ON THE HOWE DUALITY CONJECTURE IN CLASSICAL THETA CORRESPONDENCE

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to Jim Cogdell
on the occasion of his 60th birthday

Abstract. We give a proof of the Howe duality conjecture for the (almost) equal rank dual pairs. For arbitrary dual pairs, we prove the irreducibility of the (small) theta lifts for all tempered representations. Our proof works for any nonarchimedean local field of characteristic not 2 and in arbitrary residual characteristic.

1. Introduction

Let F be a nonarchimedean local field of characteristic not 2 and residue characteristic p. Let E be F itself or a quadratic field extension of F. For $\epsilon = \pm$, we consider a $-\epsilon$-Hermitian space W over E of dimension n and an ϵ-Hermitian space V of dimension m. We shall write W_n or V_m if there is a need to be specific about the dimension of the space in question. Set

$$\epsilon_0 = \begin{cases} \epsilon \text{ if } E = F; \\ 0 \text{ if } E \neq F. \end{cases}$$

Let $G(W)$ and $H(V)$ denote the isometry group of W and V respectively. Then the group $G(W) \times H(V)$ forms a dual reductive pair and possesses a Weil representation ω_ψ which depends on a nontrivial additive character ψ of F (and some other auxiliary data which we shall suppress for now). To be precise, when $E = F$ and one of the spaces, say V, is odd dimensional, one needs to consider the metaplectic double cover of $G(W)$; we shall simply denote this double cover by $G(W)$ as well. The various cases are tabulated in [GI, §3].

In the theory of local theta correspondence, one is interested in the decomposition of ω_ψ into irreducible representations of $G(W) \times H(V)$. More precisely, for any irreducible admissible representation π of $G(W)$, one may consider the maximal π-isotopic quotient of ω_ψ. This has the form $\pi \otimes \Theta_{W,V,\psi}(\pi)$ for some smooth representation $\Theta_{W,V,\psi}(\pi)$ of $H(V)$; we shall frequently suppress (W, V, ψ) from the notation if there is no cause for confusion. It was shown by Kudla [K] that $\Theta(\pi)$ has finite length (possibly zero), so we may consider

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its maximal semisimple quotient $\theta(\pi)$. One has the following fundamental conjecture due to Howe [H]:

Howe Duality Conjecture for $G(W) \times H(V)$

(i) $\theta(\pi)$ is either 0 or irreducible.

(ii) If $\theta(\pi) = \theta(\pi') \neq 0$, then $\pi = \pi'$.

We take note of the following theorem:

Theorem 1.1. *(i) If π is supercuspidal, then $\Theta(\pi)$ is either zero or irreducible (and thus is equal to $\theta(\pi)$). Moreover, for any irreducible supercuspidal π and π',*

$$\Theta(\pi) \cong \Theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

(ii) $\theta(\pi)$ is multiplicity-free.

(iii) If $p \neq 2$, the Howe duality conjecture holds.

The statement (i) is a classic theorem of Kudla [K] (see also [MVW]), whereas (iii) is a well-known result of Waldspurger [W]. The statement (ii), on the other hand, is a recent result of Li-Sun-Tian [LST]. We note that the techniques for proving the three statements in the theorems are quite disjoint from each other. For example, the proof of (i) is based on arguments using the doubling see-saw and Jacquet modules of the Weil representation: these have become standard tools in the study of local theta correspondence. The proof of (iii) is based on K-type analysis and uses various lattice models of the Weil representation. Finally, the proof of (ii) is based on an argument using the Gelfand-Kazhdan criterion for the (non-)existence of equivariant distributions.

In this paper, we shall assume statements (i) and (ii), but not statement (iii). Indeed, the purpose of this paper is to extend the results of the above theorem to the case of more general π and arbitrary residue characteristic, using the same tools in the proof of Theorem 1.1(i). More precisely, we shall prove the following two results.

Theorem 1.2. *If π is an irreducible tempered representation of $G(W)$, then $\theta(\pi)$ is either zero or irreducible. Moreover, for any irreducible tempered π and π',*

$$\theta(\pi) \cong \theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

When $m \leq (n + \epsilon_0) + 1$, $\theta(\pi)$ is tempered if it is nonzero.

Theorem 1.3. *If $|m-(n+\epsilon_0)| \leq 1$, then the Howe duality conjecture holds for $G(W) \times H(V)$.*

To be precise, Theorem 1.3 applies to the dual pairs $\mathrm{O}_{2n+1} \times \mathrm{Mp}_{2n}$, $\mathrm{O}_{2n} \times \mathrm{Sp}_{2n}$, $\mathrm{O}_{2n} \times \mathrm{Sp}_{2n+2}$, $\mathrm{U}_n \times \mathrm{U}_n$ and $\mathrm{U}_n \times \mathrm{U}_{n+1}$. We call them the (almost) equal rank dual pairs. With Theorem 1.3, the assumption that $p \neq 2$ can be totally removed from all the results in [GS] (on the local Shimura correspondence) and also parts of [GI] (those dealing with the almost equal rank case, such as the results on Prasad's conjecture in [GI, Appendix C]).

