresponds to a maximal $\mathbb{Q}_p$-split torus $\mathcal{A}_p$ of $\mathcal{C}$ and thus of $\mathcal{G}_0$. The torus $\mathcal{A}_p$ then satisfies the desired properties.

Finally, the claim that v_p lies in $B(\mathcal{C}, \mathbb{Q}_p)$ follows from a result of Prasad-Yu [PY, Prop. 1.3], which says that $B(\mathcal{C}, \mathbb{Q}_p) = B(\mathcal{G}_0, \mathbb{Q}_p)^{T_0}$. □

Note that this lemma holds for any connected $\mathbb{Q}$-isotropic semisimple $\mathcal{G}_0$; we do not need $\mathcal{G}_0$ to be simply-connected or absolutely simple.

We apply the lemma with $\mathcal{T}$ equal to the image of λ. By modifying the choice of $\mathcal{A}_p$ and K_p for a finite set of primes, we may thus assume that the conclusion of the lemma holds for all primes p. Then the element a_m lies in $\mathcal{A}_p(\mathbb{Q}_p)$ for all p and the desired upper bound for $\xi(a_m^{-1})$ follows immediately from Lemma 3.4. This gives the effective version of Theorem 5.1, i.e. with control on $m_{\Omega,\epsilon}$.

Remarks: If the diagonal torus of $\mathcal{GL}_n$ intersects $\iota(\mathcal{G}_0)$ non-trivially, then we can choose λ to take values in the diagonal torus and thus N can be taken to be 1 in Theorem 5.1. For example, this is the case when $\mathcal{G}$ is split and the representation ι is defined over $\mathbb{Z}$ for the canonical $\mathbb{Z}$-structure on $\mathcal{G}$. This gives Theorem 1.2 of the introduction. However, the sequence $\{Nm^r\}$ produced above is almost never optimal. For instance, in the example where $f = det$ and $\mathcal{G} = \mathcal{GL}_n$ discussed in the introduction, the above proof gives an equidistribution result only for the sequence $\{m^n : m > 0\}$, whereas by Linnik, one knows that Theorem 1.2 holds for the sequence $\{m : m > 0\}$. Further, the above proof does not give the more precise result (1.5). For that, one would need to consider many Hecke orbits at the same time. For the rest of the paper, we address these more refined questions in the case when $\mathcal{G}$ is a split group.

6. A technical estimate

Henceforth, let $\mathcal{G}$ be any connected $\mathbb{Q}$-split reductive algebraic group of semisimple rank $l \geq 1$. It is equipped with a canonical $\mathbb{Z}$-structure such that for each finite prime p, $\mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\mathcal{G}(\mathbb{Q}_p)$. Recall the function ξ from (3.3), which is constructed by bi-$\mathcal{G}(\mathbb{Z}_p)$-invariant functions $\xi_{\mathcal{S}_p}$ defined in Section 3. In this section, we prove the following technical statement which will be used in the next section.

Proposition 6.1. *There exists an explicit constant* $0 < c < 1$ *such that for any* $\epsilon > 0$, *there exists a constant* $C_\epsilon > 0$ *such that*

$$\xi(a) \leq C_\epsilon \cdot deg(a)^{-c+\epsilon} \quad \textit{for any } a \in G_{\mathbb{Q}}.$$

Fix a maximal split torus $\mathcal{A}$ contained in a Borel subgroup $\mathcal{B}$ of $\mathcal{G}$, both of which are defined over $\mathbb{Z}$. Then $\mathcal{G}(\mathbb{Z}_p)$ is a good maximal compact subgroup with respect to $\mathcal{A}$ [Oh2, Prop. 2.1]. Let $\Phi \subset X^\bullet(\mathcal{A})$ be the set of roots of $\mathcal{G}$ relative to $\mathcal{A}$ and set

$$2\rho = \sum_{\alpha \in \Phi^+} \alpha.$$

The set Φ^+ determines a positive Weyl chamber:

$$P^+ = \{\lambda \in X_\bullet(\mathcal{A}) : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

For each finite prime p, this gives the Cartan decomposition (cf. [Gr])

$$\mathcal{G}(\mathbb{Q}_p) = \bigcup_{\lambda \in P^+} \mathcal{G}(\mathbb{Z}_p)\lambda(p)\mathcal{G}(\mathbb{Z}_p). \tag{6.2}$$

We now have:

Lemma 6.3. *For each finite prime p and each $\epsilon > 0$, there exists a constant $C_\epsilon(p) > 0$ satisfying:*

- *for any $\lambda \in P^+$,*
$$p^{\langle\lambda,2\rho\rangle} \leq deg_p(\lambda(p)) \leq C_\epsilon(p) \cdot p^{\langle\lambda,2\rho\rangle\cdot(1+\epsilon)}.$$
- $C_\epsilon := \prod_p C_\epsilon(p) \leq \infty$.

Proof. By [Gr, Prop. 7.4], we have:
$$\deg_p(\lambda(p)) = \frac{\#(\mathcal{G}/\mathcal{P}_\lambda)(\mathbb{F}_p)}{p^{dim(\mathcal{G}/\mathcal{P}_\lambda)}} \cdot p^{\langle\lambda,2\rho\rangle},$$
where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and $\mathcal{P}_\lambda = \mathcal{M}_\lambda \cdot \mathcal{N}_\lambda$ is a standard parabolic subgroup of $\mathcal{G}$ determined by λ. It is thus clear that
$$p^{\langle\lambda,2\rho\rangle} \leq \deg_p(\lambda(p)).$$
It remains to deal with the upper bound.

If $d_i \geq 2$ are the degrees of the group $\mathcal{Z}\backslash\mathcal{G}$, and $e_i \geq 1$ those of $\mathcal{Z}\backslash\mathcal{M}_\lambda$, then by the formulas in [Ca, Pg. 75]
$$\frac{\#(\mathcal{G}/\mathcal{P}_\lambda)(\mathbb{F}_p)}{p^{dim(\mathcal{G}/\mathcal{P}_\lambda)}} = \frac{\prod_i \zeta_p(e_i)}{\prod_i \zeta_p(d_i)},$$
which is at most $\zeta_p(1)^l$, since $\zeta_p \geq 1$. Here $\zeta_p(s) = (1-p^{-s})^{-1}$ is the local factor of the Riemann ζ function. We thus see that
$$\deg_p(\lambda(p)) \leq \zeta_p(1)^l \cdot p^{\langle\lambda,2\rho\rangle}.$$
Note that if $\langle\lambda, 2\rho\rangle \neq 0$, then
$$\#\{i : e_i = 1\} = \text{ the dimension of the center of } \mathcal{Z}\backslash\mathcal{M}_\lambda \geq 1,$$
so that $\prod_p \prod_i \zeta_p(e_i)$ diverges.

Now let $\epsilon > 0$ be given and consider the function:
$$f_{p,\epsilon}(x) = \frac{\zeta_p(1)}{p^{\epsilon x} \cdot \zeta_p(1+\epsilon)} \quad \text{for } x \geq 1.$$
It is clear that $f_{p,\epsilon}$ is bounded for $x \geq 1$ and if p is sufficiently large (depending on ϵ), it is in fact bounded by 1. We let $c_{p,\epsilon} \geq 1$ be an upper bound for $f_{p,\epsilon}$, with $c_{p,\epsilon} = 1$ for almost all p.

Finally, we claim that we can take $C_{l\epsilon}(p) = (c_{p,\epsilon} \cdot \zeta_p(1+\epsilon))^l$; in other words,
$$\deg_p(\lambda(p)) \leq (c_{p,\epsilon} \cdot \zeta_p(1+\epsilon))^l \cdot p^{\langle\lambda,2\rho\rangle\cdot(1+l\epsilon)}.$$
This will prove the lemma, since $\prod_p \zeta_p(1+\epsilon) < \infty$ for $\epsilon > 0$, and $c_{p,\epsilon} = 1$ for almost all p. To prove the above inequality, note that $\langle\lambda, 2\rho\rangle$ is a natural number. If $\langle\lambda, 2\rho\rangle = 0$,

then $\deg_p(\lambda(p)) = 1$, and the result is clear. On the other hand, if $\langle \lambda, 2\rho \rangle \geq 1$, then we have:

$$\deg_p(\lambda(p)) \leq \left(\frac{\zeta_p(1)}{p^{\epsilon \cdot \langle \lambda, 2\rho \rangle}} \right)^l \cdot p^{\langle \lambda, 2\rho \rangle \cdot (1 + l\epsilon)}$$

and the factor in the parenthesis is $\leq c_{p,\epsilon} \cdot \zeta_p(1+\epsilon)$. The lemma is proved. □

Let $\coprod_p P^+$ denote the set of sequences (λ_p), indexed by the finite primes p, of elements $\lambda_p \in P^+$ with $\lambda_p = 0$ for almost all p. Each element $(\lambda_p) \in \coprod_p P^+$ gives rise to an element $a = \prod_p \lambda_p(p) \in A_{\mathbb{Q}} = \mathcal{A}(\mathbb{Q}) \cap G$. We shall denote the set of elements of $A_{\mathbb{Q}}$ obtained in this way by $A_{\mathbb{Q}}^+$. Then by Cartan decomposition (6.2), the sets of the form $G[a]$ are naturally parametrized by $A_{\mathbb{Q}}^+$. We now have:

Corollary 6.4. *Given any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for any $(\lambda_p) \in \coprod_p P^+$ with corresponding element $a \in A_{\mathbb{Q}}^+$,*

$$\prod_p p^{\langle \lambda_p, 2\rho \rangle} \leq deg(a) \leq C_\epsilon \cdot \prod_p p^{\langle \lambda_p, 2\rho \rangle \cdot (1+\epsilon)}$$

Proof of Proposition 6.1 Assume for simplicity that the rank of $\mathcal{Z}\backslash\mathcal{G}$ is ≥ 2; the rank one case can be similarly treated and so we omit the details. Further, by the discussion before the previous corollary, we may and do assume that $a \in A_{\mathbb{Q}}^+$. By Lemma 3.4, we have for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$,

$$\xi(a) \leq C_\epsilon \prod_p |\eta(a)|_p^{\frac{1}{2} - \epsilon}.$$

Here, we have written η in place of η_p since the group $\mathcal{G}$ is $\mathbb{Q}$-split. If $\{\alpha_1, ..., \alpha_l\}$ is the set of simple roots determined by Φ^+, and

$$\begin{cases} \eta = \sum_{i=1}^l n_i \alpha_i; \\ 2\rho = \sum_{i=1}^l m_i \alpha_i, \end{cases}$$

let us set

$$c_0 = \min_{1 \leq i \leq l} \frac{n_i}{m_i} \tag{6.5}$$

Note that $0 \leq c_0 \leq 1$. However η is associated to a maximal strongly orthogonal system; hence $0 < c_0 \leq 1$. If a corresponds to the element $(\lambda_p) \in \coprod_p P^+$, then

$$\prod_p |\eta(a)|_p \leq \prod_p p^{-c_0 \cdot \langle \lambda_p, 2\rho \rangle}$$

Now using the upper bound in Corollary 6.4 and Lemma 2.7(i), we obtain that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that the above is at most $C_\epsilon \cdot \deg(a)^{-c_0 + \epsilon}$.

Now it suffices to set $c = c_0/2$ to finish the proof.

Remark: We note that if $\mathcal{G}$ is in addition simply-connected, then the map

$$\mathcal{G}(\mathbb{Z})\backslash\mathcal{G}(\mathbb{Q})/\mathcal{G}(\mathbb{Z}) \to \coprod_p \mathcal{G}(\mathbb{Z}_p)\backslash\mathcal{G}(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$$

is bijective. It thus follows that every $\mathcal{G}(\mathbb{Z})$-double coset of $\mathcal{G}(\mathbb{Q})$ has a representative in $\mathcal{A}(\mathbb{Q})$. In general, we have that every $G[a]$ has a representative in $\mathcal{A}(\mathbb{Q})$ by the discussion before Corollary 6.4.

7. Equidistribution of the sets $G[m]$ on $ZH\backslash G$

As in the previous section, we let $\mathcal{G}$ be a connected $\mathbb{Q}$-split reductive group with a canonical $\mathbb{Z}$-structure. Assume that the derived group of $\mathcal{G}$ is absolutely simple of rank $l \geq 1$ and the center of $\mathcal{G}$ is of dimension 1. Then the assumptions made in section 2 are satisfied with $\Gamma = G_{\mathbb{Z}} := G \cap \mathcal{G}(\mathbb{Z})$ (see the remark there) and $\mathcal{Z} \cong \mathbb{G}_m$. We set $G_0 = [\mathcal{G},\mathcal{G}](\mathbb{R})^0$. Then $G = G_0 \times Z$ so that $G_0 \cong \overline{G}$.

Let V be a real vector space and let $V_{\mathbb{Z}}$ be a lattice in V. This endows the general linear group $\mathcal{GL}(V)$ with its canonical integral structure. Suppose that

$$\iota : \mathcal{G} \longrightarrow \mathcal{GL}(V)$$

is a representation of $\mathcal{G}$ (acting from the right) defined over $\mathbb{Z}$ such that $\mathcal{Z}$ acts by non-trivial scalars on V. The character group $X^\bullet(\mathcal{G})$ is a free $\mathbb{Z}$-module of rank 1, and we let χ_0 be the basis element such that $\chi_0|_{\mathcal{Z}}$ is a positive multiple of the central character ν of ι.

For each $m \in \mathbb{N}$, we set

$$G[m] = \{g \in G_{\mathbb{Q}} : \iota(g) \in End(V_{\mathbb{Z}}) \text{ and } \chi_0(g) = m\}.$$

We first remark that $G[m]$ depends on the representation ι, even though we have suppressed ι from the notation. It is of course possible that $G[m]$ is empty; if it is non-empty, it is clearly a union of $G_{\mathbb{Z}}$-double cosets.