We would like to point out some related results in the literature, especially the paper [R] of Roberts and the papers [M1, M2, M3, M4] of Muić:

- In the context of Theorem 1.2, the temperedness of any irreducible summand of $\theta(\pi)$ when $m \leq n + \epsilon_0 + 1$ was checked by Roberts [R, Theorem 4.2], at least for symplectic-even-orthogonal dual pairs. Further, the main idea in the proof of Theorem 1.2 can already be found in the proof of [R, Theorem 4.4] (see Proposition 3.2 below).
- In [M1], Muić established Theorem 1.2 (and much more) for discrete series representations but the results there depended on the Moeglin-Tadić (MT) classification of discrete series representations, which was conditional on some hypotheses. We are not entirely sure whether the MT classification is unconditional today. But our goal here is to give a simple proof of the theorems above without resort to classification.
- In [M3], Muić dispensed with the MT classification and proves some basic properties of $\Theta(\pi)$ for discrete series representations π. For example, he showed in the context of Theorem 1.2 that $\Theta(\pi)$ is the direct sum of discrete series representations if $m \leq n+\epsilon_0$. The techniques of proof used in [M3] are almost entirely based on the analysis of Jacquet modules. The paper [M3] does not establish Theorem 1.2, but we shall make use of some results such as [M3, Theorem 4.1, 4.2 and 6.1] in our proof of Theorem 1.2. In [GS, Prop. 8.1] and [GI, Prop. C.1], a self-contained and more streamlined proof of the relevant parts of [M3, Theorem 4.1, 4.2 and 6.1] was given for all the dual pairs considered here. We shall revisit and extend this simpler proof in Proposition 3.1.
- In [M4], assuming Theorem 1.2 for discrete series representation, Muić studied the theta lifts of tempered representations. He determined $\Theta(\pi)$ (in terms of the theta lifts of discrete series representations) and showed the irreducibility of $\theta(\pi)$ in many cases. However, the proof of parts of the main results [M4, Theorems 5.1 and 5.2] depended on the MT classification in its use of [M4, Theorem 6.6]. We do not use results from [M4] in this paper. Rather, our Theorem 1.2 renders most of [M4] unconditional, and completes some results there.

The proofs of the results in [M1, M2, M3, M4] are based on some intricate and explicit computations of Jacquet modules and some detailed knowledge (short of classification) of the discrete series representations of classical groups. On the other hand, the results of this paper are proved in a simpler and more conceptual manner, with the more intricate computations already done in [GS, GI, R]. It amounts to an attempt to prove the Howe duality conjecture using the techniques and principles found in [K] and [KR], supplemented by [LST], so as to remove the $p \neq 2$ assumption in Waldspurger's theorem.

Shortly after the completion of this paper, the authors have succeeded in giving the proof of the full Howe duality conjecture. The proof builds upon the techniques of this paper and will appear in [GT]. While the proof given in [GT] uses the doubling see-saw argument of §2, the argument in §3 is completely replaced by a different idea which originated in Minguez's thesis [Mi], with the result that the tempered representations do not play any special role in [GT]. In view of this, we have decided to keep this paper as is, even though the results here are subsumed by [GT], especially since some useful results about the theta lifts of tempered representations are shown here (such as Proposition 3.1).

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This paper is dedicated to Jim Cogdell who has inspired us not only through his mathematics but also through his lucid expositions of often technical subjects in the theory of automorphic forms, his warm personality and his attitude of service which is evident to all, especially in his interaction with his late mentor, Professor Piatetski-Shapiro. Jim's work at the beginning of his career concerns the construction of modular forms by theta correspondence and we hope this paper is an appropriate contribution to his 60th birthday volume.

2. **Special Case of Theorem 1.2**

Before beginning the proof of Theorem 1.2, let us specify the extra data needed to consider the Weil representation of $G(W) \times H(V)$; these are needed to split the metaplectic cover over the dual pair. We shall follow the setup of [GI, §3.2-3.3] in fixing a pair of splitting characters $\chi = (\chi_V, \chi_W)$, which are certain unitary characters of $E^\times$, with associated Weil representation $\omega_{W,V,\chi,\psi}$. We shall frequently suppress χ and ψ from the notation.

In this section, we shall first prove Theorem 1.2 for the case $m \le n + \epsilon_0$ or $m > 2(n + \epsilon_0)$. Indeed, what we will prove is slightly stronger than Theorem 1.2, which is stated as follows:

Theorem 2.1. *If $m \le n + \epsilon_0$ or $m > 2(n + \epsilon_0)$, then for any tempered $\pi \in \mathrm{Irr}(G(W))$, $\theta(\pi)$ is either zero or irreducible. Moreover, for tempered π and any irreducible representation π',*

$$0 \neq \theta(\pi) \subset \theta(\pi') \Longrightarrow \pi \cong \pi'.$$

When $m \le n + \epsilon_0$, $\theta(\pi)$ is tempered if it is nonzero. (Note that unlike Theorem 1.2, we do not assume π' is tempered.)

We consider the following see-saw diagram

$$\begin{tikzcd}
G(W \oplus W^-) \arrow[d, dash] \arrow[drr, dash] & & H(V) \times H(V) \arrow[d, dash] \arrow[dll, dash] \\
G(W) \times G(W^-) & & H(V)^\Delta,
\end{tikzcd}$$

where W^- denotes the space obtained from W by multiplying the form by -1, so that $G(W^-) = G(W)$. Given an irreducible tempered representation π and any irreducible representation π' of $G(W)$, the see-saw identity [GI, §6.1] gives:

$$\mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,W+W^-}(\chi_W), \pi' \otimes \pi^\vee \chi_V) = \mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi') \otimes \Theta(\pi)^{MVW}, \mathbb{C}),$$

where MVW refers to the involution on the set of smooth representations of $H(V)$ introduced in [MVW]. Here $\Theta_{V,W+W^-}(\chi_W)$ denotes the big theta lift of the character χ_W of $H(V)$ to $G(W+W^-)$. It is a result of Rallis that

$$\Theta_{V,W+W^-}(\chi_W) \hookrightarrow \mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V |\det|^{s_{m,n}} \tag{1}$$

where

- $\Delta W \subset W + W^-$ is diagonally embedded and is a maximal isotropic subspace;
- $P(\Delta W)$ is the maximal parabolic subgroup of $G(W+W^-)$ which stabilizes ΔW and has Levi factor $\mathrm{GL}(\Delta W)$;
- $\mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V |\det|^s$ denotes the degenerate principal series representation induced from the character $\chi_V |\det|^s$ of $P(\Delta W)$ (normalized induction);
- moreover,
$$s_{m,n} = \frac{m - (n + \epsilon_0)}{2}.$$

We consider the two cases in turn.

Case 1: $m \leq n + \epsilon_0$.

We first note that one can prove the temperedness of $\theta(\pi)$ (if nonzero) in the same way as in [GI, Prop. C.1 and Prop. C.4(i)]. Hence, we will focus on the rest of the theorem.

In this case, $s_{m,n} \leq 0$ and there is a surjective map (see [GI, Prop. 8.2])

$$\mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V |\det|^{-s_{m,n}} \longrightarrow \Theta_{V,W+W^-}(\chi_W). \tag{2}$$

Hence the see-saw identity gives:

$$\mathrm{Hom}_{G(W)\times G(W)}(\mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V |\det|^{-s_{m,n}}, \pi' \otimes \pi^\vee \chi_V) \supset \mathrm{Hom}_{H(V)}(\theta(\pi'), \theta(\pi)).$$

To prove the theorem, it suffices to show that the LHS has dimension ≤ 1, with equality only if $\pi = \pi'$.

For this, we need the following crucial lemma (see [KR]), which we shall state in slightly greater generality here for later use. Let $W' = W + \mathbb{H}^r$ where $\mathbb{H}$ is the hyperbolic plane (i.e. the split $-\epsilon$-Hermitian space of dimension 2), so that $n' := \dim W' = n + 2r$. Consider the split space $\mathbb{W} = W' + W^-$. For a maximal isotropic subspace Y of $\mathbb{H}^r$, the space $\Delta W \oplus Y$ is a maximal isotropic subspace of $\mathbb{W}$, whose stabilizer in $G(\mathbb{W})$ is a maximal parabolic subgroup P. Now we have:

Lemma 2.2. *As a representation of* $G(W') \times G(W^-)$, $\mathrm{Ind}_P^{G(\mathbb{W})} \chi_V \cdot |\det|^s$ *possesses an equivariant filtration*

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_q = \mathrm{Ind}_P^{G(\mathbb{W})} \chi_V \cdot |\det|^s$$

with successive quotients

$$\begin{aligned} R_t &= I_t / I_{t-1} \\ &= \mathrm{Ind}_{Q'_{t+r} \times Q_t}^{G(W') \times G(W^-)} \Big(\big(\chi_V |\det_{X_{t+r}}|^{s+\frac{t}{2}} \boxtimes \chi_V |\det_{X_t}|^{s+\frac{t}{2}} \big) \otimes \\ &\qquad\qquad (\chi_V \circ \det_{W^-_{n-2t}}) \otimes C_c^\infty(G(W_{n-2t})) \Big). \end{aligned} \tag{3}$$

Here, the induction is normalized and

- q *is the Witt index of* W*;*
- Q_t *is the maximal parabolic subgroup of* $G(W)$ *stabilizing a* t*-dimensional isotropic subspace* X_t *of* W*, with Levi subgroup* $\mathrm{GL}(X_t) \times G(W_{n-2t})$*, where* $\dim W_{n-2t} = n - 2t$*.*
- Q'_{t+r} *is the maximal parabolic subgroup stabilizing the* $(t+r)$*-dimensional isotropic subspace* $X_t + Y$ *of* W' *with Levi factor* $\mathrm{GL}(X_t + Y) \times G(W_{n-2t})$*.*
- $G(W_{n-2t}) \times G(W_{n-2t})$ *acts on* $C_c^\infty(G(W_{n-2t}))$ *by left-right translation.*

In particular,

$$R_0 = \mathrm{Ind}_{Q'(Y) \times G(W^-)}^{G(W') \times G(W^-)} \chi_V |\det_Y|^s \otimes (\chi_V \circ \det_{W^-}) \otimes C_c^\infty(G(W)).$$

We shall apply this lemma with $W' = W$, in which case $R_0 = (\chi_V \circ \det_{W^-}) \otimes C_c^\infty(G(W))$. Then we claim that the natural restriction map

$$\begin{aligned} &\mathrm{Hom}_{G(W) \times G(W)}(\mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} |\det|^{-s_{m,n}}, \pi' \otimes \pi^\vee \chi_V) \\ &\qquad\qquad \longrightarrow \mathrm{Hom}_{G(W) \times G(W)}(R_0, \pi' \otimes \pi^\vee \chi_V) \end{aligned}$$

is injective. This will imply the theorem since the RHS has dimension ≤ 1, with equality if and only if $\pi = \pi'$.

To deduce the claim, it suffices to to show that for each $0 < t \leq q$,

$$\mathrm{Hom}_{G(W) \times G(W)}(R_t, \pi' \otimes \pi^\vee \chi_V) = 0.$$

By Frobenius reciprocity, $\mathrm{Hom}_{G(W) \times G(W)}(R_t, \pi' \otimes \pi^\vee \chi_V)$ is equal to

$$\begin{aligned} \mathrm{Hom}_{L(X_t) \times L(X_t)} \Big(\big(\chi_V |\det|^{-s_{m,n}+\frac{t}{2}} \boxtimes \chi_V |\det|^{-s_{m,n}+\frac{t}{2}} \big) \otimes C_c^\infty(G(W_{n-2t})), \\ R_{\overline{Q}_t}(\pi') \otimes R_{\overline{Q}_t}(\pi^\vee) \Big) \end{aligned}$$

where

$$L(X_t) = \mathrm{GL}(X_t) \times G(W_{n-2t})$$

is the Levi factor of Q_t. Here and elsewhere $R_{\overline{Q}_t}$ indicates the normalized Jacquet functor with respect to $\overline{Q}_t$. But $-s_{m,n} + \frac{t}{2} > 0$ whereas, since π is tempered, it follows by Casselman's criterion that the center of GL_t acts on any irreducible subquotient of $R_{\overline{Q}_t}(\pi^\vee)$ by a character of the form $\mu \cdot |-|^{\alpha}$ with μ unitary and $\alpha \leq 0$. Hence we deduce that the above Hom space is 0, as desired, and we have proved Theorem 2.1 when $m \leq n + \epsilon_0$.