Lemma 7.1. *If $a \in G[m]$, then $G[a] \subset G[m]$.*

Proof. Let $b \in G[a]$. Then $b = u_1 a u_2$ for some $u_1, u_2 \in \prod_p \mathcal{G}(\mathbb{Z}_p)$. Since $\iota(b) \in End(V_{\mathbb{Q}})$ and $\iota(b) \in \bigcap_p End(V_{\mathbb{Z}_p})$, we have $\iota(b) \in End(V_{\mathbb{Z}})$. On the other hand, $\chi_0(ba^{-1}) \in \prod_p \mathbb{Z}_p^\times \cap \mathbb{Q} = \{\pm 1\}$. Note that since G is connected, $\chi_0(G) \subset \mathbb{R}^+$. Therefore, $\chi_0(ba^{-1}) = 1$ and hence $\chi_0(b) = m$. Hence $b \in G[m]$. $\square$

The lemma implies that $G[m]$ is a disjoint union of sets of the form $G[a]$ for $a \in G_{\mathbb{Q}}$. We now let $S[m]$ be a subset of $G_{\mathbb{Q}}$ such that $G[m]$ is the disjoint union of $G[a]$'s where a ranges over $S[m]$.

Lemma 7.2. *Given $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\#S[m] \leq C_\epsilon \cdot m^\epsilon$$

for any $m \in \mathbb{N}$.

Proof. By the discussion before Corollary 6.4, we may assume $S[m] \subset A^+_{\mathbb{Q}}$. Then

$$\#S[m] \leq \#\{a \in \mathcal{A}(\mathbb{Q}) : \iota(a) \in End(V_{\mathbb{Z}}) \text{ and } \chi_0(a) = m\}.$$

Now let $\mathcal{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathcal{GL}(V)$ defined over $\mathbb{Z}$ and such that $\iota(\mathcal{A}) \subset \mathcal{T}$. With respect to a suitable basis of $V_{\mathbb{Z}}$, we may assume that $\mathcal{T}$ is the diagonal torus. Since the central character of ι is non-trivial and χ_0 is a basis element of $X^\bullet(\mathcal{G})$, we deduce that $\det \circ \iota = \chi_0^k$ for a non-zero integer k. However, since $\chi_0|_{\mathcal{Z}}$ is a positive multiple of the central character ν of ι, we see that k is positive. Therefore for $C = \#\ker(\iota)$,

$$\#S[m] \leq C \cdot \#\{(d_1, \cdots, d_n) \in \mathbb{Z}^n : \prod_i d_i = m^k\} \leq C \cdot (2 \cdot \phi(m^k))^n$$

where $n = \dim V$ and $\phi(m)$ denotes the number of divisors of m. To finish the proof, it suffices to recall the well-known fact that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\phi(m) \leq C_\epsilon \cdot m^\epsilon \text{ for any } m \in \mathbb{N}.$$

□

Consider the set

$$X := \{\lambda \in P^+ : \langle \lambda, \alpha \rangle > 0 \text{ for some } \alpha \in \Phi^+ \text{ and } \iota(\lambda(t)) \in End(V_{\mathbb{Z}}) \quad \text{for all non-zero } t \in \mathbb{Z}\}.$$

Then X is non-empty and for any element λ in X, $\langle \lambda, \chi_0 \rangle$ is a positive integer and $\langle \lambda, 2\rho \rangle > 0$. Let λ_1 be an element in X such that

$$\langle \lambda_1, \chi_0 \rangle = \min_{\lambda \in X} \langle \lambda, \chi_0 \rangle.$$

We then set

$$r_0 := \langle \lambda_1, \chi_0 \rangle \tag{7.3}$$

and

$$\beta := \frac{\langle \lambda_1, 2\rho \rangle}{\langle \lambda_1, \chi_0 \rangle}. \tag{7.4}$$

Observe that $r_0 \in \mathbb{N}$ and $\beta > 0$ depend only on G and ι.

Lemma 7.5. *For any $m \in \mathbb{N}$*

$$\max\{deg(a) : a \in G[m^{r_0}]\} \geq m^{\beta \cdot r_0}.$$

Proof. If we set for each p,

$$\lambda_p = ord_p(m) \cdot \lambda_1,$$

then the element (λ_p) of $\prod_p P^+$ satisfies $\langle \lambda_p, 2\rho \rangle = \beta \cdot r_0 \cdot ord_p(m)$. Moreover, the corresponding element $a \in A^+_{\mathbb{Q}}$ lies in $G[m^{r_0}]$ and thus the result follows by the lower bound in Corollary 6.4. □

Note that the above lemma implies that $G[m^{r_0}]$ contains many $G_{\mathbb{Z}}$-single cosets:

$$\#G_{\mathbb{Z}}\backslash G[m^{r_0}] \geq m^{\beta \cdot r_0}.$$

Let $\mathcal{H} \subset \mathcal{G}$ be a $\mathbb{Q}$-algebraic subgroup such that $\mathcal{H}^0$ has no non-trivial $\mathbb{Q}$-rational character. Then $H \subset G_0$, $H \cap Z$ is trivial and we have a G_0-equivariant bijection:

$$H\backslash G_0 \cong ZH\backslash G.$$

Let $\pi : H\backslash G \to ZH\backslash G \cong H\backslash G_0$ be the natural projection and let v_0 denote the identity coset in $H\backslash G$. Using Theorem 4.7 and the results of Sections 6, we are now ready to prove the following equidistribution of the subsets $G[m^{r_0}]$'s on $ZH\backslash G$ when $m \to \infty$:

Theorem 7.6. *Let $\Omega \subset H\backslash G_0$ be a nice compact subset. There exists an explicit positive integer r_0 (7.3) depending only on G and ι such that for any $\epsilon > 0$, there exists a constant $C_{\Omega,\epsilon} > 0$ such that for any $m \in \mathbb{N}$,*

$$\left|\frac{1}{\#G_{\mathbb{Z}}\backslash G[m^{r_0}]} \cdot \left(\sum_{y \in G_{\mathbb{Z}}\backslash G[m^{r_0}]} \#\pi(v_0 G_{\mathbb{Z}} y) \cap \Omega \right) - vol(\Omega)\right| \leq C_{\Omega,\epsilon} \cdot m^{-r_0\kappa+\epsilon}.$$

Here κ (see 7.7) is a positive constant independent of Ω and m.

Proof. The right hand side of the inequality in question is equal to:

$$I = \sum_{a \in S[m^{r_0}]} \frac{\deg(a)}{\#G_{\mathbb{Z}}\backslash G[m^{r_0}]} \cdot \left(\frac{1}{\deg(a)} \cdot \left(\sum_{y \in G_{\mathbb{Z}}\backslash G[a]} \#\pi(G_{\mathbb{Z}} y) \cap \Omega \right) - vol(\Omega) \right).$$

By Theorem 4.7, this is bounded above by

$$C_\Omega \cdot \sum_{a \in S[m^{r_0}]} \frac{\deg(a) \cdot \xi(a^{-1})^{\frac{\delta}{d+2}}}{\#G_{\mathbb{Z}}\backslash G[m^{r_0}]}.$$

Note that $0 < \frac{\delta}{d+2} < 1$. Now by Proposition 6.1, there exists a constant $0 < c < 1$ such that for any $\epsilon > 0$,

$$\xi(a^{-1})^{\frac{\delta}{d+2}} \leq C_\epsilon \cdot \deg(a)^{-c+\epsilon}$$

for some $C_\epsilon > 0$. Hence there exists $C_{\Omega,\epsilon}$ such that

$$I \leq C_{\Omega,\epsilon} \cdot \sum_{a \in S[m^{r_0}]} \frac{\deg(a)^{1-c+\epsilon}}{\#G_{\mathbb{Z}}\backslash G[m^{r_0}]} \leq C_{\Omega,\epsilon} \cdot \sum_{a \in S[m^{r_0}]} \frac{\max_{a \in G[m^{r_0}]} \deg(a)^{1-c+\epsilon}}{\max_{a \in G[m^{r_0}]} \deg(a)}.$$

By Lemma 7.5, we see that

$$I \leq C_{\Omega,\epsilon} \cdot \#S[m^{r_0}] \cdot m^{(-c+\epsilon)\cdot r_0 \cdot \beta}$$

where $\beta > 0$ does not depend on m. Finally the result follows by Lemma 7.2. ∎

Remark: As for the constant κ in Theorem 7.6, we have:

$$\kappa = \begin{cases} \frac{\beta\cdot\delta}{r(\Phi)\cdot(d+2)} \text{ if the rank of } \mathcal{Z}\backslash\mathcal{G} \text{ is } \geq 2; \\ \\ \frac{\beta\cdot\delta}{4(d+2)} \text{ if the rank of } \mathcal{Z}\backslash\mathcal{G} \text{ is } 1. \end{cases} \tag{7.7}$$

Here, we recall that $d = dim(\mathcal{Z}\backslash\mathcal{G})$, δ is the exponent in Theorem 4.2 and β defined in (7.4) is a constant which depends on the representation ι. Finally, $r(\Phi)$ is a constant depending only on the root system Φ of $\mathcal{Z}\backslash\mathcal{G}$ and is defined by:

$$r(\Phi) = 2 \cdot \max_{1\leq i\leq l} \frac{m_i}{n_i},$$

where $2\rho = \sum_{1\leq i\leq l} m_i\alpha_i$ and $\eta = \sum_{1\leq i\leq l} n_i\alpha_i$. The value of $r(\Phi)$ is tabulated in [Oh2]. Among these constants, the only one which is not so explicit is δ since it depends on the subgroup H. However, when $G_{\mathbb{Z}} \cap H$ is cocompact in H, we know that $\delta = 1$.