Case 2: $m > 2(n + \epsilon_0)$**.**

In this case, $s_{m,n} > 0$ is so large that the degenerate principal series representation $\mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V \cdot |\det|^{s_{m,n}}$ is irreducible [GI, Proposition 7.1], and hence

$$\Theta_{V,W+W^-}(\chi_W) = \mathrm{Ind}_{P(\Delta W)}^{G(W+W^-)} \chi_V |\det|^{s_{m,n}}.$$

The same argument as in Case 1 completes the proof of Theorem 2.1.

3. **Proof of Theorem 1.2**

In this section, we complete the proof of Theorem 1.2. In view of Theorem 2.1, it remains to consider the case $n + \epsilon_0 < m \leq 2(n + \epsilon_0)$. For this case, we consider the theta lift of π to the Witt tower $\{V_{m'}\}$ of spaces containing $V = V_m$. The following proposition is the key technical result that we use (see [M3, Theorems 4.1 and 4.2] and [R, Theorem 4.2]):

Proposition 3.1. *Assume $m > n + \epsilon_0$. For tempered π, any irreducible quotient σ of $\Theta_{W,V_m}(\pi)$ is either tempered or is the Langlands quotient of a standard module*

$$\mathrm{Ind}_P^{H(V_m)} \tau_1 \cdot \chi_W |\det|^{s_1} \otimes \cdots \otimes \tau_k \cdot \chi_W |\det|^{s_k} \otimes \sigma'$$

with τ_i a unitary discrete series representation of some GL_{n_i}, σ' a tempered representation of $H(V_{m'})$ with $m' = m - 2\sum_i n_i$, and

$$s_1 \geq s_2 \geq \cdots \geq s_k > 0,$$

satisfying:

(a) $\tau_1 = \mathbf{1}$ *(so $n_1 = 1$) and $s_1 = \frac{m-(n+\epsilon_0)-1}{2}$, or*
(b) $\tau_1 = \mathrm{St}_{n_1}$ *and $s_1 = \frac{1}{2}$, where St_{n_1} denotes the Steinberg representation of GL_{n_1} with $n_1 = m - (n + \epsilon_0) - 1 > 1$.*

Indeed, (a) could hold only if $m > n + \epsilon_0 + 1$ and $\Theta_{W,V_{m-2}}(\pi) \neq 0$, and (b) could hold only if $m > n + \epsilon_0 + 2$ and the square-integrable support of π contains a (twisted) Steinberg representation $\chi_V \cdot \mathrm{St}_{n_1-1}$.

Here, by the square-integrable support of a tempered representation π, we mean the (unique up to association) set $\{\tau_1, \ldots, \tau_r, \pi'\}$ of essentially square-integrable representations such that π is contained in the representation parabolically induced from the representation $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi'$ of a Levi subgroup $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times G(W')$ of $G(W)$.

So as not to disrupt the proof of Theorem 1.2, we postpone the proof of this proposition to the last section. We also need the following refinement of a result of Roberts [R, Theorem 4.4]; we include the proof so as to cover all the dual pairs considered here.

Proposition 3.2. *Let π be an irreducible representation of $G(W)$ and let $V_{d'} = V_d \oplus \mathbb{H}^r$, where $\mathbb{H}$ is the split ϵ-Hermitian space of dimension 2, so that $d' \geq d \geq n + \epsilon_0$. Suppose that*

$$\sigma \subset \theta_{W,V_d}(\pi) \quad \text{and} \quad \sigma' \subset \theta_{W,V_{d'}}(\pi)$$

are irreducible representations. If $d' = d + 2r$ is sufficiently large, then σ' is a quotient of the representation

$$\mathrm{Ind}_Q^{H(V_{d'})}(\chi_W|-|^{\frac{d'-(n+\epsilon_0)-1}{2}} \otimes \chi_W|-|^{\frac{d'-(n+\epsilon_0)-3}{2}} \otimes \cdots \otimes \chi_W|-|^{\frac{d-(n+\epsilon_0)+1}{2}} \otimes \sigma)$$

induced from the parabolic subgroup Q with Levi factor $(\mathrm{GL}_1)^r \times H(V_d)$.

Furthermore, if σ is tempered, then the above conclusion holds for all $r \geq 0$, in which case the above induced representation is a standard module and σ' is its unique Langlands quotient.

Proof. Consider the see-saw diagram

$$\begin{array}{ccc} H(V_{d'} \oplus V_d^-) & & G(W) \times G(W) \\ {\scriptstyle i}\Big| & \times & \Big| \\ H(V_{d'}) \times H(V_d^-) & & G(W)^{\Delta}, \end{array}$$

and let

$$s_{d+r,n} = \frac{(d+r) - (n+\epsilon_0)}{2} > 0.$$

The see-saw identity gives:

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{G(W)^\Delta}(\Theta(\sigma') \otimes \Theta(\sigma)^{MVW}, \mathbb{C}) \\ &= \mathrm{Hom}_{H(V_{d'}) \times H(V_d^-)}(\Theta_{W,V_{d'}+V_d^-}(\chi_V), \sigma' \otimes \sigma^\vee \chi_W) \\ &\subseteq \mathrm{Hom}_{H(V_{d'}) \times H(V_d)}(\mathrm{Ind}_{P(\Delta V_d + Y)}^{H(V_{d'}+V_d)} \chi_W |\det|^{s_{d+r,n}}, \sigma' \otimes \sigma^\vee \chi_W), \end{aligned}$$

where we used the analogue of (2) for the last inclusion. Here, note that $s_{d+r,n}$ is the analogue of $-s_{m,n}$ in (2) and Y is a maximal isotropic subspace of $\mathbb{H}^r$ so that $\Delta V_d + Y$ is a maximal isotropic subspace of $V_{d'} + V_d$.

Now we apply Lemma 2.2 (or rather its analogue with the roles of W and V exchanged), which describes an $H(V_{d'}) \times H(V_d)$-equivariant filtration of the induced representation $\mathrm{Ind}_{P(\Delta V_d+Y)}^{H(V_{d'}+V_d)} \chi_W |\det|^{s_{d+r,n}}$. Note that the length of this filtration depends only on V_d and not on $V_{d'}$. When r is sufficiently large, all the characters $\chi_W|\det|^{s_{d+r,n}+\frac{t}{2}}$ which occur in the description of the successive quotients R_t of this filtration with $t > 0$ will be different from any central exponents of any Jacquet module of σ (which is a finite set). Indeed, this holds for all $r \geq 0$ when σ is tempered, as in the proof of Case 1 of Theorem 2.1.

Thus we see that when r is sufficiently large,

$$\begin{aligned}
0 &\neq \mathrm{Hom}_{H(V_{d'})\times H(V_d)}(\mathrm{Ind}_{P(\Delta V_d+Y)}^{H(V_{d'}+V_d)}\chi_W|\det|^{s_{d+r,n}}, \sigma'\otimes\sigma^\vee\chi_W)\\
&\subseteq \mathrm{Hom}_{H(V_{d'})\times H(V_d)}\left(R_0, \sigma'\otimes\sigma^\vee\chi_W\right)\\
&= \mathrm{Hom}_{H(V_{d'})\times H(V_d)}\left(\mathrm{Ind}_{Q'(Y)\times H(V_d)}^{H(V_{d'})\times H(V_d)}\chi_W|\det_Y|^{s_{d+r,n}}\otimes C_c^\infty(H(V_d)), \sigma'\otimes\sigma^\vee\right)\\
&= \mathrm{Hom}_{\mathrm{GL}(Y)\times H(V_d)\times H(V_d)}\left(\chi_W|\det_Y|^{s_{d+r,n}}\otimes C_c^\infty(H(V_d)), R_{\overline{Q}'(Y)}(\sigma')\otimes\sigma^\vee\right)\\
&= \mathrm{Hom}_{\mathrm{GL}(Y)\times H(V_d)}\left(\chi_W|\det_Y|^{s_{d+r,n}}\otimes\sigma, R_{\overline{Q}'(Y)}(\sigma')\right)\\
&= \mathrm{Hom}_{H(V_{d'})}\left(\mathrm{Ind}_{Q'(Y)}^{H(V_{d'})}(\chi_W|\det_Y|^{s_{d+r,n}}\otimes\sigma), \sigma'\right).
\end{aligned}$$

Thus, σ' is a quotient of $\mathrm{Ind}_{Q'(Y)}^{H(V_{d'})}(\chi_W|\det_Y|^{s_{d+r,n}}\otimes\sigma)$. But the latter is a quotient of the induced representation given in the proposition.

When σ is tempered, the above conclusions hold for any $r\geq 0$ and it is clear that the induced representation is a standard module. This completes the proof of the proposition.

□

We can now complete the proof of Theorem 1.2. Suppose that σ_1 and σ_2 are both irreducible summands of $\theta_{W,V_m}(\pi)$, with $m > n+\epsilon_0$. In the context of Proposition 3.2, we take $d=m$ and $d'=m+2r$ sufficiently large. Let $\sigma'=\theta_{W,V_{d'}}(\pi)$ (which is irreducible for d' sufficiently large by Theorem 2.1). By Proposition 3.2, we conclude that σ' is a quotient of

$$\Sigma_i = \mathrm{Ind}_Q^{H(V_{d'})}(\chi_W|-|^{\frac{d'-(n+\epsilon_0)-1}{2}}\otimes\chi_W|-|^{\frac{d'-(n+\epsilon_0)-3}{2}}\otimes\cdots\otimes\chi_W|-|^{\frac{d-(n+\epsilon_0)+1}{2}}\otimes\sigma_i)$$

for $i=1$ or 2. Now we claim that Σ_i is a standard module. This is clear if σ_i is tempered. On the other hand, if σ_i is nontempered, then Proposition 3.1 describes two possibilities (a) and (b) for σ_i. In either case, σ_i is the Langlands quotient of a standard module

$$\mathrm{Ind}_P^{H(V_m)}(\tau_1\cdot\chi_W|\det|^{s_1}\otimes\cdots\otimes\tau_k\cdot\chi_W|\det|^{s_k}\otimes\sigma_0)$$

with

$$0 < s_1 \leq \frac{d-(n+\epsilon_0)-1}{2}.$$

It follows from this that Σ_i is a standard module and σ' is its unique Langlands quotient. By the uniqueness of Langlands quotient data, we must have $\sigma_1\cong\sigma_2$. Hence, we conclude that $\theta_{W,V_m}(\pi)$ is isotypic, and it follows from Theorem 1.1(ii) that $\theta_{W,V}(\pi)$ is irreducible for tempered π.