8. **Proof of Theorem 1.2 and Theorem 1.4**

We now apply the results of Section 4 and the analysis of the previous two sections to prove Theorems 1.2 and 1.4 of the introduction. We continue the assumptions and notations from Section 7.

Fix $v_0 \in V_l(\mathbb{Z})$ with $l \neq 0$. Let $\mathcal{H}$ be the stabilizer in $[\mathcal{G},\mathcal{G}]$ of v_0 and assume that $\mathcal{H}^0$ has no non-trivial $\mathbb{Q}$-rational character. Now the radial projection

$$\pi : v_0G \to V_l$$

is simply the natural projection $H\backslash G \to ZH\backslash G \cong H\backslash G_0$. Further, observe that

$$v_0G[m^{r_0}] \subset v_0G \cap V_{m^r l}(\mathbb{Z})$$

where

$$r = d \cdot r_0 \cdot \frac{\langle\lambda_0, \nu\rangle}{\langle\lambda_0, \chi_0\rangle}. \tag{8.1}$$

Proof of Theorems 1.2 and 1.4: With these identifications and $l = 1$, Theorem 1.4 of the introduction is simply a restatement of Theorem 7.6. Theorem 1.2 is a simple corollary of Theorem 1.4 by the same argument as in the proof of Theorem 5.1.

9. **Examples**

In this section, we give some concrete examples to illustrate Theorems 1.2 and 1.4. In these examples, the group $\mathcal{G}$ will be GL_n so that

$$\begin{cases} G = GL_n(\mathbb{R})^+; \\ G_0 = SL_n(\mathbb{R}); \\ \Gamma = GL_n(\mathbb{Z})^+ = SL_n(\mathbb{Z}) \end{cases}$$

and $\chi_0 = \det$. Moreover,

$$G[m] = \{g \in M_n(\mathbb{Z}) : \det(g) = m\}$$

and λ_1 is the cocharacter given by $t \mapsto diag(t, 1, \cdots, 1)$ so that $r_0 = \langle \lambda_1, \chi_0 \rangle = 1$ and $\beta = n - 1$.

Example 1: Pffafian. Let $\mathcal{G} = GL_{2n}$ $(n \geq 2)$ and $V_{\mathbb{Z}}$ the lattice of skew symmetric $2n \times 2n$ matrices with entries in $\mathbb{Z}$. The representation ι is given by the action of GL_{2n} on V by $A \mapsto g^t A g$. It is not hard to see that det restricted to V is in fact a square of some integral homogeneous polynomial of degree n on V, which is called the Pffafian. Denote by $Pf(A)$ the Pffafian of a skew symmetric matrix A whose sign ambiguity is resolved by setting $Pf(v_0) = 1$ where

$$v_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in V_{\mathbb{Z}}.$$

Then

$$V_m = \{\text{skew symmetric } A : Pf(A) = m\}.$$

It is easy to see that

$$Pf(g^t A g) = \det(g) \cdot Pf(A)$$

and the stabilizer of v_0 in $\mathcal{G}_0$ is Sp_{2n}. The group $G_0 = SL_{2n}(\mathbb{R})$ acts transitively on V_1 and we have

$$V_m(\mathbb{Z}) = v_0 G[m].$$

Let $\|A\| = (\sum_{i,j} A_{ij}^2)^{\frac{1}{2}}$. Then Theorem 1.4 states that given positive numbers R and $0 < \epsilon \ll 1$, as $m \to \infty$,

$$\sum_{\gamma \in SL_{2n}(\mathbb{Z}) \backslash G[m]} \#\{A \in \gamma^t V_1(\mathbb{Z}) \gamma : \|A\| \leq m^{\frac{1}{n}} R\} = c_{2n,R} \cdot b_{m,2n} \cdot \left(1 + O_{R,\epsilon}(m^{-\kappa+\epsilon})\right)$$

where $b_{m,2n}$ is as defined in the example treated in the introduction and $c_{2n,R}$ is the volume of $\{A \in V_1 : \|A\| \leq R\}$ with respect to the measure on $V_1 \cong Sp_{2n}(\mathbb{R}) \backslash SL_{2n}(\mathbb{R})$ defined at the beginning of Section 4. Moreover, κ can be computed from the formula in (7.7) and is given by

$$\kappa = \frac{n(n+1)(4n-1)}{2(4n^2+1)(4n^3+3n^2+11n-6)}.$$

Note that $\kappa > \frac{1}{4(4n^2+1)}$ for each $n \geq 2$. In this example, the stabilizer $\mathcal{H}$ of v_0 in $\mathcal{G}_0$ is non-trivial. Hence the sets in the above sum may not be disjoint.

In the remaining examples, we let $\mathcal{G} = GL_2$ and consider the right action ι of GL_2 on the space V of binary n-forms given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : q(x, y) \mapsto q(Ax + By, Cx + Dy).$$

This is equivalent to the standard GL_2-representation on $Sym^n(\mathbb{R}^2)$ and we let $V_{\mathbb{Z}}$ be the lattice of binary n-forms with integer coefficients.