Finally, suppose that $\theta(\pi_1)\cong\theta(\pi_2)\cong\sigma\neq 0$ for two tempered representations π_1 and π_2. Since we are assuming $n+\epsilon_0+1\leq m\leq 2(n+\epsilon_0)$, the possibilities for σ are given in Proposition 3.1. Now take $d'=m+2r$ sufficiently large in Proposition 3.2. If $\sigma_1=\theta_{W,V_{d'}}(\pi_1)$ and $\sigma_2=\theta_{W,V_{d'}}(\pi_2)$, then both σ_1 and σ_2 will be the Langlands quotient of the same standard module

$$\mathrm{Ind}_Q^{H(V_{d'})}(\chi_W|-|^{\frac{d'-(n+\epsilon_0)-1}{2}}\otimes\chi_W|-|^{\frac{d'-(n+\epsilon_0)-3}{2}}\otimes\cdots\otimes\chi_W|-|^{\frac{m-(n+\epsilon_0)+1}{2}}\otimes\sigma),$$

where Q is as in Proposition 3.2. This implies that $\sigma_1 \cong \sigma_2$. By Theorem 2.1, we deduce that $\pi_1 \cong \pi_2$.

This completes the proof of Theorem 1.2.

4. **Proof of Theorem 1.3**

We shall now show Theorem 1.3 and without loss of generality, we may assume that $0 \leq m - (n + \epsilon_0) \leq 1$. By Theorem 1.2, we already know that if π is tempered, then $\theta(\pi)$ is irreducible tempered or 0. Thus it remains to treat the nontempered case. For this, we need the following lemma which gives more precise control on the big theta lift of tempered representations.

Lemma 4.1. *Assume that* $0 \leq m - (n + \epsilon_0) \leq 1$. *If* π *is tempered, then* $\Theta(\pi) = \theta(\pi)$, *so that* $\Theta(\pi)$ *is irreducible tempered or* 0.

Proof. This was shown in [GS, Prop. 8.1(i) and (ii)] and [GI, Prop. C.1 and Prop. C.4(i)]. □

Now if π is nontempered, it can be expressed uniquely as the Langlands quotient of a standard module

$$\mathrm{Ind}_{P_{r_1,\dots,r_k}}^{G(W)} \tau_1 \cdot \chi_V |\det|^{s_1} \otimes \tau_2 \cdot \chi_V |\det|^{s_2} \otimes \cdots \otimes \tau_k \cdot \chi_V |\det|^{s_k} \otimes \pi_0 \tag{4}$$

of $G(W)$, where $P_{r_1,\dots,r_k}$ is a parabolic subgroup of $G(W)$ whose Levi factor is $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \times G(W')$, so that $\dim W' = n - 2\sum_i r_i$, π_0 is a tempered representation of $G(W')$, each τ_i is a unitary tempered representation of GL_{r_i} and $s_1 > \cdots > s_k > 0$. In [GI, Prop. C.4(ii)] and [GS, Prop. 8.1(iii)], it was shown that $\Theta(\pi)$ is a quotient of the induced representation

$$\mathrm{Ind}_{Q_{r_1,\dots,r_k}}^{H(V)} \tau_1 \cdot \chi_W |\det|^{s_1} \otimes \tau_2 \cdot \chi_W |\det|^{s_2} \otimes \cdots \otimes \tau_k \cdot \chi_W |\det|^{s_k} \otimes \Theta_{W',V'}(\pi_0) \tag{5}$$

of $H(V)$, where $Q_{r_1,\dots,r_k}$ is the parabolic subgroup of $H(V)$ whose Levi factor is $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \times H(V')$, so that $\dim V' = m - 2\sum_i r_i$. Since $0 \leq \dim V' - (\dim W' + \epsilon_0) \leq 1$, Lemma 4.1 implies that $\Theta_{W',V'}(\pi_0)$ is an irreducible tempered representation, so that the above induced representation is a standard module of $H(V)$. In particular, $\theta(\pi)$ is either 0 or is the unique Langlands quotient of that standard module.

Finally, assume that $\theta(\pi) \cong \theta(\pi') \neq 0$. Express π as the Langlands quotient of a standard module as in (4), though this time we allow the case $P_{r_1,\dots,r_k} = G(W)$ (in which case π is tempered). Similarly express π' as a Langlands quotient, possibly with a different quotient data. Then by the above argument, $\theta(\pi)$ is the Langlands quotient of the induced representation (5), and similarly $\theta(\pi')$ is the Langlands quotient of the analogous induced representation. By the uniqueness of the Langlands quotient data and Theorem 1.2, we deduce that $\pi \cong \pi'$.

This completes the proof of Theorem 1.3 which establishes the Howe duality conjecture in the (almost) equal rank case.

5. **Proof of Proposition 3.1**

In this section, we give the proof of Proposition 3.1 following that of [GI, Prop. C.1].

- Suppose that σ is a nontempered irreducible quotient of $\Theta_{W,V_m}(\pi)$. Suppose that σ is the Langlands quotient of a standard module

$$\mathrm{Ind}_P^{H(V_m)}\tau_1 \cdot \chi_W|\det|^{s_1} \otimes \cdots \otimes \tau_k \cdot \chi_W|\det|^{s_k} \otimes \sigma' \tag{6}$$

with τ_i unitary discrete series representations of some GL_{n_i}, σ' a tempered representation of some $H(V_{m'})$ with $m' < m$, and

$$s_1 \geq s_2 \geq \cdots \geq s_k > 0.$$

We need to show that only possibilities (a) and (b) as given in Proposition 3.1 can occur.

- Let $t = n_1$. From the standard module above, we see that there exists a maximal parabolic subgroup $Q = Q(Y_t)$ of $H = H(V_m)$ stabilizing a t-dimensional isotropic subspace Y_t, with Levi component $L(Y_t) = \mathrm{GL}(Y_t) \times H(V_{m-2t})$, such that

$$\sigma \hookrightarrow \mathrm{Ind}_Q^H(\tau \cdot \chi_W|\det|^{-s_1} \otimes \sigma_0).$$

Here, we have written $V = Y_t \oplus V_{m-2t} \oplus Y_t^*$ with Y_t^* isotropic, $\tau = (\tau_1^c)^\vee$, where ^{c} indicates the conjugation by the generator of $\mathrm{Gal}(E/F)$, $s_1 > 0$ is the leading exponent as in (6) and σ_0 is an irreducible representation of $H(V_{m-2t})$. Thus we have a nonzero $G(W) \times H$-equivariant map

$$\omega_{V_m,W} \longrightarrow \pi \otimes \mathrm{Ind}_Q^H(\tau \cdot \chi_W|\det|^{-s_1} \otimes \sigma_0).$$

By Frobenius reciprocity, we have

$$\pi^\vee \hookrightarrow \mathrm{Hom}_{L(Y_t)}(R_Q(\omega_{V_m,W}), \tau \cdot \chi_W|\det|^{-s_1} \otimes \sigma_0).$$

- By [K], the Jacquet module $R_Q(\omega_{V_m,W})$ has an equivariant filtration

$$R_Q(\omega_{V_m,W}) = R^0 \supset R^1 \supset \cdots \supset R^t \supset R^{t+1} = 0$$

whose successive quotient $J^a = R^a/R^{a+1}$ is described in [GI, Lemma C.2]. More precisely,

$$J^a = \mathrm{Ind}_{Q(Y_{t-a},Y_t)\times H(V_{m-2t})\times P(X_a)}^{\mathrm{GL}(Y_t)\times H(V_{m-2t})\times G(W)} \left(\chi_W|\det_{Y_{t-a}}|^{\lambda_{t-a}} \otimes C_c^\infty(\mathrm{Isom}_{E,c}(X_a, Y_a)) \otimes \omega_{V_{m-2t},W_{n-2a}}\right),$$

where

- $\lambda_{t-a} = (n - m + t - a + \epsilon_0)/2$,
- $W = X_a + W_{n-2a} + X_a^*$ with X_a an a-dimensional isotropic space and $\dim W_{n-2a} = n-2a$,
- $Y_t = Y_{t-a} + Y_a'$ and $Q(Y_{t-a}, Y_a)$ is the maximal parabolic subgroup of $\mathrm{GL}(Y_t)$ stabilizing Y_{t-a}.
- $\mathrm{Isom}_{E,c}(X_a, Y_a)$ is the set of E-conjugate-linear isomorphisms from X_a to Y_a;
- $\mathrm{GL}(X_a) \times \mathrm{GL}(Y_a)$ acts on $C_c^\infty(\mathrm{Isom}_{E,c}(X_a, Y_a))$ as

$$((b, c) \cdot f)(g) = \chi_V(\det b)\chi_W(\det c)f(c^{-1}gb)$$

for $(b, c) \in \mathrm{GL}(X_a) \times \mathrm{GL}(Y_a)$, $f \in C_c^\infty(\mathrm{Isom}_{E,c}(X_a, Y_a))$ and $g \in \mathrm{GL}_a$.
- $J^a = 0$ for $a > \min\{t, q\}$, where q is the Witt index of W.

In particular, the bottom piece of the filtration (if nonzero) is:

$$J^t \cong \mathrm{Ind}_{\mathrm{GL}(Y_t)\times H(V_{m-2t})\times P(X_t)}^{\mathrm{GL}(Y_t)\times H(V_{m-2t})\times G(W)}(C_c^\infty(\mathrm{Isom}_{E,c}(X_t,Y_t))\otimes \omega_{V_{m-2t},W_{n-2t}}).$$

Thus for some $0 \le a \le t$, there is a nonzero map

$$\pi^\vee \longrightarrow \mathrm{Hom}_{L(Y_t)}(J^a, \tau\cdot\chi_W|\det|^{-s_1}\otimes\sigma_0).$$

We now consider different possibilities in turn.

- Consider first the case when $a = t$. Then

$$\begin{aligned} 0 \neq\, & \mathrm{Hom}_{L(Y_t)}(J^t, \tau\cdot\chi_W|\det|^{-s_1}\otimes\sigma_0) \\ & = \left(\mathrm{Ind}_{P(X_t)}^{G(W)}\tau^\vee\cdot\chi_V|\det|^{s_1}\otimes\Theta_{V_{m-2t},W_{n-2t}}(\sigma_0)\right)^* \end{aligned}$$

so that one has an equivariant map

$$\mathrm{Ind}_{P(X_t)}^{G(W)}\tau^\vee\cdot\chi_V|\det|^{s_1}\otimes\Theta_{V_{m-2t},W_{n-2t}}(\sigma_0)\longrightarrow\pi.$$

If this is nonzero, then by Frobenius reciprocity and Cansselman's criterion, one has a contradiction to the temperedness of π (since $s_1 > 0$).