Example 2: Binary quadratic forms. Let $d(q)$ be the discriminant of a binary quadratic form $q(x,y) = ax^2 + bxy + cy^2$ Then

$$d(q) = b^2 - 4ac$$

has degree 2 on V and is known to generate the ring of polynomial semi-invariants. We have

$$V_m = \{\text{binary quadratic form } q : d(q) = m\}$$

and

$$d(q \cdot g) = \det(g)^2 \cdot d(q).$$

Take $q_0 \in V_{d_0}(\mathbb{Z})$ for $d_0 \neq 0$. Then the stabilizer $\mathcal{H}$ in SL_2 of q_0 is isomorphic to the special orthogonal group associated to the quadratic form q_0. It is easy to see that q_0 is isotropic over $\mathbb{R}$ (resp. over $\mathbb{Q}$) if and only if d_0 is a square in $\mathbb{R}$ (resp. in $\mathbb{Q}$). Hence d_0 is not a square in $\mathbb{Q}$ if and only if $\mathcal{H}$ is $\mathbb{Q}$-anisotropic (note that if $d_0 > 0$, H is an $\mathbb{R}$-split orthogonal group.) Now fix an integer d_0 which is not a square in $\mathbb{Q}$ and $q_0 \in V_{d_0}(\mathbb{Z})$. Then $\mathcal{H}$ has no non-trivial $\mathbb{Q}$-rational characters and hence Theorem 1.4 is applicable. Note that we have $r = 2$ and hence

$$q_0 G[m] \subset V_{m^2 d_0}(\mathbb{Z}).$$

In general, one would not have equality above. For example, when $q_0 = x^2 + y^2$, $q_0 G[m] \neq V_{-4m^2}(\mathbb{Z})$ for any $m \equiv 3 \pmod 4$.

If we set $\|q\| = \max\{|a|, |b|, |c|\}$, then Theorem 1.4 says that for any positive numbers R and $0 < \epsilon \ll 1$,

$$\sum_{\gamma \in SL_2(\mathbb{Z}) \backslash G[m]} \#\{q \in q_0 SL_2(\mathbb{Z})\gamma : \|q\| \leq mR\} = c_R \cdot b_{m,2} \cdot \left(1 + O_{R,\epsilon}(m^{-\frac{1}{20}+\epsilon})\right)$$

as $m \to \infty$. Here where $b_{m,2}$ is defined as in the example treated in the introduction and c_R is the volume of $\{q \in V_{d_0} : \|q\| \leq R\}$ with respect to the measure defined at the beginning of Section 4. Hence we obtain an equidistribution result such as Theorem 1.2 for the radial projection of $V_{m^2 d_0}(\mathbb{Z})$ on V_{d_0} as $m \to \infty$. Note that for $V_{d_0}(\mathbb{Z})$ to be non-empty, it is necessary and sufficient that $d_0 \equiv 0$ or $1 \pmod 4$. Therefore, there is an obvious obstruction to having an equidistribution result for the radial projection of $V_{m_i}(\mathbb{Z})$ for any sequence $\{m_i\}$ tending to infinity.

An integer d is a fundamental discriminant if and only if d is either a square-free integer congruent to 1 mod 4 or 4 times of a square-free integer which is 2 or 3 mod 4. It was shown by Duke [Du, Thm. 1] that the radial projection of $V_d(\mathbb{Z})$ to a fixed variety becomes equidistributed as $d \to \infty$ (or $d \to -\infty$) along fundamental discriminants. This result depends on his proof of a non-trivial bound on the Fourier coefficients of

Maass cusp forms of half-integral weight, whereas our result depends on a known non-trivial bound towards the Ramanujan conjecture for the Fourier coefficients of cusp forms on GL_2 of integral weight.

Combining the two results, one obtains an equidistribution result for the radial projection of $V_m(\mathbb{Z})$ on a fixed variety for any sequence $m \equiv 0$ or $1 \pmod 4$ as $m \to \infty$ (or $m \to -\infty$); see [CU, §2.3].

Example 3: Binary cubic forms. Let $d(q)$ be the discriminant of a binary cubic form $q(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$. Then

$$d(q) = b^2c^2 + 18abcd - 4ac^3 - 4db^3 - 27a^2d^2$$

has degree 4 on V and the ring of polynomial semi-invariants of ι is generated by d. Moreover,

$$d(q \cdot g) = \det(g)^6 d(q).$$

If we let q_0 be the binary cubic form

$$q_0(x, y) = x^2y - xy^2,$$

then $d(q_0) = 1$,

$$V_m = \{\text{binary cubic form } q : d(q) = m\},$$

and $G_0 \cong SL_2(\mathbb{R})$ acts transitively on V_1. Note that the stabilizer $\mathcal{H}$ in $\mathcal{G}_0$ of any q with $d(q) \neq 0$ is finite. In this case, we have $r = 6$ and hence

$$q_0G[m] \subset V_{m^6}(\mathbb{Z}).$$

If we set $\|q\| = \max\{|a|, |b|, |c|, |d|\}$, then Theorem 1.4 says that for any positive numbers R and $0 < \epsilon \ll 1$,

$$\sum_{\gamma \in SL_2(\mathbb{Z})\backslash G[m]} \#\{q \in q_0SL_2(\mathbb{Z})\gamma : \|q\| \leq m^{\frac{3}{2}}R\} = c_R \cdot b_{m,2} \cdot \left(1 + O_{R,\epsilon}(m^{-\frac{1}{20}+\epsilon})\right)$$

as $m \to \infty$, where $b_{m,2}$ is as defined in Example 2 and c_R is the volume of $\{q \in V_1 : \|q\| \leq R\}$ with respect to the measure defined at the beginning of Section 4. Thus we obtain an equidistribution result such as Theorem 1.2 for the radial projection of $V_{m^6}(\mathbb{Z})$ as $m \to \infty$. We note here that the number of $SL_2(\mathbb{Z})$-orbits contained in $v_0G[m] \subset V_{m^6}(\mathbb{Z})$ is at least of order m. On the other hand, it is known [Sh, Prop. 2.17(i), Pg. 186] that if $h(m)$ denotes the number of $SL_2(\mathbb{Z})$-orbits in $V_m(\mathbb{Z})$, then

$$\frac{1}{N} \sum_{m \leq N} h(m) \sim \frac{\pi^2}{9} \quad \text{as } N \to \infty,$$

so that the average of the $h(m)$'s are bounded. This shows that for some sequence $\{m_i\}$ of positive integers tending to infinity, the sequence $\{h(m_i)\}$ is bounded. It easily follows that we can find a nice compact subset $\Omega \subset V_1$ such that the number of radial projections of $V_{m_i}(\mathbb{Z})$ into Ω is uniformly bounded for all m_i, and hence one cannot have an equidistribution result as in Thm. 1.2 for such a sequence $\{m_i\}$.

Example 4: Binary quartic forms We conclude this section with an example in which the ring of semi-invariants is a polynomial ring with two generators. Consider the representation of GL_2 on binary quartic forms. It is known [Ol, Pg. 29] that the ring of semi-invariants is a polynomial ring with generators

$$\begin{cases} f_1(q) = 12ae - 3bd + c^2; \\ f_2(q) = 72ace - 27eb^2 - 27ad^2 + 9bcd - 2c^2. \end{cases}$$

where $q(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$. We remark that the discriminant d of q is (up to scaling) given by

$$d(q) = 4f_1(q)^3 - f_2(q)^2.$$

For any $g \in GL_2$,

$$f_1(q \cdot g) = \det(g)^4 \cdot f_1(q) \text{ and } f_2(q \cdot g) = \det(g)^6 \cdot f_2(q)$$

Therefore we see that $r = 2$ in this case. If one takes any $q_0 \in V_{l_1,l_2}(\mathbb{Z})$, then

$$q_0 G[m] \subset V_{m^4 l_1, m^6 l_2}(\mathbb{Z}).$$

If q_0 has 4 different roots in $\mathbb{P}^1$, then the stabilizer of q_0 in SL_2 is finite.

Putting $\|q\| = \max\{|a|, |b|, |c|, |d|, |e|\}$, Theorem 1.4 says that for positive numbers R and $0 < \epsilon \ll 1$,

$$\sum_{\gamma \in SL_2(\mathbb{Z}) \backslash G[m]} \#\{q \in q_0 SL_2(\mathbb{Z})\gamma : \|q\| \le m^2 R\} = c_{q_0,R} \cdot b_{m,2} \cdot (1 + O_{R,\epsilon}(m^{-\frac{1}{20}+\epsilon}))$$

as $m \to \infty$. Here $b_{m,2}$ is as in Example 2 and $c_{q_0,R}$ is the volume of $\{q \in q_0 SL_2(\mathbb{R}) : \|q\| \le R\}$ with respect to the measure used at the beginning of Section 4.

Note that since the ring of semi-invariants is a polynomial ring, V_{l_1,l_2} is the union of finitely many $SL_2(\mathbb{R})$ orbits for any l_1 and l_2 [MF, Pg. 160-161]. For a generic choice of l_1 and l_2, the stabilizer of any point in V_{l_1,l_2} is finite and so if each of these orbits has an integer point, we have an equidistribution result on V_{l_1,l_2} (instead of just $q_0 SL_2(\mathbb{R})$) as in Theorem 7.6.

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