- Now suppose that $t = 1$ and $a = 0$. Then

$$\mathrm{Hom}_{L(Y_1)}(J^0, \tau\cdot\chi_W|\det|^{-s_1}\otimes\sigma_0) = \mathrm{Hom}_{L(Y_1)}(\chi_W|-|^{\lambda_1}\otimes\omega_{V_{m-2},W}, \tau\cdot\chi_W|-|^{-s_1}\otimes\sigma_0).$$

This Hom space is nonzero if and only if

$$\tau = \mathbf{1}, \quad s_1 = -\lambda_1 = \frac{m-(n+\epsilon_0)-1}{2} \quad \text{and} \quad \Theta_{V_{m-2},W}(\sigma_0)\neq 0.$$

For $\pi^\vee$ to embed into this Hom space, we need π to be a quotient of $\Theta_{V_{m-2},W}(\sigma_0)$, or equivalently σ_0 is a quotient of $\Theta_{W,V_{m-2}}(\pi)$. This gives the possibility (a) of the proposition.

- The remaining case is $t > a$ and $t > 1$. Note that $t - a \ge 1$. In this case, the non-vanishing of $\mathrm{Hom}_{L(Y_t)}(J^a, \tau\cdot\chi_W|\det|^{-s_1}\otimes\sigma_0)$ is equivalent to

$$\begin{aligned} \mathrm{Hom}_{\mathrm{GL}(Y_{t-a})\times\mathrm{GL}(Y_a')\times H(V_{m-2t})}\Big(& \chi_W|\det_{Y_{t-a}}|^{\lambda_{t-a}}\otimes C_c^\infty(\mathrm{Isom}_{E,c}(X_a,Y_a)) \\ & \otimes\omega_{V_{m-2t},W_{n-2a}}, R_{\overline{Q(Y_{t-a},Y_t)}}(\tau)\cdot\chi_W|\det_{Y_t}|^{-s_1}\otimes\sigma_0\Big)\neq 0. \end{aligned} \tag{7}$$

- Since τ is an irreducible (unitary) discrete series representation of $\mathrm{GL}(Y_t)$, by results of Zelevinsky (see [M3, Pg. 105]) we have

$$R_{\overline{Q(Y_{t-a},Y_t)}}(\tau) = \delta_1|\det|^{e_1}\otimes\delta_2|\det|^{e_2}$$

for some irreducible (unitary) discrete series representations δ_1 and δ_2 of $\mathrm{GL}(Y_{t-a})$ and $\mathrm{GL}(Y_a')$ respectively, and some $e_1, e_2 \in \mathbb{R}$ such that

$$e_1 < e_2 \quad \text{and} \quad e_1\cdot(t-a)+e_2\cdot a = 0. \tag{8}$$

In particular, we must have $e_1 \le 0$. Note that if $a = 0$, then $e_1 = 0$.

- Now, the center of $\mathrm{GL}(Y_{t-a})$ acts on $R_{\overline{Q(Y_{t-a},Y_t)}}(\tau)\cdot\chi_W|\det_{Y_t}|^{-s_1}$ by the character $\omega_{\delta_1}\cdot\chi_W|\det|^{e_1-s_1}$, whereas $\mathrm{GL}(Y_{t-a})$ acts on $\chi_W|\det|^{\lambda_{t-a}}\otimes C_c^\infty(\mathrm{Isom}_{E,c}(X_a,Y_a))\otimes\omega_{V_{m-2t},W_{n-2a}}$ by the character $\chi_W|\det|^{\lambda_{t-a}}$. Here ω_{δ_1} is the central character of δ_1 which is a unitary character. For (7) to hold, we must have $t-a=1$ (so that $a>0$), δ_1 equal to the trivial character and
$$e_1-s_1=\lambda_1=(n+\epsilon_0+1-m)/2\in\frac{1}{2}\mathbb{Z}.$$
This has a chance of holding because both e_1-s_1 and λ_1 are <0.
- Moreover, by results of Zelevinsky (see [M3, Pg. 105]), we must have
$$\tau=\mathrm{St}_t$$
so that
$$e_1=-(t-1)/2,\quad e_2=1/2\quad\text{and}\quad\delta_2=\mathrm{St}_{t-1}. \tag{9}$$
Then we deduce that
$$\pi^\vee\hookrightarrow\left(\mathrm{Ind}_{P(X_a)}^{G(W)}\mathrm{St}_a\chi_V|\det|^{s_1-e_2}\otimes\Theta_{V_{m-2t},W_{n-2a}}(\sigma_0)\right)^\vee. \tag{10}$$
Since π is tempered, we have
$$s_1\le e_2=1/2. \tag{11}$$
- To summarize, we have shown that $a>0$,
$$s_1=e_1-\lambda_1=e_1+\frac{m_0-(n+\epsilon_0)-1}{2}\in\frac{1}{2}\mathbb{Z}$$
and
$$0<s_1\le\frac{1}{2}.$$
Hence $s_1=1/2$ and $\tau=\mathrm{St}_t$. Together with (9), we deduce that
$$t=m-(n+\epsilon_0)-1>1.$$
This gives possibility (b) in Proposition 3.1. Moreover, (10) shows that when (b) holds,
$$\mathrm{Ind}_{P(X_a)}^{G(W)}\chi_V\cdot\mathrm{St}_a\otimes\Theta_{V_{m-2t},W_{n-2a}}(\sigma_0)\twoheadrightarrow\pi,$$
with $a=t-1>0$. Hence there is an irreducible subquotient π_0 of $\Theta_{V_{m-2t},W_{n-2a}}(\sigma_0)$ such that
$$\mathrm{Ind}_{P(X_a)}^{G(W)}\chi_V\cdot\mathrm{St}_a\otimes\pi_0\twoheadrightarrow\pi.$$
Then it follows by Casselman's temperedness criterion that π_0 is itself tempered. Hence, $\chi_V\cdot\mathrm{St}_a$ is contained in the square-integrable support of π.

This completes the proof of Proposition 3.1.

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